§6. Weak Turbulence theory: Introduction

This is the next step: we now want to do better in describing the fluctuations than just linear theory.

We had

$$f_s = f_{0s}(t,\vec{v}) + \sum_k e^{i\vec{k}\cdot\vec{r}}\,\delta f_{ks}(t,\vec{v})$$

equilibrium (time-averaged) — fluctuations

$$\frac{\partial f_{0s}}{\partial t} = -\frac{q_s}{m_s}\sum_k \overline{\varphi_k^* \, i\vec{k}\cdot\frac{\partial \delta f_{ks}}{\partial \vec{v}}} \qquad (7)\ (p.4)$$

$$\frac{\partial \delta f_{ks}}{\partial t} + i\vec{k}\cdot\vec{v}\,\delta f_{ks} = \frac{q_s}{m_s}\varphi_k\, i\vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}} + \underbrace{\frac{q_s}{m_s}\sum_{k'\neq 0}\varphi_{k'}\, i\vec{k}'\cdot\frac{\partial \delta f_{k-k',s}}{\partial \vec{v}}}_{\text{this was neglected in QLT}} \qquad (5)$$

where $\varphi_k = \frac{4\pi}{k^2}\sum_s q_s \int d^3\vec{v}\,\delta f_{ks}$ (4) ← (p.3)

__Note__: We are assuming a uniform equilibrium, i.e.

$$f_{0s}(t,\vec{v}) = \overline{f_s} = \langle f_s \rangle$$

↑ time average; ↑ spatial average (i.e. average over small scales)

This means that in eq. (5), we have, after averaging,

$$\frac{q_s}{m_s}\sum_{k'}\overline{\varphi_{k'}\, i\vec{k}'\cdot\frac{\partial \delta f_{k-k',s}}{\partial \vec{v}}} = 0 \text{ unless } k=0.$$

[redacted]

In more general situations, where (slow) spatial variation of the equilibrium is allowed, we have to subtract from eq. (5) its average.

For small amplitudes, the last term in eq. (5) was neglected. Now we are going to include it perturbatively:

let $$\delta f_{ks} = \delta f_{ks}^{(0)} + \delta f_{ks}^{(1)} + \delta f_{ks}^{(2)} + \dots \tag{59}$$

To lowest order, eq. (5) is

$$\frac{\partial \delta f_{ks}^{(0)}}{\partial t} + i\vec{k}\cdot\vec{v}\,\delta f_{ks}^{(0)} = \frac{q_s}{m_s}\varphi_k\, i\vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}} \tag{60}$$

↑ formally, we should write $\varphi_k^{(0)}$ here, but we will keep the exact φ_k and consider (60) to be the definition of $\delta f_{ks}^{(0)}$

If we now subtract eq. (60) from eq. (5), we have

$$\frac{\partial \delta f_{ks}^{(1)}}{\partial t} + i\vec{k}\cdot\vec{v}\,\delta f_{ks}^{(1)} = \frac{q_s}{m_s}\sum_{k'}\varphi_{k'}\, i\vec{k}'\cdot\frac{\partial \delta f_{k-k',s}^{(0)}}{\partial \vec{v}} \tag{61}$$

to lowest order, but again keeping the exact $\varphi_{k'}$.

Finally, subtract (60) and (61) from (5) and get

$$\frac{\partial \delta f_{ks}^{(2)}}{\partial t} + i\vec{k}\cdot\vec{v}\,\delta f_{ks}^{(2)} = \frac{q_s}{m_s}\sum_{k'}\varphi_{k'}\, i\vec{k}'\cdot\frac{\partial \delta f_{k-k',s}^{(1)}}{\partial \vec{v}} \tag{62}$$

↑ still exact. etc...

While keeping the exact φ_k in these equations was an overkill, it is actually more convenient to work this way and the resulting dist. function (59) is correct up to 2nd order in amplitudes.

What is now the equation for φ_k? – it's the Poisson equation, but here it is convenient to write it as an evolution equation by taking time derivative of the Poisson equation – as I did on p. 14 and on p. 28:

$$\frac{\partial \varphi_k}{\partial t} = -\frac{4\pi}{k^2}\sum_s q_s \int d^3\vec{v}\, i\vec{k}\cdot\vec{v}\left[\delta f_{ks}^{(0)} + \delta f_{ks}^{(1)} + \delta f_{ks}^{(2)} + \dots\right] \quad (63)$$

(the three terms: ↑ eq. (60), ↑ eq. (61), ↑ eq. (62))

If we solved eqs. (60), (61), (62) ... and substituted the solutions into eq. (63), this will give a nonlinear equation for φ. Nonlinear because, clearly,

$$\delta f^{(0)} \sim \varphi, \quad \delta f^{(1)} \sim \varphi\,\delta f^{(0)} \sim \varphi^2, \quad \delta f^{(2)} \sim \varphi\,\delta f^{(1)} \sim \varphi^3 \dots$$

so as we go to ever higher orders, ever-higher-order nonlinearity appears in eq. (63).

In the linear order, we already know that eq. (63) reduces to a set of noninteracting linear modes:

$$\frac{\partial \varphi_k^{(i)}}{\partial t} = \left[-i\omega_k^{(i)} + \gamma_k\right]\varphi_k^{(i)} + \text{nonlinear terms} \quad (64)$$

We expect that nonlinearities will introduce <u>interactions</u>: both between different wave numbers (because $\varphi^2 = \sum_{k'} \varphi_{k'}\varphi_{k-k'}$ etc...) and maybe also between different wave modes ($\varphi_k^{(i)}$ with $\varphi_k^{(j)}$).

These interactions will change amplitudes of the waves: in <u>weak turbulence</u> theory, this change is slow

$$\omega_k^{(i)} \gg \tau_{NL}^{-1} \quad (\propto \varphi \text{ or } \varphi^2 \text{ if 2-order interactions are empty as happens occasionally})$$

(τ_{NL}: nonlinear interaction time)

Thus, waves interact, but remain waves: in each interaction, the amplitude changes slightly and the wave speeds on to the next interaction.

[NB: In contrast, in <u>strong turbulence</u>, oscillation and interaction times are comparable (or interaction is stronger), so waves decorrelate in one period.]

Based on the small-amplitude expansion, one can derive closed equations for the spectra of the waves. The full kinetic calculation is very involved, so I will first show the general scheme of the weak-turbulence theory. Eq. (63) for a wave mode $\varphi_k^{(i)}$ [derived from eq. (64)] will have the following general form:

$$\frac{\partial \varphi_k^{(i)}}{\partial t} = \left[-i\omega_k^{(i)} + \gamma_k^{(i)}\right]\varphi_k^{(i)} + \sum_{mn}\sum_{pq} \delta_{k,p+q} V_{kpq}^{(imn)} \varphi_p^{(m)} \varphi_q^{(n)} +$$

$$+ \sum_{mnj}\sum_{pql} \delta_{k,p+q+l} V_{kpql}^{(imnj)} \varphi_p^{(m)} \varphi_q^{(n)} \varphi_l^{(j)} \qquad (65)$$

I will derive the explicit form of the coupling coefficients later, but now let me demonstrate how we go about solving this equation.

Note that eq. (65) is a generic form of a nonlinear eqn with 2nd and 3rd-order nonlinearities. Many systems look like this. E.g., fluid systems are often in the form (65) with just the 2nd-order terms.

E.g., for Alfvén waves in RMHD, we have

$$(i) = \pm \; ; \; \varphi_k^{(i)} = \zeta_k^{\pm} \; ; \; \omega_k^{(i)} = \mp k_\parallel v_A \; ; \; \gamma_k^{(i)} = 0$$

and $V_{kpq}^{\pm\mp\pm} = -\hat{z}\cdot(\vec{k}\times\vec{p})\,\dfrac{\vec{k}_\perp\cdot\vec{q}_\perp}{k_\perp^2}$, other couplings $= 0$

Exercise: Work this out! [I have a set of notes on weak turbulence of Alfvén waves that I can give you]

Let us seek the solution of eq. (65) in the form

$$\varphi_k^{(i)} = a_k^{(i)}(t)\, e^{-i\omega_k^{(i)} t + \gamma_k^{(i)} t} \qquad (66)$$

(annotations: $a_k^{(i)}(t)$ — slow, nonlinear; $\omega_k^{(i)}$ — fast, $\gamma_k^{(i)}$ — slow; together: linear)

The lowest order (linear) solution is simply $a_k^{(i)}(t) = a_k^{(i)}(0) = \varphi_k^{(i)}(0)$.

With inclusion of nonlinearities, amplitudes satisfy

$$\frac{\partial a_k^{(i)}}{\partial t} = \sum_{mn}\sum_{pq} \delta_{k,p+q} V_{kpq}^{(imn)} e^{-i[\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}]t + \Gamma_{kpq}^{(imn)} t} a_p^{(m)} a_q^{(n)} + \sum_{mnj}\sum_{pql} \delta_{k,p+q+l} V_{kpql}^{(imnj)} e^{-i[\omega_p^{(m)}+\omega_q^{(n)}+\omega_l^{(j)}-\omega_k^{(i)}]t + \Gamma_{kpql}^{(imnj)} t} a_p^{(m)} a_q^{(n)} a_l^{(j)} \qquad (67)$$

where

$$\Gamma_{kpq}^{(imn)} = \gamma_p^{(m)} + \gamma_q^{(n)} - \gamma_k^{(i)} \qquad (67a)$$

$$\Gamma_{kpql}^{(imnj)} = \gamma_p^{(m)} + \gamma_q^{(n)} + \gamma_l^{(j)} - \gamma_k^{(i)} \qquad (67b)$$

[illegible]

If we consider a statistical ensemble of waves, what we really want is their spectra:

$$C_k^{(i)}(t) = \underbrace{\langle |a_k^{(i)}|^2 \rangle}_{\text{ensemble avg.}} = \langle |\varphi_k^{(i)}|^2 \rangle \, e^{-2\gamma_k^{(i)} t} \qquad (68a)$$

NB: $a_k^{(i)*} = a_{-k}^{(i)}$ because $\varphi_k^{(i)}$ is a F. transform of a real field

Then, from (67),

$$\frac{\partial C_k^{(i)}}{\partial t} = \left\langle a_k^{(i)*} \frac{\partial a_k^{(i)}}{\partial t} \right\rangle + \text{c.c.} =$$

$$= \sum_{mn} \sum_{pq} \delta_{k,p+q} V_{kpq}^{(imn)} e^{-i[\omega_p^{(m)} + \omega_q^{(n)} - \omega_k^{(i)}]t + \Gamma_{kpq}^{(imn)} t} \langle a_p^{(m)} a_q^{(n)} a_{-k}^{(i)} \rangle$$

$$+ \sum_{mnj} \sum_{pql} \delta_{k,p+q+l} V_{kpql}^{(imnj)} e^{-i[\omega_p^{(m)} + \omega_q^{(n)} + \omega_l^{(j)} - \omega_k^{(i)}]t + \Gamma_{kpql}^{(imnj)} t} \langle a_p^{(m)} a_q^{(n)} a_l^{(j)} a_{-k}^{(i)} \rangle$$

$$+ \text{c.c.} \qquad (68)$$

So we now have to calculate 3rd and 4th-order corr. fns. If we tried writing evolution eqns for them, we would get even higher-order corr. fns in the rhs.

In a general nonlinear system, this is where the main difficulty lies: the full set is infinite; this is known as the <u>closure problem</u>. In weak turbulence, this is resolved <u>perturbatively</u>: let us go back to (67) and solve it up to 2nd order in amplitudes:

$$a_k^{(i)}(t) = a_k^{(i)}(0) + \sum_{mn} \sum_{pq} \delta_{k,p+q} V_{kpq}^{(imn)} \int_0^t dt' \, e^{-i[\omega_p^{(m)} + \omega_q^{(n)} - \omega_k^{(i)}]t' + \Gamma_{kpq}^{(imn)} t'} \times$$

$$\times \underbrace{a_p^{(m)}(t') \, a_q^{(n)}(t')}_{} + \ldots$$

(to lowest order, these can be replaced by $a_p^{(m)}(0) \, a_q^{(n)}(0)$)

So

3rd order and higher ↓

$$a_k^{(i)}(t) = a_k^{(i)}(0) + \sum_{mn}\sum_{pq} \delta_{k,p+q} V_{kpq}^{(imn)} a_p^{(m)}(0)\, a_q^{(n)}(0)\, \Delta_{kpq}^{(imn)} + \dots \qquad (69)$$

where

$$\Delta_{kpq}^{(imn)} \equiv \frac{e^{-i[\omega_p^{(m)} + \omega_q^{(n)} - \omega_k^{(i)}]t + \Gamma_{kpq}^{(imn)} t} - 1}{-i[\omega_p^{(m)} + \omega_q^{(n)} - \omega_k^{(i)}] + \Gamma_{kpq}^{(imn)}} \qquad (69a)$$

– all the t dependence is in here.

Then the 3rd-order correlator is

$$\langle a_p^{(m)} a_q^{(n)} a_{-k}^{(i)} \rangle = \langle a_p^{(m)}(0)\, a_q^{(n)}(0)\, a_{-k}^{(i)}(0) \rangle +$$

$$+ \sum_{rs}\sum_{p'q'} \Big[\delta_{p,p'+q'} V_{pp'q'}^{(mrs)} \Delta_{pp'q'}^{(mrs)} \langle a_{p'}^{(r)}(0)\, a_{q'}^{(s)}(0)\, a_q^{(n)}\, a_{-k}^{(i)}(0) \rangle$$

$$+ \delta_{q,p'+q'} V_{qp'q'}^{(nrs)} \Delta_{qp'q'}^{(nrs)} \langle a_p^{(m)}(0)\, a_{p'}^{(r)}(0)\, a_{q'}^{(s)}(0)\, a_{-k}^{(i)}(0) \rangle$$

$$+ \delta_{-k,p'+q'} V_{-kp'q'}^{(irs)} \Delta_{-kp'q'}^{(irs)} \langle a_p^{(m)}(0)\, a_q^{(n)}(0)\, a_{p'}^{(r)}(0)\, a_{q'}^{(s)}(0) \rangle \Big] \qquad (70)$$

and the 4th-order correlator

$$\langle a_p^{(m)} a_q^{(n)} a_\ell^{(j)} a_{-k}^{(i)} \rangle = \langle a_p^{(m)}(0)\, a_q^{(n)}(0)\, a_\ell^{(j)}(0)\, a_{-k}^{(i)}(0) \rangle \qquad (71)$$

if we keep only powers of the amplitude up to 4th.

Let us now assume that to lowest order the amplitudes for different wave numbers and different wave modes are independent of each other (statistically). This is sometimes referred to as the Random Phase Approximation. It amounts to assuming that $a_k^{(i)}(0) = |a_k^{(i)}(0)| e^{i\phi_k^{(i)}}$ ← independent random phases.

Since $a_k^{(i)}(0)^* = |a_k^{(i)}(0)| e^{-i\phi_k^{(i)}} = a_{-k}^{(i)}(0)$, we have

$$\langle a_p^{(m)}(0) a_q^{(n)}(0)\rangle = \delta_{mn}\delta_{p,-q}\underbrace{\langle |a_p^{(m)}(0)|^2\rangle}_{= C_p^{(m)}} \qquad (72)$$

and

$$\langle a_p^{(m)}(0) a_q^{(n)}(0) a_\ell^{(j)}(0) a_{-k}^{(i)}(0)\rangle = \delta_{mn}\delta_{p,-q}\delta_{ji}\delta_{\ell k} C_p^{(m)} C_\ell^{(j)} +$$

$$+\delta_{mj}\delta_{p,-\ell}\delta_{ni}\delta_{q,k} C_p^{(m)} C_q^{(n)} +$$

$$+\delta_{mi}\delta_{pk}\delta_{nj}\delta_{q,-\ell} C_p^{(m)} C_q^{(n)} \qquad (73)$$

– the correlator in (71)

Then

3rd-order term in eq. (68) =

see (67b)

$$= \sum_m\sum_p \cancel{\delta_{k,k}}_{1}\, V_{kp-pk}^{(immi)} e^{-i[\cancel{\omega_p^{(m)}}+\cancel{\omega_{-p}^{(m)}}+\cancel{\omega_k^{(i)}}-\cancel{\omega_k^{(i)}}]t+2\gamma_p^{(m)}t} C_p^{(m)} C_k^{(i)}$$

$$+\sum_m\sum_p \cancel{\delta_{k,k}}_{1}\, V_{kpk-p}^{(imim)} e^{-i[\cancel{\omega_p^{(m)}}+\cancel{\omega_k^{(i)}}+\cancel{\omega_{-p}^{(m)}}-\cancel{\omega_k^{(i)}}]t+2\gamma_p^{(m)}t} C_p^{(m)} C_k^{(i)}$$

$$+\sum_n\sum_q \cancel{\delta_{k,k}}_{1}\, V_{kkq-q}^{(iinn)} e^{-i[\cancel{\omega_k^{(i)}}+\cancel{\omega_q^{(n)}}+\cancel{\omega_{-q}^{(n)}}-\cancel{\omega_k^{(i)}}]t+2\gamma_q^{(n)}t} C_k^{(i)} C_q^{(n)}$$

$$+ \text{c.c.} = \sum_m\sum_p C_p^{(m)} C_k^{(i)} e^{2\gamma_p^{(m)}t}\Big[\underbrace{V_{kp-pk}^{(immi)} + V_{kpk-p}^{(imim)} + V_{kkp-p}^{(iimm)}}+\text{c.c.}\Big]$$

From (65) and the reality condition $\varphi_k^{(i)*} = \varphi_{-k}^{(i)}$, we have

$$V^*_{kpq\ell} = V_{-k-p-q-\ell} \quad \text{and} \quad C_{-p} = C_p, \text{ so}$$

$$\text{c.c.} = V_{-k-pp-k}^{(immi)} + V_{-k-p-kp}^{(imim)} + V_{-k-k-pp}^{(iimm)} \qquad (74)$$

all these are actually the same because from eq. (65), $V_{kpq\ell}^{(imnj)}$ is symm. wrt last 3 indices

The correlator (70) has 3 such terms:

~~[struck-out line]~~ → 0 because of random phases all odd correlators = 0

$$\langle a_p^{(m)} a_q^{(n)} a_{-k}^{(i)} \rangle = \cancel{\langle a_p^{(m)}(0) a_q^{(n)}(0) a_{-k}^{(i)}(0) \rangle}$$

$$+ \sum_{rs} \sum_{p'q'} \Big[\delta_{p,p'+q'} V_{pp'q'}^{(mrs)} \Delta_{pp'q'}^{(mrs)} \Big(\cancel{\delta_{rs} \delta_{p',-q'} \delta_{ni} \delta_{q,k} C_{p'}^{(r)} C_q^{(n)}} +$$

$$+ \delta_{rn} \delta_{p',-q} \delta_{si} \delta_{q',k} C_{p'}^{(r)} C_{q'}^{(s)} + \delta_{ri} \delta_{p',k} \delta_{sn} \delta_{q',-q} C_{p'}^{(r)} C_{q'}^{(s)} \Big)$$

$$+ \delta_{q,p'+q'} V_{qp'q'}^{(nrs)} \Delta_{qp'q'}^{(nrs)} \Big(\delta_{mr} \delta_{p,-p'} \delta_{si} \delta_{q',k} C_p^{(m)} C_{q'}^{(s)} +$$

$$+ \delta_{ms} \delta_{p,-q'} \delta_{ri} \delta_{p',k} C_p^{(m)} C_{p'}^{(r)} + \cancel{\delta_{mi} \delta_{p,k} \delta_{rs} \delta_{p',-q'} C_p^{(m)} C_{p'}^{(r)}} \Big)$$

$$+ \delta_{-k,p'+q'} V_{-kp'q'}^{(irs)} \Delta_{-kp'q'}^{(irs)} \Big(\cancel{\delta_{mn} \delta_{p,-q} \delta_{rs} \delta_{p',q'} C_p^{(m)} C_{p'}^{(r)}} +$$

$$+ \delta_{mr} \delta_{p,-p'} \delta_{ns} \delta_{q,-q'} C_p^{(m)} C_q^{(n)} + \delta_{ms} \delta_{p,-q'} \delta_{nr} \delta_{q,-p'} C_p^{(m)} C_q^{(n)} \Big) \Big]$$

(Vanish (see below) — pointing to the struck-out terms)

$$= \sum_r \sum_{p'} \Big[\delta_{p,0} \cancel{V_{0p'-p'}^{(mrr)}} \Delta_{0p'-p'}^{(mrr)} \delta_{ni} \delta_{qk} C_{p'}^{(r)} C_q^{(n)} \Big] +$$

0 (because $\varphi_{k=0} = 0$)

$C_{-q}^{(n)} = C_q^{(n)}$

$$+ \delta_{p,-q+k} V_{p-qk}^{(mni)} \Delta_{p-qk}^{(mni)} C_{-q}^{(n)} C_k^{(i)} + \delta_{p,k-q} V_{pk-q}^{(min)} \Delta_{pk-q}^{(min)} C_k^{(i)} C_{-q}^{(n)}$$

(the same)

$$+ \delta_{q,-p+k} V_{q-pk}^{(nmi)} \Delta_{q-pk}^{(nmi)} C_p^{(m)} C_k^{(i)} + \delta_{q,k-p} V_{qk-p}^{(nim)} \Delta_{qk-p}^{(nim)} C_p^{(m)} C_k^{(i)}$$

(the same)

$$+ \sum_r \sum_{p'} \Big[\delta_{q,0} \cancel{V_{0p'-p'}^{(nrr)}} \Delta_{0p'-p'}^{(nrr)} \delta_{mi} \delta_{pk} C_p^{(m)} C_{p'}^{(r)} \Big]$$

$$+ \sum_r \sum_{p'} \Big[\delta_{-k,0} \cancel{V_{0p'-p'}^{(irr)}} \Delta_{0p'-p'}^{(irr)} \delta_{mn} \delta_{p,-q} C_p^{(m)} C_{p'}^{(r)} \Big]$$

0

$$+ \delta_{-k,-p-q} V_{-k-p-q}^{(imn)} \Delta_{-k-p-q}^{(imn)} C_p^{(m)} C_q^{(n)} + \delta_{-k,-q-p} V_{-k-q-p}^{(inm)} \Delta_{-k-q-p}^{(inm)} C_p^{(m)} C_q^{(n)}$$

(the same)

From eq. (65), we know that $V^{(imn)}_{kpq} = V^{(inm)}_{kqp}$ (nothing changes if we permute $\varphi^{(m)}_p$ and $\varphi^{(n)}_q$ in the 2-order term – – in the same way, $V^{(imnj)}_{kpql}$ is symmetric wrt permutations of its last 3 indices, a fact that was used in (74)).

Therefore, there are actually only 3 terms in the above expression: (these are also symmetric – see eq. (69a))

$$\langle a^{(m)}_p a^{(n)}_q a^{(i)}_{-k} \rangle = 2\,\delta_{p,k-q}\, V^{(mni)}_{p-qk}\, \Delta^{(mni)}_{p-qk}\, C^{(i)}_k C^{(n)}_q +$$
$$+ 2\,\delta_{q,k-p}\, V^{(nmi)}_{q-pk}\, \Delta^{(nmi)}_{q-pk}\, C^{(i)}_k C^{(m)}_p$$
$$+ 2\,\delta_{k,p+q}\, V^{(imn)}_{-k-p-q}\, \Delta^{(imn)}_{-k-p-q}\, C^{(m)}_p C^{(n)}_q \qquad (75)$$

This is now substituted into

2nd-order term in eq. (68) =

[illegible]

[illegible]

($V^{(imn)*}_{kpq}$ from eq. (65) (reality condn))

$$= 2 \sum_{mn}\sum_{pq} \delta_{k,p+q}\, V^{(imn)}_{kpq}\, e^{-i[\omega^{(m)}_p + \omega^{(n)}_q - \omega^{(i)}_k]t + \Gamma^{(imn)}_{kpq} t} \Big[V^{(imn)}_{-k-p-q}\, \Delta^{(imn)}_{-k-p-q}\, C^{(m)}_p C^{(n)}_q +$$
$$+ 2\, V^{(nmi)}_{q-pk}\, \Delta^{(nmi)}_{q-pk}\, C^{(m)}_p C^{(i)}_k \Big] + \text{c.c.} \qquad (76)$$

(this is the 1st and 2nd term in (75) after changing $n \leftrightarrow m$, $p \leftrightarrow q$ in the 1st term)

Let us now sort out the coupling coefficients.

- In the 1st term in (76) everything is real except the time-dependent factors:

$$e^{-i[\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}]t+\Gamma_{kpq}^{(imn)}t}\,\Delta^{(imn)}_{-k-p-q}(t)+\text{c.c.} =$$

(use def. (69a) and $\omega_{-k}^{(i)}=-\omega_k^{(i)}$, $\gamma_{-k}^{(i)}=\gamma_k^{(i)}$)

$$= e^{2\Gamma_{kpq}^{(imn)}t}\left[\frac{1-e^{-i[\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}]t-\Gamma_{kpq}^{(imn)}t}}{i[\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}]+\Gamma_{kpq}^{(imn)}}+\text{c.c.}\right] =$$

$$= e^{2\Gamma_{kpq}^{(imn)}t}\,2\pi\,\delta(\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}) \quad \text{for } \Gamma_{kpq}^{(imn)}t\ll 1 \text{ and } \omega_k^{(i)}t\gg 1 \qquad (77)$$

because, generally,

$$\frac{1-e^{-i(x-i\epsilon)t}}{i(x-i\epsilon)}\pm\text{c.c.} = \int_0^t dt'\,e^{-i(x-i\epsilon)t'}+\text{c.c.} = \int_0^t dt'\,e^{-\epsilon t'}\left(e^{+ixt'}+e^{-ixt'}\right)$$

$$\underset{\text{if } t\ll\epsilon^{-1}}{\approx} \int_{-t}^{t} dt'\,e^{ixt'} \underset{\text{if } t\gg\frac{1}{x}}{\approx} \int_{-\infty}^{+\infty} dt'\,e^{ixt'} = 2\pi\,\delta(x)$$

Note that $\Gamma_{kpq}^{(imn)}$ is effectively the width of this δ function.

- Now work out the 2nd term in (76):

$$V_{kpq}^{(imn)}\,V_{q-pk}^{(nmi)}\,\underbrace{e^{-i[\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}]t+\Gamma_{kpq}^{(imn)}t}\,\Delta_{q-pk}^{(nmi)}}+\text{c.c.} =$$

$$\frac{e^{-i[\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}+\omega_{-p}^{(m)}+\omega_k^{(i)}-\omega_q^{(n)}]t+2\gamma_p^{(m)}t}-e^{-i[\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}]t+\Gamma_{kpq}^{(imn)}t}}{-i[\omega_{-p}^{(m)}+\omega_k^{(i)}-\omega_q^{(n)}]+\Gamma_{q-pk}^{(nmi)}}$$

$$= e^{2\gamma_p^{(m)}t}\,\frac{1-e^{-i[\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}]t-\Gamma_{q-pk}^{(nmi)}t}}{i[\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}]+\Gamma_{q-pk}^{(nmi)}},$$

where we used $\Gamma_{kpq}^{(imn)}-2\gamma_p^{(m)}=-\Gamma_{q-pk}^{(nmi)}t$

$$= e^{2\gamma_p^{(m)}t}\,\mathrm{Re}\left(V^{(imn)}_{kpq}V^{(nmi)}_{q-pk}\right)\left[\frac{1-e^{-i[\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}]t-\Gamma^{(nmi)}_{q-pk}t}}{i\,[\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}]+\Gamma^{(nmi)}_{q-pk}}+c.c.\right]$$

$$+\,e^{2\gamma_p^{(m)}t}\,\mathrm{Im}\left(V^{(imn)}_{kpq}V^{(nmi)}_{q-pk}\right)i\left[\quad \text{———''———} \quad - c.c.\right]$$

$$= e^{2\gamma_p^{(m)}t}\left[\mathrm{Re}\left(V^{(imn)}_{kpq}V^{(nmi)}_{q-pk}\right)2\pi\,\delta(\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)})\right.$$

$$\left.+\,\mathrm{Im}\left(V^{(imn)}_{kpq}V^{(nmi)}_{q-pk}\right)2P\frac{1}{\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}}\right] \qquad (78)$$

↑ this means principal value to be taken under the integrals.

The first term was obtained just like in (77); the second term works out as follows:

$$\frac{1-e^{-i(x-i\epsilon)t}}{i(x-i\epsilon)}-c.c. \approx \frac{1-e^{-ixt}}{ix}-c.c. = \frac{2}{ix}(1-\cos xt)$$

(↑ at "≈": $t \ll \epsilon^{-1}$, $x \gg \epsilon$; ↑ at $(1-\cos xt)$: averages out under time averaging)

OK, we are now ready to put everything together:

(77) → (76), (78) → (76); (76) → (68), (74) → (68); (68) → gives us the desired equation for $C_k^{(i)}$

I'd like to write it for

$$I_k^{(i)}(t) \doteq \langle|\varphi_k^{(i)}|^2\rangle = C_k^{(i)}(t)\,e^{2\gamma_k^{(i)}t}$$

(see definition (68a) on p.45)

The result is

coupling to other waves

$$\frac{\partial I_k^{(i)}}{\partial t} = 2\gamma_k^{(i)} I_k^{(i)} + \overbrace{4\pi \sum_{mn}\sum_{pq} \delta_{k,p+q} \left|V_{kpq}^{(imn)}\right|^2 \delta(\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}) I_p^{(m)} I_q^{(n)}}$$

$$+2\Big\{4\pi \sum_{mn}\sum_{pq} \delta_{k,p+q}\Big[\mathrm{Re}\left(V_{kpq}^{(imn)} V_{q-pk}^{(nmi)}\right)\pi\,\delta(\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}) +$$

$$+\frac{1}{\pi}\mathrm{Im}\left(V_{kpq}^{(imn)} V_{q-pk}^{(nmi)}\right)\mathcal{P}\frac{1}{\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)}}\Big] I_p^{(m)} +$$

$$+3\sum_m\sum_p \mathrm{Re}\, V_{kkpp-p}^{(iimm)} I_p^{(m)}\Big\} I_k^{(i)} \qquad (79)$$

nonlinear damping of fluctuations [~like eddy damping]

In fluid systems, where kinetic effects do not matter, i.e. where we consider only wave-wave interactions and no particle effects, one usually has $V_{kpq\ell}^{(imnj)}=0$ and $V_{kpq}^{(imn)}$ is real, so eq. (79) has a simpler form:

$$\frac{\partial I_k^{(i)}}{\partial t} = 2\gamma_k^{(i)} I_k^{(i)} +$$

$$+4\pi \sum_{mn}\sum_{pq} \delta_{k,p+q}\,\delta(\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)})\left(V_{kpq}^{(imn)} V_{q-pk}^{(nmi)} I_p^{(m)} I_k^{(i)} + \left|V_{kpq}^{(imn)}\right|^2 I_p^{(m)} I_q^{(n)}\right) \qquad (80)$$

Waves interact only if these conditions are satisfied:

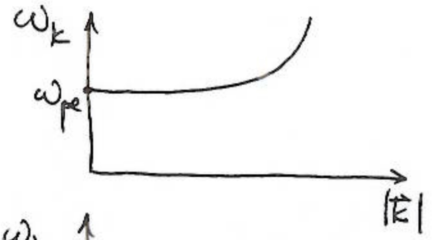
$$\left\{\begin{array}{ll} \vec{p}+\vec{q}=\vec{k} & \text{"momentum conservation"} \\ \omega_p^{(m)}+\omega_q^{(n)}=\omega_k^{(i)} & \text{"energy conservation"} \end{array}\right\} \qquad (81)$$

Remark 1.

- Note that the resonance condition (81) is often impossible to satisfy. Embarassingly, 2 examples of this are

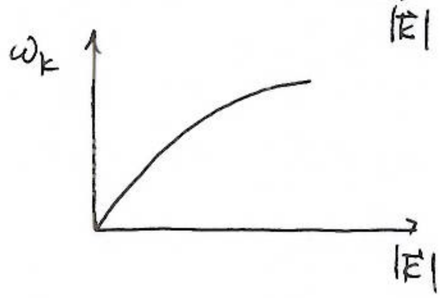
– Langmuir waves $\omega_k^2 = \omega_{pe}^2\left(1+\frac{3}{2}k^2\lambda_{De}^2\right)$ (↑ small)

[eq. (18) p. 10]

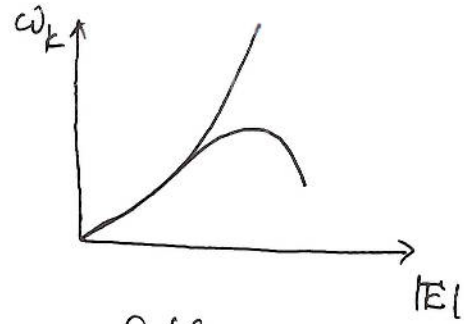


– Ion-acoustic waves $\omega_k^2 = \dfrac{k^2 c_s^2}{1+k^2\lambda_{De}^2/2}$ (↑ small)

[eq. (20), p. 12]



For isotropic systems, it is possible to argue generally that waves with dispersion relations like these are of non-decay type – i.e., (81) cannot be satisfied, while dispersion relations that look like this:



are of decay type, i.e. (81) can be satisfy and 3-wave interactions are permitted.

When the resonance (81) is forbidden, one of the following may help:

1) If many wave modes are present in the system, 3-wave interaction may be possible between waves of different types: e.g. Langmuir waves can interact with ion-acoustic waves

2) Wave-particle interactions may be more important than wave-wave interactions (more of this later)

3) All else failing, the perturbation theory in small amplitudes carried out above has to be continued to next order to include 4-wave interactions. It becomes obvious how to do this if I summarise again the general procedure that I followed:

I had $\varphi \sim a e^{-i\omega t}$ [eq. (66) ignoring γ]

and $\frac{\partial a}{\partial t} \sim V a a$ [eq. (67)] (omitting writing the freq.-dependent oscillatory factors)

so $\frac{\partial \langle aa \rangle}{\partial t} \sim V \langle aaa \rangle$ [eq. (68)]

Integrating (67), $a \sim a(0) + \int^t V a a$ ← approximate by a(0) [eq. (69)]

Substituting into (68),

$$\frac{\partial \langle aa \rangle}{\partial t} \sim VV\langle aaaa \rangle \sim VV \langle aa \rangle \langle aa \rangle \quad \text{[eq. (79)]}$$

with the rhs only non-zero if the 3-wave resonance conditions (81) were satisfied.

Now if (81) cannot be satisfied, the rhs of (79) = 0.

So we go to next order in integrating (67):

$$a \sim a(0) + \int^t V a a \sim a(0) + \int^t V a(0) a(0) + \\ + \iint^t VV aaa + \dots$$

← approximate by a(0)

Substituting into (68),

$$\frac{\partial \langle aa \rangle}{\partial t} \sim VVV\langle aaaaaa \rangle + \dots$$

$$\sim VVV \langle aa \rangle \langle aa \rangle \langle aa \rangle$$

with the rhs $\neq 0$ if the 4-wave resonance condition is satisfied:

$$\left.\begin{array}{l} \vec{p}+\vec{q} = \vec{k}+\vec{\ell} \\ \omega_p^{(m)}+\omega_q^{(n)} = \omega_k^{(i)}+\omega_\ell^{(j)} \end{array}\right\} \qquad (82)$$

As a rule these new conditions can be satisfied, so one need not go to even higher orders in the expansion [except for some other purposes that I will not dwell on here].

As you might imagine, working out 4-wave interactions is a lot of work, although once you get the hang of it, it is not dificult conceptually – just algebra!

Remark 2. Resonance conditions (81) or (82) only need to be satisfied with a certain precision, i.e. the δ-fn in eq. (79) has a certain width:

$$\omega_p^{(m)}+\omega_q^{(n)}-\omega_k^{(i)} \sim \Delta\omega \sim \Gamma_{kpq}^{(imn)} \sim \gamma$$ as we argued on p.50. In fact, we could argue that not only the linear γ but also the nonlinear interaction broadens the resonance:

$$\gamma_{NL} \sim V\varphi \quad \text{– grows with amplitude.}$$

When $\gamma_{NL} \sim \tau_{NL}^{-1} \sim \omega$, eq. (81) no longer needs to be satisfied in order for intraction to happen, ~~[illegible]~~ and the weak-turbulence theory ~~[illegible]~~ breaks down.

For example, for Alfvén waves in RMHD (p.44), we have

$$V^{\pm\mp\pm}_{kpq} = -\hat{z}\cdot(\vec{k}\times\vec{p})\,\frac{\vec{k}_\perp\cdot\vec{q}_\perp}{k_\perp^2}$$

$$V^{\pm\mp\pm}_{q-pk} = -\hat{z}\cdot\left[\vec{q}\times(-\vec{p})\right]\frac{\vec{q}_\perp\cdot\vec{k}_\perp}{q_\perp^2} = -\frac{k_\perp^2}{q_\perp^2}V^{\pm\mp\pm}_{kpq}$$

so eq. (80) becomes

$$\frac{\partial I_k^\pm}{\partial t} = 4\pi\sum_{pq}\delta_{k,p+q}\,\delta\big(v_A(\underbrace{\pm p_\parallel \mp q_\parallel \pm k_\parallel}_{=\,\pm 2p_\parallel})\big)\,\left|V^{\pm\mp\pm}_{kpq}\right|^2 I_p^\mp\left(I_q^\pm - \frac{k_\perp^2}{q_\perp^2}I_k^\pm\right) \qquad (80a)$$

$\uparrow$ so this is the spectrum of fluctuations with $p_\parallel = 0$, i.e. not really Alfvén waves, but 2D motions.

In the weak MHD turbulence literature, this problem is dealt with by assuming that the spectra will be continuously extendable to $k_\parallel = 0$.
It is not clear whether this is OK.
[see papers by Galtier et al.]

If $k_\parallel = 0$ motions are forbidden (e.g. by boundary conditions), one could work out 4-wave interactions like explained above [Sridhar & Goldreich 1994].
Of course all this is of limited use because the turbulence is, in fact, strong, so resonance conditions need not be satisfied and weak turbulence theory cannot be done.