Intrinsic Ambipolarity & Edge Gyrokinetics

Peter J. Catto
Plasma Science & Fusion Center, MIT
with Felix Parra, Grisha Kagan & Andrei Simakov

Overview

- δf or local gyrokinetics has proven useful to treat local core turbulence at $k_{\perp}\rho \sim 1$ on turbulent saturation time scales
- BUT there are global or full f subtleties and complications
  - Global axisymmetyric radial electric field in a tokamak
  - Turbulent calculations in the pedestal and SOL
  - Turbulent calculations on transport time scales
- Topics
  - Today: Intrinsic ambipolarity & edge gyrokinetics
  - Next time: Transport time scale gyrokinetics

Terminology

- Full Fokker-Planck (FP) equation:

$$df/dt \equiv \partial f/\partial t + \vec{v}\cdot\nabla f + (Ze/M)(-\nabla\Phi + c^{-1}\vec{v}\times\vec{B})\cdot\nabla_v f = C$$

- Drift kinetic equation (DKE): $E=v^2/2 + Ze\Phi/M$ & $\langle d\mu/dt\rangle_\varphi = 0$

$$\partial f/\partial t + (v_{||}\vec{n} + \vec{v}_d)\cdot\nabla f + (Ze/M)(\partial\Phi/\partial t)\partial f/\partial E = \overline{C}\{f\}$$

- Gyrokinetic equation (GKE): $E=v^2/2 + Ze\Phi/M$ & $\langle d\mu/dt\rangle_\varphi = 0$

$$\partial f/\partial t + (v_{||}\vec{n} + \langle\vec{v}_d\rangle_\varphi)\cdot\nabla_R f + (Ze/M)(\partial\langle\Phi\rangle_\varphi/\partial t)\partial f/\partial E = \langle C\{f\}\rangle_\varphi$$

- Drift kinetic gyroaverage holds $\vec{r}$ or (r, θ, ζ) fixed

- Gyrokinetic gyroaverage holds $\vec{R} = \vec{r} + \Omega^{-1}\vec{v}\times\vec{n}$ fixed

Typical drift kinetic orderings

- Small parameters: 1 >> $k_\perp \rho \sim \delta \sim \rho/L_\perp \sim \nu/\Omega$
  - Assumes $k_\perp L_\perp \sim 1 \sim k_{||} L_{||}$ (allows $L_\perp/\rho >> k_\perp L_\perp >> 1$)
  - Drift kinetics can order $\Omega^{-1}\partial/\partial t \sim \delta$ but typically $\Omega^{-1}\partial/\partial t \sim \delta^2$
  - For zonal flow $e\Phi_\kappa/T \sim \delta$ so $e\partial\Phi_\kappa/\partial t \sim T\Omega\delta^2$
- Global f and Φ: $f \approx f_M \equiv$ Maxwellian & $e\Phi/T \sim 1$ with $e\partial\Phi/\partial t \sim T\Omega\delta^2$
- Fluctuations: $e\Phi_k/T \sim f_k / f_M \sim \delta << 1$
- Allows $\nabla\Phi \sim T/eL_\perp \sim k_\perp\Phi_k$ and $\nabla f_k \sim \nabla f_M$
- Drift ordering: $V_{ExB} \sim \delta v_i << v_i$

Drift kinetics in tokamak core

- Using canonical angular momentum
  $\psi_* = \psi - (Mc/Ze)R^2 \nabla\zeta \cdot \vec{v}$ streamlines derivation of DKE
- Let $f = f_0 + f_1 + f_2 + \ldots$ & gyroaverage at fixed $\vec{r}$
- Lowest order: $\Omega \vec{v} \times \vec{n} \cdot \nabla_v f_0 = -\Omega \partial f_0 / \partial\varphi = 0$
- Lowest order Maxwellian: $\bar{\dot{f}}_0 = v_{||}\vec{n} \cdot \nabla f_0 = C_0\{f_0\} = 0$
  $f_0 = f_M = f_M(\psi, E)$ with $E = v^2/2 + (Ze/M)\Phi$
- But $\psi \approx \psi_*$ suggests using $f = f_* + h$ with
  $f_* = f_M(\psi_*, E) = f_M(\psi, E) + (\psi_* - \psi)\, \partial f_M(\psi, E)/\partial\psi + \ldots$

Axisymmetric B ion drift kinetics

- $\vec{B} = I\nabla\zeta + \nabla\zeta \times \nabla\psi = B\vec{n}$ and electrostatically
  $df_*/dt = c(\partial\Phi/\partial\zeta)\partial f_M/\partial\psi_* + (Ze/M)(\partial\Phi/\partial t)\partial f_M/\partial E$
- Fokker-Planck equation becomes
  $dh/dt + (Ze/M)(\partial\Phi/\partial t)\partial f_M/\partial E + c(\partial\Phi/\partial\zeta)\partial f_M/\partial\psi_* = C\{f_* + h\}$
- Lowest order using $h << f_M$ & $\vec{r}, E, \mu, \varphi$ variables gives
  $-\Omega\partial h_1/\partial\varphi = C_0\{f_M(\psi.E)\} = 0$ with $h = h_1 + h_2 + \ldots$
- Next order: using $\partial f_M/\partial\psi \Rightarrow f_M(Mv^2/2T^2)\partial T/\partial\psi$
  $-\Omega\partial h_2/\partial\varphi + dh_1/dt = C_1\{h_1 - (Mc/Ze)R^2\nabla\zeta\cdot\vec{v}\partial f_M/\partial\psi\}$
  $+ (Zef_M/T)\partial\Phi/\partial t - c(\partial\Phi/\partial\zeta)\partial f_M/\partial\psi_*$
- Gyroaveraging gives desired $O(\delta)$ DKE:
  $$\partial\bar{h}_1/\partial t + v_{||}\vec{n}\cdot\nabla\bar{h}_1 = C_1\{\bar{h}_1 - f_M(Iv_{||}Mv^2/2T^2\Omega)\partial T/\partial\psi\} + (Zef_M/T)\partial\Phi/\partial t - c(\partial\Phi/\partial\zeta)\partial f_M/\partial\psi_*$$

MIT PSFC

Intrinsic ambipolarity

- Use $I v_{\|} \vec{n} \cdot \nabla|_E (v_{\|}/\Omega) = \vec{v}_d \cdot \nabla\psi$ & $\bar{f}_1 = \bar{h}_1 - (I v_{\|}/\Omega)\partial f_M/\partial\psi$ to recover standard $O(\delta)$ form in steady axisymmetric state

$$v_{\|}\vec{n}\cdot\nabla\bar{f}_1 - C_1\{\bar{f}_1\} = -\vec{v}_d\cdot\nabla\psi\,\partial f_M/\partial\psi = -\vec{v}_d\cdot\nabla f_M$$

- First form more convenient in steady axisymmetric state:

$$v_{\|}\vec{n}\cdot\nabla\bar{h}_1 = C_1\{\bar{h}_1 - (I v_{\|}/\Omega) f_M (M v^2/2T - 5/2)\partial \ell n T/\partial\psi\}$$

- Only a $\partial T/\partial\psi$ drive: no $\partial\Phi/\partial\psi$ appears!

- In axisymmetric systems for $k_\perp L_\perp \sim 1$, n & T evolution does not depend on or in any way determine $\langle\Phi\rangle_\theta$ through $O(\delta^2)$

- Intrinsically ambipolar to $O(\delta)$ so far

MIT PSFC

Toroidal angular momentum

- Flux surface averaging source free conservation of total toroidal angular momentum in a quasineutral plasma

$$\left\langle \vec{J}\cdot\nabla\psi \right\rangle_\theta = \frac{c}{V'}\frac{\partial}{\partial\psi}V'\left\langle R^2\nabla\zeta\cdot\overleftrightarrow{\pi}_i\cdot\nabla\psi\right\rangle_\theta + Mc\frac{\partial}{\partial t}\left\langle nR^2\vec{V}\cdot\nabla\zeta\right\rangle_\theta$$

with $\overleftrightarrow{\pi}_i = M\int d^3v f(\vec{v}\vec{v} - v^2\overleftrightarrow{I}/3)$ $\qquad R^2\nabla\zeta\cdot\vec{J}\times\vec{B} = \vec{J}\cdot\nabla\psi$

$$\left\langle X\right\rangle_\theta \equiv (1/V')\oint d\theta d\zeta X/\vec{B}\cdot\nabla\theta$$

- In the steady state must be consistent with charge conservation & Ampere's law

$$(c/4\pi)\left\langle \nabla\psi\cdot\nabla\times\vec{B}\right\rangle_\theta = 0 = \left\langle \vec{J}\cdot\nabla\psi\right\rangle_\theta$$

- Axisymmetric, steady state radial electric field determined by

$$\left\langle R^2\nabla\zeta\cdot\overleftrightarrow{\pi}_i\cdot\nabla\psi\right\rangle_\theta = 0$$

MIT PSFC

Intrinsic ambipolarity to $O(\delta^2)$

- Direct evaluation of $\overset{\leftrightarrow}{\pi}_i = M\int d^3v f(\vec{v}\vec{v} - v^2\overset{\leftrightarrow}{I}/3)$ using $f_1 = \bar{h}_1 - (Mc/Ze)R^2\nabla\zeta\cdot\vec{v}\partial f_M/\partial\psi + O(\delta^2)$ gives $\nabla\psi\cdot\overset{\leftrightarrow}{\pi}_i\cdot\nabla\zeta = 0$ since $\bar{h}_1$ doesn't matter
- Using $\tilde{f}$ to $O(\delta^2)$ can show (notice $\bar{f}$ doesn't matter)
  $$\left\langle R^2\nabla\psi\cdot\overset{\leftrightarrow}{\pi}_i\cdot\nabla\zeta\right\rangle_\theta \to \left\langle (MI/B)\int d^3v f_1 v_\parallel \vec{v}_d\cdot\nabla\psi\right\rangle_\theta + \text{ small} \to 0$$
  (this is non-trivial to prove!)
- $\partial\Phi/\partial\psi$ does not enter to $O(\rho_p\rho/L^2) \sim O(\delta^2)$
- To determine $\partial\Phi/\partial\psi$ need to evaluate $\left\langle R^2\nabla\psi\cdot\overset{\leftrightarrow}{\pi}_i\cdot\nabla\zeta\right\rangle_\theta$ to $O(\rho_p\rho\nu/\Omega L^2)$ neoclassically $\Rightarrow$ need f to $O(\rho_p\rho\nu/\Omega L^2) \sim O(\delta^3)$

Gyrokinetic $\Phi(\psi)$

- Sources
  - Neoclassical + Reynolds stress: $\nabla\Phi \sim T/eL_\perp \sim k_\perp \Phi_k$
  - Zonal flow generated by turbulence: $\nabla_\perp \Phi_k \sim k_\perp T/ek_\perp L_\perp \sim T/eL_\perp$

- Gyrokinetic quasineutrality presumably gets zonal flow contribution correct, but not the neoclassical since gyrokinetic equation only good through $O(\rho_p/L_\perp)$ $C\{f_0\}$

- Gyrokinetics gives correct neoclassical relation between poloidal ion flow & $\partial\Phi/\partial\psi$ since it calculates f to $O(\rho_p/L_\perp)$ [coefficient sensitive to collision operator]

- "Potential" problem if slowly varying part of $\Phi(\psi)$ helps to regulate turbulence since it violates intrinsic ambipolarity

Gyrokinetic implications

- Gyrokinetics is normally only good to $O(\delta)$ for $k_\perp\rho \sim 1$
  - Therefore, it should not determine the axisymmetric, long radial wavelength portion of $\Phi(\psi)$ - zonal flow is short wavelength so ok
  - If it does determine global Φ, then you can't believe it and must make sure your results are insensitive to it!
- Global (or full f) gyrokinetics should not determine the axisymmetric, long wavelength portion of $\Phi(\psi)$ to $O(\delta^2)$
  - Can we check this?
  - How does gyrokinetics get into trouble?

Gyrokinetic orderings

- Small parameters: $\delta = \dfrac{\rho}{L} \sim \dfrac{\omega_*}{\Omega} \sim \dfrac{\nu}{\Omega} << 1$
- f and Φ have $k_\perp \rho \sim 1$ but $k_{||} L \sim 1$
- For $k_\perp L \sim 1$, $e\Phi/T \sim 1$ and $f \approx f_M$ ≡ Maxwellian
- For $k_\perp \rho \sim 1$, $e\Phi_k/T \sim f_k / f_M \sim \delta$
- For general $k_\perp$: $\dfrac{e\phi_k}{T} \sim \dfrac{f_k}{f_M} \sim \dfrac{1}{k_\perp L}$
  - Note $\nabla\Phi \sim T/eL \sim k_\perp \Phi_k$ and $\nabla f_k \sim \nabla f_M$
  - Drift ordering: $V_{ExB} \sim \delta v_i << v_i$

Gyrokinetic details

- Evaluate the GK variables $G = G_0 + G_1 + G_2 + \ldots$ by removing gyrophase dependence order by order using

$$\Omega \partial G_{j+1}/\partial\varphi = dG_j/dt - \langle dG_j/dt \rangle_\varphi$$

- To keep μ an adiabatic invariant must retain the gyrophase independent piece that makes $\langle d\mu/dt \rangle_\varphi = 0$

- The μ variable is only obtained to $O(\delta)$ since it is unclear how to make $\langle d\mu/dt \rangle_\varphi = 0$ to higher order and the lowest order f is presumed to be near Maxwellian

Gyrokinetic variable R

- Define $\mathbf{R}$ such that $d\mathbf{R}/dt = \langle d\mathbf{R}/dt \rangle_\varphi$ + small
  where $d/dt \equiv \partial/\partial t + \mathbf{v}\cdot\nabla - (Ze/M)\nabla\Phi \cdot \nabla_v - \Omega \partial/\partial\varphi$
  - with $\langle \ldots \rangle_\varphi \equiv$ gyroaverage at fixed $\mathbf{R}$
- $\mathbf{R} = \mathbf{r} + \mathbf{R}_1 + \mathbf{R}_2$, $\mathbf{R}_1 = O(\delta L)$ and $\mathbf{R}_2 = O(\delta^2 L)$
  - To first order $\dot{\vec{R}} \cong \dot{\vec{r}} + \dot{\vec{R}}_1 \cong \vec{v} - \Omega\, \partial \vec{R}_1/\partial\varphi$
- Imposing $d\mathbf{R}/dt = \langle d\mathbf{R}/dt \rangle_\varphi$ to first order

$$\dot{\vec{R}} \cong \vec{v} - \Omega\, \partial \vec{R}_1/\partial\varphi = \langle \dot{\vec{R}} \rangle = \langle \vec{v} \rangle = v_{||}\vec{n}$$

Then $\vec{R}_1 = \Omega^{-1} \int d\varphi\, (\dot{\vec{r}} - \langle \dot{\vec{r}} \rangle) = \Omega^{-1} \vec{v} \times \vec{n}$

Similarly $\vec{R}_2 = \Omega^{-1} \int d\varphi\, (\dot{\vec{r}} + \dot{\vec{R}}_1 - \langle \dot{\vec{r}} + \dot{\vec{R}}_1 \rangle)$

Gyrokinetic validity

- GKE normally derived using $\vec{R} = \vec{r} + \Omega^{-1}\vec{v}\times\vec{n}$ for which $\langle d\vec{R}/dt\rangle_\varphi = \vec{v}_d + u\vec{n}$ and $d\vec{R}/dt - \langle d\vec{R}/dt\rangle_\varphi \sim \delta v_i \sim \vec{v}_d \sim v_p$

- Therefore
$$df/dt - \langle df/dt\rangle_\varphi = -\Omega\partial\tilde{f}/\partial\varphi + (\dot{\vec{R}} - \langle\dot{\vec{R}}\rangle_\varphi)\cdot\nabla f + \ldots$$
gives
$$\tilde{f} \sim \Omega^{-1}\int d\varphi(\dot{\vec{R}} - \langle\dot{\vec{R}}\rangle_\varphi)\cdot\nabla f_M + \ldots \sim \delta^2 f_M$$

- GKE normally gives $f(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{R}, E, \mu, t) + O(\delta^2)$ error even though GKs good for arbitrary $k_\perp\rho$: only good to $O(\delta)$

- Desire GK variables to $O(\delta^2)$ at $k_\perp L\sim1$ with leading collisional gyrophase dependence [it can be evaluated to $O(\delta^2)$] : then can evaluate $f(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{R}, E, \mu, t) + O(\delta^3)$

MIT PSFC

Gyrokinetic equation

- Variables G $\Rightarrow \vec{R}$, E = $v^2/2 + Ze\Phi/M$, μ, φ
- Changing variables, Fokker-Planck equation becomes:

$$\frac{\partial f}{\partial t} + \dot{\vec{R}} \cdot \nabla_R f + \dot{\varphi}\frac{\partial f}{\partial \varphi} + \dot{\mu}\frac{\partial f}{\partial \mu} + \frac{Ze}{M}\frac{\partial \Phi}{\partial t}\frac{\partial f}{\partial E} = C\{f\}$$

- Variables G constructed so $dG_j/dt - \langle dG_j/dt \rangle_\varphi$ + small.
- Leading φ dependence from $-\Omega \partial \tilde{f}/\partial \varphi = C\{f\} - \langle C\{f\} \rangle_\varphi$
- Gyroaveraging at fixed $\vec{R}$,E,μ (recall $\langle d\mu/dt \rangle_\varphi = 0$) gives

$$\frac{\partial f}{\partial t} + \dot{\vec{R}} \cdot \nabla_R f + \frac{Ze}{M}\frac{\partial \langle \Phi \rangle_\varphi}{\partial t}\frac{\partial f}{\partial E} = \langle C\{f\} \rangle_\varphi$$

to $O(\delta)$ when we ignore $O(\delta^2)$ from f & variable change
Here $\dot{\vec{R}} = u\vec{n}(\vec{R}) + \vec{v}_d$ with u parallel velocity & $\vec{v}_d$ drift velocity

Alternate gyrokinetic forms

- Numerically often easier to use kinetic energy K or parallel velocity $u = [2(K - \mu B(R)]^{1/2}$
- Using kinetic energy $K = v^2/2 + Ze(\Phi - \langle \Phi \rangle_\varphi)/M + \ldots$

$$\frac{\partial f}{\partial t} + (u\vec{n} + \vec{v}_d) \cdot [\nabla_R f - \frac{Ze}{M} \nabla_R (\Phi - \langle \Phi \rangle_\varphi) \frac{\partial f}{\partial K}] = \langle C\{f\} \rangle_\varphi$$

$$\vec{v}_d = \vec{v}_M - (c/B) \nabla_R \langle \Phi \rangle_\varphi \times \vec{n} \qquad \langle \Phi \rangle_\varphi = (2\pi)^{-1} \oint d\varphi \, \Phi\left(\vec{R} - \vec{R}_1 - \vec{R}_2, t\right)$$

- Also possible to write in conservative form
- Will use the K form for discussion of quasineutrality

Quasineutrality (QN): $Zn_i = n_e$

- Taylor expanding for ions to $O(\delta)$

$$f_i(\vec{R},K,\mu,t) \cong f_i(\vec{r} + \Omega^{-1}\vec{v} \times \vec{n}, v^2/2, \mu_0, t) - \frac{Ze}{T_i}(\Phi - \langle\Phi\rangle_\varphi) f_M$$

- For electrons (ITG ordering), $n_e = n_0 + \dfrac{en_0}{T_e}(\Phi - \langle\Phi\rangle_\theta)$
  - with $\langle\ldots\rangle_\theta \equiv$ flux surface average
- For $k_\perp \rho \sim 1$ and to $O(\delta n)$

$$\frac{Z^2 e}{T_i}\int d^3v (\Phi - \langle\Phi\rangle_\varphi) f_M + \frac{en_0}{T_e}(\Phi - \langle\Phi\rangle_\theta) = Z\hat{N}_i - n_0$$

  - with $\hat{N}_i = \int d^3v f_i(\vec{r} + \Omega^{-1}\vec{v} \times \vec{n}, v^2/2, \mu_0, t)$
- For $k_\perp L \sim 1$ & axisymmetry, need QN independent of $\langle\Phi\rangle_\theta$ to $O(\delta^2 n)$ due to intrinsic ambipolarity!

θ - pinch solution to $O(\delta^2)$

- Use Krook $C\{f\} = -\nu\,(f - f_M)$ and $\langle \ldots \rangle_\varphi$ to $O(\delta^2 f_M)$

$$f_i = \langle f_M \rangle = f_{M0}\left[1 - \frac{Mv_\perp^2}{2p_i}\nabla\cdot\left(\frac{cn_i}{B\Omega}\nabla_\perp\Phi\right) + \left(2 - \frac{Mv_\perp^2}{2T_i}\right)\frac{Mc^2}{2T_iB^2}\left|\nabla_\perp\Phi\right|^2 + \ldots\right]$$

with

$$f_M = n_i\left(\frac{M}{2\pi T_i}\right)^{3/2}\exp\left(-\frac{M(\vec{v} - \vec{V}_i)^2}{2T_i}\right), \qquad f_{M0} = n_i\left(\frac{M}{2\pi T_i}\right)^{3/2}\exp\left(-\frac{MK}{T_i}\right)$$

- To find $\langle \Phi \rangle_\theta$, need QN to $O(\delta^2 n)$ (valid for any n_e)

$$-\nabla\cdot\left(\frac{Zcn_i}{B\Omega}\nabla_\perp\Phi\right) + \frac{Zn_iMc^2}{2T_iB^2}\left|\nabla_\perp\Phi\right|^2 = Z\hat{N}_i - n_e$$

with

$$\hat{N}_i = \int d^3v\,(1 + \frac{v_\parallel}{\Omega}\vec{n}\cdot\nabla\times\vec{n})f_i(\vec{r}, v^2/2, \mu_0) + (\overleftrightarrow{I} - \vec{n}\vec{n}) : \frac{\nabla\nabla p_i}{2M\Omega^2}$$

- Yellow $O(\delta^2)$ terms in f_i result in exact cancellation: 0 = 0

θ - pinch and tokamak potential

- Any global axisymmetric, long wavelength $\langle\Phi\rangle_\theta$ should satisfy QN to $O(\delta^2)$
- Typically $\delta^2 f_M$ terms MISSING in QN $\Rightarrow$ giving a non-physical $\langle\Phi\rangle_\theta$
- Even with full $\delta^2 f_M$ terms, $\langle\Phi\rangle_\theta$ must be undetermined: any initial guess works!
- Only need f_i to $O(\delta^2 f_M)$ if use $\left\langle R^2 \nabla\psi \cdot \vec{\vec{\pi}}_i \cdot \nabla\zeta \right\rangle_\theta = 0$

$$Ze\frac{\partial\Phi}{\partial r} + \frac{1}{n_i}\frac{\partial p_i}{\partial r} = rB\int dr \frac{3}{rB}\frac{\partial T_i}{\partial r}\left[\frac{5}{3}\frac{\partial}{\partial r}\ln B - \frac{\partial}{\partial r}\ln\left(\frac{p_i}{r}\frac{\partial T_i}{\partial r}\right)\right] \sim \frac{\partial T_i}{\partial r}$$

- Same in tokamaks: $\left\langle R^2 \nabla\psi \cdot \vec{\vec{\pi}}_i \cdot \nabla\zeta \right\rangle_\theta = 0$ gives $\langle\Phi\rangle_\theta$ at $O(\delta^3 p)$ for f_i to $O(\delta^2 f_M)$

Bottom line!

Gyrokinetic quasineutrality works for $k_\perp \rho \sim 1$

BUT it cannot determine the self-consistent axisymmetric electric field in long wavelength limit [see Felix Parra for more details]

Need an alternative equation for $k_\perp L \sim 1$: probably a moment approach similar to drift kinetics (next time)

Edge gyrokinetics

- Simplification: electrostatic gyrokinetics
  $\vec{B}$ slowly varying and time independent

- To handle $\rho_p \sim L_\perp$ conveniently replace radial gyrokinetic variable by canonical angular momentum

$$\psi_* = \psi - (Mc/Ze)R^2\vec{v}\cdot\nabla\zeta = \psi + \Omega^{-1}\vec{v}\times\vec{n}\cdot\nabla\psi - (Iv_\parallel/\Omega)$$

$C\{f_0\}$

- Variables $\vec{R} \to \psi_*, \vartheta_*, \zeta_*$, E_* and μ_* defined with $d\vec{R}/dt$, dE_*/dt and $d\mu_*/dt$ independent of gyrophase φ

  $\Rightarrow$ fast gyromotion absorbed in GK variables
  - $d/dt \equiv$ Vlasov operator
  - Gyrophase dependence from $-\Omega\partial\tilde{f}/\partial\varphi = C\{f\} - \langle C\{f\}\rangle_\varphi$

- Need to find $f(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{R}, E, \mu, t) + \tilde{f}$ & $\tilde{f} \sim O(f_M \delta\nu/\Omega)$

Exact isothermal ion solution

- An exact solution to the ion kinetic equation exists in the isothermal limit when ion-electron collisions are neglected
- Function of total energy & canonical angular momentum to make Vlasov operator vanish

$$f_0 = f_0(\psi_*, E)$$

- Must be Maxwellian to make ion-ion collision operator vanish

$$f_0 = f_M(\psi_*, E)$$

- Therefore (T, η, ω constants)

$$f_M(\psi_*, E) = \eta (M/2\pi T)^{3/2} \exp(-ME/T - e\omega\psi_*/cT)$$

$$f_M(\psi_*, E) = n (M/2\pi T)^{3/2} \exp[-M(\vec{v} - \omega R^2 \nabla\zeta)^2/2T]$$

Axisymmetric steady state edge GKs

- Conveniently retains finite poloidal gyroradius effects
- Preserves ψ_* and total energy = E as constants of the motion in steady state axisymmetric limit to exactly recover isothermal limit
- Axisymmetric steady state: $\dot{\vartheta}_* \partial f_0 / \partial \vartheta_* = \langle C\{f_0\} \rangle_\varphi$
- In axisymmetric steady state can prove the ion temperature must vary slowly compared to a poloidal ion gyroradius
  - $\rho_p \to 0$: $\langle \int d^3v \ell n f_0 \langle C\{f_0\} \rangle_\varphi \rangle_\vartheta = 0$ gives $f_0 = f_M$ in core
  - $\rho_p \to L_\perp$: $\int_{ped} d^3r \int d^3v \ell n f_0 \langle C\{f_0\} \rangle_\varphi = 0$ with $\partial f_0 / \partial \vartheta_* = 0$ gives rigidly rotating Maxwellian $f_0 = f_0(\psi_*, E) = f_M$ so $\rho_p \nabla \ell n T << 1$ when $\rho_p \nabla \ell n n \sim 1$ in **banana** regime

Pedestal pressure balance

- Assume pedestal flow subsonic (as in C-Mod): $|\vec{V}_i| << v_i$
- Since banana T variation slow: $\vec{V}_i \approx \omega_i R^2 \nabla\zeta$ where

$$\omega_i = -c\left(\frac{d\Phi}{d\psi} + \frac{1}{en}\frac{dp_i}{d\psi}\right) \approx 0 \quad \text{and} \quad \frac{cT_i R}{v_i en}\frac{dn}{d\psi} \sim \frac{\rho_p}{L_\perp} \sim 1$$

- Ions electrostatically confined: $\frac{d\Phi}{d\psi} \approx -\frac{1}{en}\frac{dp_i}{d\psi} \approx -\frac{T_i}{en}\frac{dn}{d\psi}$
- Electrons magnetically confined: $\vec{V}_e = \omega_e R^2 \nabla\zeta + u_e(\psi)\vec{B}$

$$\omega_e = -c\left(\frac{d\Phi}{d\psi} - \frac{1}{en}\frac{dp_e}{d\psi}\right) \approx \frac{c}{en}\frac{d(p_e + p_i)}{d\psi} \quad \text{and} \quad \frac{\omega_e R}{v_i} \sim \frac{\rho_p}{L_\perp} \sim 1$$

- Not clear what establishes a $\rho_p \sim L$ pedestal
- Another reason sonic ordering inappropriate!

Edge zonal flow GKE

- Subsonic zonal flow gyrokinetic equation (axisymmetric):

  Let $f_0 = f_M(\psi_*, E; T(\psi)) + h(\psi_*, \vartheta_*, \zeta_*, E, \mu_*, t)$

  then

$$\frac{\partial h}{\partial t} + \dot{\vartheta}_* \frac{\partial h}{\partial \vartheta_*} - \left\langle C_{ii}^{\ell}\left\{ g - \frac{I v_{\|}}{\Omega} \frac{M v^2}{2T^2} \frac{\partial T}{\partial \psi} f_M \right\} \right\rangle_{\varphi} = -\frac{e}{T} \frac{\partial \Phi_*}{\partial t} f_M J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) e^{iQ}$$

  with $\Phi(\psi,t) = \Phi_0(t)\exp[iS(\psi)]$ & $\Phi_*(\psi_*,t) = \Phi_0(t)\exp[iS(\psi_*)]$

  Taylor expanding S leads to $Q = S' I v_{\|}/\Omega$

  Same as Hinton & Rosenbluth

  Can retain finite orbit effects in $\dot{\vartheta}_*$ and Φ_* [see Kagan for more details]

Full f edge gyokinetic equation

- Full electrostatic full f gyrokinetic equation:

$$\frac{\partial f}{\partial t} + c\frac{\partial\langle\Phi\rangle_\varphi}{\partial\zeta_*}\frac{\partial f}{\partial\psi_*} + \dot{\vartheta}_*\frac{\partial f}{\partial\vartheta_*} + \dot{\zeta}_*\frac{\partial f}{\partial\zeta_*} + \frac{e}{M}\frac{\partial\langle\Phi\rangle_\varphi}{\partial t}\frac{\partial f}{\partial E} = \langle C\{f\}\rangle_\varphi$$

with gyroaverage holding ψ_* fixed

$$\dot{\vartheta}_* = (v_\parallel^*\vec{n}_* + \vec{v}_d)\cdot(\nabla\vartheta)_* + (Iv_\parallel/\Omega)\partial(v_\parallel\vec{n}\cdot\nabla\vartheta)/\partial\psi$$

$$\dot{\zeta}_* = (v_\parallel^*\vec{n}_* + \vec{v}_d)\cdot(\nabla\zeta)_* + (Iv_\parallel/\Omega)\partial(v_\parallel\vec{n}\cdot\nabla\zeta)/\partial\psi$$

$$\vec{v}_d = \vec{v}_m + (c/B)\vec{n}\times\langle\nabla\Phi\rangle_\varphi$$

Can use different energy variable or parallel velocity

Edge gyrokinetic subtleties

- In a subsonic $\rho_p \sim L_\perp$ with global $\Phi_0(\psi)$ satisfying

$$e\partial\Phi/\partial\psi = -T_i\partial\ell n\, n/\partial\psi + O(\partial\ell n\, T_i/\partial\psi) \approx -T_i\partial\ell n\, n/\partial\psi$$

  - Zonal flow $\Phi_1(\psi,t)$ can have $k_\perp\rho_p > 1 >> k_\perp\rho$
  - Poloidal ExB drift can be significant: $\dot{\vartheta}_* \approx (v_{||} + cI\Phi_0'/B)/qR$ since $cI\Phi_O'/B \approx -(cIT/eBn)\partial n/\partial\psi \sim v_i\rho_p/L_\perp \sim v_i$
  - Poloidal ExB and orbit squeezing due to Φ_O'' alter zonal flow!

- Poloidal ExB and orbit squeezing effects on neoclassical
  - Use $f = f_* + h$ with $f_* = f_M(\psi_*,E)$ and expand T_i about ψ

$$\frac{(v_{||} + cI\Phi_0'/B)}{qR}\frac{\partial \bar{h}_1}{\partial\theta} = C_1\{\bar{h}_1 - f_M \frac{IMv^2}{2T\Omega}(v_{||} + \frac{cI\Phi_0'}{B})\frac{\partial\ell nT}{\partial\psi}\}$$

  - Transit average of C_1 involves $cI\Phi_O'/B \approx -(cIT/eBn)\partial n/\partial\psi$ altering ion flow and heat flux, but not altering ion = electron particle xport

Discussion

- Gyrokinetics should be made to satisfy intrinsic ambipolarity
  - Can only turbulently evolve n & T; GKs can't evolve the full Φ
- Edge gyrokinetics conveniently formulated using canonical angular momentum as radial variable
  - In the banana regime radial ion temperature variation must be slow compared to the poloidal ion gyroradius
  - Subsonic pedestal: ions electrostatic & electrons magnetic
  - Zonal flow in pedestal different than in core
  - Also works on axis and in internal transport barrier
- Next time: Hybrid gyrokinetic-fluid description
  - Density, temperatures, potential, ion flow, current evolved by conservation equations
  - Gyrokinetic f only used for closure and (almost) anyones will do!