§4. Quasilinear Theory.

In §1, we decomposed the time evolution of our wave-particle system into slow evolution of equilibrium plus fast fluctuations. We had

$$f_s = f_{0s}(\epsilon t, \vec{v}) + \sum_k e^{i\vec{k}\cdot\vec{r}}\, \delta f_{ks}(\epsilon t, t, \vec{v}),$$

slow-evolving homogeneous eq. — small scales in space — slow amplitude evolution — fast oscillation

where $f_{0s}(\epsilon t) = \overline{f_s(t)}$ — fast time average over $\frac{1}{\omega} \ll T \ll \frac{1}{\epsilon\omega}$

$\epsilon \sim \frac{1}{\omega_k \tau_{eq}}$ fluct. freq.

Then

$$\begin{cases} \dfrac{\partial f_{0s}}{\partial t} = -\dfrac{q_s}{m_s}\displaystyle\sum_k \overline{\varphi_{-k}\, i\vec{k}\cdot\frac{\partial \delta f_k}{\partial \vec{v}}} & (7) \\[2ex] \dfrac{\partial \delta f_{ks}}{\partial t} + i\vec{k}\cdot\vec{v}\,\delta f_{ks} = \dfrac{q_s}{m_s}\varphi_k\, i\vec{k}\cdot\dfrac{\partial f_{0s}}{\partial \vec{v}} + \left(\dfrac{\partial \delta f_{ks}}{\partial t}\right)_c & (8) \\[2ex] \varphi_k = \dfrac{4\pi}{k^2}\displaystyle\sum_s q_s \int d^3\vec{v}\, \delta f_{ks} & (4) \end{cases}$$

coll. integral ignored except for coarse-graining δf

To work out the rhs of (7), we need explicit solutions of eq. (8) and (4) — these come from linear theory:

$$\varphi_k(t) = \sum_i c_i\, e^{p_i t} \equiv \sum_i \varphi_k^{(i)}(0)\, e^{-i\omega_k^{(i)} t + \gamma_k^{(i)} t} \qquad (12)$$

(p.8)

slns of $\epsilon(p_i, \vec{k}) = 0$

$$\delta f_{ks}(t) = \frac{i q_s}{m_s}\sum_i \frac{\varphi_k^{(i)}(0)\, e^{-i\omega_k^{(i)} t + \gamma_k^{(i)} t}}{i(\vec{k}\cdot\vec{v} - \omega_k^{(i)}) + \gamma_k^{(i)}}\, \vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}} + \cancel{\text{ball. term}} \qquad (32)$$

p.21

ignored because we use coarse-grained distr. fn. — eq. (33)

Substituting this into (7), we get

$$\frac{\partial f_{0s}}{\partial t} = -\frac{i q_s^2}{m_s^2} \sum_k \sum_{ij} \underbrace{\varphi_k^{(j)*}(0)\, e^{i\omega_k^{(j)}t+\gamma_k^{(j)}t}}_{\text{because } \varphi_{-k}=\varphi_k^*}\; i\vec{k}\cdot\frac{\partial}{\partial \vec{v}} \frac{\varphi_k^{(i)}(0)\, e^{-i\omega_k^{(i)}t+\gamma_k^{(i)}t}}{i(\vec{k}\cdot\vec{v}-\omega_k^{(i)})+\gamma_k^{(i)}}\, \vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}}$$

time average eliminates $i \neq j$ terms

$$= \frac{q_s^2}{m_s^2} \sum_i \sum_k \vec{k}\cdot\frac{\partial}{\partial \vec{v}} \frac{\overbrace{|\varphi_k^{(i)}(0)|^2 e^{2\gamma_k^{(i)}t}}^{=|\varphi_k^{(i)}(t)|^2}}{i(\vec{k}\cdot\vec{v}-\omega_k^{(i)})+\gamma_k^{(i)}}\, \vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}} =$$

$$= \frac{q_s^2}{m_s} \sum_i \frac{1}{2} \sum_k \vec{k}\cdot\frac{\partial}{\partial \vec{v}} |\varphi_k^{(i)}(t)|^2 \left[\frac{1}{i(\vec{k}\cdot\vec{v}-\omega_k^{(i)})+\gamma_k^{(i)}} + \frac{1}{i(-\vec{k}\cdot\vec{v}-\underbrace{\omega_{-k}^{(i)}}_{=-\omega_k^{(i)}})+\underbrace{\gamma_{-k}^{(i)}}_{=\gamma_k^{(i)}}}\right] \vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}}$$

(because $\varphi_{-k}=\varphi_k^*$)

$$= \frac{q_s^2}{m_s^2} \sum_i \sum_k \vec{k}\cdot\frac{\partial}{\partial \vec{v}} \frac{\gamma_k^{(i)} |\varphi_k^{(i)}(t)|^2}{(\vec{k}\cdot\vec{v}-\omega_k^{(i)})^2+\gamma_k^{(i)2}}\, \vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}}$$

This can now be rewritten as a diffusion equation:

$$\boxed{\frac{\partial f_{0s}}{\partial t} = \frac{\partial}{\partial \vec{v}}\cdot \hat{D}_s \cdot \frac{\partial f_{0s}}{\partial \vec{v}} \equiv \frac{\partial}{\partial v_\alpha} D_{s\alpha\beta} \frac{\partial}{\partial v_\beta} f_{0s}} \qquad (38)$$

where the diffusion tensor is

$$\hat{D}_s = \frac{q_s^2}{m_s^2} \sum_i \sum_k \vec{k}\vec{k}\, |\varphi_k^{(i)}|^2 \frac{\gamma_k^{(i)}}{(\omega_k^{(i)}-\vec{k}\cdot\vec{v})^2+\gamma_k^{(i)2}} \qquad (39)$$

Together with

$$\boxed{\frac{\partial |\varphi_k^{(i)}|^2}{\partial t} = 2\gamma_k^{(i)} |\varphi_k^{(i)}|^2} \qquad (40)$$

These equations form a closed system.

Here $\omega_k^{(i)}$ and $\gamma_k^{(i)}$ are obtained from the (linear) dispersion relation

$$\epsilon(-i\omega_k^{(i)}+\gamma_k^{(i)}, \vec{k}) = 1 - \sum_s \frac{4\pi q_s^2}{m_s k^2} \int d^3\vec{v} \frac{1}{(\vec{k}\cdot\vec{v}-\omega_k^{(i)})-i\gamma_k^{(i)}} \vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}} = 0 \qquad (41)$$

with f_{0s} satisfying eq. (38).

Remark:

Ex.1

In §2, we already had an explicit formula for $\gamma_k^{(i)}$ [Eq. (13)], but it is useful to give here another derivation coming from the quasilinear "derivation" (or, rather, confirmation) of eq. (40): recall that the time derivative of Poisson's law is

$$\frac{\partial \varphi_k}{\partial t} = \frac{4\pi}{k^2}\sum_s q_s \int d^3\vec{v}\, \frac{\partial \delta f_{ks}}{\partial t} =$$ substitute from kin. eq. (8) - see p. 26

$$= -\frac{4\pi}{k^2}\sum_s q_s \int d^3\vec{v}\, i\vec{k}\cdot\vec{v}\,\delta f_{ks}$$ [we already used this on p. 14]

Then $$\frac{\partial}{\partial t}\overline{|\varphi_k|^2} = \left(\overline{\varphi_k^* \frac{\partial \varphi_k}{\partial t}} + \overline{\varphi_k \frac{\partial \varphi_k^*}{\partial t}}\right) =$$

$$= -\sum_s \frac{4\pi q_s}{k^2}\int d^3\vec{v}\left(i\vec{k}\cdot\vec{v}\,\overline{\varphi_k^*\delta f_{ks}} - i\vec{k}\cdot\vec{v}\,\overline{\varphi_k \delta f_{ks}^*}\right) =$$

$$= 2\sum_s \frac{4\pi q_s}{k^2}\int d^3\vec{v}\,(\vec{k}\cdot\vec{v})\,\mathrm{Im}\,\overline{\varphi_k^*\delta f_{ks}} =$$ substitute (32) and (12)

$$= 2\,\mathrm{Im}\sum_s \frac{4\pi q_s^2\, i}{k^2 m_s}\int d^3\vec{v}\,(\vec{k}\cdot\vec{v})\,\overline{\sum_{ij}\varphi_k^{(j)*}(0)e^{i\omega_k^{(j)}t+\gamma_k^{(j)}t}\frac{\varphi_k^{(i)}(0)e^{-i\omega_k^{(i)}t+\gamma_k^{(i)}t}}{i(\vec{k}\cdot\vec{v}-\omega_k^{(i)})+\gamma_k^{(i)}}}\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}}$$

$$= 2\,\mathrm{Im}\sum_i\sum_s \frac{4\pi q_s^2}{k^2 m_s}\, i\int d^3\vec{v}\,\frac{|\varphi_k^{(i)}(t)|^2\,\vec{k}\cdot\vec{v}}{i(\vec{k}\cdot\vec{v}-\omega_k^{(i)})+\gamma_k^{(i)}}\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}} =$$

$$= \sum_i \underbrace{\left[2\,\mathrm{Im}\sum_s \frac{4\pi q_s^2}{k^2 m_s}\int d^3\vec{v}\,\frac{\vec{k}\cdot\vec{v}}{\vec{k}\cdot\vec{v}-\omega_k^{(i)}-i\gamma_k^{(i)}}\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}}\right]}_{2\gamma_k^{(i)}}|\varphi_k^{(i)}|^2$$

So, we have $\frac{\partial}{\partial t}\overline{|\vec{E}_k|^2} = \sum_i 2\gamma_k^{(i)} |\vec{E}_k^{(i)}|^2$ (42) $\to$ same as (40)

and
$$\boxed{\gamma_k^{(i)} = \sum_s \frac{4\pi q_s^2}{k^2 m_s}\int d^3\vec{v}\,(\vec{k}\cdot\vec{v})\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}}\,\underbrace{\mathrm{Im}\frac{1}{\vec{k}\cdot\vec{v}-\omega_k^{(i)}-i\gamma_k^{(i)}}}_{=\frac{\gamma_k^{(i)}}{(\vec{k}\cdot\vec{v}-\omega_k^{(i)})^2+\gamma_k^{(i)2}}}} \quad (43)$$

same expression as in eq. (39)

This is a useful expression for $\gamma_k^{(i)}$ that ~~could~~ will come handy for some of the proofs.

Note ~~however~~ that it is actually an equation consistent with what we did in the linear theory (§2):

$$\gamma_k^{(i)} = \sum_s \frac{4\pi q_s^2}{k^2 m_s}\int d^3\vec{v}\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}}\,\mathrm{Im}\frac{\vec{k}\cdot\vec{v}-\omega_k^{(i)}-i\gamma_k^{(i)}+\omega_k^{(i)}+i\gamma_k^{(i)}}{\vec{k}\cdot\vec{v}-\omega_k^{(i)}-i\gamma_k^{(i)}} =$$

$$= \mathrm{Im}\sum_s \frac{4\pi q_s^2}{k^2 m_s}\int d^3\vec{v}\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}}\left[\cancel{1} + (\omega_k^{(i)}+i\gamma_k^{(i)})\frac{1}{\vec{k}\cdot\vec{v}-\omega_k^{(i)}-i\gamma_k^{(i)}}\right] =$$

velocity integral vanishes because $f_{0s}(\pm\infty)=0$

$$= \mathrm{Im}\left[(\omega_k^{(i)}+i\gamma_k^{(i)})\underbrace{\sum_s\frac{4\pi q_s^2}{k^2 m_s}\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}}\,\frac{1}{\vec{k}\cdot\vec{v}-\omega_k^{(i)}-i\gamma_k^{(i)}}}_{=1-\underbrace{\epsilon(-i\omega_k^{(i)}+\gamma_k^{(i)},\vec{k})}_{0} = 1}\right] = \gamma_k^{(i)}$$

by def of $\omega_k^{(i)}$ and $\gamma_k^{(i)}$ – see eq. (41)

So, eq. (43) and eq. (41) are consistent.

To solve them, we still have to use eqs. (14) and (13):

$$\mathrm{Re}\,\epsilon(-i\omega_k^{(i)},\vec{k}) = 0 \;\Rightarrow\; \text{get } \omega_k^{(i)} \quad (14)$$

$$\gamma_k^{(i)} = -\frac{\mathrm{Im}\,\epsilon(-i\omega_k^{(i)},\vec{k})}{\left[\frac{\partial\,\mathrm{Re}\,\epsilon(-i\omega,\vec{k})}{\partial\omega}\right]_{\omega=\omega_k^{(i)}}} \quad (13)$$

So the scheme of the QL theory is

equilibrium f_{0s} $\xrightarrow{\text{eqs. (13-14)}}$ damped or unstable waves ω_k, γ_k $\xrightarrow{\text{eq. (40)}}$ fluctuation grow $\xrightarrow{\text{eq. (38)}}$ equilibrium is slowly modified

Before we start looking at specific examples, let us make sure that QLT satisfies the particle and energy conservation laws.

Particle #. $\int d^3\vec{v}$ Eq. (38):

$$\frac{\partial n_{0s}}{\partial t} = 0 \quad \text{OK (for homogeneous eq.)}$$

Ex. 2 Energy. $\sum_s \int d^3\vec{v}\, \frac{m_s v^2}{2}$ Eq. (38):

$$\frac{\partial \mathcal{E}_0}{\partial t} = \sum_s \int d^3\vec{v}\, \frac{m_s v^2}{2} \frac{\partial}{\partial \vec{v}} \cdot D_s \cdot \frac{\partial f_{0s}}{\partial \vec{v}} \overset{\text{by parts}}{=}$$

$$= -\sum_s \int d^3\vec{v}\, \frac{q_s^2}{m_s} \sum_i \sum_k \vec{k}\cdot\vec{v}\, |\varphi_k^{(i)}|^2\, \vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}}\, \frac{\gamma_k^{(i)}}{(\vec{k}\cdot\vec{v}-\omega_k^{(i)})^2 + \gamma_k^{(i)2}}$$

$$= -\sum_k \sum_i \underbrace{\frac{k^2 |\varphi_k^{(i)}|^2}{8\pi}}_{= \frac{|\vec{E}_k^{(i)}|^2}{8\pi}}\, 2\, \underbrace{\sum_s \frac{4\pi q_s^2}{m_s k^2} \int d^3\vec{v}\, \vec{k}\cdot\vec{v}\, \vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}}\, \frac{\gamma_k^{(i)}}{(\vec{k}\cdot\vec{v}-\omega_k^{(i)})^2 + \gamma_k^{(i)2}}}_{= \gamma_k^{(i)} \text{ eq. (43)}}$$

$$= -\sum_k 2\gamma_k^{(i)} \frac{|\vec{E}_k^{(i)}|^2}{8\pi} \overset{\text{eq. (42)}}{=} -\frac{\partial}{\partial t} \frac{\overline{|\vec{E}_k|^2}}{8\pi}$$

So
$$\frac{\partial}{\partial t}\left[\mathcal{E}_0 + \frac{\overline{|\vec{E}_k|^2}}{8\pi}\right] = 0 \quad \text{OK}$$