L3. 30.07.08

§3. Phase Mixing

[and related stories: energy conservation, heating, entropy, irreversibility of Landau damping, role of collisions]

From the linear theory of §2, we have learned that in a collisionless plasma, small-amplitude (electrostatic) fluctuations will have the character of waves (Langmuir, ion-acoustic) that are either Landau-damped or unstable − depending on the eq. distribution f_{0s}.

In the linear theory, f_{0s} was considered to be const in time − this was OK for small-amplitude fluctuations and <u>only for a limited time</u>.

Clearly, after some time (related to the fluctuation amplitude), the fluctuations will start affecting the equilibrium via the rhs in eq. (7):

$$\frac{\partial f_{0s}}{\partial t} = -\frac{q_s}{m_s}\sum_{k'}\left\langle \varphi_{-k'}\, i\vec{k}'\cdot\frac{\partial \delta f_{k's}}{\partial \vec{v}}\right\rangle \qquad (7)$$

So now we could take the solutions for $\varphi_{-k'}$ and $\delta f_{k's}$ calculated in the linear theory, substitute into (7), average over time and get the evolution of f_{0s}.

This is the algorithm for deriving quasilinear theory, but before we do this, I would like to explain some nuances about what it actually means that fluctuations are "collisionlessly" damped (or unstable). It is understood that in the process of such damping, energy is transferred from fluctuations to particles (or vice versa for unstable equilibria).

Let us examine, what actually happens to energy.

Let us go back to the original system of eqs (2-3):

$$\left\{\begin{array}{ll} \dfrac{\partial f_s}{\partial t} + \vec{v}\cdot\nabla f_s - \dfrac{q_s}{m_s}\overbrace{(\nabla\varphi)}^{=-\vec{E}}\cdot\dfrac{\partial f_s}{\partial \vec{v}} = \left(\dfrac{\partial f_s}{\partial t}\right)_c & (2) \\ -\nabla^2\varphi = 4\pi\sum_s q_s\int d^3\vec{v}\, f_s & (3) \end{array}\right.$$

keep collisions this time

From (3), we have

substitute (2)

$$-\nabla^2\frac{\partial\varphi}{\partial t} = 4\pi\sum_s q_s\int d^3\vec{v}\,\frac{\partial f_s}{\partial t} =$$

$$= 4\pi\sum_s q_s\int d^3\vec{v}\left[-\vec{v}\cdot\nabla f_s + \cancel{\frac{q_s}{m_s}(\nabla\varphi)\cdot\frac{\partial f_s}{\partial \vec{v}}} + \cancel{\left(\frac{\partial f_s}{\partial t}\right)_c}\right]$$

$$= -\nabla\cdot 4\pi\underbrace{\sum_s q_s\int d^3\vec{v}\,\vec{v}\,f_s}_{=\vec{j}\ \text{current density}}$$

0 because $f_s(\pm\infty)=0$

0 because particle # is conserved

– obviously, for this is just $+\nabla\cdot$ [Ampère-Maxwell law with $\vec{B}=0$]:

$$\frac{\vec{E}}{4\pi}\cdot\ \Big|\quad \frac{\partial\vec{E}}{\partial t} = -4\pi\vec{j} \qquad (22)$$

$$\frac{\partial}{\partial t}\frac{|\vec{E}|^2}{8\pi} = -\vec{E}\cdot\vec{j} = -\sum_s q_s\int d^3\vec{v}\,\vec{E}\cdot\vec{v}\,f_s \qquad (23)$$

work done on the particles – "heating"

Where does all this energy go?

Consider the evolution of the total particle energy:

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\sum_s \int d^3\vec{r} \int d^3\vec{v}\, \frac{m_s v^2}{2} f_s \overset{\text{use eq. (2)}}{=}$$

$$= \sum_s \int d^3\vec{r} \int d^3\vec{v}\, \frac{m_s v^2}{2} \Big[\underbrace{\cancel{-\vec{v}\cdot\nabla f_s}}_{\substack{0 \text{ if we neglect} \\ \text{fluxes through the boundary} \\ \text{(OK for inf. or periodic system)}}} - \frac{q_s}{m_s}\vec{E}\cdot \overbrace{\frac{\partial f_s}{\partial \vec{v}}}^{\text{by parts}} + \underbrace{\cancel{\left(\frac{\partial f_s}{\partial t}\right)_c}}_{\substack{0 \text{ because} \\ \text{collisions conserve} \\ \text{energy}}} \Big] =$$

$$= \sum_s q_s \int d^3\vec{r} \int d^3\vec{v}\, \vec{E}\cdot\vec{v}\, f_s \quad \text{– OK, here's the energy} \qquad (24)$$

Energy conservation:

$$\boxed{\frac{d}{dt}\left(\mathcal{E} + \int d^3\vec{r}\, \frac{|\vec{E}|^2}{8\pi}\right) = 0} \qquad (25)$$

Now recall that $\vec{E}_k = -i\vec{k}\varphi_k$ and $\varphi_k(t) = \sum_i c_i e^{p_i t}$ [eq. (12)]

where $p_i(\vec{k}) = -i\omega_k^{(i)} + \gamma_k^{(i)}$ are solutions of $\epsilon(p_i, \vec{k}) = 0$

and $c_i(\vec{k}) = \varphi_k^{(i)}(0)$ is the initial amplitude for mode #i.

Then

$$\frac{|\vec{E}_k|^2}{8\pi} = \frac{k^2|\varphi_k|^2}{8\pi} =$$

$$= \frac{k^2}{8\pi}\sum_{ij} \varphi_k^{(i)}(0)\,\varphi_k^{(j)*}(0)\, e^{\overbrace{-i(\omega_k^{(i)}-\omega_k^{(j)})t}^{\text{oscillatory}} + (\gamma_k^{(i)}+\gamma_k^{(j)})t}$$

Now, if we time average this, we'll get 0 for all terms except $i=j$, so

$$\overline{\frac{|\vec{E}_k|^2}{8\pi}} = \sum_i \frac{k^2}{8\pi}|\varphi_k^{(i)}(0)|^2 e^{2\gamma_k^{(i)}t}$$

el. field corresponding to ith mode

Then $$\boxed{\frac{d}{dt}\frac{\overline{|\vec{E}_k|^2}}{8\pi} = \sum_i 2\gamma_k^{(i)} \frac{|\vec{E}_k^{(i)}|^2}{8\pi}} \qquad (26)$$

system volume

From Parceval's theorem, $\int d^3\vec{r}\,\frac{|\vec{E}|^2}{8\pi} = V \sum_k \frac{|\vec{E}_k|^2}{8\pi}$.

We can now time average the energy conservation law (25) and use eq. (26) for the avg. change in el. energy:

$$\frac{d\overline{\mathcal{E}}}{dt} = -\frac{d}{dt}\int d^3\vec{r}\,\frac{\overline{|\vec{E}|^2}}{8\pi} = -V\sum_k\sum_i 2\gamma_k^{(i)}\frac{|\vec{E}_k^{(i)}|^2}{8\pi} > 0 \text{ if damping } (\gamma_k^{(i)}<0) \qquad (27)$$

negative if damping

So, if waves are damped,
particle energy increases ("heating").
If there is an instability, particles can lose energy.

Let us now consider the simplest specific example:

- electrons in homogeneous Maxwellian equilibrium

$$f_{0e} = \frac{n_{0e}}{(\pi v_{the}^2)^{3/2}} e^{-\frac{v^2}{v_{the}^2}} = n_{0e}\left(\frac{m_e}{2\pi T_{0e}}\right)^{3/2} e^{-\frac{m_e v^2}{2T_{0e}}}$$

- ions motionless ($T_i = 0$, $\delta f_i = 0$)

In such a system, we just have Landau-damped Langmuir waves. The electron energy is

$$\overline{\mathcal{E}} = V\int d^3\vec{v}\,\frac{m_e v^2}{2} f_{0e} = \frac{3}{2} V n_{0e} T_{0e}$$

Since, in a homogeneous system, $n_{0e} = \text{const}$, we have

$$\frac{d\overline{\mathcal{E}}}{dt} = \frac{3}{2} V n_{0e}\frac{dT_{0e}}{dt} = -V\sum_k 2\gamma_k \frac{|\vec{E}_k|^2}{8\pi} \qquad (28)$$

So damping $\Rightarrow$ heating of the equilibrium

There is another very important conservation law that all kinetic systems must respect.

Consider the entropy of the plasma

$$S = -\sum_s \int d^3\vec{r} \int d^3\vec{v}\, f_s \ln f_s$$

The evolution eqn for S is derived by multiplying the kinetic equation by $-(1+\ln f_s)$ and integrating over the entire phase space.

Since $(1+\ln f_s)\,\partial f_s = \partial(f_s \ln f_s)$, [derivative wrt anything]

we have

$$\frac{dS}{dt} = -\sum_s \int d^3\vec{r} \int d^3\vec{v} \left(\frac{\partial f_s}{\partial t}\right)_c \ln f_s \qquad (29)$$

It is possible the prove that

1) rhs of (29) ≥ 0 $\Rightarrow$ entropy is never decreased by collisions (and, in eq.(29), is only changed by them)

– Boltzmann's H theorem

2) rhs of (29) $= 0$ $\Leftrightarrow$ f_s is a Maxwellian

The law of entropy increase is related to irreversibility. In an exactly collisionless system, entropy is conserved and everything is reversible – so how can we have collisionless damping?

Let us go back to our simple example:

$$f_e = \underbrace{f_{oe}}_{\text{homogeneous Maxwellian}} + \underbrace{\delta f_e}_{\text{small perturbation}}$$

Then

$$S = -\int d^3\vec{r}\int d^3\vec{v}\,(f_{oe}+\delta f_e)\underbrace{\ln(f_{oe}+\delta f_e)}_{\ln f_{oe} + \ln\left(1+\frac{\delta f_e}{f_{oe}}\right) = \ln f_{oe} + \frac{\delta f_e}{f_{oe}} - \frac{1}{2}\frac{\delta f_e^2}{f_{oe}^2} + \dots} =$$

$$= -\int d^3\vec{r}\int d^3\vec{v}\,\Big(f_{oe}\ln f_{oe} + \delta f_e - \frac{1}{2}\frac{\delta f_e^2}{f_{oe}^2} + \delta f_e \ln f_{oe} + \frac{\delta f_e^2}{f_{oe}} + \dots\Big)$$

$$= -\int d^3\vec{r}\int d^3\vec{v}\,\Big[\underbrace{f_{oe}\ln f_{oe}}_{S_0} + \underbrace{(1+\ln f_{oe})\delta f_e}_{\text{vanishes upon time averaging}} + \frac{1}{2}\frac{\delta f_e^2}{f_{oe}} + \dots\Big]$$

Time average:

$$\bar{S} = S_0 + \overline{\delta S}$$

where

$$S_0 = -\int d^3\vec{r}\int d^3\vec{v}\, f_{oe}\Big[\ln n_{oe}\Big(\frac{m_e}{2\pi}\Big)^{3/2} - \frac{3}{2}\ln T_{oe} - \frac{m_e v^2}{2T_{oe}}\Big] =$$

$$= -V n_{oe}\ln\Big[n_{oe}\Big(\frac{m_e}{2\pi}\Big)^{3/2}\Big] + \frac{3}{2}V n_{oe}\ln T_{oe} + \frac{V}{\cancel{T_{oe}}}\frac{3}{2}n_{oe}\cancel{T_{oe}}$$

and

$$\overline{\delta S} = -\int d^3\vec{r}\int d^3\vec{v}\,\overline{\frac{\delta f_e^2}{2f_{oe}}}$$

Now time average eq. (29) and substitute the above:

$$\frac{d\bar{S}}{dt} = \frac{dS_0}{dt} + \frac{d\overline{\delta S}}{dt} = -\int d^3\vec{r}\int d^3\vec{v}\,\overline{\Big[\underbrace{\Big(\frac{\partial f_{oe}}{\partial t}\Big)_c}_{0 \text{ because Maxwellian (for e-e collisions)}} + \Big(\frac{\partial \delta f_e}{\partial t}\Big)_c + \dots\Big]\Big(\ln f_{oe} + \frac{\delta f_e}{f_{oe}} + \dots\Big)}$$

$$\frac{dS_0}{dt} = \frac{3}{2}V n_{oe}\frac{1}{T_{oe}}\frac{dT_{oe}}{dt} \qquad (30)$$

(heating ⇔ eq. entropy increase)

$$= -\int d^3\vec{r}\int d^3\vec{v}\,\overline{\frac{\delta f_e}{f_{oe}}\Big(\frac{\partial \delta f_e}{\partial t}\Big)_c}$$

So we have derived

$$\underbrace{\frac{3}{2} V n_{0e} \frac{1}{T_{0e}} \frac{dT_{0e}}{dt}}_{= -\frac{1}{T_{0e}} \frac{d}{dt} \int d^3\vec{r} \frac{|\vec{E}|^2}{8\pi} \text{ from (28)}} - \frac{d}{dt} \int d^3\vec{r} \int d^3\vec{v}\, \overline{\frac{\delta f_e^2}{2 f_{0e}}} = \underbrace{- \int d^3\vec{r} \int d^3\vec{v}\, \overline{\frac{\delta f_e}{f_{0e}} \left(\frac{\partial \delta f_e}{\partial t}\right)_c}}_{\text{negative because lin. coll. op. is negative-definite}}$$

and we have the foll. cons. law:

$$\boxed{\frac{d}{dt} \underbrace{\int d^3\vec{r} \left[\int d^3\vec{v}\, \overline{\frac{T_{0e} \delta f_e^2}{2 f_{0e}}} + \frac{\overline{|\vec{E}|^2}}{8\pi} \right]}_{-T_{0e} \overline{\delta S} + \iint \frac{|\vec{E}|^2}{8\pi} \text{ free energy}} = \int d^3\vec{r} \int d^3\vec{v}\, \overline{\frac{\delta f_e}{f_{0e}} \left(\frac{\partial \delta f_e}{\partial t}\right)_c} \leq 0} \quad (31)$$

This is directly generalisable to multiple species

So what would happen if we had an exactly collisionless system?

RHS of (31) $\to 0$ $\Rightarrow$ $$\frac{d}{dt} \left[\underbrace{\int d^3\vec{r} \int d^3\vec{v}\, \overline{\frac{T_{0e} \delta f_e^2}{2 f_{0e}}}}_{\text{this must be growing then}} + \underbrace{\int d^3\vec{r}\, \frac{\overline{|\vec{E}|^2}}{8\pi}}_{\text{this is decaying if there is damping}} \right] = 0 \quad (31a)$$

Indeed, damping $\Rightarrow$ heating [Eq. (28)] $\Rightarrow$ eq. entropy grows [Eq. (30)]

But total entropy is conserved [Eq. (29)] $\Rightarrow$ growth of eq. entropy must be compensated by decrease of the entropy of the perturbed distribution:

$$-\overline{\delta S} = \int d^3\vec{r} \int d^3\vec{v}\, \overline{\frac{\delta f_e^2}{2 f_{0e}}} \text{ grows } \Rightarrow \delta f_e \text{ is growing.}$$

So the perturbation does not decay away and stays forever... We also need some sort of special arrangement because

$$\underset{\text{damped}}{\varphi_k} = -\frac{4\pi}{k^2} e \int d^3\vec{v}\, \underset{\text{growing}}{\delta f_{ek}}$$

One way to achieve this is by having δf_{ek} oscillate rapidly in $\vec{v}$ space — this is indeed what happens — p. 20

In fact, what turns out to happen (as we are about to see) is that δf_e tends to develop ever smaller scales in velocity space — "phase mixing". But then, sooner or later, the collision term will become important because it is a 2-order diffusion operator in velocity space:

$$\left(\frac{\partial \delta f_e}{\partial t}\right)_c \sim \underbrace{\nu_{coll}}_{\text{small coll. freq.}} v_{the}^2 \underbrace{\frac{\partial^2}{\partial v^2}}_{\text{large velocity grad}} \delta f_e \quad \text{— finite!}$$

cf: neutral fluid with small viscosity: $Re \gg 1$, large grads!

→ so rhs of (31) < 0 and both the field and the perturbed distr. function can decay.

L4-5
6.02.08

Where do small scales in velocity space come from?

Consider an ultra-simplified kinetic eqn: $\varphi_k = 0$, 1D:

$$\frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial x} = 0, \quad \delta f\big|_{t=0} = g(x,v)$$

some benign dependence

Solution: $\delta f(t,x,v) = g(x-vt, v)$

Typical scale in v space: $\frac{\partial g}{\partial v} = -t\frac{\partial g}{\partial x}$, so $\frac{\partial}{\partial v} \sim kt$ ← grows

This is because particles with small velocity difference $v-v'$ at pt x at time t came from widely separated points $x_0 - x_0' = -(v-v')t$ at time 0 (a long time ago)

In k space this is explicit:

$$\frac{\partial \delta f_k}{\partial t} + ikv\delta f_k = 0 \quad \Rightarrow \quad \delta f_k(t,v) = g_k(v)e^{-ikvt}$$

ballistic response

Let us now go back to the linear theory and see if we can trace the ballistic response.

The Laplace-transformed distribution function was:

$$\delta\hat{f}_{ks}(p,\vec{v}) = \frac{1}{p+i\underline{k}\cdot\vec{v}}\left[i\frac{q_s}{m_s}\hat{\varphi}_k(p)\,\underline{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}} + g_{ks}(\vec{v})\right] \qquad (9)$$

Taking the inverse Laplace transform in the way described on p. 8, we get

$$\delta f_{ks}(t,\vec{v}) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma} dp\, e^{pt}\frac{1}{p+i\underline{k}\cdot\vec{v}}\left[i\frac{q_s}{m_s}\underbrace{\hat{\varphi}_k(p)}_{=\sum_i \frac{c_i}{p-p_i} + \text{analytic part (see p. 8)}}\underline{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}} + g_{ks}(\vec{v})\right] =$$

this pole

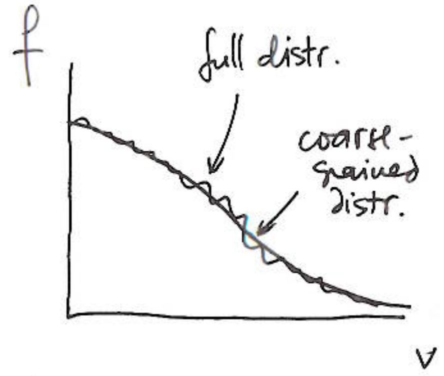
these poles

$$= \underbrace{e^{-i\underline{k}\cdot\vec{v}t}\left[i\frac{q_s}{m_s}\hat{\varphi}_k(-i\underline{k}\cdot\vec{v})\,\underline{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}} + g_{ks}(\vec{v})\right]}_{\text{ballistic response } ①} + \underbrace{\frac{iq_s}{m_s}\sum_i\frac{c_i e^{p_i t}}{p_i+i\underline{k}\cdot\vec{v}}\,\underline{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}}}_{②} \qquad (32)$$

large gradients in $\vec{v}$!

no large grads wrt $\vec{v}$ here

In the presence of collisions even with a small coll. frequency, the ballistic part of the perturbed distribution will get smeared out by the collisional diffusion in the velocity space. Incorporating this effect exactly would require solving the kinetic equation with a collision operator — quite difficult. Instead, we can take this effect into account by "coarse-graining" the distribution function in velocity space and thus averaging out the small scales; so only term ② is left in (32):



$$\langle \delta f_{ks}(t,\vec{v})\rangle_{\nu} = \frac{iq_s}{m_s}\sum_i \frac{c_i e^{p_i t}}{p_i + i\vec{k}\cdot\vec{v}}\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}} \qquad (33)$$

Let us calculate the el. field from this:

$$\varphi_k = \frac{4\pi}{k^2}\sum_s q_s \int d^3\vec{v}\,\langle \delta f_{ks}(t,\vec{v})\rangle_{\nu} =$$

$$= \sum_i c_i e^{p_i t}\Big[\underbrace{\sum_s i\,\frac{4\pi q_s^2}{m_s k^2}\int d^3\vec{v}\,\frac{1}{p_i + i\vec{k}\cdot\vec{v}}\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}} - 1}_{-\epsilon(p_i,\vec{k}) = 0\,!} + 1\Big] = \qquad (34)$$

$$= \sum_i c_i e^{p_i t}$$

as, indeed, we claimed in §2.

This is why irreversible damping is possible: we (in fact) are dealing with a smoothed-out (coarse-grained) distribution, whose entropy does not need to be conserved!

What this procedure means is that we are first taking the limit $t\to\infty$, keeping ν_{coll} finite and allowing collisions to get rid of the ballistic term (smooth the distr. function) and then take $\nu_{coll}\to 0$, allowing the damping due to wave-particle resonance to dominate.

Note. You might have spotted that in the entropy calculation on p.18 I neglected the interspecies collisions, which tend to equalise temperatures of different species — this is OK because those terms, unlike $\nu_{coll}\frac{\delta f}{f_{0}}\left(\frac{\partial \delta f}{\partial t}\right)_c$, are not singular in ν_{coll}, i.e. taking $\nu_{coll}\to 0$ does make them small (they only involve $\frac{\partial}{\partial v}$ of the eq. distr., so $\frac{\partial}{\partial v}$ stays finite).

Let us estimate how long it takes collisions to eliminate the ballistic term [term ① in (32)]:

$$\frac{\partial}{\partial v} ① \sim kt ① \Rightarrow \nu_{coll} v_{th}^2 \frac{\partial^2}{\partial v^2} ① \sim \underbrace{\nu_{coll} v_{th}^2 k^2 t^2}_{} ①$$

This has to become big:

$$\nu_{coll} v_{th}^2 k^2 t^2 \gg kv \sim \omega_k \Rightarrow t \gg \left(\frac{\omega_k}{\nu_{coll} k^2 v_{th}^2}\right)^{1/2} \quad (35)$$

Compare this with what happens to term ②:

$$\frac{\partial}{\partial v} ② \sim \frac{1}{v_{th}} ② \Rightarrow \nu_{coll} v_{th}^2 \frac{\partial^2}{\partial v^2} ② \sim \nu_{coll} ②$$

So we can neglect the effect of collisions on this term as long as $t \ll \nu_{coll}^{-1}$, which is consistent with (35) if

$$\frac{1}{\nu_{coll}} \gg \left(\frac{\omega_k}{\nu_{coll} k^2 v_{th}^2}\right)^{1/2} \quad \text{or} \quad \nu_{coll} \ll \frac{k^2 v_{th}^2}{\omega_k} \quad \text{(no problem)}$$

Note that (35) is in fact a very conservative estimate. It is possible to argue that in nonlinear systems containing many waves, phase mixing can be accomplished in much faster way due to phase-space "turbulent diffusion" – the collisional scales in velocity space are then reached on ν_{coll}-independent time scales of turbulent diffusion [Dupree 1966, Phys. Fluids 9, 1773]

Let me illustrate the effect of collisions and the non-interchangeability of limits using our ultra-simplified example: consider $\varphi_k = 0$, 1D, so

("model coll. operator" (retain fastest derivatives))

$$\frac{\partial \delta f_k}{\partial t} + ikv\,\delta f_k = \nu_{coll} v_{th}^2 \frac{\partial^2 \delta f_k}{\partial v^2}$$

[redacted] [magic!]

The solution of this equation can be sought in the form:

$$\delta f_k(t,v) = \Phi(t,u)\, e^{-ikvt - \frac{1}{3}\nu_{coll} v_{th}^2 k^2 t^3}$$

where $u = v - ik\nu_{coll} v_{th}^2 t^2$. Then

$$\frac{\partial \delta f_k}{\partial t} + ikv\,\delta f_k = \left[\frac{\partial \Phi}{\partial t} \cancel{- 2ik\nu_{coll} v_{th}^2 t \frac{\partial \Phi}{\partial u}} - (\cancel{ikv} + \cancel{\nu_{coll} v_{th}^2 k^2 t^2})\Phi + \cancel{ikv\Phi}\right] e^{-(\dots)}$$

$$= \nu_{coll} v_{th}^2 \frac{\partial}{\partial v}\left[\frac{\partial \Phi}{\partial u} - ikt\Phi\right] e^{-(\dots)} =$$

$$= \nu_{coll} v_{th}^2 \left[\frac{\partial^2 \Phi}{\partial u^2} \cancel{- ikt\frac{\partial \Phi}{\partial u}} \cancel{- ikt\frac{\partial \Phi}{\partial u}} \cancel{- k^2 t^2 \Phi}\right] e^{-(\dots)}$$

So we get $\quad \dfrac{\partial \Phi}{\partial t} = \nu_{coll} v_{th}^2 \dfrac{\partial^2 \Phi}{\partial u^2}$ – usual diffusion equation

Sln: $$\delta f_k(t,v) = e^{-ikvt - \frac{1}{3}\nu_{coll} v_{th}^2 k^2 t^3} \int_{-\infty}^{+\infty} dv'\, g_k(v') \frac{e^{-\frac{(v-v'-ik\nu_{coll} v_{th}^2 t^2)^2}{4\nu_{coll} v_{th}^2 t}}}{\sqrt{4\pi \nu_{coll} v_{th}^2 t}} =$$

(initial distr. function)

$$= e^{-ikvt - \frac{1}{12}\nu_{coll} v_{th}^2 k^2 t^3} \,*$$

$$* \int_{-\infty}^{+\infty} dv'\, g_k(v') \frac{e^{-\frac{(v-v')^2}{4\nu_{coll} v_{th}^2 t} + \frac{ik(v-v')t}{2}}}{\sqrt{4\pi \nu_{coll} v_{th}^2 t}}$$

$\rightarrow g_k(v)\, e^{-ikvt}$ if $\nu_{coll} \to 0$

$\rightarrow 0$ if $t \to \infty$ [redacted]

Collisions damp out the velocity structure after

$$\nu_{coll} v_{th}^2 k^2 t^3 \gg kvt \quad \text{[redacted]}$$

or $$t \gg \left(\frac{v}{\nu_{coll} k v_{th}^2}\right)^{1/2} \sim \left(\frac{\omega_k}{\nu_{coll} k^2 v_{th}^2}\right)^{1/2}$$ cf eq. (35)

So what happens if our plasma is, after all, exactly collisionless or so weakly collisional that eq. (35) cannot be satisfied for $t \sim \gamma_k^{-1}$?

Let us go back to eq. (32) and substitute into it

$$\hat{\varphi}_k(-i\mathbf{k}\cdot\vec{v}) = \sum_i \frac{c_i}{-i\mathbf{k}\cdot\vec{v} - p_i} + \text{analytic part of } \hat{\varphi}_k(p) \text{ @ } p = -i\mathbf{k}\cdot\vec{v} \quad :$$

$$\delta f_{ks}(t,\vec{v}) = e^{-i\mathbf{k}\cdot\vec{v}t}\left[\frac{iq_s}{m_s}\left(-\sum_i \frac{c_i}{p_i + i\mathbf{k}\cdot\vec{v}} + \cancel{\text{analytic part}}\right)\mathbf{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}} + g_{ks}(\vec{v})\right] +$$

$$+ \frac{iq_s}{m_s}\sum_i \frac{c_i e^{p_i t}}{p_i + i\mathbf{k}\cdot\vec{v}}\,\mathbf{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}} \qquad (36)$$

this is $=0$ because it is independent of time and we must have $\delta f_{ks}(t=0,\vec{v}) = g_{ks}(\vec{v})$

Then the electric field is given by

$$\varphi_k(t) = \frac{4\pi}{k^2}\sum_s q_s \int d^3\vec{v}\, e^{-i\mathbf{k}\cdot\vec{v}t}\left[-\frac{iq_s}{m_s}\sum_i \frac{c_i}{p_i + i\mathbf{k}\cdot\vec{v}}\,\mathbf{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}} + g_{ks}(\vec{v})\right] + \underbrace{\sum_i c_i e^{p_i t}}_{\text{like in eq. (34) (coarse-grained part)}}$$

$$\frac{1}{p_i + i\mathbf{k}\cdot\vec{v}} \to P\frac{1}{p_i + i\mathbf{k}\cdot\vec{v}} + i\pi\delta(p_i + i\mathbf{k}\cdot\vec{v})$$

$$= +\sum_i c_i\left[-i\sum_s \frac{4\pi q_s^2}{k^2 m_s} P\int d^3\vec{v}\,\frac{e^{-i\mathbf{k}\cdot\vec{v}t}}{p_i + i\mathbf{k}\cdot\vec{v}}\,\mathbf{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}}\right.$$

nonresonant part, vanishes because of fast oscillation in $\vec{v}$ space: $t \gg \frac{1}{kv} \sim \frac{1}{kv_{ths}}$

$$\left. - \underbrace{\sum_s \frac{4\pi q_s^2}{k^2 m_s} e^{p_i t}\pi F'_{0s}\left(\frac{\omega_k}{k}\right)}_{\ll\, c_i e^{p_i t}\,\mathrm{Im}\,\epsilon \text{ - small}}\right] + \frac{4\pi}{k^2}\sum_s q_s \underbrace{\int d^3\vec{v}\, e^{-i\mathbf{k}\cdot\vec{v}t} g_{ks}(\vec{v})} \qquad (37)$$

vanishes because of $\vec{v}$-space oscillation as long as width of initial distribution not too small: $t \gg \frac{1}{k\Delta v}$

$t \ll \frac{1}{\gamma_k} \Rightarrow \Delta v \gg \frac{\gamma_k}{k}$

smaller than this

$$+\sum_i c_i e^{p_i t} \approx \sum_i c_i e^{p_i t} \text{ still}$$

(but the perturbed distr. fn does not decay)