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§1. Introduction

$\boxed{\text{L1-2, 23.01.08}}$

A plasma is fully described by

- distribution functions for particles of each species s

$$f_s = f_s(t, \vec{r}, \vec{v}) \quad \text{6D phase space}$$

satisfies Vlasov-Landau (Boltzmann) kinetic equation, which expresses conservation of probability in phase space:

$$\frac{Df_s}{Dt} = \frac{\partial f_s}{\partial t} + \underbrace{\frac{d\vec{r}}{dt}}_{=\vec{v}} \cdot \frac{\partial f_s}{\partial \vec{r}} + \underbrace{\frac{d\vec{v}}{dt}}_{=\vec{F}} \cdot \frac{\partial f_s}{\partial \vec{v}} = \left(\frac{\partial f_s}{\partial t}\right)_c$$

collision integral (particle interactions)

$$\vec{F} = \frac{q_s}{m_s}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) \qquad (1)$$

particle eqns of motion

- electric and magnetic fields, which satisfy Maxwell's equations with charge densities and currents calculated from the particle distribution.

Let us start with a simple example of such a system:

- ion-electron plasma: $s = e, i$; $q_e = -e$, $q_i = Ze$
- unmagnetised: $\vec{B} = 0$

$$\vec{E} = -\nabla\varphi \quad \text{(electrostatic)}$$

- collisionless/weakly collisional, i.e. $\nu_{coll} \ll \omega$, so ignore the coll. term.

This is described by the foll. system of eqns:

$$\begin{cases} \dfrac{\partial f_s}{\partial t} + \vec{v}\cdot\nabla f_s - \dfrac{q_s}{m_s}(\nabla\varphi)\cdot\dfrac{\partial f_s}{\partial \vec{v}} = \underset{\substack{\uparrow \\ \text{collision term} \\ \text{neglected}}}{0} & \text{Vlasov} \quad (2) \\ -\nabla^2\varphi = 4\pi \underbrace{\sum_s q_s \int d^3\vec{v}\, f_s}_{\text{charge density}} & \text{Poisson} \quad (3) \end{cases}$$

It is a generic situation in plasma physics that one considers a state that is a superposition of some equilibrium state (slowly varying in space and time, determined by the global system properties – lab set up or astro. object – so varying on system scales) plus fluctuations (externally stirred or excited by instabilities of the eq. state, fast varying in space and time – usually waves).

So, let

$$f_s = \underset{\text{slow}}{f_{0s}(t,\vec{r},\vec{v})} + \underset{\text{fast}}{\delta f_s(t,\vec{r},\vec{v})}$$

For simplicity, consider a homogeneous equilibrium, i.e. $f_{0s} = f_{0s}(t,\vec{v})$ [this means we are looking at scales $\ll$ eq. scale] with no fields ($\vec{E}_0 = 0$).

Note that we can think of the eq. distribution here as the average of the full distribution over fast timescales [and fast spatial scales].

Formally,

$$f_{0s}^{(1)} \doteq \overline{f_s} = \frac{1}{T}\int_{t-T/2}^{t+T/2} dt'\, f_s(t') \,, \quad \delta f_s \doteq f_s - \overline{f_s}$$

where $T \gg \frac{1}{\omega}$ (char. fluctuation timescale)

and $T \ll$ eq. time scale

[Spatial average can be defined similarly]. ← for inhom. equilibria.

Let us decompose the fluctuations into Fourier modes (formally, can do this in a periodic box or infinite space)

$$\varphi(t,\vec r) = \sum_k e^{i\vec k\cdot\vec r}\varphi_k(t) \quad \Big[= \frac{V}{(2\pi)^3}\int d^3\vec k\, e^{i\vec k\cdot\vec r}\varphi_k(t) \Big]$$

(system volume) in the continuous form

and

$$f_s = f_{0s}(t,\vec v) + \sum_k e^{i\vec k\cdot\vec r}\,\delta f_{ks}(t,\vec v)$$

Then we can write

Poisson: $$\varphi_k = \frac{4\pi}{k^2}\sum_s q_s \int d^3\vec v\, \delta f_{ks} \qquad (4)$$

Vlasov for $k \neq 0$:

$$\frac{\partial \delta f_{ks}}{\partial t} + i\vec k\cdot\vec v\,\delta f_{ks} = \frac{q_s}{m_s}\varphi_k\, i\vec k\cdot\frac{\partial f_{0s}}{\partial \vec v} + \frac{q_s}{m_s}\sum_{k'}\varphi_{k-k'}\, i(\vec k-\vec k')\cdot\frac{\partial \delta f_{k',s}}{\partial \vec v} \qquad (5)$$

wave-particle interaction – linear

nonlinear interactions (negligible for small amplitudes)

Vlasov for $k=0$:

$$\frac{\partial f_{0s}}{\partial t} + \frac{\partial \delta f_{0s}}{\partial t} = -\frac{q_s}{m_s} \sum_{k'} \varphi_{-k'}\, i\vec{k}' \cdot \frac{\partial \delta f_{k's}}{\partial \vec{v}} \qquad (6)$$

Average out fast time variation:

$$\frac{\partial f_{0s}}{\partial t} = -\frac{q_s}{m_s} \sum_{k'} \overline{\left\langle \varphi_{-k'}\, i\vec{k}' \cdot \frac{\partial \delta f_{k's}}{\partial \vec{v}} \right\rangle} \qquad (7)$$

- When fluctuations are assumed infinitesimal, all nonlinear terms in these eqns can be dropped, so f_{0s} = const and δf_{ks} satisfies the <u>linearised</u> equation

$$\frac{\partial \delta f_{ks}}{\partial t} + i\vec{k}\cdot\vec{v}\, \delta f_{ks} = \frac{q_s}{m_s} \varphi_k\, i\vec{k} \cdot \frac{\partial f_{0s}}{\partial \vec{v}} \qquad (8)$$

In practice, this works when the amplitudes of the fluctuations are small and we are only interested in short term evolution: on timescales

$$\underset{\text{fluct.}}{\frac{1}{\omega_k}} \ll t \ll \text{eq. timescale [long, TBD]}$$

- The next step is to consider the amplitudes small but ask for longer-term (slow) evolution of the equilibrium. Mathematically, one retains the rhs in eq. (7) but uses linearised eq. (8), neglecting the nonlinear interaction term in eq. (5). This is called <u>quasilinear theory</u>.

This only works for very small amplitudes such that the nonlinear interaction between fluctuations (last term in eq. (5)) is negligible. In real life, fluctuations rarely saturate at amplitudes so small that this is a good approximation.

- When amplitudes are still small so that

  linear time scale $\sim \frac{1}{\omega_k} \ll$ nonlinear time scale, ($\propto$ amplitude)

  there is still a small parameter $\sim \frac{\tau_L}{\tau_{NL}}$ and one can do analytical theory: retain the nonlinearity in eq. (5) and solve perturbatively for the slow evolution of the fluctuation amplitudes.

  This is called weak turbulence theory.

  This already ~~incl~~ includes interactions between spatial scales, so one gets broad (power-law) fluctuation spectra, while in quasilinear theory, spectra are usually peaked around the wave# of max growth rate.

- If fluctuations fail to saturate at the weak level, so we get $\tau_{NL} \sim \tau_L$, we are dealing with strong turbulence. Only phenomenological scaling/dimensional theories exist – or closure theories, which involve ~~[illegible]~~ hand waving and non-rigorous assumptions.

Note that weak turbulence approach is often fruitful in plasmas and, more generally, in systems that support waves because propagating structures often do not meet for long enough time to interact strongly — like eddies in a neutral fluid, e.g.

However, even in wave systems, one often encounters strong turbulence: nature seems to like it when linear propagation and nonlinear interaction timescales are balanced (e.g. MHD Alfvén-wave turbulence)

In what follow, we shall work through some (relatively) simple examples of all of the approaches introduced above and discuss

- methods
- physics that emerges
- conditions of validity

NB: In (many) applications, one is fundamentally interested in the dynamics at slow/long scales (evolution of the equilibrium) — so the aim is to be able to systematically average over turbulent fluctuations and produce transport equations like (7).

Analytical theory: aim - express the rhs in terms of fos

Numerical simulations: compute the average and parametrise it dependence on the system's large-scale properties.