Multiple Integrals and Vector Calculus

Oxford Physics
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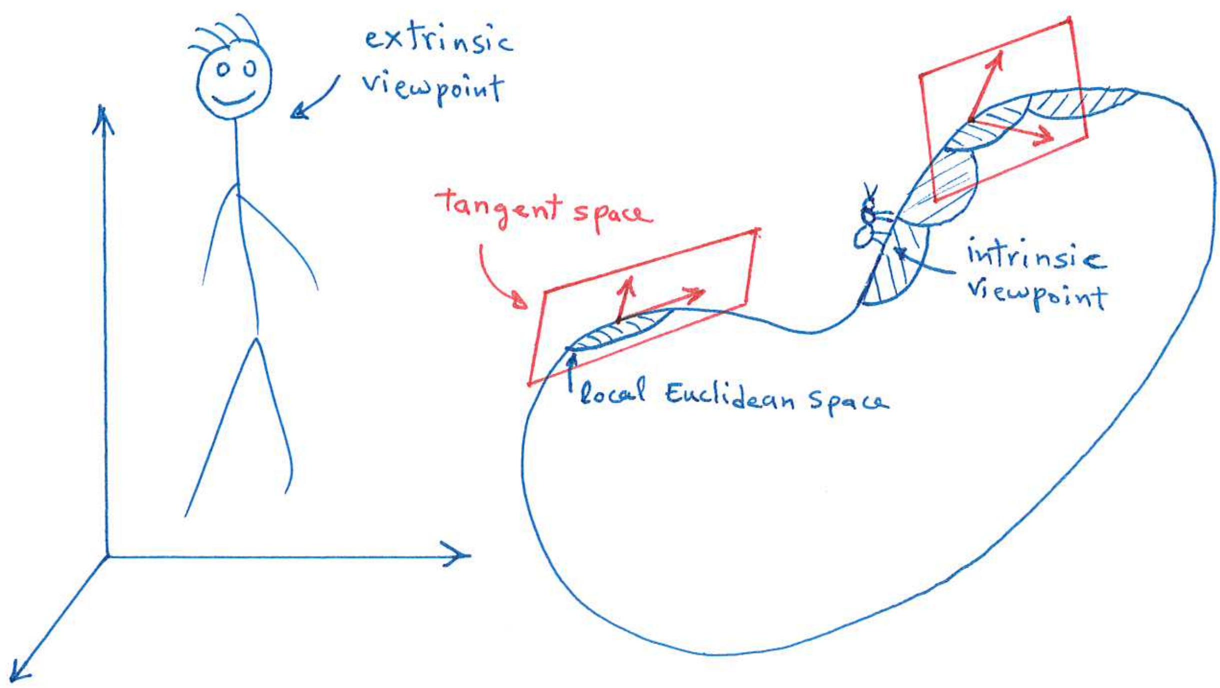
1. Introduction and Outlook

Why "vector calculus and multiple integrals"?

To develop mathematical tools that allow us to study the properties of physical (scalar, vectorial, tensorial) quantities, and determine how they are generated from sources, how they vary in space and evolve with time, and how they comply with boundary conditions.

1.1. Elementary Differential Geometry

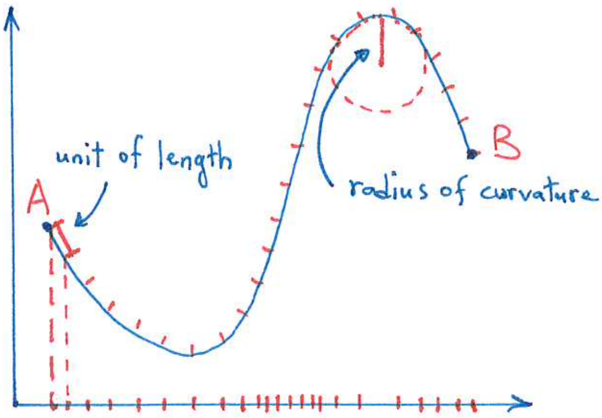
To have a better perspective of the scope of the mathematical tools we are going to develop and use, and how they will be helpful in physics, we introduce a number of concepts from the mathematical field of differential geometry. We are used to a Euclidean position space in which we can define the position of each point as a vector, and use the vector algebra to perform operations and manipulations with these vectors. In general, the space (or the spacetime in the case of Einstein's theory of general relativity) might only be locally describable as a Euclidean space due to curvature. To characterize the geometry of such a space, which is called a manifold, we will need a patchwork of such local descriptions,



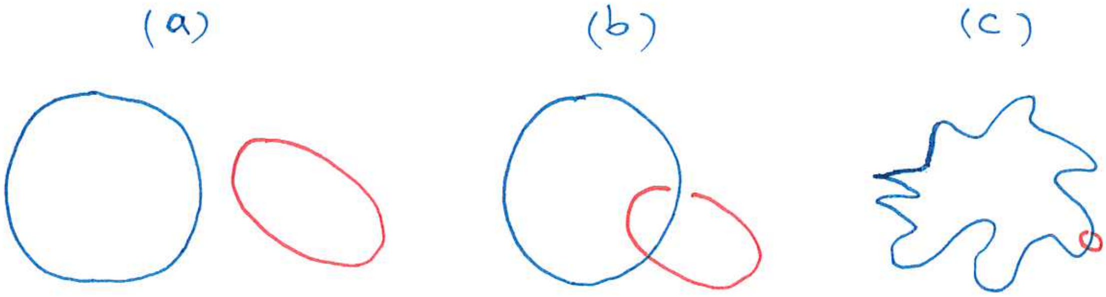
Using this intrinsic description of the geometry, we can naturally define a local vector space at each point, that is called the tangent space, using the underlying local Euclidean space structure. However, vectors belonging to the tangent spaces corresponding to different locations cannot be easily compared, and a global notion of a vector space will not in general be applicable. Vectors from neighbouring tangent spaces can be compared via parallel transport, although a vector will not necessarily be parallel to itself if parallel-transported along a closed trajectory; the difference is a measure of the intrinsic curvature (try parallel-transporting a pen around your head along a closed trajectory, and you'll see!)

An extrinsic description of the same geometry can be achieved by embedding the manifold in a higher dimensional Euclidean space. This allows us to compare vectors belonging to (the tangent spaces at) different locations, in a global scheme.

If a manifold has a metric, which is a measure of length or a ruler, we can quantify geometric characteristics, such as distance



between different points and radius of curvature. A manifold that is embedded in a higher dimensional Euclidean space, has an induced-metric, which is defined using the original Euclidean metric (normal distance). Note that the shortest distance between points (A & B in the above figure) along an embedded manifold could be different from the shortest Euclidean distance. Also, a projection factor leads to a pattern of distances on any axis, for points that are equidistant on the manifold (or alternatively equidistant points on an axis do not correspond to equidistant points on the manifold).

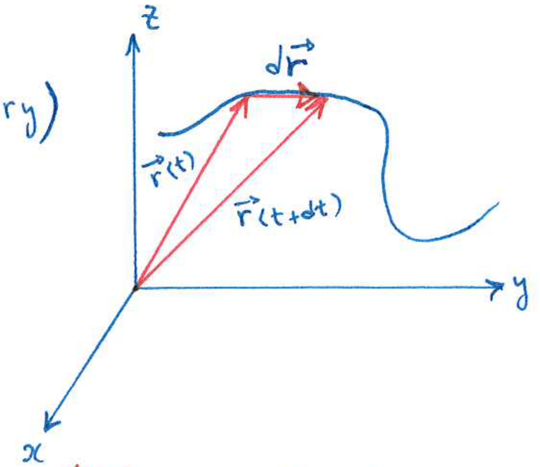


(a) & (b) are:
- geometrically similar (though not identical)
- topologically different

(b) & (c) are:
- geometrically different
- topologically identical

Geometric characteristics, which are sensitive to distances and length scales, are to be contrasted with topological characteristics, which are not. Two situations are topologically identical if they can be continuously deformed onto one another. Distinct topological states require discontinuous transformations, such as cutting, so that they can be mapped to each other.

1.2. Geometry of curves



A curve in 3D (e.g. a path or trajectory)

$$\vec{r}(t) = x(t)\,\hat{e}_x + y(t)\,\hat{e}_y + z(t)\,\hat{e}_z$$

$\dfrac{d\vec{r}}{dt}$ is tangent to the curve.

length of a segment

$$ds^2 = |d\vec{r}\,|^2 = d\vec{r}\cdot d\vec{r} = dx^2 + dy^2 + dz^2$$

Euclidean metric

$$ds = dt\,\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

The parameter t could be time if the curve is a trajectory.

The parameter that defines a curve can be chosen with some freedom.

For example, we can choose one of the coordinates, say x, to be the parameter,

$$\vec{r}(x) = x\hat{e}_x + y(x)\,\hat{e}_y + z(x)\,\hat{e}_z$$

$$ds = dx\,\sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} \qquad \text{(induced metric)}$$

or the arc length s: $\vec{r}(s) = x(s)\,\hat{e}_x + y(s)\,\hat{e}_y + z(s)\,\hat{e}_z$

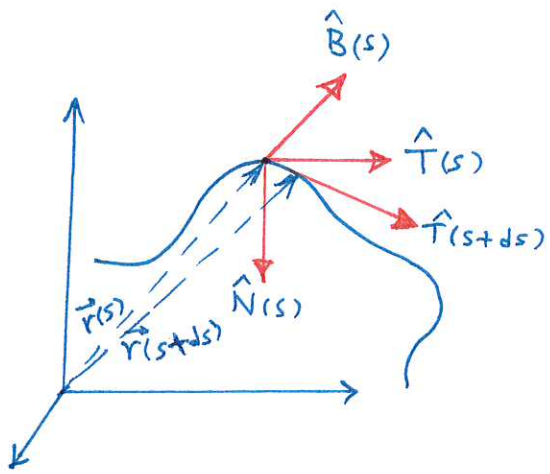
which makes the tangent vector a unit vector:

$$\hat{T}(s) = \frac{d\vec{r}(s)}{ds} \quad ; \quad \hat{T}(s)^2 = 1$$

The direction of $\hat{T}(s)$ changes due to curvature, along a perpendicular direction $\left(\hat{T}^2 = 1 \Rightarrow \hat{T}\cdot\dfrac{d\hat{T}}{ds} = 0 \Rightarrow \dfrac{d\hat{T}}{ds} \perp \hat{T} \right)$

represented by a unit vector $\hat{N}(s)$ called the principal normal vector.

The magnitude of the rate of change of the tangent defines the curvature κ ($\kappa = \frac{1}{\rho}$; ρ = radius of curvature):



$$\boxed{\frac{d\hat{T}(s)}{ds} = \frac{d^2\vec{r}(s)}{ds^2} = \kappa\,\hat{N}(s)}$$

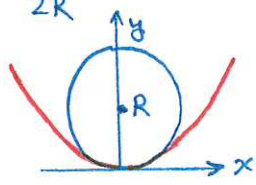
Defining the binormal vector $\hat{B}(s) = \hat{T}(s) \times \hat{N}(s)$ completes the construction of a comoving orthonormal basis set $\{\hat{T}(s), \hat{N}(s), \hat{B}(s)\}$.

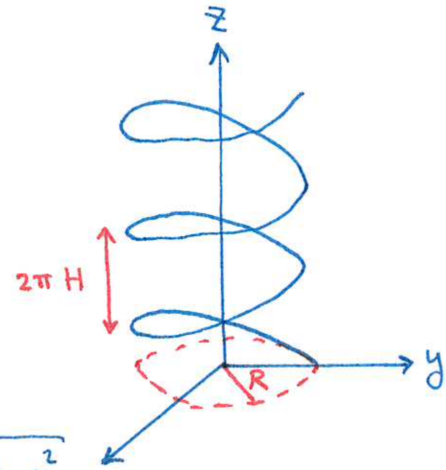
If a curve is planar (i.e. it fits entirely in a plane) $\hat{B}(s)$ is constant, and for non-planar curve $\frac{d\hat{B}}{ds}$ gives a measure of the torsion τ. Since $\frac{d\hat{B}}{ds} \perp \hat{B}$ and $\frac{d\hat{B}}{ds} \perp \hat{T}$ ($\hat{B}\cdot\hat{T} = 0 \Rightarrow \frac{d\hat{B}}{ds}\cdot\hat{T} + \hat{B}\cdot\underbrace{\overbrace{\frac{d\hat{T}}{ds}}^{\kappa\hat{N}}}_{0} = 0$), we have $\boxed{\frac{d\hat{B}}{ds} = -\tau\,\hat{N}(s)}$.

Using the orthonormality of $\{\hat{T}, \hat{N}, \hat{B}\}$, and the above boxed equations we can obtain $\boxed{\frac{d\hat{N}}{ds} = \tau\,\hat{B} - \kappa\,\hat{T}}$. These are called the Frenet-Serret equations.

Exercise: Show that for any trajectory $\vec{r}(t)$, the acceleration can be written as $\vec{a}(t) = \frac{dv}{dt}\hat{T}(t) + \frac{v^2}{\rho}\hat{N}(t)$ (tangential + centripetal).

Exercise: Show that the radius of curvature of the parabola $y = \frac{x^2}{2R}$ at $x=0$ is R, by starting from the equation that describes the following circle and expanding in $x \ll R$.





Example: A helix of (projected) base radius R and helical pitch $2\pi H$.

$$\vec{r}(\theta) = (R\cos\theta,\ R\sin\theta,\ H\theta)$$

$$ds = d\theta\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2} = d\theta\sqrt{R^2\sin^2\theta + R^2\cos^2\theta + H^2}$$

$$ds = d\theta\sqrt{R^2+H^2} \longrightarrow s = \theta\sqrt{R^2+H^2} \longrightarrow \theta = \frac{s}{\sqrt{R^2+H^2}}$$

contour length in a full turn $L = 2\pi\sqrt{R^2+H^2}$

$$\vec{r}(s) = \left(R\cos\left(\frac{s}{\sqrt{R^2+H^2}}\right),\ R\sin\left(\frac{s}{\sqrt{R^2+H^2}}\right),\ \frac{H}{\sqrt{R^2+H^2}}\,s\right)$$

$$\hat{T}(s) = \frac{d\vec{r}}{ds} = \left(-\frac{R}{\sqrt{R^2+H^2}}\sin\left(\frac{s}{\sqrt{R^2+H^2}}\right),\ \frac{R}{\sqrt{R^2+H^2}}\cos\left(\frac{s}{\sqrt{R^2+H^2}}\right),\ \frac{H}{\sqrt{R^2+H^2}}\right) \qquad \hat{T}^2 = 1 \checkmark$$

$$\frac{d\hat{T}}{ds} = \left(-\frac{R}{R^2+H^2}\cos\left(\frac{s}{\sqrt{R^2+H^2}}\right),\ -\frac{R}{R^2+H^2}\sin\left(\frac{s}{\sqrt{R^2+H^2}}\right),\ 0\right)$$

$$K = \left|\frac{d\hat{T}}{ds}\right| = \frac{R}{R^2+H^2} \longrightarrow \rho = \frac{1}{K} = R + \frac{H^2}{R} > R$$

$$\hat{N}(s) = \left(-\cos\left(\frac{s}{\sqrt{R^2+H^2}}\right),\ -\sin\left(\frac{s}{\sqrt{R^2+H^2}}\right),\ 0\right) \qquad \hat{N}^2 = 1 \checkmark$$

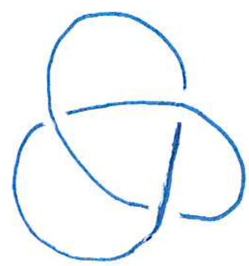
$$\hat{B}(s) = \hat{T}\times\hat{N} = \left(\frac{H}{\sqrt{R^2+H^2}}\sin\left(\frac{s}{\sqrt{R^2+H^2}}\right),\ -\frac{H}{\sqrt{R^2+H^2}}\cos\left(\frac{s}{\sqrt{R^2+H^2}}\right),\ \frac{R}{\sqrt{R^2+H^2}}\right) \qquad \hat{B}^2 = 1 \checkmark$$

$$\frac{d\hat{B}}{ds} = \frac{H}{R^2+H^2}\left(\cos\left(\frac{s}{\sqrt{R^2+H^2}}\right),\ \sin\left(\frac{s}{\sqrt{R^2+H^2}}\right),\ 0\right)$$

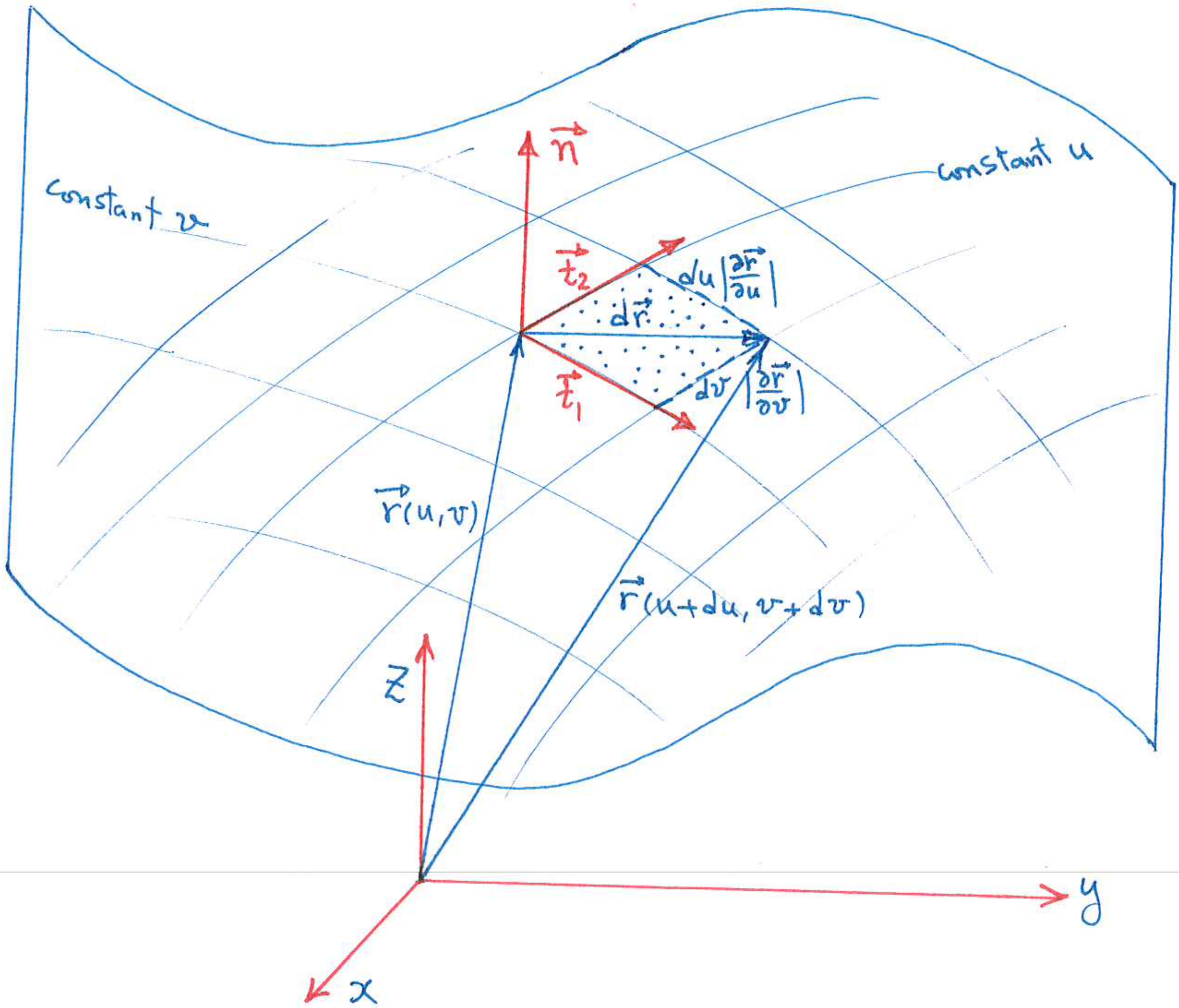
$$\tau = \frac{H}{R^2+H^2}$$

Exercise: Try picturing the 3D shape of a trefoil knot, defined via

$$\begin{cases} x(\theta) = (2+\cos 3\theta)\cos 2\theta \\ y(\theta) = (2+\cos 3\theta)\sin 2\theta \\ z(\theta) = \sin 3\theta \end{cases}$$



1.3. Geometry of Surfaces



A surface in 3D (parametrized by u and v)

$$\vec{r}(u,v) = x(u,v)\,\hat{e}_x + y(u,v)\,\hat{e}_y + z(u,v)\,\hat{e}_z$$

tangent vectors: $\vec{t}_1(u,v) = \vec{t}_u(u,v) = \frac{\partial \vec{r}}{\partial u}$, $\vec{t}_2(u,v) = \vec{t}_v = \frac{\partial \vec{r}}{\partial v}$

normal vector: $\vec{n} = \vec{t}_1 \times \vec{t}_2 = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$

unit vectors: $\hat{t}_1 = \frac{\partial \vec{r}/\partial u}{|\partial \vec{r}/\partial u|}$, $\hat{t}_2 = \frac{\partial \vec{r}/\partial v}{|\partial \vec{r}/\partial v|}$, $\hat{n} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|}$

A displacement along the surface: $d\vec{r} = \frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv$

Euclidean metric

Line element: $ds^2 = d\vec{r}\cdot d\vec{r} = \left(\frac{\partial \vec{r}}{\partial u}du + \frac{\partial \vec{r}}{\partial v}dv\right)\cdot\left(\frac{\partial \vec{r}}{\partial u}du + \frac{\partial \vec{r}}{\partial v}dv\right)$

$$ds^2 = \left(\frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial u}\right)(du)^2 + 2\left(\frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial v}\right)du\,dv + \left(\frac{\partial \vec{r}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial v}\right)(dv)^2$$

non-examinable

$$ds^2 = g_{11}(du)^2 + 2g_{12}\,du\,dv + g_{22}(dv)^2 \qquad \text{(induced metric)}$$

$$\left[\text{or } ds^2 = g_{uu}(du)^2 + 2g_{uv}\,du\,dv + g_{vv}(dv)^2\right]$$

$$g_{11} = g_{uu} = \frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial u} = \vec{t}_1\cdot\vec{t}_1$$

$$g_{12} = g_{21} = g_{uv} = g_{vu} = \frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial v} = \vec{t}_1\cdot\vec{t}_2$$

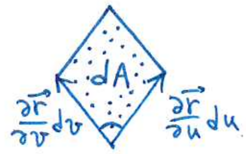
$$g_{22} = g_{vv} = \frac{\partial \vec{r}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial v} = \vec{t}_2\cdot\vec{t}_2$$

$$\underline{g} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

or in shorthand: $g_{ij} = \vec{t}_i\cdot\vec{t}_j \quad (i,j = 1,2)$ so-called metric tensor

Area element: $dA = \left|\left(\frac{\partial \vec{r}}{\partial u}du\right)\times\left(\frac{\partial \vec{r}}{\partial v}dv\right)\right|$ area of the spotty region

$$dA = \left|\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right|du\,dv = |\vec{t}_1\times\vec{t}_2|\,du\,dv = |\vec{n}|\,du\,dv$$



A vector area element can be defined, as a vector along $\hat{n}$:

$$d\vec{S} = \hat{n}\,dA = \vec{n}\,du\,dv = \frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\,du\,dv$$

There is freedom in choosing the two parameters that describe a surface. One way to describe a surface in 3D is to introduce a constraint between x, y, and z: $\phi(x,y,z) = 0$.

An example is the so-called Monge representation that uses a height function: $\vec{r}(x,y) = x\,\hat{e}_x + y\,\hat{e}_y + h(x,y)\,\hat{e}_z$

or $\vec{r}(x,y) = (x, y, h(x,y))$

which corresponds to $\phi(x,y,z) = z - h(x,y) = 0$.

$$\vec{t}_1 = \frac{\partial \vec{r}}{\partial x} = \hat{e}_x + \frac{\partial h}{\partial x}\,\hat{e}_z \qquad \vec{t}_1 = \left(1, 0, \frac{\partial h}{\partial x}\right)$$

$$\vec{t}_2 = \frac{\partial \vec{r}}{\partial y} = \hat{e}_y + \frac{\partial h}{\partial y}\,\hat{e}_z \qquad \vec{t}_2 = \left(0, 1, \frac{\partial h}{\partial y}\right)$$

$$\vec{n} = \vec{t}_1 \times \vec{t}_2 = -\frac{\partial h}{\partial x}\,\hat{e}_x - \frac{\partial h}{\partial y}\,\hat{e}_y + \hat{e}_z \qquad \vec{n} = \left(-\frac{\partial h}{\partial x}, -\frac{\partial h}{\partial y}, 1\right)$$

$$\hat{t}_1 = \frac{\left(1, 0, \frac{\partial h}{\partial x}\right)}{\sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2}}, \quad \hat{t}_2 = \frac{\left(0, 1, \frac{\partial h}{\partial y}\right)}{\sqrt{1+\left(\frac{\partial h}{\partial y}\right)^2}}, \quad \hat{n} = \frac{\left(-\frac{\partial h}{\partial x}, -\frac{\partial h}{\partial y}, 1\right)}{\sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2}}$$

non-examinable

metric components:

$$g_{11} = \vec{t}_1 \cdot \vec{t}_1 = 1 + \left(\frac{\partial h}{\partial x}\right)^2$$

$$g_{12} = \vec{t}_1 \cdot \vec{t}_2 = \left(\frac{\partial h}{\partial x}\right)\left(\frac{\partial h}{\partial y}\right)$$

$$g_{22} = \vec{t}_2 \cdot \vec{t}_2 = 1 + \left(\frac{\partial h}{\partial y}\right)^2$$

$$\underline{g} = \begin{bmatrix} 1+\left(\frac{\partial h}{\partial x}\right)^2 & \left(\frac{\partial h}{\partial x}\right)\left(\frac{\partial h}{\partial y}\right) \\ \left(\frac{\partial h}{\partial x}\right)\left(\frac{\partial h}{\partial y}\right) & 1+\left(\frac{\partial h}{\partial y}\right)^2 \end{bmatrix}$$

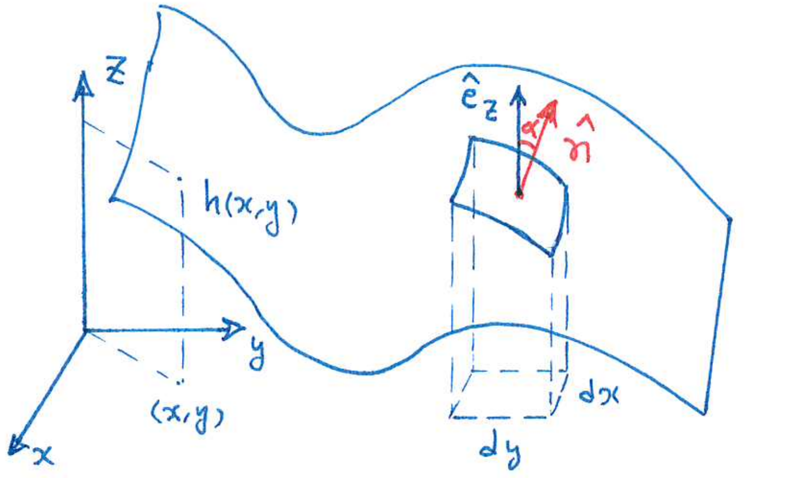
$$g = \det(\underline{g}) = 1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2$$

Area element: $dA = \left|\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}\right| dx\,dy = |\vec{n}|\,dx\,dy$

$$|\vec{n}| = \sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2} \longrightarrow \boxed{dA = \sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2}\,dx\,dy}$$

Note that $dA = \sqrt{g}\,dx\,dy$.

Method of projection:



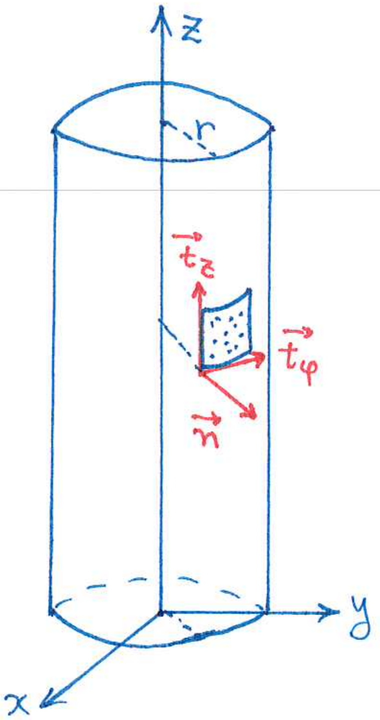
$$dx\,dy = dA \cos\alpha$$

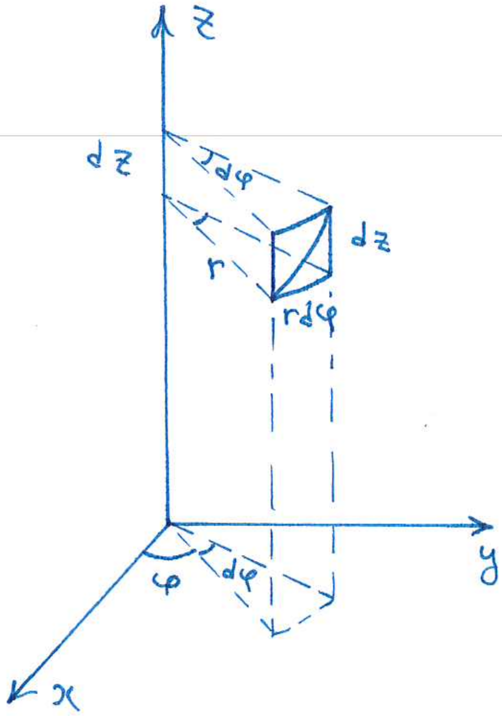
$$dA = \frac{1}{\cos\alpha}\, dx\,dy$$

$$\cos\alpha = \hat{e}_z \cdot \hat{n}$$

$$\cos\alpha = \frac{1}{\sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2}} \quad \rightarrow \quad dA = \sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2}\; dx\,dy \quad \checkmark$$

Example: A cylindrical surface of radius r.





$\vec{R}(\varphi, z) = r\cos\varphi\,\hat{e}_x + r\sin\varphi\,\hat{e}_y + z\,\hat{e}_z$ or $\vec{R}(\varphi, z) = (r\cos\varphi, r\sin\varphi, z)$

from geometry: area element $dA = (r\,d\varphi)\,dz$

line element $ds^2 = (r\,d\varphi)^2 + dz^2$

$$\vec{t}_1 = \vec{t}_\varphi = \frac{\partial \vec{R}}{\partial \varphi} = -r\sin\varphi\,\hat{e}_x + r\cos\varphi\,\hat{e}_y \qquad ; \ |\vec{t}_1| = r$$

$$\vec{t}_2 = \vec{t}_z = \frac{\partial \vec{R}}{\partial z} = \hat{e}_z \qquad ; \ |\vec{t}_2| = 1$$

$$\hat{t}_1 = \frac{1}{r}\frac{\partial \vec{R}}{\partial \varphi} = -\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y = \hat{e}_\varphi$$

$$\vec{n} = \vec{t}_\varphi \times \vec{t}_z = r\cos\varphi\,\hat{e}_x + r\sin\varphi\,\hat{e}_y \qquad ; \ |\vec{n}| = r$$

$$\hat{n} = \frac{1}{r}(\vec{t}_\varphi \times \vec{t}_z) = \cos\varphi\,\hat{e}_x + \sin\varphi\,\hat{e}_y = \hat{e}_r$$

area element $\quad dA = |\vec{n}|\,d\varphi\,dz = r\,d\varphi\,dz$ ✓

non-examinable

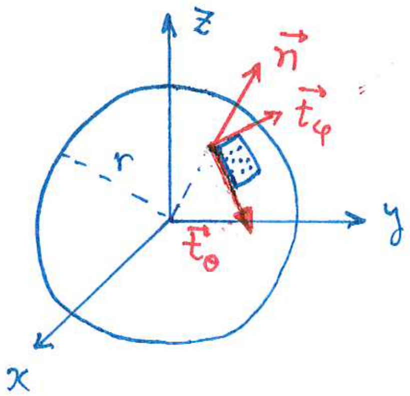
metric components

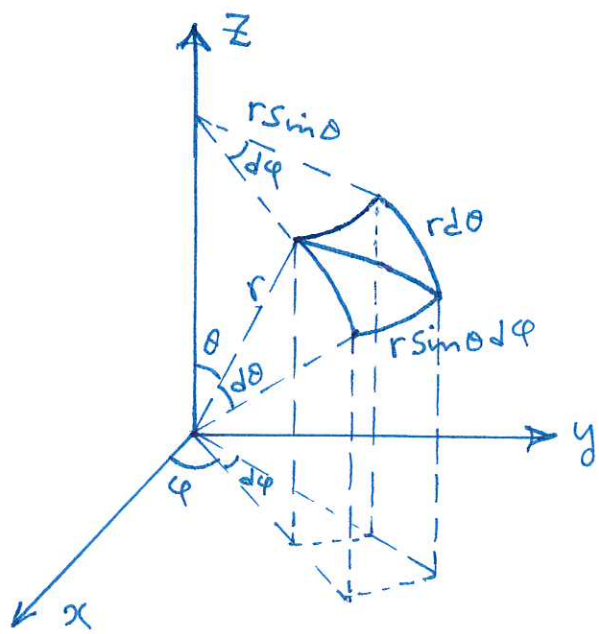
$$g_{\varphi\varphi} = \vec{t}_\varphi\cdot\vec{t}_\varphi = r^2, \quad g_{\varphi z} = \vec{t}_\varphi\cdot\vec{t}_z = 0, \quad g_{zz} = \vec{t}_z\cdot\vec{t}_z = 1$$

line element $\quad ds^2 = g_{\varphi\varphi}\,d\varphi^2 + 2g_{\varphi z}\,d\varphi\,dz + g_{zz}\,dz^2 = r^2 d\varphi^2 + dz^2$ ✓

$$g = \det(\underline{g}) = r^2 \ \rightarrow \ \sqrt{g} = r \ \rightarrow \ dA = \sqrt{g}\,d\varphi\,dz$$ ✓

<u>Example</u>: A spherical surface of radius r.





$$\vec{R}(\theta,\varphi) = r\sin\theta\cos\varphi\,\hat{e}_x + r\sin\theta\sin\varphi\,\hat{e}_y + r\cos\theta\,\hat{e}_z$$

or $\vec{R}(\theta,\varphi) = (r\sin\theta\cos\varphi,\, r\sin\theta\sin\varphi,\, r\cos\theta)$

from geometry: area element $dA = (r\,d\theta)(r\sin\theta\,d\varphi) = r^2\sin\theta\,d\theta\,d\varphi$

line element $ds^2 = (r\,d\theta)^2 + (r\sin\theta\,d\varphi)^2$

$$\vec{t}_1 = \vec{t}_\theta = \frac{\partial\vec{R}}{\partial\theta} = r\cos\theta\cos\varphi\,\hat{e}_x + r\cos\theta\sin\varphi\,\hat{e}_y - r\sin\theta\,\hat{e}_z \quad ; |\vec{t}_1| = r$$

$$\vec{t}_2 = \vec{t}_\varphi = \frac{\partial\vec{R}}{\partial\varphi} = -r\sin\theta\sin\varphi\,\hat{e}_x + r\sin\theta\cos\varphi\,\hat{e}_y \quad ; |\vec{t}_2| = r\sin\theta$$

$$\vec{n} = \vec{t}_\theta \times \vec{t}_\varphi = r^2\sin\theta\,(\sin\theta\cos\varphi\,\hat{e}_x + \sin\theta\sin\varphi\,\hat{e}_y + \cos\theta\,\hat{e}_z); |\vec{n}| = r^2\sin\theta$$

$$\hat{t}_1 = \frac{1}{r}\frac{\partial\vec{R}}{\partial\theta} = \cos\theta\cos\varphi\,\hat{e}_x + \cos\theta\sin\varphi\,\hat{e}_y - \sin\theta\,\hat{e}_z = \hat{e}_\theta$$

$$\hat{t}_2 = \frac{1}{r\sin\theta}\frac{\partial\vec{R}}{\partial\varphi} = -\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y = \hat{e}_\varphi$$

$$\hat{n} = \frac{1}{r^2\sin\theta}(\vec{t}_\theta \times \vec{t}_\varphi) = \sin\theta\cos\varphi\,\hat{e}_x + \sin\theta\sin\varphi\,\hat{e}_y + \cos\theta\,\hat{e}_z = \hat{e}_r$$

area element $dA = |\vec{n}|\,d\theta\,d\varphi = r^2\sin\theta\,d\theta\,d\varphi$ ✓

non-examinable

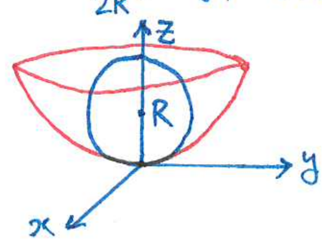
metric components

$$g_{\theta\theta} = \vec{t}_\theta\cdot\vec{t}_\theta = r^2,\quad g_{\theta\varphi} = \vec{t}_\theta\cdot\vec{t}_\varphi = 0,\quad g_{\varphi\varphi} = \vec{t}_\varphi\cdot\vec{t}_\varphi = r^2\sin^2\theta$$

line element $ds^2 = g_{\theta\theta}\,d\theta^2 + 2g_{\theta\varphi}\,d\theta\,d\varphi + g_{\varphi\varphi}\,d\varphi^2 = r^2 d\theta^2 + r^2\sin^2\theta\,d\varphi^2$ ✓

$g = \det(\underline{g}) = r^4\sin^2\theta \rightarrow \sqrt{g} = r^2\sin\theta \rightarrow dA = \sqrt{g}\,d\theta\,d\varphi$ ✓

Exercise: show that the radius of curvature of the paraboloid $z = \frac{1}{2R}(x^2+y^2)$ is R by starting from the equation that describes this sphere and expanding in $x, y \ll R$.



Exercise: Show that for the most general embedding of a 2D surface in 3D, which is described by the metric

$$\underline{g} = \begin{bmatrix} \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial u} & \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v} \\ \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial u} & \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial v} \end{bmatrix} , \qquad g = \det(\underline{g})$$

the area element $dA = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\, dv$, could be written as

$$dA = \sqrt{g}\, du\, dv$$

2. Scalar and Vector Fields

A scalar is a quantity that is not changed by coordinate transformations such as rotation.

Examples (from physics): charge, mass, temperature, ...

A vector is a quantity that is characterized by a magnitude and a direction, and is defined in terms of components that transform like (Euclidean) coordinates.

Example: position vector and rotation of coordinate system

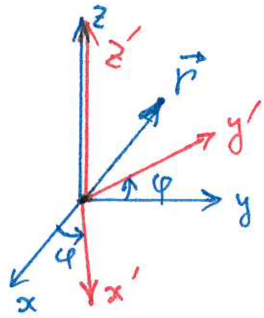
$$\vec{r} = x\,\hat{e}_x + y\,\hat{e}_y + z\,\hat{e}_z = x'\,\hat{e}'_x + y'\,\hat{e}'_y + z'\,\hat{e}'_z$$

$\vec{r}$ is not changing

components and basis

vectors are changing $\rightarrow$ Passive transformation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



(we could also have active transformation, e.g. time evolution of $\vec{r}(t)$)

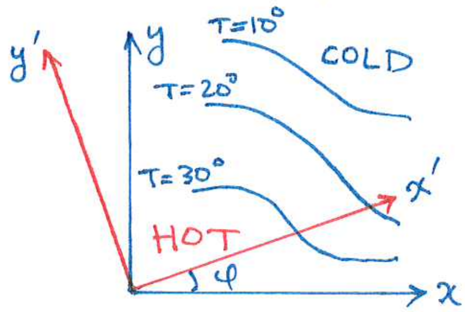
A vector $\vec{F} = F_x\,\hat{e}_x + F_y\,\hat{e}_y + F_z\,\hat{e}_z = F'_x\,\hat{e}'_x + F'_y\,\hat{e}'_y + F'_z\,\hat{e}'_z$

Since the basis vectors transform as before, the components of $\vec{F}$ should transform as the components of $\vec{r}$.

$$\begin{bmatrix} F'_x \\ F'_y \\ F'_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

A scalar field is a scalar quantity that is defined at every point in space.

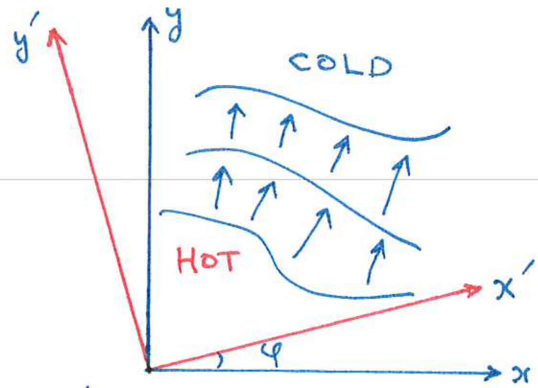
Example: temperature field in 2D: $T(x,y)$



coordinate transformation $(x,y) \mapsto (x',y')$

$$T'(x',y') = T(x,y)$$

A vector field is a vector quantity that is defined at every point in space.



Example: heat flux in 2D

$$\vec{h}(x,y) = h_x(x,y)\,\hat{e}_x + h_y(x,y)\,\hat{e}_y$$

coordinate transformation $(x,y) \mapsto (x',y')$

$$\vec{h}'(x',y') = \vec{h}(x,y)$$

$$\vec{h}'(x',y') = h'_x(x',y')\,\hat{e}'_x + h'_y(x',y')\,\hat{e}'_y$$

$$\begin{bmatrix} h'_x \\ h'_y \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} h_x \\ h_y \end{bmatrix}$$

Exercise: Show that the basis vectors are transformed as follows under the rotation of coordinate system.

$$\begin{bmatrix} \hat{e}'_x \\ \hat{e}'_y \\ \hat{e}'_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

How come the rotation matrix is not the inverse of that used for the components?

Example: The Euclidean line element in 3D is a scalar under arbitrary rotation.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad X' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$ds^2 = d\vec{r}\cdot d\vec{r} = dx^2 + dy^2 + dz^2 = dX^T \cdot dX$$

$$ds'^2 = d\vec{r}'\cdot d\vec{r}' = dx'^2 + dy'^2 + dz'^2 = dX'^T \cdot dX'$$

$$ds'^2 = dX^T \cdot \underbrace{R^T \cdot R}_{I} \cdot dX = dX^T \cdot dX = ds^2 \checkmark$$

$X' = RX$ ← arbitrary rotation matrix

$RR^T = R^TR = I$

$dX' = R\,dX \longrightarrow dX'^T = dX^T R^T$

We can do the same calculation using a helpful index-notation by going from

$$(x, y, z) \overset{R}{\longmapsto} (x', y', z')$$

to

$$(x_1, x_2, x_3) \overset{R}{\longmapsto} (x_1', x_2', x_3')$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$x_i' = \sum_j R_{ij}\, x_j$$

$$dx_i' = \sum_j R_{ij}\, dx_j$$

$$R_{ij} = \frac{\partial x_i'}{\partial x_j}$$

$R^TR = I$ in index notation: $\sum_k R^T_{ik} R_{kj} = I_{ij}$

$R^T_{ik} = R_{ki}$; $I_{ij} \equiv \delta_{ij}$ ij component of identity matrix

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Kronecker delta

$$ds'^2 = \sum_k dx_k'\, dx_k' = \sum_k \left(\sum_i R_{ki} dx_i\right)\left(\sum_j R_{kj} dx_j\right)$$

$$ds'^2 = \sum_{i,j} dx_i\, dx_j \underbrace{\left(\sum_k \underbrace{R_{ki}}_{R^T_{ik}} R_{kj}\right)}_{\delta_{ij}} = \sum_{i,j} \underbrace{dx_i\, dx_j\, \delta_{ij}}_{\text{only non-zero when } i=j} = \sum_i dx_i\, dx_i = ds^2 \checkmark$$

3. Gradient

For a smooth (differentiable) scalar field $\phi(x,y,z)$, its gradient is defined by

$$\text{grad}\,\phi = \vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{e}_x + \frac{\partial\phi}{\partial y}\hat{e}_y + \frac{\partial\phi}{\partial z}\hat{e}_z$$

which is compiled as a "vector" using "components" that appear in the complete differential of ϕ: $\boxed{d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \vec{\nabla}\phi\cdot d\vec{r}}$

in component notation:

$$\vec{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

Example: $\phi(x,y,z) = xy^2 + 3z \rightarrow \vec{\nabla}\phi = (y^2, 2xy, 3)$.

The notation $\vec{\nabla}\phi$ suggests the use of the "del operator"

$$\vec{\nabla} = \hat{e}_x\frac{\partial}{\partial x} + \hat{e}_y\frac{\partial}{\partial y} + \hat{e}_z\frac{\partial}{\partial z}$$

as an operator that acts on functions and returns their gradients.

3.1. Transformation Properties of Gradient

We need to show that $\vec{\nabla}\phi$ is a vector as per our definition.

$$(x,y,z) \mapsto (x',y',z')$$

(chain rule) $\frac{\partial\phi}{\partial x'} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x'}$

$\frac{\partial\phi}{\partial y'} = \cdots$

$\frac{\partial\phi}{\partial z'} = \cdots$

$$\rightarrow \begin{bmatrix} \frac{\partial\phi}{\partial x'} \\ \frac{\partial\phi}{\partial y'} \\ \frac{\partial\phi}{\partial z'} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & \frac{\partial z}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & \frac{\partial z}{\partial y'} \\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'} \end{bmatrix} \begin{bmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \\ \frac{\partial\phi}{\partial z} \end{bmatrix}$$

It is convenient to use index-notation: $(x_1, x_2, x_3) \mapsto (x'_1, x'_2, x'_3)$

$$\frac{\partial \phi}{\partial x'_i} = \sum_m \frac{\partial \phi}{\partial x_m}\left(\frac{\partial x_m}{\partial x'_i}\right) \qquad K_{mi} = \frac{\partial x_m}{\partial x'_i} = L^{-1}_{mi} \text{ (see below)}$$

$$dx'_j = \sum_n dx_n \left(\frac{\partial x'_j}{\partial x_n}\right) \qquad L_{jn} = \frac{\partial x'_j}{\partial x_n} = K^{-1}_{jn} \text{ (see below)}$$

First, note: $\sum_i K_{mi} L_{in} = \sum_i \frac{\partial x_m}{\partial x'_i} \frac{\partial x'_i}{\partial x_n} = \frac{\partial x_m}{\partial x_n} = \delta_{mn} = I_{mn}$

$$\rightarrow \quad L = K^{-1}$$

We have met K^{-1} before, in the context of rotation of coordinate system (orthogonal transformation) where (we called it R, and) all the components were constants. In general, K could represent any arbitrary coordinate transformation, and its components could depend on x_i. K is called the Jacobian matrix.

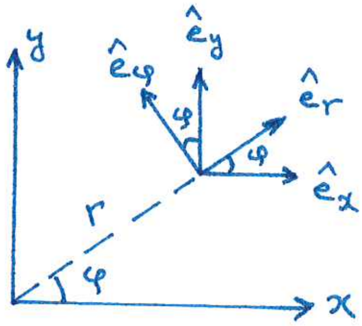
Proof that $\vec{\nabla}\phi$ is a vector:

$$d\phi' = \sum_i \frac{\partial \phi}{\partial x'_i} dx'_i = \sum_i \left[\sum_m \frac{\partial \phi}{\partial x_m}\left(\frac{\partial x_m}{\partial x'_i}\right)\right]\left[\sum_n dx_n \left(\frac{\partial x'_i}{\partial x_n}\right)\right]$$

$$= \sum_{m,n} \frac{\partial \phi}{\partial x_m} dx_n \underbrace{\left[\sum_i \left(\frac{\partial x_m}{\partial x'_i}\right)\left(\frac{\partial x'_i}{\partial x_n}\right)\right]}_{\frac{\partial x_m}{\partial x_n} = \delta_{mn}} = \sum_n \frac{\partial \phi}{\partial x_n} dx_n = d\phi \quad \checkmark$$

in other words: $d\phi' = \vec{\nabla}'\phi \cdot d\vec{r}\,' = \vec{\nabla}\phi \cdot d\vec{r} = d\phi$.

We can use the Jacobian matrix to find the components of $\vec{\nabla}\phi$ in any coordinate system.



Example: Gradient in polar coordinates.

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad \begin{cases} \hat{e}_r = \cos\varphi\,\hat{e}_x + \sin\varphi\,\hat{e}_y \\ \hat{e}_\varphi = -\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y \end{cases}$$

$$\begin{bmatrix} \frac{\partial\psi}{\partial r} \\ \frac{\partial\psi}{\partial\varphi} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial\varphi} & \frac{\partial y}{\partial\varphi} \end{bmatrix} \begin{bmatrix} \frac{\partial\psi}{\partial x} \\ \frac{\partial\psi}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{bmatrix} \begin{bmatrix} \frac{\partial\psi}{\partial x} \\ \frac{\partial\psi}{\partial y} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial\psi}{\partial x} \\ \frac{\partial\psi}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\frac{1}{r}\sin\varphi \\ \sin\varphi & \frac{1}{r}\cos\varphi \end{bmatrix} \begin{bmatrix} \frac{\partial\psi}{\partial r} \\ \frac{\partial\psi}{\partial\varphi} \end{bmatrix}$$

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial x}\hat{e}_x + \frac{\partial\psi}{\partial y}\hat{e}_y$$

$$\vec{\nabla}\psi = \left(\cos\varphi\frac{\partial\psi}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial\psi}{\partial\varphi}\right)\hat{e}_x + \left(\sin\varphi\frac{\partial\psi}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial\psi}{\partial\varphi}\right)\hat{e}_y$$

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\left(\cos\varphi\,\hat{e}_x + \sin\varphi\,\hat{e}_y\right) + \frac{1}{r}\frac{\partial\psi}{\partial\varphi}\left(-\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y\right)$$

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\varphi}\hat{e}_\varphi$$

It is straightforward to show that in cylindrical coordinates, we have

$$\boxed{\vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\varphi}\hat{e}_\varphi + \frac{\partial\psi}{\partial z}\hat{e}_z}$$

Exercise: Show that in spherical coordinates, we have

$$\boxed{\vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\hat{e}_\varphi}$$

Exercise: Use $\vec{\nabla}\psi\cdot d\vec{r}$ with $d\vec{r}$ in cylindrical and spherical coordinates to obtain the above (boxed) formulas for $\vec{\nabla}\psi$ in each coordinate system, respectively.

3.2. Geometric Properties of Gradient

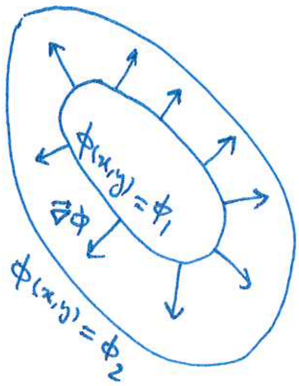
The rate of change of $\phi(\vec{r})$ along a curve $\vec{r}(s)$

$$\frac{d\phi}{ds} = \vec{\nabla}\phi \cdot \frac{d\vec{r}}{ds} = \vec{\nabla}\phi \cdot \hat{T}(s)$$

In 2D: $\phi(x,y) = C$ defines constant-ϕ contour lines.

If $d\vec{r}$ ($\hat{T}$) is along a contour line, ϕ does not change $\rightarrow \vec{\nabla}\phi \cdot d\vec{r} = 0$

$\rightarrow \vec{\nabla}\phi \perp d\vec{r}$; $\vec{\nabla}\phi$ is normal to $d\vec{r}$

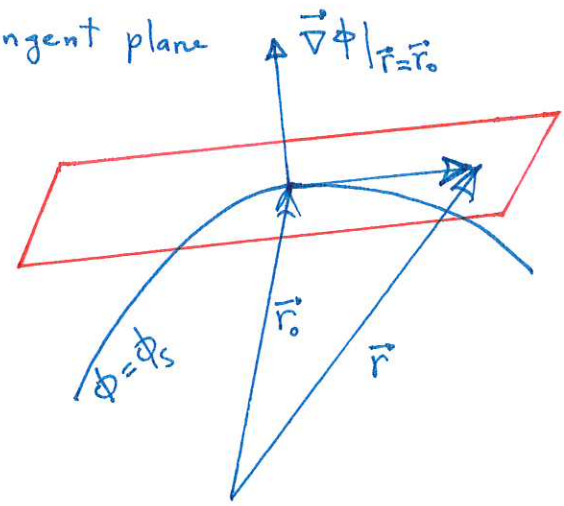


In 3D: $\phi(x,y,z) = C$ defines a surface.

$\rightarrow \vec{\nabla}\phi$ is normal to the surface

$$\rightarrow \boxed{\hat{n} = \frac{\vec{\nabla}\phi}{|\nabla\phi|}} \quad \text{normal unit vector}$$

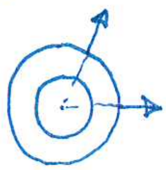
We can use this to describe the tangent plane to the surface at a point $\vec{r}_0$:



$$(\vec{r}-\vec{r}_0)\cdot\vec{\nabla}\phi|_{\vec{r}=\vec{r}_0} = 0 \quad \left[(\vec{r}-\vec{r}_0)\cdot\hat{n} = 0\right]$$

Example: $\phi(r) = x^2+y^2+z^2 = r^2$

$\nabla\phi = 2(x,y,z) = 2\vec{r}$



tangent plane: $(\vec{r}-\vec{r}_0)\cdot\vec{r}_0 = 0 \rightarrow xx_0 + yy_0 + zz_0 = r_0^2$.

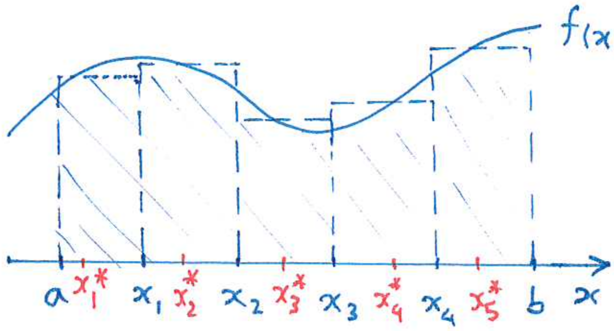
Exercise: Use the Monge representation of a surface $\phi(x,y,z) = z - h(x,y) = 0$ to show that the normal vector is
$$\hat{n} = \frac{\left(-\frac{\partial h}{\partial x}, -\frac{\partial h}{\partial y}, 1\right)}{\sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2}}.$$

4. Multiple Integrals

Recall the definition of a 1D integral: $\int_a^b dx\, f(x) = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i^*)(x_i - x_{i-1})$

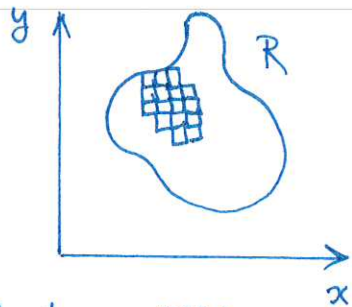
where $a = x_0 < x_1 < \cdots < x_N = b$; $x_{i-1} \le x_i^* \le x_i$; $\lim_{N\to\infty}(x_i - x_{i-1}) = 0$

The above definition is subject to the existence of the limit and its insensitivity to the choice of x_j^*'s. Geometric interpretation = area under $f(x)$.



4.1. 2D Integrals

A 2D integral is defined analogously:

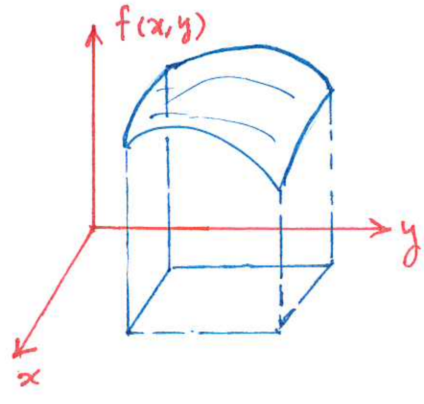


$$\int_R dA\, f(x,y) = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i)\, \Delta A_i$$

where ΔA_i are N subregions (tiles) that partition R, and (x_i, y_i) corresponds to a point in each subregion. $\lim_{N\to\infty} \Delta A_i = 0$.

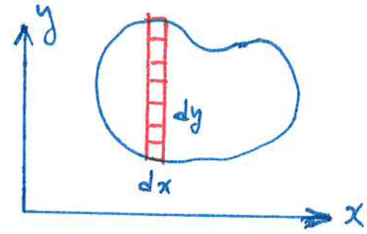
The definition is subject to the existence of the limit and its insensitivity to the choice of (x_i, y_i).

Geometric interpretation = volume under $f(x,y)$.



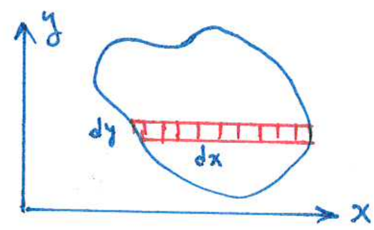
2D integrals can be written as iterated 1D integrals

(either) sum over contributions from vertical slices



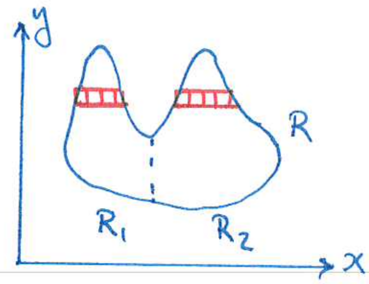
$$\int_R dA\, f(x,y) = \int dx \int_{y_1(x)}^{y_2(x)} dy\, f(x,y)$$

(or) sum over contributions from horizontal slices



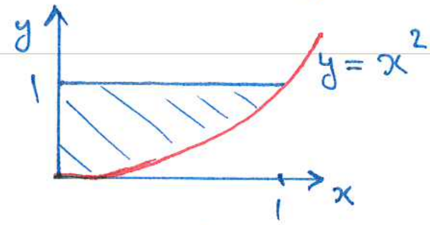
$$\int_R dA\, f(x,y) = \int dy \int_{x_1(y)}^{x_2(y)} dx\, f(x,y)$$

Note: sometimes we have to split the region R for this to work.

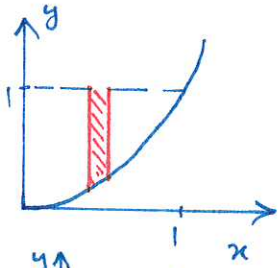


Example: calculate $I = \int_R dA\; xy$.

where R is the hashed region below.

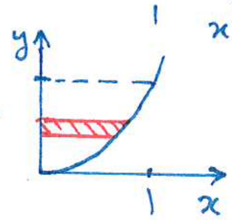


(either) $I = \int_0^1 dx \int_{x^2}^1 dy\; xy = \int_0^1 dx\; x \int_{x^2}^1 dy\; y = \int_0^1 dx\; x \left(\frac{y^2}{2} \Big|_{x^2}^1 \right)$



$$= \int_0^1 dx\, \frac{x}{2} \left(1 - x^4\right) = \left(\frac{x^2}{4} - \frac{x^6}{12} \right)\Big|_0^1 = \frac{1}{6}$$

(or) $I = \int_0^1 dy \int_0^{\sqrt{y}} dx\; xy = \int_0^1 dy\; y \left(\frac{1}{2} x^2 \Big|_0^{\sqrt{y}} \right) = \frac{1}{2} \int_0^1 dy\; y^2 = \frac{1}{6}$



4.2. 3D Integrals

$$\int_{V_0} dV\, f(x,y,z) = \lim_{N \to \infty} \sum_{p=1}^{N} f(x_p, y_p, z_p)\, \Delta V_p$$

$\lim_{N\to\infty} \Delta V_p = 0$

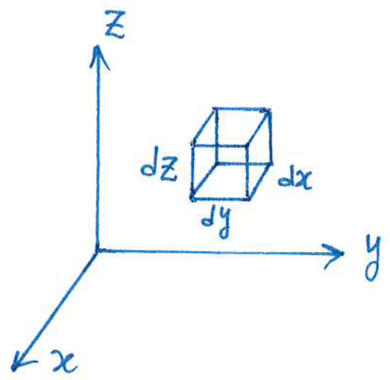
ΔV_p partition V_0

provided the limit exists and is independent of the choice of (x_p, y_p, z_p)

similar to the 2D case, a 3D integral can be carried out by iterating 1D integrals.

4.3. Choice of Coordinate System

- Cartesian coordinates

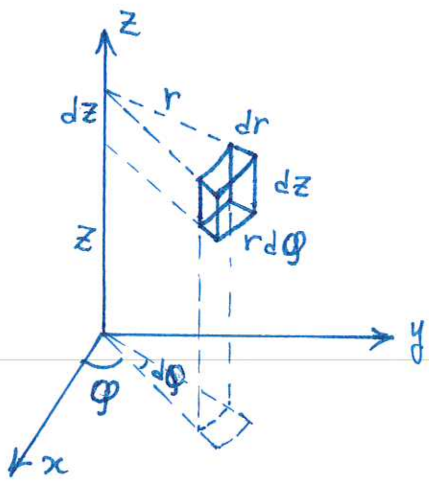


– Line element: $ds^2 = dx^2 + dy^2 + dz^2$

– Area element: $dA = \{dx\,dy,\ dx\,dz,\ dy\,dz\}$

– Volume element: $dV = dx\,dy\,dz$

- Cylindrical coordinates

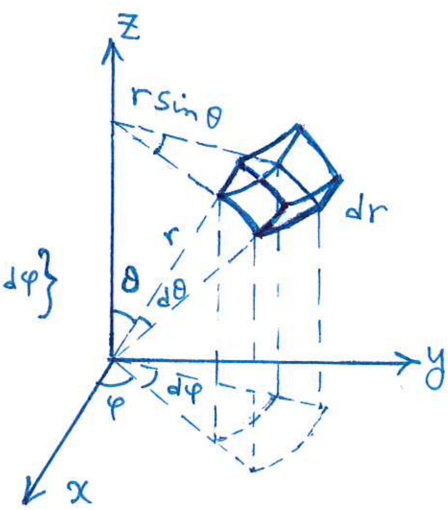


– Line element: $ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$

– Area element: $dA = \{dr\,dz,\ r\,d\varphi\,dz,\ r\,d\varphi\,dr\}$

– Volume element: $dV = r\,dr\,d\varphi\,dz$

- Spherical coordinates

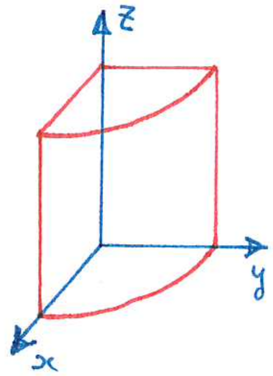


– Line element: $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$

– Area element: $dA = \{r\,dr\,d\theta,\ r\sin\theta\, d\varphi\, dr,\ r^2 \sin\theta\, d\theta\, d\varphi\}$

– Volume element: $dV = r^2 \sin\theta\, d\theta\, d\varphi\, dr$

Example: calculate the volume bounded by the planes: $z=0$, $z=6$, $x=0$, $y=0$, and the surface: $x^2+y^2=4$.



Cartesian coordinates:

$$V=\int_{V_0} dV=\underbrace{\int_0^6 dz\,\underbrace{\int_0^2 dy\,\underbrace{\int_0^{\sqrt{4-y^2}} dx}_{\text{stripes}}}_{\text{quarter-disks}}}_{\text{stack of quarter disks}}=\overbrace{\int_0^6 dz}^{6}\,\underbrace{\int_0^2 dy\,\sqrt{4-y^2}}_{\text{change of var needed}}=6\times4\times\underbrace{\int_0^{\frac{\pi}{2}} d\varphi\,\cos^2\varphi}_{\pi/4}=6\pi$$

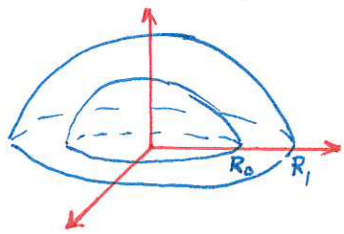
$y=2\sin\varphi \qquad 0\leq\varphi\leq\frac{\pi}{2}$

$dy=2\cos\varphi\,d\varphi$

Cylindrical coordinates:

$$V=\int_0^6 dz\,\underbrace{\int_0^2 r\,dr\,\underbrace{\int_0^{\frac{\pi}{2}} d\varphi}_{\text{quarter circle arcs}}}_{\text{quarter-disks}}=6\times\frac{r^2}{2}\Big|_0^2\times\frac{\pi}{2}=6\pi \checkmark$$

Example: Centre of mass of a hemi-spherical shell with $\rho(\vec{r})=\frac{\alpha}{r}$.



Total mass $\quad M=\int_{V_0} dV\,\rho(\vec{r}) \qquad \left(M=\int dm\right)$

$$=\int_{R_0}^{R_1} dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{2\pi} d\varphi\; r^2\sin\theta\;\rho(r)$$

$$=2\pi\alpha\int_0^{\frac{\pi}{2}} d\theta\,\sin\theta\int_{R_0}^{R_1} dr\,r=\pi\alpha\left(R_1^2-R_0^2\right)$$

Centre of mass $\quad x_{CM}=\frac{1}{M}\int dV\,x\,\rho(\vec{r}) \qquad \left(\vec{r}_{CM}=\frac{\int dm\,\vec{r}}{\int dm}\right)$

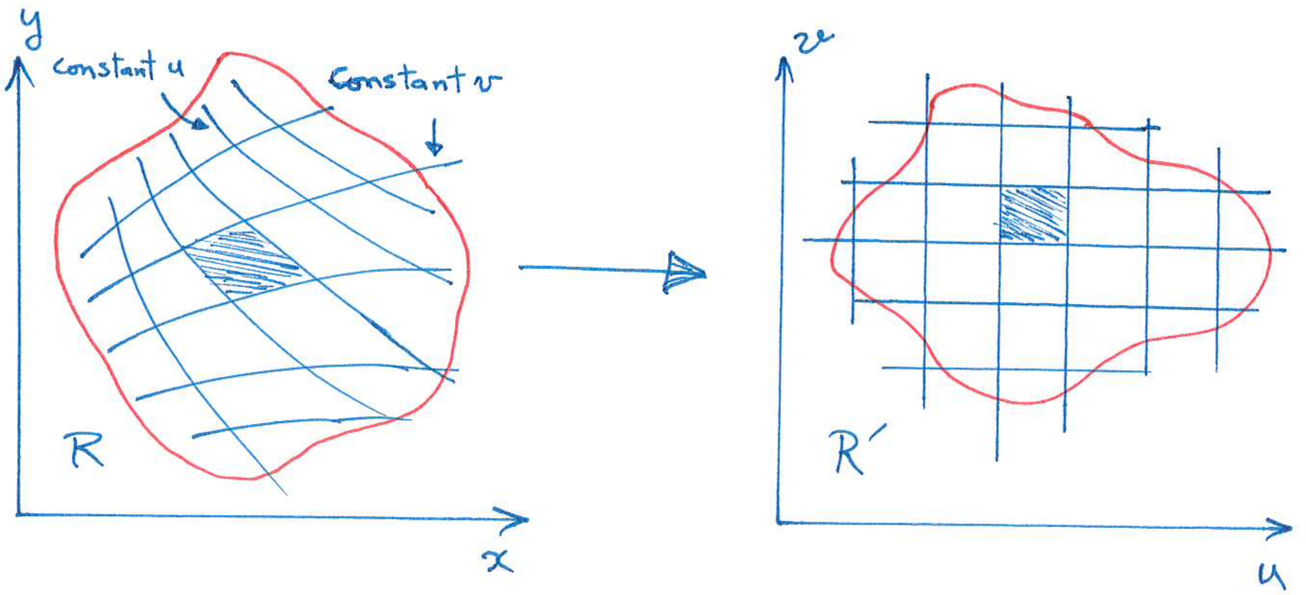
$$x_{CM}=\frac{1}{M}\int_{R_0}^{R_1} dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{2\pi} d\varphi\; r^2\sin\theta\cdot\frac{\alpha}{r}\cdot\overbrace{r\sin\theta\cos\varphi}^{x}=0 \quad \text{as} \quad \int_0^{2\pi} d\varphi\cos\varphi=0$$

$$y_{CM}=0,\quad z_{CM}=\frac{1}{M}\int_{R_0}^{R_1} dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{2\pi} d\varphi\; r^2\sin\theta\cdot\frac{\alpha}{r}\cdot\overbrace{r\cos\theta}^{z}=\frac{1}{3}\,\frac{R_1^3-R_0^3}{R_1^2-R_0^2}$$

4.4. Change of Variables in 2D Integrals

How can we set up a given integral $I = \int_R dx\, dy\, f(x,y)$

in terms of new variables u and v?



Change of variables $(x,y) \mapsto (u,v)$ can be regarded as an embedding of a 2D patch (surface) in a 2D space.

Using $\vec{r}(u,v) = (x(u,v), y(u,v), 0)$, we can calculate the new area element: $dA = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\, dv$

in components: $dA = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| du\, dv$

The conversion factor is the determinant of the Jacobian matrix

$$\frac{\partial(x,y)}{\partial(u,v)} \equiv \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Therefore:

$$dx\, dy = J \cdot du\, dv \qquad \overbrace{J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|}$$

This is called The Jacobian

The domain of integration also changes from R to R'.

$$I = \int_R dx\,dy\, f(x,y) = \int_{R'} du\,dv \cdot J \cdot f(x(u,v), y(u,v))$$

Example: Polar coordinates $\quad dx\,dy = r\,dr\,d\varphi$

$$x = r\cos\varphi, \quad y = r\sin\varphi \quad \rightarrow \quad \frac{\partial(x,y)}{\partial(r,\varphi)} = \det\begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix} = r$$

Example: The Gaussian Integral $\quad I = \int_{-\infty}^{\infty} dx\, e^{-x^2}$

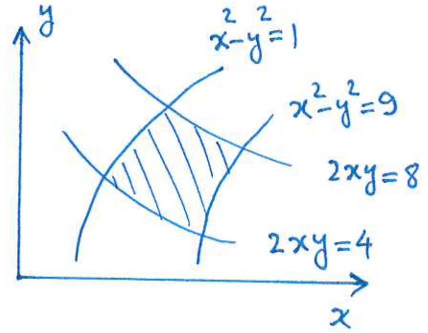
Square it! $\quad I^2 = \int_{-\infty}^{\infty} dx\, e^{-x^2} \int_{-\infty}^{\infty} dy\, e^{-y^2} = \int_{-\infty}^{\infty} dx\,dy\, e^{-(x^2+y^2)}$

Now change to polar coordinates: $\quad I^2 = \int_0^{\infty} dr\, r \int_0^{2\pi} d\varphi\, e^{-r^2}$

$$I^2 = 2\pi \times \frac{1}{2} \underbrace{\int_0^{\infty} d(r^2)\, e^{-(r^2)}}_{1} = \pi \quad \rightarrow \quad I = \sqrt{\pi} \quad (\text{as the integrand} > 0)$$

Exercise: Prove that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$.

Example: Integration over a finite region



$$I = \int_R dx\,dy\, (x^2+y^2)$$

change of variables: $u = x^2-y^2, \quad v = 2xy$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix} = 4(x^2+y^2) \quad \rightarrow \quad du\,dv = 4(x^2+y^2)\,dx\,dy$$

Limits: $x^2-y^2=1 \rightarrow u=1$, $x^2-y^2=9 \rightarrow u=9$, $2xy=4 \rightarrow v=4$, $2xy=8 \rightarrow v=8$

$$I = \int_1^9 du \int_4^8 dv \cdot \frac{1}{4} = 8.$$

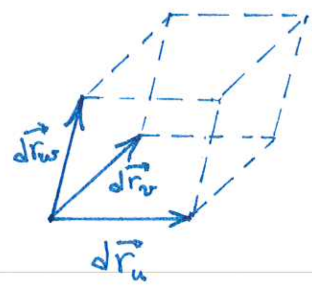
$\boxed{\text{Look at what the borders in } (x,y) \text{ map to in } (u,v) \text{ plane.}}$

4.5. Change of Variables in 3D Integrals

As in the 2D case, change of variables $(x,y,z) \mapsto (u,v,w)$ can be regarded as an embedding of a 3D domain (volume) in a 3D space. Using $\vec{r}(u,v,w) = \big(x(u,v,w), y(u,v,w), z(u,v,w)\big)$, we can calculate the new volume element:

$$dV = \left| \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) \cdot \frac{\partial \vec{r}}{\partial w} \right| du\, dv\, dw$$

where the triple product gives the volume of the parallelepiped made from



$$d\vec{r}_u = \frac{\partial \vec{r}}{\partial u} du, \quad d\vec{r}_v = \frac{\partial \vec{r}}{\partial v} dv, \quad d\vec{r}_w = \frac{\partial \vec{r}}{\partial w} dw.$$

In components, the conversion factor is again equal to the determinant of the Jacobian matrix, which will be shown as $\frac{\partial(x,y,z)}{\partial(u,v,w)}$:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \equiv \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} = \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) \cdot \frac{\partial \vec{r}}{\partial w}$$

therefore:

$$dx\,dy\,dz = J \cdot du\,dv\,dw \qquad J = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right|$$

is the 3D Jacobian

Exercise: By calculating the Jacobian, show that the volume element in spherical coordinates is: $dV = r^2 \sin\theta \, d\varphi \, d\theta \, dr$

non-examinable

<u>Example</u>: Relationship between Jacobian and induced-metric.

$(x_1, x_2, x_3) \mapsto (u_1, u_2, u_3)$ index-notation

u_i arbitrary "curvilinear" coordinates. $K_{\ell i} = \frac{\partial x_\ell}{\partial u_i}$ Jacobian matrix

what is the induced-metric?

$$ds^2 = \sum_k dx_k dx_k = \sum_k \left[\sum_i \frac{\partial x_k}{\partial u_i} du_i\right]\left[\sum_j \frac{\partial x_k}{\partial u_j} du_j\right] = \sum_{i,j}\left[\sum_k \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j}\right] du_i du_j$$

$$ds^2 = \sum_{i,j} g_{ij}\, du_i du_j \qquad g_{ij} = \sum_k \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j} \quad \text{induced-metric}$$

Now note: $$g_{ij} = \sum_k \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j} = \sum_{k,\ell} \frac{\partial x_k}{\partial u_i}\delta_{k\ell}\frac{\partial x_\ell}{\partial u_j} = \sum_{k,\ell} K_{ki} I_{k\ell} K_{\ell j}$$

in matrix form $\underline{g} = K^T \cdot I \cdot K$

$$g \equiv \det(\underline{g}) = \underbrace{\det(K^T)}_{=\det(K)} \cdot \underbrace{\det(I)}_{=1} \cdot \underbrace{\det(K)}_{=J} = J^2$$

$\rightarrow \quad J = \sqrt{\det(\underline{g})}$ this is valid in any dimension

Therefore, $dV = \sqrt{g}\, du_1 du_2 \cdots du_d$ is the so-called invariant volume element in d-dimensions in any arbitrary coordinate system.

5. Line Integrals

A line integral is an integral along a path that is defined by a curve $\vec{r}(t)$.

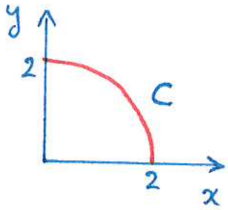
The measure of integral could be the line element ds, or the infinitesimal displacement vector $d\vec{r}$.

5.1. Line Integral of Scalar Fields

$$\int_C ds\ \phi(\vec{r})\Big|_{\vec{r}=\vec{r}(t)} = \underbrace{\int_{t_i}^{t_f}}_{\text{limits on } t} dt\ \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}}_{\text{function of } t}\ \underbrace{\phi(\vec{r}(t))}_{\text{function of } t}$$

$\vec{r}(t) = (x(t), y(t), z(t))$

Example: $\phi(x,y) = \alpha x y$



$\vec{r}(t) = (x(t), y(t)) = (2\cos t, 2\sin t)$ $\qquad 0 \le t \le \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} dt\ \sqrt{(-2\sin t)^2 + (2\cos t)^2}\ \ 4\alpha \cos t \sin t = 8\alpha \int_0^{\frac{\pi}{2}} dt \cos t \sin t = 4\alpha$$

Example: Gravitational energy stored in a helix with n turns and mass density λ. $\vec{r}(\theta) = (R\cos\theta, R\sin\theta, H\theta)$ [see page 7]

$$U = \int dm\ g\ z = \lambda g \int ds\ z = \lambda g \int_0^{2\pi n} d\theta \sqrt{R^2+H^2} \cdot H\theta = \lambda g H \sqrt{R^2+H^2}\,(2\pi^2 n^2)$$

$$M = \int dm = \lambda \int ds = \lambda \int_0^{2\pi n} d\theta \sqrt{R^2+H^2} = \lambda (2\pi n) \sqrt{R^2+H^2}$$

$$\rightarrow \quad U = Mg(\pi n H) = \frac{1}{2} M g H_{max} \qquad , \ H_{max} = 2\pi n H$$

5.2. Line Integral of Vector Fields

$$\int_C d\vec{r}\cdot\vec{F}(\vec{r}) = \int_C dt\,\frac{d\vec{r}}{dt}\cdot\vec{F}(r)\Big|_{\vec{r}=r(t)} = \int_C dx\,F_x + \int_C dy\,F_y + \int_C dz\,F_z$$

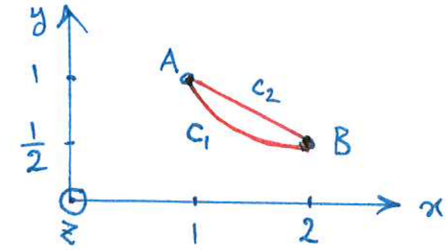
An example in physics is the work done by a force $\vec{F}(\vec{r})$.

Example: $I = \int_C d\vec{r}\cdot\vec{F}(\vec{r})$ with $\vec{F}(x,y,z) = (xy^2+z,\ x^2y+2,\ x)$

along two paths going from $A = (1,1,0)$ to $B = (2,\frac{1}{2},0)$

C_1: $y = \frac{1}{x}$, $z=0$

C_2: straight line $y = \frac{3}{2} - \frac{1}{2}x$, $z=0$



we choose x as parameter.

C_1: $d\vec{r} = (dx, -\frac{1}{x^2}dx, 0)$, $\vec{F} = (\frac{1}{x}, x+2, x)$

$$I = \int_1^2 dx\left[1\times\frac{1}{x} - \frac{1}{x^2}\times(x+2)\right] = -2\int_1^2 \frac{dx}{x^2} = \frac{2}{x}\Big|_1^2 = -1$$

C_2: $d\vec{r} = (dx, -\frac{1}{2}dx, 0)$, $\vec{F} = \left(x(\frac{3}{2}-\frac{x}{2})^2,\ x^2(\frac{3}{2}-\frac{x}{2})+2,\ x\right)$

$$I = \int_1^2 dx\left[x\left(\frac{3-x}{2}\right)^2 - \frac{x^2}{2}\left(\frac{3-x}{2}\right) - 1\right] = \int_1^2 dx\left[\frac{x^3}{2} - \frac{9x^2}{4} + \frac{9x}{4} - 1\right] = -1$$

In this example, we obtain the same result for both paths.

Exercise: Calculate the above integral along a third path of your own choice.

5.3. Conservative Vector Fields

In general, the result of a line integral between two points A and B depends on the integration path. However, there is a special class of vector fields for which the result is independent of the path for ANY two points A and B, i.e.

$$\boxed{\int_{C_{A\to B}} d\vec{r}\cdot\vec{F}(\vec{r}) = \phi(B) - \phi(A)}$$

where $\phi(\vec{r})$ is a scalar field. Such vector fields are called conservative.

Since A and B are arbitrary, we can consider a case where B is infinitesimally close to A, $\quad d\vec{r}\cdot\vec{F}(\vec{r}) = \phi(B) - \phi(A) = d\phi$

on the other hand, we know that $\quad d\phi = \vec{\nabla}\phi\cdot d\vec{r}$

therefore $\quad \vec{F}(\vec{r})\cdot d\vec{r} = \vec{\nabla}\phi\cdot d\vec{r}$, and since $d\vec{r}$ is arbitrary

$$\boxed{\vec{F}(\vec{r}) = \vec{\nabla}\phi(\vec{r})}$$

So, conservative vector fields can always be written as the gradient a scalar "potential".

A consequence of the path-independence of the line integral of conservative vector fields is that the integral over any closed path vanishes: $\boxed{\oint d\vec{r}\cdot\vec{F} = 0}$

If $\vec{F} = \vec{\nabla}\phi$, then

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \phi}{\partial x} = \frac{\partial^2\phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial \phi}{\partial y} = \frac{\partial F_y}{\partial x} \rightarrow \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = 0$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial F_z}{\partial x} \rightarrow \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} = 0$$

Second derivatives commute.

$$\frac{\partial F_y}{\partial z} = \frac{\partial^2 \phi}{\partial z \partial y} = \frac{\partial F_z}{\partial y} \rightarrow \frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial y} = 0$$

The three quantities in the box can be viewed as components of

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

vector product of the "del operator" $\vec{\nabla}$ and $\vec{F}$, formally called: curl $\vec{F}$

So, in short: if $\vec{F} = \vec{\nabla}\phi$, then $\vec{\nabla} \times \vec{F} = 0$.

The reverse is also true: if $\vec{\nabla} \times \vec{F} = 0$, then $\vec{F} = \vec{\nabla}\phi$

as we will see later. [see page 58]

A consequence of the above is that the general differential

$$F_x(x,y,z)\,dx + F_y(x,y,z)\,dy + F_z(x,y,z)\,dz = \vec{F}\cdot d\vec{r}$$

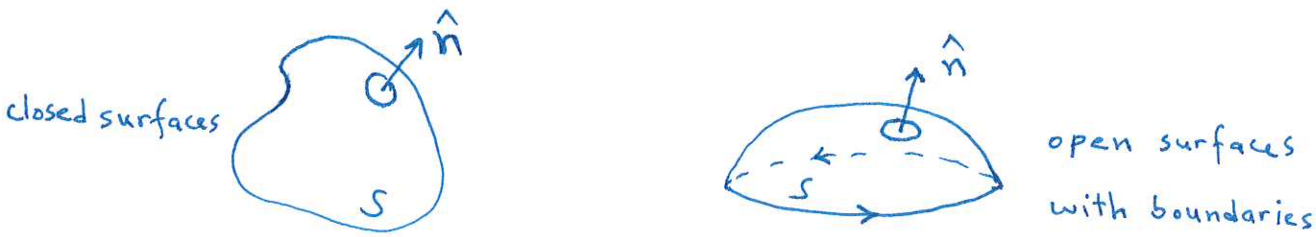
is exact, i.e. can be written as a total differential $d\phi$

iff $\vec{\nabla} \times \vec{F} = 0$.

Exercise: Show that $2xyz\,dx + x^2 z\,dy + x^2 y\,dz$ is an exact differential.

6. Surface Integrals

A surface integral is an integral along a surface that is defined by an embedding $\vec{r}(u,v)$ or a constraint $\psi(x,y,z)=0$. The measure of integral could be either the scalar area element dA, or the vector area element $d\vec{S} = dA\,\hat{n}$, which points towards the normal vector.



We use the following convention to fix the direction of the normal $\hat{n}$:

(i) If S is a closed surface then $\hat{n}$ is taken to always point outwards.

(ii) If S is open, we orient the boundary and obtain the direction of $\hat{n}$ by the ride-hand rule.

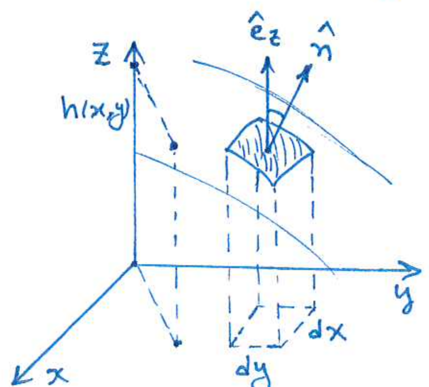
6.1. Surface Integral of Scalar Fields

$$\int_S dA\,\phi(\vec{r})\Big|_{\vec{r}=\vec{r}(u,v)} = \int du\,dv \left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right| \phi(\vec{r}(u,v))$$

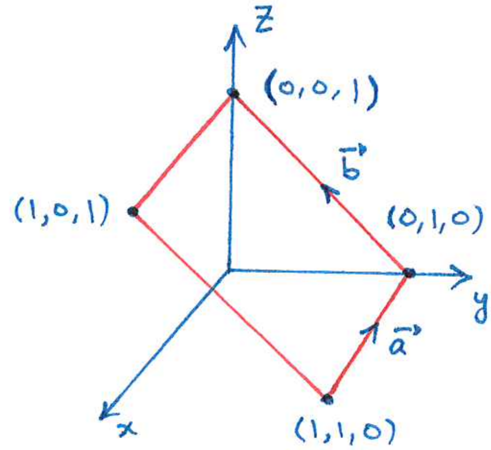
or in Monge representation [the method of projection]

$$\int dx\,dy \sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2}\;\phi(x,y,h(x,y))$$

when $\vec{r}(x,y) = (x,y,h(x,y))$.



<u>Example</u>: Tilted plate



$$z = h(x,y) = 1-y$$

$$\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{1}{\sqrt{2}}(0,1,1)$$

$$dA = \sqrt{2}\,dx\,dy$$

$$S = \int_0^1 dx \int_0^1 dy\,\sqrt{2} = \sqrt{2}$$

$$z_{CM} = \frac{1}{S}\int dx\,dy\sqrt{2}\;z(x,y) = \frac{1}{\sqrt{2}}\int_0^1 dx\int_0^1 dy\,\sqrt{2}\,(1-y) = \frac{1}{2} \quad \text{(uniform density)}$$

<u>Example</u>: A Pringles Crisp

can be described as a hyperbolic paraboloid

$z = h(x,y) = \frac{1}{2R}(x^2 - y^2)$, where R is the radius of curvature,

bounded by a cylindrical surface of radius a (the container); $x^2+y^2 \le a^2$.

The total surface area of the crisp can be calculated as

$$S = \int dx\,dy\sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2} = \int dx\,dy\sqrt{1+\frac{x^2}{R^2}+\frac{y^2}{R^2}} = \int_0^a 2\pi r\,dr\sqrt{1+\frac{r^2}{R^2}}$$

$$S = \pi R^2\int_0^{\frac{a^2}{R^2}} du\sqrt{1+u} = \pi R^2\,\frac{2}{3}(1+u)^{3/2}\Big|_0^{\frac{a^2}{R^2}} = \frac{2\pi R^2}{3}\left[\left(1+\frac{a^2}{R^2}\right)^{3/2}-1\right]$$

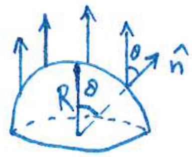
6.2. Surface Integral of Vector Fields

$$\int_S d\vec{S}\cdot\vec{F}(\vec{r}) = \int_S dA\ \hat{n}\cdot\vec{F}(\vec{r})\Big|_{\vec{r}=\vec{r}(u,v)} = \int_S du\,dv\left(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right)\cdot\vec{F}(\vec{r}(u,v))$$

This integral gives us the flux of the vector field $\vec{F}(\vec{r})$ going through the surface S.

Example: Flux of the (uniform) vector field.

$\vec{F}(\vec{r}) = F_0\,\hat{e}_z$ through the upper hemispherical cap



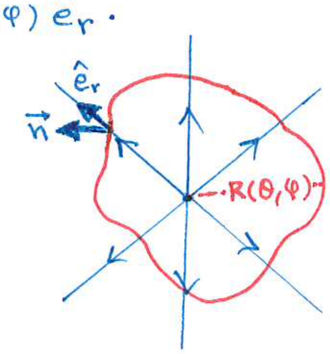
$$\Phi = \int_S d\vec{S}\cdot\vec{F} = \int R^2\sin\theta\, d\varphi\, d\theta\ \underbrace{\overset{\hat{n}}{\hat{e}_r}\cdot\hat{e}_z}_{\cos\theta}\ F_0$$

$$= 2\pi R^2 F_0 \int_0^{\frac{\pi}{2}} d\theta\,\sin\theta\cos\theta = \pi R^2 F_0 = \text{flux through the base.}$$

Example: Proof of Gauss' law in electrostatics.

Surface integral of the electric field due to a point charge q at the origin $\vec{E}(\vec{r}) = \dfrac{q}{4\pi\varepsilon_0}\dfrac{\hat{e}_r}{r^2}$

around an arbitrary closed surface $\vec{r}(\theta,\varphi) = R(\theta,\varphi)\,\hat{e}_r$.



$$\frac{\partial\vec{r}}{\partial\theta} = \frac{\partial R}{\partial\theta}\hat{e}_r + R\frac{\partial\hat{e}_r}{\partial\theta} = \frac{\partial R}{\partial\theta}\hat{e}_r + R\hat{e}_\theta$$

$$\frac{\partial\vec{r}}{\partial\varphi} = \frac{\partial R}{\partial\varphi}\hat{e}_r + R\frac{\partial\hat{e}_r}{\partial\varphi} = \frac{\partial R}{\partial\varphi}\hat{e}_r + R\sin\theta\,\hat{e}_\varphi$$

$$\vec{n} = \frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial\varphi} = R^2\sin\theta\,\hat{e}_r - \frac{\partial R}{\partial\theta}R\sin\theta\,\hat{e}_\theta - \frac{\partial R}{\partial\varphi}R\,\hat{e}_\varphi$$

$$\int d\theta\, d\varphi\ \vec{n}\cdot\vec{E}\Big|_{\vec{r}=R(\theta,\varphi)\hat{e}_r} = \int d\theta\, d\varphi\ R(\theta,\varphi)^2\sin\theta\cdot\frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{R(\theta,\varphi)^2} = \frac{q}{4\pi\varepsilon_0}\overbrace{\int\sin\theta\, d\theta\, d\varphi}^{4\pi} = \frac{q}{\varepsilon_0}$$

Notes: $\dfrac{\partial\hat{e}_r}{\partial\theta} = \hat{e}_\theta$, $\dfrac{\partial\hat{e}_r}{\partial\varphi} = \sin\theta\,\hat{e}_\varphi$ [see page 13]

$$\vec{n}(\theta,\varphi)\cdot\hat{e}_r = R(\theta,\varphi)^2\sin\theta$$

7. Divergence and Curl

We have already met the "del operator" $\vec{\nabla}$, which is a vector differential operator, through the definition of gradient:

$$\vec{\nabla}\phi(x,y,z) = \frac{\partial\phi}{\partial x}\hat{e}_x + \frac{\partial\phi}{\partial y}\hat{e}_y + \frac{\partial\phi}{\partial z}\hat{e}_z$$

$$\Rightarrow \qquad \vec{\nabla} \equiv \hat{e}_x\frac{\partial}{\partial x} + \hat{e}_y\frac{\partial}{\partial y} + \hat{e}_z\frac{\partial}{\partial z}$$

The operator acts on a scalar function ϕ and returns a vector $\vec{\nabla}\phi$, and satisfies identities inherited from usual derivatives, e.g.

$$\vec{\nabla}(\phi+\psi) = \vec{\nabla}\phi + \vec{\nabla}\psi$$

$$\vec{\nabla}(\alpha\phi) = \alpha\vec{\nabla}\phi \qquad (\text{for constant } \alpha)$$

Using $\vec{\nabla}$ we can define two other important operations involving vector fields: **divergence** and curl.

7.1. Divergence

Let $\vec{F}(x,y,z)$ be a differentiable vector field

$$\vec{F}(x,y,z) = F_x(x,y,z)\hat{e}_x + F_y(x,y,z)\hat{e}_y + F_z(x,y,z)\hat{e}_z$$

then the divergence of $\vec{F}$ is defined as

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

which is a <u>scalar</u> field itself.

Example:

$$\vec{F} = (y^2 z + x^3)\,\hat{e}_x + x\,\hat{e}_y + z^3\,\hat{e}_z$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial(y^2 z + x^3)}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial z^3}{\partial z} = 3x^2 + 3z^2$$

Later, using Gauss' Theorem we will give a physical interpretation of $\vec{\nabla}\cdot\vec{F}$ as the measure of the flux of $\vec{F}$ out of (or into) a small volume surrounding a given point (per unit volume).

Example: vector fields with non-zero divergence

Basic identities:

$$\vec{\nabla}\cdot(\vec{F} + \vec{G}) = \vec{\nabla}\cdot\vec{F} + \vec{\nabla}\cdot\vec{G}$$

$$\vec{\nabla}\cdot(\alpha\vec{F}) = \alpha\,\vec{\nabla}\cdot\vec{F} \qquad \text{(for constant } \alpha\text{)}$$

7.2. Curl

For a differentiable vector field $\vec{F}$, the curl of $\vec{F}$ is defined as

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{e}_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{e}_y + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{e}_z$$

which is a vector field itself.

As can be done with a vector (cross) product, we can also write $\vec{\nabla}\times\vec{F}$ as a determinant with the understanding that components of $\vec{\nabla}$ act on components of $\vec{F}$:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

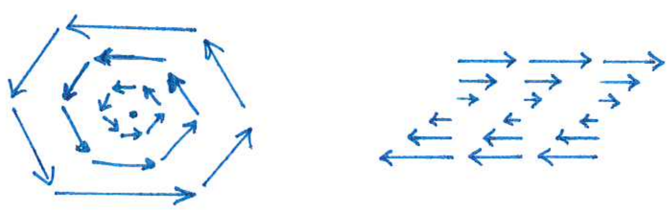
Example:

$$\vec{F} = -y\,\hat{e}_x + x\,\hat{e}_y \quad \rightarrow \quad \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$\vec{\nabla}\times\vec{F} = \hat{e}_x\left(-\underbrace{\frac{\partial x}{\partial z}}_{0}\right) - \hat{e}_y\left(-\underbrace{\frac{\partial(-y)}{\partial z}}_{0}\right) + \hat{e}_z\underbrace{\left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right)}_{2} = 2\,\hat{e}_z$$

Later, using Stokes' theorem we will give a physical interpretation of $\vec{\nabla}\times\vec{F}$ as a measure of the rate (and the direction of the axis) of rotation of $\vec{F}$ about a given point.

Example: vector fields with non-zero curl



Basic identities:

$$\vec{\nabla}\times(\vec{F}+\vec{G}) = \vec{\nabla}\times\vec{F} + \vec{\nabla}\times\vec{G}$$

$$\vec{\nabla}\times(\alpha\vec{F}) = \alpha\,\vec{\nabla}\times\vec{F} \qquad (\text{for constant } \alpha)$$

7.3. Index-Notation

As we have already experienced, index-notation is a very useful tool for matrix manipulations, coordinate transformations, and proving identities involving vector operators. We label Cartesian directions with indices (subscripts) that take the values $i = \overset{x}{1}, \overset{y}{2}, \overset{z}{3}$ (same with $j, k, l, \dots$).

Then $\vec{F} = F_x \hat{e}_x + F_y \hat{e}_y + F_z \hat{e}_z = F_1 \hat{e}_1 + F_2 \hat{e}_2 + F_3 \hat{e}_3 = \sum_i F_i \hat{e}_i$

To see how this notation facilitates calculations, consider another vector $\vec{G} = \sum_j G_j \hat{e}_j$ (indices are "dummy"; we could use anything):

$$\vec{F} \cdot \vec{G} = \left[\sum_i F_i \hat{e}_i\right]\left[\sum_j G_j \hat{e}_j\right] = \sum_{ij} F_i G_j \underbrace{(\hat{e}_i \cdot \hat{e}_j)}_{\delta_{ij}} = \sum_i F_i \overbrace{\sum_j \delta_{ij} G_j}^{G_i} = \sum_i F_i G_i$$

Note how we used the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

to replace any occurrence of j with i within a j-summation cycle.

$$\boxed{\hat{e}_i \cdot \hat{e}_j = \delta_{ij}}$$

A further immense simplification results if we adopt "Einstein Summation Convention" that says

whenever indices are repeated on the same side of an equation a summation over the indices is implied. (say $\vec{F} = F_i \hat{e}_i$ then $\vec{F} \cdot \vec{G} = [F_i \hat{e}_i][G_j \hat{e}_j] = F_i G_j \underbrace{\hat{e}_i \cdot \hat{e}_j}_{\delta_{ij}} = F_i G_i$)

The notation can be applied to derivatives:

a further simplificatio[n]

$$\vec{\nabla} = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} = \hat{e}_i \frac{\partial}{\partial x_i} = \hat{e}_i \partial_i \qquad \left[\partial_i \equiv \frac{\partial}{\partial x_i} \right]$$

Then $\vec{\nabla}\phi = \hat{e}_i \partial_i \phi$, and

$$\vec{\nabla}\cdot\vec{F} = \left[\hat{e}_i \partial_i\right]\left[F_j \hat{e}_j\right] = \partial_i F_j \overbrace{\hat{e}_i \cdot \hat{e}_j}^{\delta_{ij}} = \partial_i F_i$$

Note: we have used $\partial_i \hat{e}_j = 0$ in the above [valid for Cartesian coo[rds]]

What about the vector (cross) product? and curl?

To represent this in index-notation we need to introduce the Levi-Civita "epsilon symbol" denoted as ϵ_{ijk}, and defined as

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{in all other cases} \end{cases}$$

In other words, $\boxed{\epsilon_{ijk} \text{ is antisymmetric in all its indices.}}$

$\epsilon_{123} = +1$, $\epsilon_{132} = -1$ [as needs one swap to get back to 123]

$\epsilon_{321} = -1$ [as needs 3 adjacent pairwise swaps to get back to 123]

$\epsilon_{121} = 0$ [as neither even nor odd permutation of 123]

Any component with repeated indices vanishes:

$$\epsilon_{111} = \epsilon_{222} = \epsilon_{333} = \epsilon_{112} = \epsilon_{113} = \epsilon_{223} = \epsilon_{221} = \cdots = 0$$

Using ϵ_{ijk}, we can write the components of the vector product between two vectors $\vec{F} = F_i \hat{e}_i$ and $\vec{G} = G_j \hat{e}_j$ as

$$(\vec{F} \times \vec{G})_i = \epsilon_{ijk} F_j G_k$$

$$i=1 ; \quad (\vec{F}\times\vec{G})_1 = \epsilon_{1jk} F_j G_k = \overbrace{\epsilon_{123}}^{+1} F_2 G_3 + \overbrace{\epsilon_{132}}^{-1} F_3 G_2 = F_2 G_3 - F_3 G_2$$

$$i=2 ; \quad (\vec{F}\times\vec{G})_2 = \epsilon_{2jk} F_j G_k = \overbrace{\epsilon_{213}}^{-1} F_1 G_3 + \overbrace{\epsilon_{231}}^{+1} F_3 G_1 = F_3 G_1 - F_1 G_3$$

$$i=3 ; \quad (\vec{F}\times\vec{G})_3 = \epsilon_{3jk} F_j G_k = \overbrace{\epsilon_{312}}^{+1} F_1 G_2 + \overbrace{\epsilon_{321}}^{-1} F_2 G_1 = F_1 G_2 - F_2 G_1$$

Equivalently, we can write this as

$$\boxed{\vec{F} \times \vec{G} = \epsilon_{ijk} \hat{e}_i F_j G_k}$$

Note that we can represent the ϵ-symbol as the triple product of the basis vectors: $\boxed{\hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k) = \epsilon_{ijk}}$

or $\hat{e}_j \times \hat{e}_k = \epsilon_{ijk} \hat{e}_i$.

Now, for curl we have

$$\vec{\nabla} \times \vec{F} = [\hat{e}_j \partial_j] \times [F_k \hat{e}_k] = \overbrace{\hat{e}_j \times \hat{e}_k}^{\epsilon_{ijk} \hat{e}_i} \partial_j F_k = \epsilon_{ijk} \hat{e}_i \partial_j F_k$$

or in components: $\boxed{(\vec{\nabla} \times \vec{F})_i = \epsilon_{ijk} \partial_j F_k}$

Note: we have used $\partial_j \hat{e}_k = 0$, which is valid for Cartesian coordinates.

Let us now use index-notation to prove a number of identities linear in $\vec{\nabla}$:

$$* \quad \vec{\nabla}\cdot(\vec{F}\times\vec{G}) = \partial_i\left[\epsilon_{ijk} F_j G_k\right] = \epsilon_{ijk}\,\partial_i\left(F_j G_k\right)$$

$$= \epsilon_{ijk}(\partial_i F_j)G_k + \epsilon_{ijk} F_j(\partial_i G_k)$$

$$= +\epsilon_{kij}\, G_k\,(\partial_i F_j) - \epsilon_{jik}\, F_j(\partial_i G_k)$$

'+' because 2 swaps in $\epsilon_{ijk}\to\epsilon_{ikj}\to\epsilon_{kij}$

'−' because 1 swap in $\epsilon_{ijk}\to\epsilon_{jik}$

$$= \vec{G}\cdot(\vec{\nabla}\times\vec{F}) - \vec{F}\cdot(\vec{\nabla}\times\vec{G})$$

$$* \quad \vec{\nabla}\cdot(\phi\vec{F}) = \partial_i(\phi F_i) = \phi\,\partial_i F_i + F_i\,\partial_i\phi = \phi(\vec{\nabla}\cdot\vec{F}) + \vec{F}\cdot(\vec{\nabla}\phi)$$

$$* \quad \vec{\nabla}(\phi\psi) = \phi\vec{\nabla}\psi + \psi\vec{\nabla}\phi$$

<u>Exercise:</u> show that $\vec{\nabla}\times(\phi\vec{F}) = \phi\,\vec{\nabla}\times\vec{F} - \vec{F}\times\vec{\nabla}\phi$.

A particular class of identities that involve two cross products requires special attention. Let us first look at this structure in vectors:

$$\left[\vec{A}\times(\vec{B}\times\vec{C})\right]_i = \epsilon_{ijk}\,A_j\,(\vec{B}\times\vec{C})_k = \epsilon_{ijk}\,\epsilon_{klm}\,A_j\,B_l\,C_m$$

To proceed further, we need to work out what $\epsilon_{ijk}\,\epsilon_{klm}$ is. (k-summed over)

$$\epsilon_{ijk}\,\epsilon_{klm} \overset{\text{2 swaps}}{=} \epsilon_{kij}\,\epsilon_{klm} = \begin{cases} +1 & \begin{bmatrix}i \,\&\, j \text{ are different}\\ l \,\&\, m \text{ are different}\end{bmatrix} \,\&\, \begin{bmatrix} i=l \\ j=m\end{bmatrix} \\ -1 & \qquad\qquad // \qquad\qquad \&\, \begin{bmatrix} i=m \\ j=l\end{bmatrix} \\ 0 & \text{otherwise}\end{cases}$$

This can be written as $\boxed{\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}}$

The combination $\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$ reflects all the symmetry properties of $\epsilon_{kij}\epsilon_{k\ell m}$: antisymmetry with respect to $i \leftrightarrow j$ swap or $\ell \leftrightarrow m$ swap seperately and symmetry with respect to simultaneous $i \leftrightarrow j$ & $\ell \leftrightarrow m$ swaps.

Going back to our vector identity:

$$\left[\vec{A} \times (\vec{B} \times \vec{C})\right]_i = \epsilon_{kij}\epsilon_{k\ell m} A_j B_\ell C_m = \left(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}\right) A_j B_\ell C_m$$

$$= \left(\delta_{i\ell} B_\ell\right)\left(\delta_{jm} A_j C_m\right) - \left(\delta_{im} C_m\right)\left(\delta_{j\ell} A_j B_\ell\right)$$

$$= B_i \left(A_j C_j\right) - C_i \left(A_j B_j\right)$$

$$\rightarrow \boxed{\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A}\cdot\vec{C}) - C\,(\vec{A}\cdot\vec{B})}$$

Now we are equipped with the tool needed to tackle $\vec{\nabla} \times (\vec{F} \times \vec{G})$.

$$* \left[\vec{\nabla} \times (\vec{F} \times \vec{G})\right]_i = \epsilon_{ijk}\,\partial_j\,(\vec{F} \times \vec{G})_k = \epsilon_{ijk}\epsilon_{k\ell m}\,\partial_j\left(F_\ell G_m\right)$$

$$= \left(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}\right)\partial_j\left(F_\ell G_m\right) = \partial_j\left(F_i G_j\right) - \partial_j\left(F_j G_i\right)$$

$$= G_j\partial_j F_i + F_i\partial_j G_j - G_i\partial_j F_j - F_j\partial_j G_i$$

$$= (\vec{G}\cdot\vec{\nabla})F_i + F_i\,\vec{\nabla}\cdot\vec{G} - G_i\,\vec{\nabla}\cdot\vec{F} - (\vec{F}\cdot\vec{\nabla})G_i$$

in other words

$$\boxed{\vec{\nabla} \times (\vec{F} \times \vec{G}) = (\vec{G}\cdot\vec{\nabla})\vec{F} + \vec{F}\,(\vec{\nabla}\cdot\vec{G}) - \vec{G}\,(\vec{\nabla}\cdot\vec{F}) - (\vec{F}\cdot\vec{\nabla})\,\vec{G}}$$

8. Second Order Vector Operators

Many key equations in physics — such as wave equation, Schrödinger's eqn, diffusion eqn, Poisson's eqn — involve expressions quadratic in $\vec{\nabla}$. Using the tools we developed, we can look at all possible combinations of $\vec{\nabla}\,\vec{\nabla}$.

8.1. Two $\vec{\nabla}$'s Acting on Scalar Fields

* $\vec{\nabla}\cdot(\vec{\nabla}\phi)$, which is a scalar.

$$\vec{\nabla}\cdot(\vec{\nabla}\phi) = \hat{e}_i\partial_i\cdot(\hat{e}_j\partial_j\phi) = \overbrace{(\hat{e}_i\cdot\hat{e}_j)}^{\delta_{ij}}\partial_i\partial_j\phi = \delta_{ij}\partial_i\partial_j\phi = \partial_i\partial_i\phi = \partial_i^2\phi$$

(this step possible in Cartesian coordinates since $\partial_i\hat{e}_j = 0$)

This operator is called the Laplacian: $\vec{\nabla}\cdot(\vec{\nabla}\phi) \equiv \nabla^2\phi$

$$\boxed{\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi}$$

in standard notation

* $\vec{\nabla}\times(\vec{\nabla}\phi)$, which is a vector.

$$\underbrace{[\vec{\nabla}\times(\vec{\nabla}\phi)]_i = \underset{\text{antisymmetric}}{\epsilon_{ijk}}\,\overset{\text{symmetric}}{\partial_j\partial_k}\phi}_{A} \underset{\text{relabel } j\,\&\,k}{=} \epsilon_{ikj}\partial_k\partial_j\phi = \underbrace{-\epsilon_{ijk}\overset{\text{derivatives commute}}{\partial_j\partial_k}\phi}_{-A \ (1\text{ swap} \to -)} \to A = 0$$

$\to$

$$\boxed{\vec{\nabla}\times(\vec{\nabla}\phi) = 0}$$

The curl of any gradient is identically zero.

8.2. Two $\vec{\nabla}$'s Acting on Vector Fields

$*$ $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})$, which is a scalar.

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = \partial_i\left(\epsilon_{ijk}\,\partial_j F_k\right) = \underbrace{\epsilon_{ijk}}_{\text{anti-symmetric}}\,\overbrace{\partial_i\partial_j}^{\text{symmetric}} F_k = 0 \qquad [\text{as in } \vec{\nabla}\times\vec{\nabla}\phi]$$

$\rightarrow$ $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$

The divergence of any curl is identically zero.

$*$ $\vec{\nabla}(\vec{\nabla}\cdot\vec{F})$, which is a vector.

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{F}) = \hat{e}_i\,\partial_i\left(\partial_j F_j\right)$$

$*$ $(\vec{\nabla}\cdot\vec{\nabla})\vec{F}$, which is a vector.

$$(\vec{\nabla}\cdot\vec{\nabla})\vec{F} = \partial_j\partial_j\,(F_i\,\hat{e}_i) \underset{\text{because } \partial_j \hat{e}_i = 0 \text{ in Cartesian coordinates}}{=} \hat{e}_i\,\partial_j^2 F_i$$

$$\boxed{(\vec{\nabla}\cdot\vec{\nabla})\vec{F} = \nabla^2\vec{F} = \hat{e}_x\,\nabla^2 F_x + \hat{e}_y\,\nabla^2 F_y + \hat{e}_z\,\nabla^2 F_z}$$ in standard notation

$*$ $\vec{\nabla}\times(\vec{\nabla}\times\vec{F})$, which is a vector.

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \hat{e}_i\,\epsilon_{ijk}\,\partial_j\left(\epsilon_{klm}\,\partial_l F_m\right) = \hat{e}_i\,\overbrace{\epsilon_{ijk}\,\epsilon_{klm}}^{(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})}\,\partial_j\partial_l F_m$$

$$= \hat{e}_i\,\delta_{il}\,\delta_{jm}\,\partial_j\partial_l F_m - \hat{e}_i\,\delta_{im}\,\delta_{jl}\,\partial_j\partial_l F_m$$

$$= \hat{e}_i\,\partial_i\,\partial_j F_j - \hat{e}_i\,\partial_j^2 F_i$$

$$\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \nabla^2\vec{F}}$$

$* \; (\vec{\nabla}\times\vec{\nabla})\cdot\vec{F} = \epsilon_{ijk}\partial_j\partial_k F_i = 0$ (a scalar)

__Note 1__: we found that if $\vec{F} = \nabla\phi$, then $\vec{\nabla}\times\vec{F} = 0$. The reverse is also true: if $\vec{\nabla}\times\vec{F} = 0$, then $\vec{F} = \nabla\phi$, for some ϕ. [page 33]

__Note 2__: we found that if $\vec{F} = \vec{\nabla}\times\vec{G}$, then $\vec{\nabla}\cdot\vec{F} = 0$. The reverse is also true: if $\vec{\nabla}\cdot\vec{F} = 0$, then $\vec{F} = \vec{\nabla}\times\vec{G}$, for some $\vec{G}$.

__Example__: Spherical symmetry

If $\phi(\vec{r}) = \phi(|\vec{r}|) = \phi(r)$, we have $\left[r^2 = x_i x_i \; ; \; r = \sqrt{x_i x_i} \; ; \; \partial_i r = \frac{x_i}{r} \right]$

$$\partial_i \phi(r) = \phi'(r)\,\partial_i r = \phi' \frac{x_i}{r} \quad \rightarrow \quad \boxed{\vec{\nabla}\phi = \frac{d\phi}{dr}\,\hat{e}_r}$$

$$\partial_i\partial_j\phi = \frac{d}{dr}\left[\frac{1}{r}\left(\frac{d\phi}{dr}\right)\right]\frac{x_i x_j}{r} + \frac{1}{r}\frac{d\phi}{dr}\delta_{ij} = \left(\phi'' - \frac{\phi'}{r}\right)\frac{x_i x_j}{r^2} + \frac{\phi'}{r}\delta_{ij}$$

$$\nabla^2\phi = \delta_{ij}\partial_i\partial_j\phi = \left(\phi'' - \frac{\phi'}{r}\right)\underbrace{\delta_{ij}\frac{x_i x_j}{r^2}}_{1} + \frac{\phi'}{r}\underbrace{\delta_{ij}\delta_{ij}}_{3} \quad \rightarrow \quad \boxed{\nabla^2\phi = \phi'' + \frac{2}{r}\phi'}$$

$$\phi(r) = \frac{1}{r} \rightarrow \partial_i\partial_j\left(\frac{1}{r}\right) = -\frac{1}{r^3}\left[\delta_{ij} - 3\frac{x_i x_j}{r^2}\right] \quad \text{(dipolar field)}$$

$$\nabla^2\left(\frac{1}{r}\right) = -\frac{3}{r^3} + \frac{3}{r^3} = 0$$

[every where except $r=0$, where the cancellation might not be well defined]

$$\vec{F}(r) = \psi(r)\,\hat{e}_r \quad , \quad F_i = \psi(r)\frac{x_i}{r}$$

$$\partial_i F_j = \partial_i\left[\frac{\psi(r)}{r}\right]x_j + \left(\frac{\psi}{r}\right)\partial_i x_j = \frac{1}{r}\frac{d}{dr}\left(\frac{\psi}{r}\right)x_i x_j + \left(\frac{\psi}{r}\right)\delta_{ij}$$

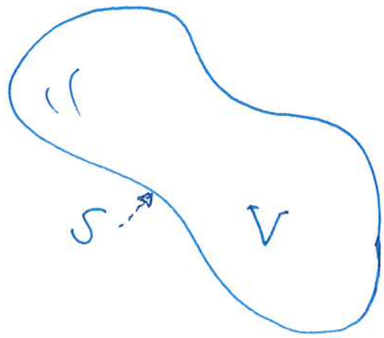
$$\vec{\nabla}\cdot\vec{F} = \partial_i F_i = r\frac{d}{dr}\left(\frac{\psi}{r}\right) + 3\left(\frac{\psi}{r}\right) \quad \rightarrow \quad \boxed{\vec{\nabla}\cdot\vec{F} = \psi' + \frac{2}{r}\psi}$$

$$\vec{\nabla}\times\vec{F} = \hat{e}_k\,\epsilon_{kij}\,\partial_i F_j = \hat{e}_k\left[\frac{1}{r}\frac{d}{dr}\left(\frac{\psi}{r}\right)\underbrace{\epsilon_{kij}x_i x_j}_{0} + \left(\frac{\psi}{r}\right)\underbrace{\epsilon_{kij}\delta_{ij}}_{0}\right] \rightarrow \boxed{\vec{\nabla}\times\vec{F} = 0}$$

9. Gauss' Theorem

The statement of Gauss' Divergence Theorem:

Let V be a volume in space that is bounded by the closed surface S, and $\vec{F}(x,y,z)$ be a continuous and differentiable vector field defined in this region. Then the total flux of $\vec{F}$ coming out of the closed surface S is equal to the volume integral of the divergence of $\vec{F}$:



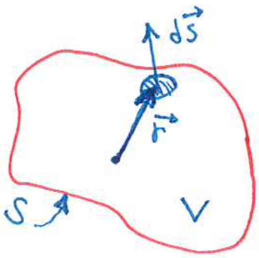
$$\boxed{\oint_S d\vec{S}\cdot\vec{F} = \int_V dV\ \vec{\nabla}\cdot\vec{F}}$$

$\oint_S$ symbol denotes integration over closed surface S.

Let us see how it works in practice, helping us convert certain volume and surface integrals onto one another, before we prove it.

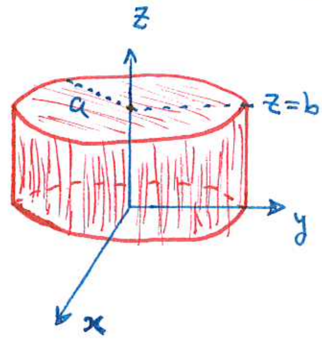
Example: The volume enclosed by an arbitrary surface S can be calculated as $V = \frac{1}{3}\oint_S d\vec{S}\cdot\vec{r}$. $\left[\vec{\nabla}\cdot\vec{r} = \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 3\right]$

$$\oint_S d\vec{S}\cdot\vec{r} = \int_V dV\ \underbrace{\vec{\nabla}\cdot\vec{r}}_{3} = 3\int_V dV = 3V.$$



Example: $\oint_S d\vec{S}\cdot(\vec{\nabla}\times\vec{G}) = 0$ identically for arbitrary S, and any $\vec{G}$.

<u>Example</u>: Evaluate I over the cylindrical shell S



$$I = \oint_S \left[x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy \right]$$

we have $F_x = x^3$, $F_y = x^2 y$, $F_z = x^2 z$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial x^3}{\partial x} + \frac{\partial (x^2 y)}{\partial y} + \frac{\partial (x^2 z)}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2$$

$I = \int dV\, 5x^2$; in cylindrical coordinates $dV = r\,dr\,d\varphi\,dz$, $x = r\cos\varphi$

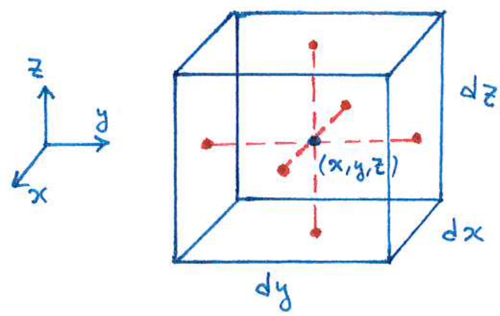
$$I = \int_0^b dz \int_0^a r\,dr \int_0^{2\pi} d\varphi\; 5 r^2 \cos^2\varphi = 5b \underbrace{\int_0^a r^3\, dr}_{a^4/4} \underbrace{\int_0^{2\pi} d\varphi \cos^2\varphi}_{\frac{1}{2}\times 2\pi = \pi} = \frac{5\pi}{4} b a^4 .$$

The proof of Gauss' Divergence Theorem:

Let us first look at a special simple case, where the volume is an infinitesimal cuboid at (x, y, z), with sides of length dx, dy, and dz.

$\oint_S d\vec{S}\cdot\vec{F}$ has six contributions:

which face?



$\simeq +\, dy\, dz\, F_x\left(x + \frac{dx}{2}, y, z\right)$	front
$-\, dy\, dz\, F_x\left(x - \frac{dx}{2}, y, z\right)$	back
$+\, dx\, dz\, F_y\left(x, y + \frac{dy}{2}, z\right)$	right
$-\, dx\, dz\, F_y\left(x, y - \frac{dy}{2}, z\right)$	left
$+\, dx\, dy\, F_z\left(x, y, z + \frac{dz}{2}\right)$	top
$-\, dx\, dy\, F_z\left(x, y, z - \frac{dz}{2}\right)$	bottom

Now we can use <u>Taylor expansion</u> to simplify the result.

sign determined by outward normal vs $\hat{e}_i$.

$$\text{front} + \text{back} = dy\,dz \left[\overbrace{F_x\left(x + \frac{dx}{2}, y, z\right)}^{\simeq F_x(x,y,z) + \frac{dx}{2} \partial_x F_x(x,y,z)} - \underbrace{F_x\left(x - \frac{dx}{2}, y, z\right)}_{\simeq F_x(x,y,z) - \frac{dx}{2} \partial_x F_x(x,y,z)} \right] \simeq dx\,dy\,dz \; \frac{\partial F_x}{\partial x}$$

$$\text{right} + \text{left} = dx\,dz \left[\overbrace{F_y\left(x, y + \frac{dy}{2}, z\right)}^{\simeq F_y(x,y,z) + \frac{dy}{2} \partial_y F_y(x,y,z)} - \underbrace{F_y\left(x, y - \frac{dy}{2}, z\right)}_{\simeq F_y(x,y,z) - \frac{dy}{2} \partial_y F_y(x,y,z)} \right] \simeq dx\,dy\,dz \; \frac{\partial F_y}{\partial y}$$

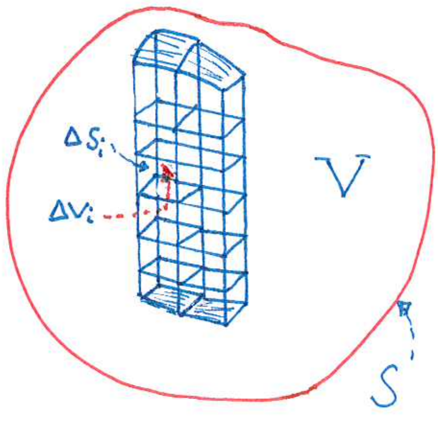
$$\text{top} + \text{bottom} = dx\,dy \left[\overbrace{F_z\left(x, y, z + \frac{dz}{2}\right)}^{\simeq F_z(x,y,z) + \frac{dz}{2} \partial_z F_z(x,y,z)} - \underbrace{F_z\left(x, y, z - \frac{dz}{2}\right)}_{\simeq F_z(x,y,z) - \frac{dz}{2} \partial_z F_z(x,y,z)} \right] \simeq dx\,dy\,dz \; \frac{\partial F_z}{\partial z}$$

Putting them all together:

$$\oint_S d\vec{S} \cdot \vec{F} = dx\,dy\,dz \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) = dV \;\; \vec{\nabla} \cdot \vec{F}$$

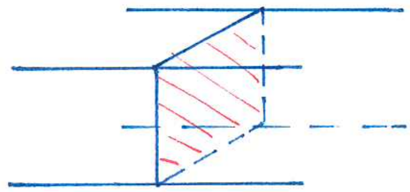
which proves the theorem for the infinitesimal cuboid.

To complete the proof, we need to extend the argument to arbitrary finite volumes. This can be achieved by approximating the volume by infinitesimal cuboids, and adding up their contributions.

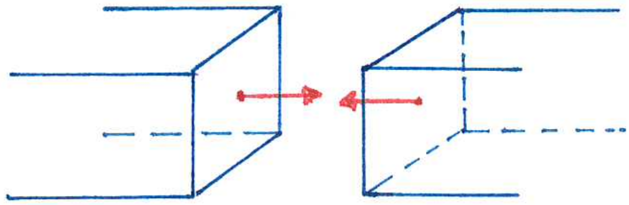


Now note:

$$\sum_i \oint_{\Delta S_i} d\vec{S}\cdot\vec{F} = \oint_S d\vec{S}\cdot\vec{F}$$



The reason is that all internal surfaces give two contributions of equal magnitude and opposite sign, and thus cancel out.



External plaquettes, however, contribute only once, and add up to form the exterior boundary S.

Naturally, we also have $\displaystyle\sum_i \int_{\Delta V_i} dV\ \vec{\nabla}\cdot\vec{F} = \int_V dV\ \vec{\nabla}\cdot\vec{F}$.

To make the argument precise, we need to take the limit in which ΔS_i and ΔV_i tend to zero, and the number of the cuboids tends to infinity. This proves the theorem.

<u>Note</u>: The infinitesimal version of the Divergence Theorem [page 50] provides us with an alternative definition of div $\vec{F}$ at a point P:

$$\vec{\nabla}\cdot\vec{F}\Big|_P = \lim_{\Delta V\to 0} \frac{1}{\Delta V} \oint_{\Delta S = \text{boundary of } \Delta V} d\vec{S}\cdot\vec{F}$$

This definition reaffirms our intuition for div $\vec{F}$ as the outward flux of $\vec{F}$ (per unit volume) at any given point.

Moreover, the definition does not explicitly refer to Cartesian coordinates, and in fact provides a local coordinate-independent definition of div $\vec{F}$.

<u>Exercise</u>: Apply the infinitesimal version of the Divergence theorem to a volume element in cylindrical coordinates [page 24] by using the relevant area elements, and show that div $\vec{F}$ in cylindrical coordinates reads:

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r}\partial_r\left(rF_r\right) + \frac{1}{r}\partial_\varphi F_\varphi + \partial_z F_z \qquad \left[\vec{F} = F_r\hat{e}_r + F_\varphi\hat{e}_\varphi + F_z\hat{e}_z\right]$$

Do the same thing in spherical coordinates, and show that div $\vec{F}$ in spherical coordinates is given as: $\left[\vec{F} = F_r\hat{e}_r + F_\theta\hat{e}_\theta + F_\varphi\hat{e}_\varphi\right]$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\partial_r\left(r^2F_r\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta\, F_\theta\right) + \frac{1}{r\sin\theta}\partial_\varphi F_\varphi$$

<u>Exercise</u>: Discovering "Dirac Delta Function" I

Apply Gauss' theorem to $\vec{F} = -\nabla\left(\frac{1}{r}\right) = \frac{1}{r^2}\hat{e}_r$ in a spherical volume of radius R.

$$\text{LHS} = \oint_S d\vec{S}\cdot\vec{F} = \int \cancel{R^2}\sin\theta\, d\theta\, d\varphi\ \hat{e}_r\cdot\frac{1}{\cancel{R^2}}\hat{e}_r = \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi = 4\pi$$

$$\text{RHS} = \int r^2\sin\theta\, dr\, d\theta\, d\varphi \underbrace{\left(\vec{\nabla}\cdot\vec{F}\right)}_{=0\ \text{[see page 47, or previous Exercise]}} = 0 \qquad \text{LHS}\ (4\pi) \neq \text{RHS}\ (0)$$

What is happening here? How can we "save" Gauss' theorem?

Example: Differential form of Gauss' law in electrostatics

$$\oint_S d\vec{S}\cdot\vec{E} = \frac{Q}{\epsilon_0}$$

$$\oint_S d\vec{S}\cdot\vec{E} = \int_V dV\, \vec{\nabla}.\vec{E} \qquad\qquad Q = \int_V dV\, \rho(\vec{r})$$

$$\rightarrow \quad \int_V dV\, \vec{\nabla}.\vec{E} = \int_V dV\, \frac{\rho}{\epsilon_0} \quad \rightarrow \quad \int_V dV \left[\vec{\nabla}.\vec{E} - \frac{\rho}{\epsilon_0}\right] = 0$$

since V is arbitrary (because S was arbitrary), we must have

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$$

In words, this tells us that electric charge density acts as the source (when +) and sink (when −) of the electric field lines.

Lack of existence of magnetic monopoles, which gives us

$$\oint_S d\vec{S}\cdot\vec{B} = 0$$

can be similarly transformed into its differential form:

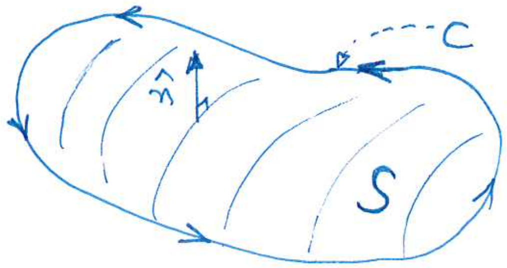
$$\vec{\nabla}.\vec{B} = 0$$

These are two of the four Maxwell equations. The other two will involve curl $\vec{E}$ and curl $\vec{B}$.

10. Stokes' Theorem

The statement of Stokes' Theorem:

Let S be an oriented surface with boundary curve C, and $\vec{F}(x,y,z)$ be a continuous and differentiable vector field defined in this region. Then the line integral of $\vec{F}$ around the boundary is equal to the flux of **curl** $\vec{F}$ going through the surface:



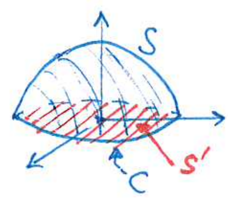
$$\boxed{\oint_C d\vec{l}\cdot\vec{F} = \int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{F})}$$

$\oint$ symbol denotes integration over closed curve C.

Let us see how it works in practice, helping us convert certain surface and line integrals onto one another, before we prove it.

<u>Example</u>: Suppose $\vec{F} = 4y\,\hat{e}_x + x\,\hat{e}_y + 2z\,\hat{e}_z$. Evaluate $I = \int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{F})$ over the upper hemisphere of radius a defined via $x^2+y^2+z^2=a^2$ and $z\geq 0$.

$$\oint d\vec{l}\cdot\vec{F} = \int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{F}) = I$$



C: $\vec{r}(\varphi) = (a\cos\varphi,\ a\sin\varphi,\ 0) \rightarrow d\vec{l} = a\,d\varphi\,(-\sin\varphi, \cos\varphi, 0)$

$\vec{F}\big|_C = (4a\sin\varphi,\ a\cos\varphi,\ 0)$

$$I = \int_0^{2\pi} a^2 d\varphi\left[-4\sin^2\varphi + \cos^2\varphi\right] = -3\pi a^2$$

Note: $\displaystyle\int_{S'} d\vec{S}\cdot(\vec{\nabla}\times\vec{F}) = \int_{S'} dx\,dy\,\hat{e}_z\cdot(\vec{\nabla}\times\vec{F}) = \int_{S'} dx\,dy\,\overbrace{\left[\frac{\partial x}{\partial x} - \frac{\partial(4y)}{\partial y}\right]}^{-3} = -3\pi a^2 = I$

This suggests that for Stokes' theorem any surface bounded by C could be used.

The proof of Stokes' Theorem:

Let us first look at a special simple case, where the surface is an infinitesimal rectangle (plaquette) at (x, y, z) oriented within the x-y plane (i.e. normal is $\hat{e}_z$), with sides of length dx and dy.

$\oint_{\Delta C} d\vec{\ell} \cdot \vec{F}$ has four contributions:

which side?

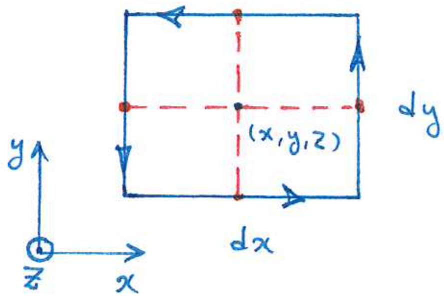
$$\simeq + dy\, F_y\left(x + \frac{dx}{2}, y, z\right) \qquad \text{right}$$

$$- dy\, F_y\left(x - \frac{dx}{2}, y, z\right) \qquad \text{left}$$

$$- dx\, F_x\left(x, y + \frac{dy}{2}, z\right) \qquad \text{top}$$

$$+ dx\, F_x\left(x, y - \frac{dy}{2}, z\right) \qquad \text{bottom}$$

sign determined by $d\vec{\ell}$ vs $\hat{e}_i$.



Now we can use Taylor expansion to simplify the result.

$$\text{right} + \text{left} = dy \left[\overbrace{F_y\left(x + \frac{dx}{2}, y, z\right)}^{F_y(x,y,z) + \frac{dx}{2}\partial_x F_y(x,y,z)} - \underbrace{F_y\left(x - \frac{dx}{2}, y, z\right)}_{F_y(x,y,z) - \frac{dx}{2}\partial_x F_y(x,y,z)} \right] \simeq dx\,dy \left(\frac{\partial F_y}{\partial x}\right)$$

$$\text{top} + \text{bottom} = dx \left[\overbrace{-F_x\left(x, y + \frac{dy}{2}, z\right)}^{F_x(x,y,z) + \frac{dy}{2}\partial_y F_x(x,y,z)} + \underbrace{F_x\left(x, y - \frac{dy}{2}, z\right)}_{F_x(x,y,z) - \frac{dy}{2}\partial_y F_x(x,y,z)} \right] \simeq dx\,dy \left(-\frac{\partial F_x}{\partial y}\right)$$

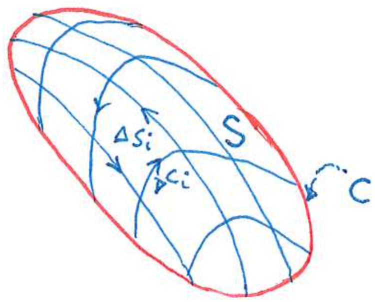
Putting them all together:

$$\oint_{\Delta C} d\vec{\ell} \cdot \vec{F} = dx\,dy \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = dA\, \hat{e}_z \cdot (\vec{\nabla} \times \vec{F}) = d\vec{S} \cdot (\vec{\nabla} \times \vec{F})$$

which proves the theorem for the infinitesimal plaquette.

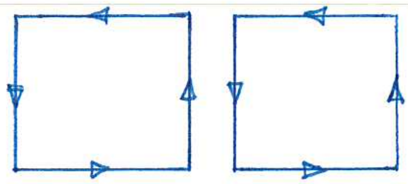
In the last step of the argument, we made use of the fact that there was nothing special about the specific orientation we had chosen, and wrote the result in a coordinate-independent form.

To complete the proof, we need to extend the argument to arbitrary finite surfaces. This can be achieved by approximating the surface by infinitesimal plaquettes, and adding up their contributions.



Now note:

$$\sum_i \oint_{\Delta C_i} d\vec{l}\cdot\vec{F} = \oint_C d\vec{l}\cdot\vec{F}$$



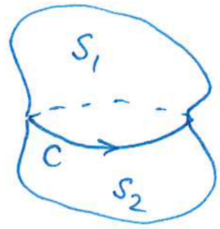
The reason is that all internal sides give two contributions of equal magnitude and opposite sign, and thus cancel out.

External sides, however, contribute to the line integral only once, and add up to form the exterior boundary curve C.

Naturally, we also have $\sum_i \int_{\Delta S_i} d\vec{S}\cdot(\vec{\nabla}\times\vec{F}) = \int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{F})$.

To make the argument precise, we need to take the limit in which ΔC_i and ΔS_i tend to zero, and the number of plaquettes tends to infinity. This proves the theorem.

From the proof it is easy to deduce that the surface integral gives the same value for any surface bounded by a given curve C:



$$\oint_C d\vec{l}\cdot\vec{F} = \int_{S_1} d\vec{S}\cdot(\vec{\nabla}\times\vec{F}) = \int_{S_2} d\vec{S}\cdot(\vec{\nabla}\times\vec{F})$$

Exercise: Prove the above statement using the corollary of Gauss' Theorem that gave us $\oint_S d\vec{S}\cdot(\vec{\nabla}\times\vec{F}) = 0$ for any closed surface S.

Note 1: The infinitesimal version of Stokes' Theorem [page 55] provides us with an alternative definition of curl $\vec{F}$ at a point P:

$$\underbrace{(\vec{\nabla}\times\vec{F})\cdot\hat{n}}_{\hat{n} = \text{normal to } \Delta S \text{ at } P}\Big|_P = \lim_{\Delta S\to 0} \frac{1}{\Delta S}\oint_{\Delta C} d\vec{l}\cdot\vec{F}$$

ΔC = boundary of ΔS

This definition reaffirms our intuition for curl $\vec{F}$ as the measure of the local rotation (circulation) of $\vec{F}$ (per unit area) at a point.

Moreover, the definition does not explicitly refer to Cartesian coordinates and in fact provides a local coordinate-independent definition of curl $\vec{F}$.

Exercise: Apply the infinitesimal version of Stokes' theorem to area elements in cylindrical coordinates [page 24] by using the relevant line integral measures, and show that curl $\vec{F}$ in cylindrical coordinates reads:

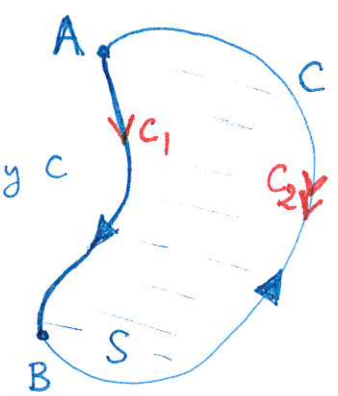
$$\left[\vec{F} = F_r\,\hat{e}_r + F_\varphi\,\hat{e}_\varphi + F_z\,\hat{e}_z\right]$$

$$\vec{\nabla}\times\vec{F} = \hat{e}_r\left[\frac{1}{r}\partial_\varphi F_z - \partial_z F_\varphi\right] + \hat{e}_\varphi\left[\partial_z F_r - \partial_r F_z\right] + \hat{e}_z\,\frac{1}{r}\left[\partial_r(rF_\varphi) - \partial_\varphi F_r\right]$$

Do the same thing in spherical coordinates, and show that curl $\vec{F}$ in spherical coordinates is given as: $\left[\vec{F} = F_r\hat{e}_r + F_\theta\,\hat{e}_\theta + F_\varphi\,\hat{e}_\varphi\right]$

$$\vec{\nabla}\times\vec{F} = \hat{e}_r\,\frac{1}{r\sin\theta}\left[\partial_\theta\left(\sin\theta\,F_\varphi\right) - \partial_\varphi F_\theta\right]$$
$$+\,\hat{e}_\theta\left[\frac{1}{r\sin\theta}\partial_\varphi F_r - \frac{1}{r}\partial_r\left(rF_\varphi\right)\right]$$
$$+\,\hat{e}_\varphi\,\frac{1}{r}\left[\partial_r\left(rF_\theta\right) - \partial_\theta F_r\right]$$

Note 2: Stokes' theorem can be used to prove that if $\vec{\nabla}\times\vec{F}=0$ then $\vec{F}=\vec{\nabla}\phi$ [see page 33].

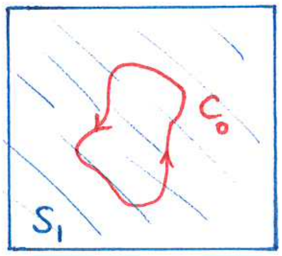


$$\oint_C d\vec{l}\cdot\vec{F} = \int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{F}) = 0 \qquad \text{for arbitrary } C$$

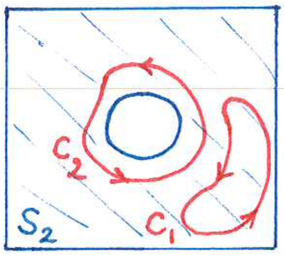
consider a closed path C that contains two points of A and B. We can break it into two paths C_1 and C_2 that connect A to B: $C = C_1 + (-C_2) = C_1 - C_2$

$$0 = \oint_C d\vec{l}\cdot\vec{F} = \int_{C_1} d\vec{l}\cdot\vec{F} + \int_{-C_2} d\vec{l}\cdot\vec{F} = \int_{C_1} d\vec{l}\cdot\vec{F} - \int_{C_2} d\vec{l}\cdot\vec{F} \Rightarrow \int_{C_1} d\vec{l}\cdot\vec{F} = \int_{C_2} d\vec{l}\cdot\vec{F}$$

we have thus proved that the integral between two arbitrary points A and B is path-independent, and hence a function of end-points only, if $\vec{\nabla}\times\vec{F}=0$ everywhere:
$$\int_{C_{A\to B}} d\vec{l}\cdot\vec{F} = \phi(B)-\phi(A)$$
choosing the points to be infinitesimally close to each other, we can prove that $\vec{F}=\vec{\nabla}\phi$ as in page 32.



There is, however, a mathematical subtlety about this statement: it only holds in regions where the space is "simply connected".



The criterion: a domain is simply connected if a closed loop surrounding it can be continuously deformed (and shrunk) to zero. For example, all of S_1 is simply connected, as seen from any closed loop like C_0. In S_2, C_1 encloses a simply connected region, but C_2 does not. In the case of C_2, we can see that implementing Stokes' theorem will have problems, as we do not have the possibility to choose a smooth surface bounded by C_2.

Exercise: Discovering "Dirac Delta Function" II

Apply Stokes' theorem to $\vec{F} = \left(\frac{-y}{x^2+y^2}\right)\hat{e}_x + \left(\frac{x}{x^2+y^2}\right)\hat{e}_y = \frac{1}{r}\,\hat{e}_\varphi$

in a circular region of radius **R**.

$$\vec{\nabla}\times\vec{F} = 0 \xleftarrow{\text{or see page 58}} \quad \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \hat{e}_z\left[\frac{2}{x^2+y^2} - \frac{2}{x^2+y^2}\right] = 0$$

beware: cancellation might not work at the origin!

$$\text{LHS} = \oint_C d\vec{\ell}\cdot\vec{F} = \int_0^{2\pi} \cancel{R}\,d\varphi\,\hat{e}_\varphi\cdot\frac{1}{\cancel{R}}\,\hat{e}_\varphi = \int_0^{2\pi} d\varphi = 2\pi$$

$$\text{RHS} = \int_S d\vec{S}\cdot\underbrace{(\vec{\nabla}\times\vec{F})}_{=0} = 0 \qquad\qquad \text{LHS }(2\pi) \neq \text{RHS }(0)$$

What is happening here? How can we "save" Stokes' theorem?

Example: Differential forms of Ampere's law and Faraday's law

Ampere's law: $\oint_C d\vec{\ell}\cdot\vec{B} = \mu_0 I$

$$\oint_C d\vec{\ell}\cdot\vec{B} = \int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{B}) \qquad\qquad I = \int_S d\vec{S}\cdot\vec{J}$$

$$\rightarrow \int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{B}) = \int d\vec{S}\cdot\mu_0\vec{J} \quad\rightarrow\quad \int_S d\vec{S}\cdot\left[\vec{\nabla}\times\vec{B} - \mu_0\vec{J}\right] = 0$$

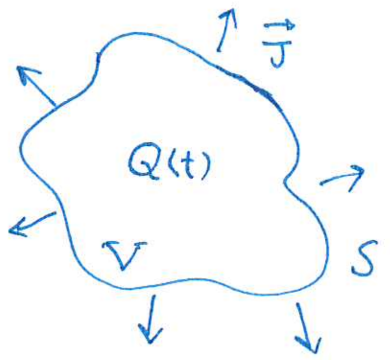
since S is arbitrary (because C was arbitrary), we must have

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

Faraday's law: $\oint_C d\vec{\ell}\cdot\vec{E} = -\frac{d}{dt}\int_S d\vec{S}\cdot\vec{B}$

$$\rightarrow \int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{E}) = \int d\vec{S}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) \quad\rightarrow\quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Example: conservation of charge and the need for displacement current in Ampere's law



conservation of charge in volume V

enclosed by surface S: $\displaystyle \frac{dQ}{dt} = -\oint_S d\vec{S}\cdot\vec{J}$

$$\frac{d}{dt}\overbrace{\int_V dV\,\rho}^{Q} = \int_V dV\,\frac{\partial\rho}{\partial t} = -\int_V dV\,\vec{\nabla}\cdot\vec{J}$$

$$\rightarrow \int_V dV\left[\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J}\right] = 0 \quad ; \text{ since } V \text{ is arbitrary}$$

$$\rightarrow \qquad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

Ampere's law: $\qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

inconsistency: $\quad \underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{0} = \mu_0\,\underbrace{\vec{\nabla}\cdot\vec{J}}_{=-\frac{\partial\rho}{\partial t}\neq 0}$!!

since $\rho = \epsilon_0\vec{\nabla}\cdot\vec{E}$, we can augment Ampere's law as

$$\vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \vec{J}_D\right] = \mu_0\left[\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right]$$

which is now consistent with the divergence $= 0$ requirement.

The addition of this term by Maxwell helped him discover in 1865 that electromagnetic wave can propagate "in vacuum" with the speed of light:

$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \longrightarrow \vec{\nabla}\times\underbrace{\frac{\partial\vec{B}}{\partial t}}_{-\vec{\nabla}\times\vec{E}} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \longrightarrow -\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\underbrace{\cancel{\vec{\nabla}\cdot\vec{E}}}_{0}) - \nabla^2\vec{E} \longrightarrow \nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{wave eqn.}$$

$$\mu_0\epsilon_0 = \frac{1}{c^2}$$

<u>Exercise</u>: "Gauss Invariant" for closed curves in 3D

Show that the following double line integral

$$G(C_1, C_2) = \frac{1}{4\pi} \oint_{C_1} \oint_{C_2} \frac{(d\vec{r}_1 \times d\vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}$$

along two **separate** loops C_1 and C_2 does not depend on the geometry of the loops, and **is** only sensitive to the degree of their entanglement.

Show that for the examples of loops sketched on page 4

$G_a = 0$ and $G_b = G_c = 1$.

$G(C_1, C_2)$ is a <u>topological invariant</u> that characterizes the degree of entanglement of closed loops in 3D.