§2. Linear Theory

[Langmuir waves, Landau damping/instability, ion-acoustic waves]

So, consider eqns (8) and (4):

Vlasov
$$\frac{\partial \delta f_{ks}}{\partial t} + i\vec{k}\cdot\vec{v}\,\delta f_{ks} = \frac{q_s}{m_s}\varphi_k i\vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}} \qquad (8)$$

Poisson
$$\varphi_k = \frac{4\pi}{k^2}\sum_s q_s \int d^3\vec{v}\,\delta f_{ks} \qquad (4)$$

Consider an initial value problem (Landau):

$$\delta f_{ks}(t=0, \vec{v}) = g_{ks}(\vec{v})$$

and solve by Laplace transform:

$$\left.\begin{aligned}\delta\hat{f}_{ks}(p,\vec{v}) &= \int_0^\infty dt\, e^{-pt}\,\delta f_{ks}(t,\vec{v})\\ \hat{\varphi}_k(p) &= \int_0^\infty dt\, e^{-pt}\,\varphi_k(t)\end{aligned}\right\}\ \text{exist for } \mathrm{Re}\,p \geq \sigma \text{ if } |\varphi|, |\delta f| < e^{\sigma t}$$

Eq. (8) becomes

$$p\,\delta\hat{f}_{ks} - g_{ks} + i\vec{k}\cdot\vec{v}\,\delta\hat{f}_{ks} = \frac{q_s}{m_s}\hat{\varphi}_k i\vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}}$$

$$\delta\hat{f}_{ks} = \frac{1}{p+i\vec{k}\cdot\vec{v}}\left(i\frac{q_s}{m_s}\hat{\varphi}_k(p)\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}} + g_{ks}\right) \qquad (9)$$

Eq. (4):
$$\hat{\varphi}_k(p) = \frac{4\pi}{k^2}\sum_s q_s\int d^3\vec{v}\,\delta\hat{f}_{ks}$$

$$\hat{\varphi}_k(p) = \frac{4\pi}{k^2\epsilon(p,\vec{k})}\sum_s q_s\int d^3\vec{v}\,\frac{g_{ks}(\vec{v})}{p+i\vec{k}\cdot\vec{v}} \qquad (10)$$

where

$$\epsilon(p,\vec{k}) = 1 - \sum_s \frac{4\pi q_s^2}{k^2 m_s}\, i\int d^3\vec{v}\,\frac{1}{p+i\vec{k}\cdot\vec{v}}\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}}$$

– ~~dispersion function~~ dielectric function.

Let $\omega_{ps}^2 = \dfrac{4\pi q_s^2 n_{os}}{m_s}$ plasma frequency,

$n_{os} = \int d^3\vec{v}\, f_{os}$ eq. density

Then

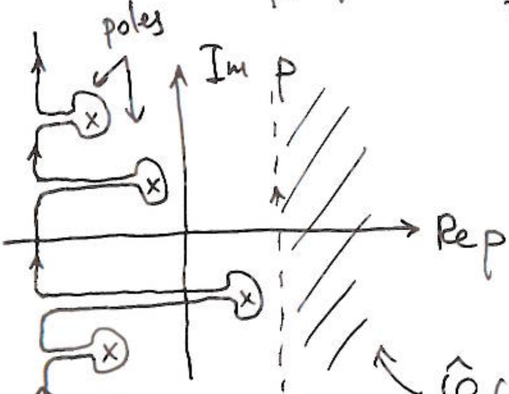
$$\epsilon(p,\vec{k}) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \frac{i}{n_{os}} \int d^3\vec{v} \frac{1}{p + i\vec{k}\cdot\vec{v}}\, \vec{k}\cdot\frac{\partial f_{os}}{\partial \vec{v}} \qquad (11)$$

The solution — inverse Laplace transform of $\hat{\varphi}_k(p)$ — is determined by the poles of $\hat{\varphi}_k(p)$, which the zeros of $\epsilon(p,\vec{k})$: indeed, if

disp. relation $\boxed{\epsilon(p_i,\vec{k}) = 0}$ for some set of solutions $p_i = p_i(\vec{k})$

then $\hat{\varphi}_k(p) = \sum_i \dfrac{c_i}{p - p_i}$ + analytic part

coefficients



$$\varphi_k(t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{+i\infty+\sigma} dp\, e^{pt}\, \hat{\varphi}_k(p) =$$

$$= \sum_i c_i e^{p_i t} \qquad (12)$$

coefficients that can be calculated from the initial distribution.

Now solve the dispersion relation. We shall look for weakly damped or growing waves: $p = -i\omega + \gamma$, where $\gamma \ll \omega$

expand in small γ

Then $\epsilon(p,\vec{k}) = \epsilon(-i\omega+\gamma,\vec{k}) \approx \epsilon(-i\omega,\vec{k}) + i\gamma\frac{\partial}{\partial\omega}\epsilon(-i\omega,k)$

$$= \mathrm{Re}\,\epsilon + i\,\mathrm{Im}\,\epsilon + i\gamma\frac{\partial}{\partial\omega}\mathrm{Re}\,\epsilon - \underbrace{\gamma\frac{\partial}{\partial\omega}\mathrm{Im}\,\epsilon}_{\text{small}} = 0$$

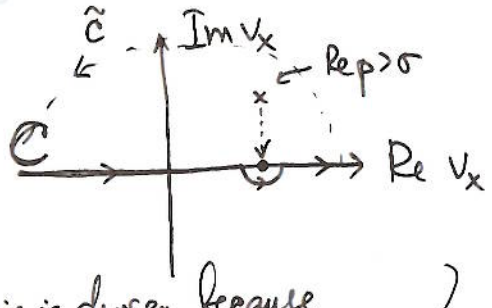
Im of this $=0$: $$\boxed{\gamma = -\frac{\mathrm{Im}\,\epsilon}{\frac{\partial\,\mathrm{Re}\,\epsilon}{\partial\omega}}} \qquad (13)$$

Re of this $=0$: $\boxed{\mathrm{Re}\,\epsilon = 0}$ – equ for ω (using $\gamma \ll \omega$) (14)

Now, from eq. (11),

$$\epsilon(-i\omega,\vec{k}) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2}\frac{1}{n_{0s}}\int d^3\vec{v}\,\frac{1}{\vec{k}\cdot\vec{v}-\omega}\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}}$$

We do the velocity integral by analytically continuing from $\mathrm{Re}\,p \geq \sigma > 0$ (where Laplace transform existed and so does the integral above because the integration is over real velocities) to $\mathrm{Re}\,p \leq 0$ by using Landau contour: if $\vec{k}$ is along x axis,



This is chosen because

$$\int_C (\dots) + \int_{\tilde{C}} (\dots) = 2\pi i\, F_{0s}'\left(\frac{ip}{k}\right)$$

analytic for all p $\Leftarrow$ analytic for all p

$$\int_C d^3\vec{v}\,\frac{1}{\vec{k}\cdot\vec{v}-\omega}\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}} = \int_C dv_x\,\frac{1}{v_x-\omega/k}\,F_{0s}'(v_x)$$

$F_{0s} \doteq \iint dv_y\,dv_z\, f_{0s}(\vec{v})$

$$= \mathcal{P}\int_{-\infty}^{+\infty} dv_x\,\frac{1}{v_x-\omega/k}\,F_{0s}'(v_x) + i\pi F_{0s}'\left(\frac{\omega}{k}\right)$$

principal value (meaning approaching the pole symmetrically from both sides $-\epsilon \leftrightarrow +\epsilon$)

So, now we have

$$\mathrm{Re}\,\epsilon = 1 - \sum_s \frac{\omega_{ps}^2}{k^2}\frac{1}{n_{os}} \mathrm{P}\int_{-\infty}^{+\infty} dv_x \frac{1}{v_x - \omega/k} F_{os}'(v_x) \qquad (15)$$

$$\mathrm{Im}\,\epsilon = -\sum_s \frac{\omega_{ps}^2}{k^2}\frac{\pi}{n_{os}} F_{os}'\left(\frac{\omega}{k}\right) \qquad (16)$$

We now consider two special cases:

1) high-freq.: $\omega \gg k v_{the}$ ← thermal speed of electrons (for arbitrary eq. distributions, "typical" speed)

2) low-freq.: $\underset{\text{hot electrons}}{k v_{the}} \gg \omega \gg \underset{\text{cold ions}}{k v_{thi}}$

So, we have

1) $\frac{\omega}{k} \gg v_{the}$: expand eq. (15):

$$1 + \frac{k v_x}{\omega} + \frac{k^2 v_x^2}{\omega^2} + \frac{k^3 v_x^3}{\omega^3} + \ldots$$

$$\mathrm{Re}\,\epsilon \simeq 1 + \sum_s \frac{\omega_{ps}^2}{k^2}\frac{1}{n_{os}}\frac{k}{\omega} \mathrm{P}\int_{-\infty}^{+\infty} dv_x \frac{1}{1 - \frac{k v_x}{\omega}} F_{os}'(v_x) \simeq$$

$$\simeq 1 + \sum_s \frac{\omega_{ps}^2}{k\omega}\frac{1}{n_{os}}\Big[\cancelto{0}{\int dv_x F_o'(v_x)} - \frac{k}{\omega}\underbrace{\int dv_x F_{os}(v_x)}_{=n_{os}} -$$

$$- \frac{k^2}{\omega^2}\int dv_x v_x F_{os}(v_x) - 3\frac{k^3}{\omega^3}\int dv_x v_x^2 F_{os}(v_x) - \ldots\Big] =$$

$$= 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\Big[1 - \underbrace{\frac{k}{\omega}\frac{1}{n_{os}}\int dv_x v_x F_{os}(v_x) - 3\frac{k^2}{\omega^2}\frac{1}{n_{os}}\int dv_x v_x^2 F_{os}(v_x) \ldots}_{\text{small corrections}}\Big]$$

Since $\frac{\omega_{pi}^2}{\omega_{pe}^2} = \frac{Z m_e}{m_i} \ll 1$, the ion contribution is negligible.

Solution to lowest order: $\boxed{\omega^2 = \omega_{pe}^2 = \frac{4\pi e^2 n_{oe}}{m_e}}$ Langmuir wave (17)

To next order, assuming no electron eq. current $\int dv_x v_x F_{oe}(v_x) = 0$,

(18) $\omega^2 = \omega_{pe}^2\left(1 + \frac{3}{2}k^2\lambda_{De}^2\right)$, where $\lambda_{De}^2 = \frac{2}{\omega_{pe}^2}\underbrace{\frac{1}{n_{oe}}\int dv_x v_x^2}_{= v_{the}^2/2} F_{oe}(v_x)$ Debye length

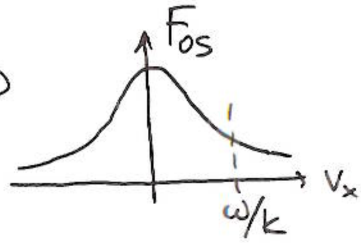
Now calculate damping rate from (13) and (16):

$$\frac{\partial \,\mathrm{Re}\,\epsilon}{\partial \omega} \approx 2\frac{\omega_{pe}^2}{\omega^3}$$

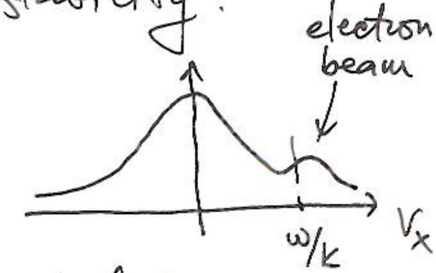
$$\gamma \approx \frac{\omega^3}{2\omega_{pe}^2}\cdot\frac{\omega_{pe}^2}{k^2}\,\frac{\pi}{n_{0e}}\,F'_{0e}\left(\frac{\omega}{k}\right) = \frac{\pi}{2}\,\frac{\omega^3}{k^2}\,\frac{1}{n_{0e}}\,F'_{0e}\left(\frac{\omega}{k}\right) \qquad (19)$$

$F'_{0e}\left(\frac{\omega}{k}\right) < 0 \Rightarrow$ Landau damping $\gamma < 0$

$F'_{0e}\left(\frac{\omega}{k}\right) > 0 \Rightarrow$ instability



A typical example is bump-on-tail instability: we shall consider this example in great detail in what follows.



$\boxed{\text{L3, 30.01.08}}$

2) $v_{the} \gg \frac{\omega}{k} \gg v_{thi}$: expand the ion contribution same way we did electrons above, while the electron term in eq. (15) is expanded neglecting $\frac{\omega}{k}$.

$$\frac{\omega_{pe}^2}{k^2}\,\frac{1}{n_{0e}}\,\mathcal{P}\int_{-\infty}^{+\infty} dv_x\,\frac{1}{v_x-\omega/k}\,F'_{0e}(v_x) \approx$$

$$\approx \frac{\omega_{pe}^2}{k^2}\,\frac{1}{n_{0e}}\,\mathcal{P}\int_{-\infty}^{+\infty} \frac{dv_x}{v_x}\,F'_{0e}(v_x) = -\frac{2\omega_{pe}^2}{k^2 v_{the}^2} + \dots$$

assume this to be a Maxwellian, possibly with a current: $F_{0e}(v_x) = \frac{n_{0e}}{\sqrt{\pi v_{the}^2}}\,e^{-\frac{(v_x-u_e)^2}{v_{the}^2}}$, $u_e \sim v_{thi} \ll v_{the}$

So, $\mathrm{Re}\,\epsilon \approx \underset{\text{small}}{1} + \frac{2\omega_{pe}^2}{k^2 v_{the}^2} - \frac{\omega_{pi}^2}{\omega^2}\left(1 - \underset{\text{small as long as } k^2\lambda_{Di}^2 \ll 1}{\ldots}\right) = 0$

$$\omega^2 = \frac{\omega_{pi}^2}{1 + \frac{2\omega_{pe}^2}{k^2 v_{the}^2}} = \frac{1}{2} k^2 \underbrace{v_{the}^2 \frac{\omega_{pi}^2}{\omega_{pe}^2}}_{=} \frac{1}{1 + k^2\lambda_{De}^2/2} = \frac{k^2 c_s^2}{1 + \underset{\text{small}}{k^2\lambda_{De}^2/2}} \qquad (20)$$

ion-acoustic wave

$$\frac{2T_{oe}}{m_e}\frac{m_e Z}{m_i} = \frac{2T_e Z}{m_i} \equiv 2c_s^2 \quad \text{speed of sound}$$

Now calculate damping/growth:

$$\frac{\partial\,\mathrm{Re}\,\epsilon}{\partial\omega} \approx \frac{2\omega_{pi}^2}{\omega^3}$$

since $\frac{\omega}{k} \gg v_{thi}$, exp-lly small if F_{oi} is at all similar to a Maxwellian

$$\gamma \approx \frac{\omega^3}{2\omega_{pi}^2}\left[\frac{\omega_{pe}^2}{k^2}\frac{\pi}{n_{oe}}\underbrace{F_{oe}'\left(\frac{\omega}{k}\right)}_{\text{ss}} + \cancel{\frac{\omega_{pi}^2}{k^2}\frac{\pi}{n_{oi}}F_{oi}'\left(\frac{\omega}{k}\right)}\right] =$$

$$-\frac{2}{v_{the}^2}\left(\frac{\omega}{k} - u_e\right)F_{oe}\left(\frac{\omega}{k}\right) \approx -\frac{2}{v_{the}^2}\left(\frac{\omega}{k} - u_e\right)\frac{n_{oe}}{\sqrt{\pi v_{the}^2}}$$

for a Maxwellian with a current

$$= -\frac{\omega^3}{\cancel{2}\omega_{pi}^2}\frac{\omega_{pe}^2}{k^2}\frac{\pi}{\cancel{n_{oe}}}\frac{\cancel{2}}{v_{the}^3}\frac{\cancel{n_{oe}}}{\sqrt{\pi}}\left(\frac{\omega}{k} - u_e\right) =$$

$$= -\frac{m_i}{Zm_e}\sqrt{\pi}\frac{c_s^3}{v_{the}^3} k c_s\left(1 - \frac{u_e}{c_s}\right) = -\sqrt{\pi}\, k c_s \sqrt{\frac{Zm_e}{m_i}}\left(1 - \frac{u_e}{c_s}\right)$$

When $u_e < c_s \Rightarrow$ Landau damping of ion acoustic waves on electrons (21)

When $u_e > c_s \Rightarrow$ Ion-acoustic instability: excitation of ion acoustic waves by electron current.