Stochasticity of Magnetic Field Lines.

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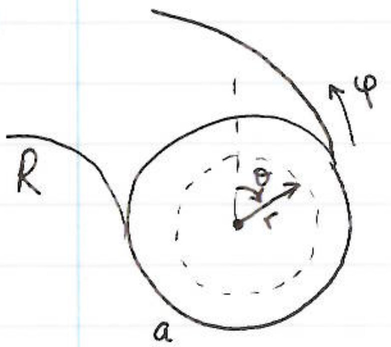
UCLA Jan 2006

Consider the following magnetic field:

$$\vec{B} = \underbrace{B_{0z}\hat{z}}_{\text{field on the rational surface}} + \underbrace{B_{0y}(x)}_{\text{field away from rational surface}}\hat{y} + \underbrace{\delta\vec{B}(x,y,z)}_{\text{perturbation}}$$

x: coordinate $\perp$ rational surface ($x=0$)



x is similar to r (not quite: actually coord. $\perp$ surface)

$\frac{y}{a}$ " to θ (poloidal)

$\frac{z}{R}$ " to φ (toroidal)

Flux function ψ: $\quad B_{0y}(x) + \delta B_y = \frac{\partial\psi}{\partial x}$, $\quad \delta B_x = -\frac{\partial\psi}{\partial y}$

Field lines: $\quad \frac{dx}{dz} = \frac{B_x}{B_z} \approx \frac{\delta B_x}{B_{0z}} = -\frac{1}{B_{0z}}\frac{\partial\psi}{\partial y}$

$$\frac{dy}{dz} = \frac{B_y}{B_z} \approx \frac{1}{B_{0z}}\frac{\partial\psi}{\partial x}$$

Note that this is mathematically similar to Hamiltonian mechanics with $H=\psi$, $p = B_{0z}x$, $q = y$, $t = z$,

so $\dot{p} = -\frac{\partial H}{\partial q}$, $\dot{q} = \frac{\partial H}{\partial p}$

(this is true in general: $H = \psi_{\text{poloidal}}$, $p = \psi_{\text{toroidal}}$)

In the vicinity of the rational surface,

$B_{oy}(x) \approx B_{oy}' x$, so we may write

$$\psi = \frac{1}{2} B_{oy}' x^2 + \sum_{m,n} \underbrace{\delta\psi_{mn}(x)}_{\substack{\shortparallel \\ \delta\psi_{mn}(0)+\dots}} \cos\left(\frac{my}{a} - \frac{nz}{R}\right)$$

$\frac{m}{a} = k$ is the poloidal wave# of perturbation

$n = 0$ corresponds to perturbations at the surface we are on

$n \neq 0$ comes from perturbations at other rational surfaces.

For simplicity, keep only $n = 0, 1$.

Then

$$\psi = \frac{1}{2} B_{oy}' x^2 + \delta\psi_{10} \cos ky + \delta\psi_{11} \cos\left(ky - \frac{z}{R}\right)$$

$$\begin{cases} \dfrac{dx}{dz} = -\dfrac{\delta\psi_{10} k}{B_{oz}} \sin ky - \dfrac{\delta\psi_{11} k}{B_{oz}} \sin\left(ky - \dfrac{z}{R}\right) \\[2ex] \dfrac{dy}{dz} = \dfrac{B_{oy}'}{B_{oz}} x \end{cases}$$

Denote $ky \equiv \tilde{y}$, $kz \equiv \tilde{z}$, $\frac{B_{oy}'}{B_{oz}} x \equiv \tilde{x}$, $\frac{B_{oy}'}{B_{oz}^2} \psi \equiv \tilde{\psi}$

Then
$$\frac{d\tilde{x}}{d\tilde{z}} = \frac{B_{oy}'}{B_{oz}} \frac{1}{k} \frac{dx}{dz} = -\frac{B_{oy}'}{B_{oz}^2} \frac{1}{k} \frac{\partial \psi}{\partial y} = -\frac{\partial \tilde{\psi}}{\partial \tilde{y}}$$

$$\frac{d\tilde{y}}{d\tilde{z}} = \frac{dy}{dz} = \frac{1}{B_{oz}} \frac{\partial \psi}{\partial x} = \frac{B_{oy}'}{B_{oz}^2} \frac{\partial \psi}{\partial \tilde{x}} = \frac{\partial \tilde{\psi}}{\partial \tilde{x}}$$

Also denote

$$\frac{\delta\psi_{10} B_{oy}'}{B_{oz}^2} = K \quad , \quad \frac{\delta\psi_{11}}{\delta\psi_{10}} = \epsilon \quad , \quad \frac{1}{kR} = \alpha$$

With these definitions, and dropping tildes,

$$\psi = \frac{1}{2}x^2 - K\cos y - \epsilon K\cos(y - \alpha z)$$

$$\begin{cases} \dfrac{dx}{dz} = -K\sin y - \epsilon K\sin(y-\alpha z) \\[2ex] \dfrac{dy}{dz} = x \end{cases}$$

Let us treat ϵ as small, so we have a hamiltonian of a nonlinear oscillator plus perturbation.

<u>Unperturbed solution.</u>

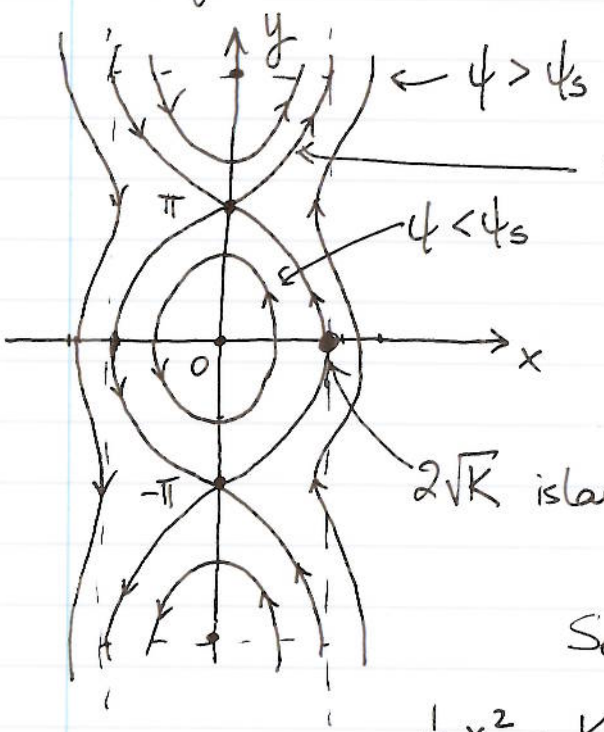
$$\psi_0 = \frac{1}{2}x^2 - K\cos y$$

Fixed pts:

$$\frac{dy}{dz} = x = 0$$

$$\frac{dx}{dz} = -K\sin y = 0$$

$$x = 0, \; y = \pi l$$



separatrix: $\psi = \psi_s = K$

$2\sqrt{K}$ island half-width.

Solve on the separatrix:

$$\frac{1}{2}x^2 - K\cos y = \psi_s = K$$

$$x^2 = 2K(1+\cos y) = 4K\cos^2\left(\frac{y}{2}\right)$$

So $x = \pm 2\sqrt{K}\cos\frac{y}{2} = \frac{dy}{dz}$

$$\pm\sqrt{K}\,dz = \frac{dy}{2\cos\frac{y}{2}}$$

$$\pm\sqrt{K}(z-z_0) = \frac{1}{2}\int_{y_0}^{y}\frac{dy}{\cos\frac{y}{2}} = \frac{1}{2}\int_{y_0}^{y}\frac{dy}{\sin\left(\frac{y+\pi}{2}\right)} = \frac{1}{4}\int_{y_0}^{y}\frac{dy}{\sin\frac{y+\pi}{2}\cos\frac{y+\pi}{2}}$$

$$= \frac{1}{4}\int_{y_0}^{y}\frac{dy}{\tan\frac{y+\pi}{4}\cos^2\frac{y+\pi}{4}} = \int_{y_0}^{y}\frac{d\tan\frac{y+\pi}{4}}{\tan\frac{y+\pi}{4}} = \ln\tan\frac{y+\pi}{4}\Big|_{y_0}^{y}$$

Take $y_0 = 0$ $(x = \pm 2\sqrt{K})$. Then

soliton

$$\boxed{\begin{aligned} y &= 4\,\mathrm{atan}\, e^{\pm\sqrt{K}(z-z_0)} - \pi \quad (1) \\ x &= \frac{dy}{dz} = \pm\frac{2\sqrt{K}}{\cosh\left[\sqrt{K}(z-z_0)\right]} \end{aligned}}$$

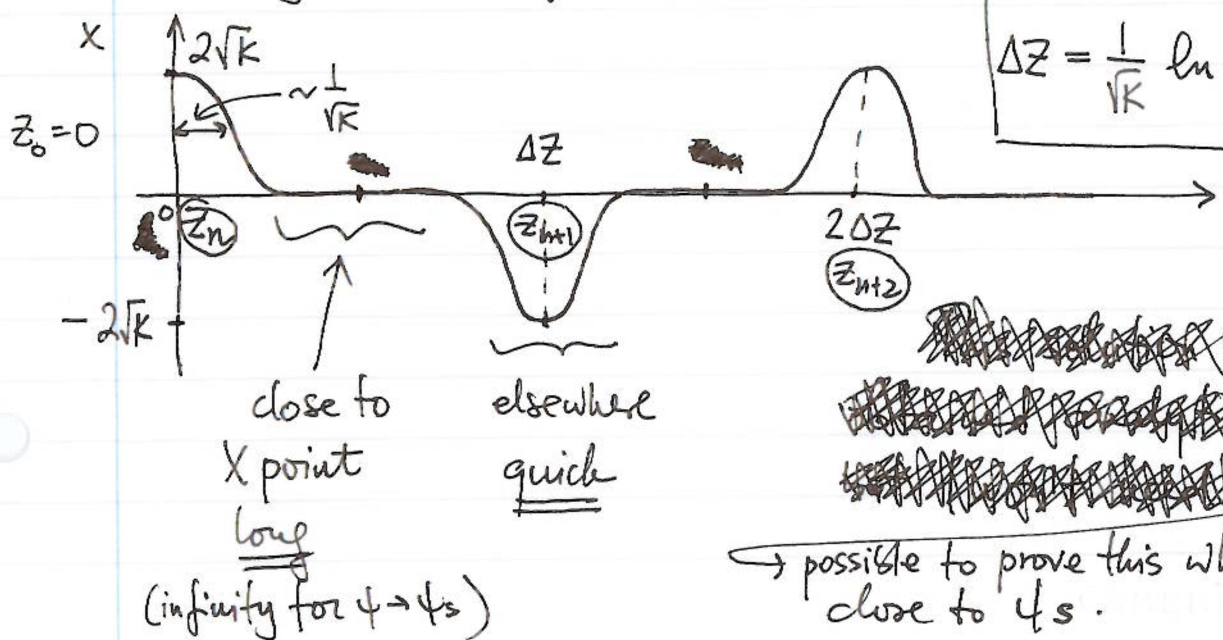
"Velocity" (2)

Note: $z\to\infty$ $y\to\pm\pi$ $x\to 0$

– infinitely long approach to the x points $(0,\pm\pi)$.

"Velocity" of the "particle" close to the separatrix:

$$\boxed{\Delta z = \frac{1}{\sqrt{K}}\ln\frac{32\psi_s}{|\psi-\psi_s|}} \quad (3)$$



→ possible to prove this when ψ is close to ψ_s.

Now introduce the perturbation.

In the general form, it looks like this:

$$\psi = \psi_0 + \delta\psi$$

where (in our case) $\psi_0 = \frac{1}{2}x^2 - K\cos y$

$$\delta\psi = -\epsilon K\cos(y - \alpha z)$$

How does $\overset{\text{"energy"}}{\psi_0}(z)$ – formerly conserved – change as we follow z?

$$\frac{d\psi_0}{dz} = \frac{\partial\psi_0}{\partial x}\frac{dx}{dz} + \frac{\partial\psi_0}{\partial y}\frac{dy}{dz} = -\frac{\partial\psi_0}{\partial x}\frac{\partial\psi}{\partial y} + \frac{\partial\psi_0}{\partial y}\frac{\partial\psi}{\partial x} =$$

$$= -\frac{\partial\psi_0}{\partial x}\frac{\partial\delta\psi}{\partial y} + \frac{\partial\psi_0}{\partial y}\underbrace{\cancel{\frac{\partial\delta\psi}{\partial x}}}_{0} = -x\frac{\partial\delta\psi}{\partial y} =$$

(perturbation only in y)

$$= -x\epsilon K\sin(y - \alpha z) \text{ for our example.}$$

If we are close to the separatrix, $x = 0$ most of the "time" (as we increase z) except for short bursts of width

$$\sim \frac{1}{\sqrt{K}} \ll \Delta z = \frac{1}{\sqrt{K}}\ln\frac{32\psi_{0s}}{|\psi_0 - \psi_{0s}|} \qquad \psi_{0s} = K$$

We can, therefore, describe our field lines close to the separatrix in terms of a discrete map that takes it from one burst to the next $(z_n \to z_{n+1})$

We have $\boxed{z_{n+1} = z_n + \Delta z_n}$

where $\Delta z_n = \frac{1}{\sqrt{K}} \ln \frac{32\psi_{os}}{|\psi_{on+1} - \psi_{os}|}$, $\psi_{os} = K$

value of ψ_0 during (z_n, z_{n+1})

Take $\psi_{on+1} = \psi_0\left(z_n + \frac{\Delta z_n}{2}\right)$

Then $\boxed{\psi_{on+1} = \psi_{on} + \Delta \psi_{on}}$

where $$\Delta\psi_{on} = \int_{z_n - \frac{\Delta z_n}{2}}^{z_n + \frac{\Delta z_n}{2}} dz \, \frac{d\psi_0}{dz} = -\epsilon K \int_{-\frac{\Delta z_n}{2}}^{\frac{\Delta z_n}{2}} dz' \, x(z_n + z') \sin\left[y(z_n + z') - \alpha(z' + z_n)\right]$$

$z' = z - z_n$

As long as $\Delta\psi_{on} \ll \psi_{on}$, we may use the solution (1-2) for $x(z)$ and $y(z)$ on the separatrix $(z_0 \to z_n)$

~~Then~~ Also take integration limits to $\pm\infty$:

$$\Delta\psi_{on} = \underbrace{(\mp 1)}_{(-1)^n} \epsilon K \int_{-\infty}^{+\infty} dz' \frac{2\sqrt{K}}{\cosh[\sqrt{K}(z - z_n)]} \sin\left[y(z' + z_n) - \alpha z' - \underbrace{\alpha z_n}_{\varphi_n \text{ phase}}\right]$$

$$= 2(-1)^n \epsilon K \int_{-\infty}^{+\infty} d\zeta \, \frac{\sin\left[y - \frac{\alpha}{\sqrt{K}}\zeta - \varphi_n\right]}{\cosh \zeta}$$

$\zeta = \sqrt{K} z'$

Note that

$$x = \pm 2\sqrt{K} \cos\frac{y}{2} = \frac{\pm 2\sqrt{K}}{\cosh[\sqrt{K}(z - z_0)]} \Rightarrow \begin{cases} \cos\frac{y}{2} = \frac{1}{\cosh\zeta} \\ \sin\frac{y}{2} = \sqrt{1 - \cos^2\frac{y}{2}} = \frac{\sinh\zeta}{\cosh\zeta} \end{cases}$$

use these to do the integral

The integral can be done and we get

$$\Delta\psi_{on} = 4\pi\epsilon\alpha^2(-1)^n \frac{e^{\pi\alpha/2\sqrt{K}}}{\sinh\frac{\pi\alpha}{2\sqrt{K}}} \sin\varphi_n \equiv$$

$$\equiv \Delta\psi_{os}(-1)^n \sin\varphi_n$$

So we have derived the following map:

$$\boxed{\begin{aligned} \psi_{on+1} &= \psi_{on} + (-1)^n \Delta\psi_{os}\sin\varphi_n \\ \varphi_{n+1} &= \varphi_n + \frac{\alpha}{\sqrt{K}} \ln \frac{32\psi_{os}}{|\psi_{on+1}-\psi_{os}|} \end{aligned}}$$

$$\psi_{os} = K$$
$$\varphi_n = \alpha z_n$$

Change of ψ_o is always small ($\Delta\psi_{os} \ll \psi_{on}$).

Change of φ can be large because $\psi_{on+1}-\psi_{os}$ is small.

Furthermore, $\Delta\varphi_n$ can vary a lot even for small variations of ψ_{on} (because we are close to the separatrix and $\Delta z \to \infty$).

Instability develops if, roughly, the change of phase, $\Delta\varphi_n$ depends significantly on φ_n:

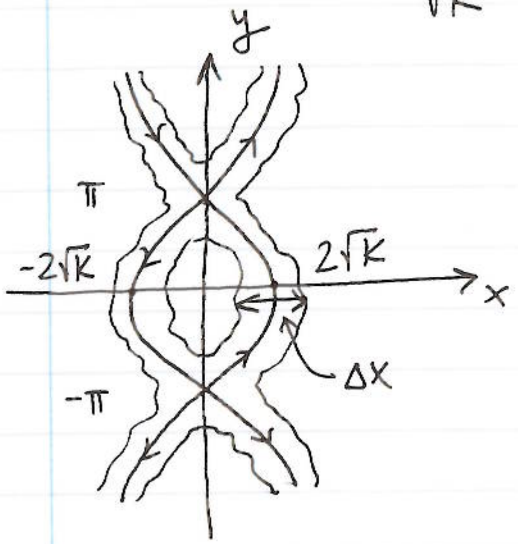
$$\boxed{\left|\frac{\partial\varphi_{n+1}}{\partial\varphi_n} - 1\right| \gtrsim 1}$$

$$\left|\frac{\partial\varphi_{n+1}}{\partial\varphi_n} - 1\right| = \left|\frac{\partial\Delta\varphi_n}{\partial\varphi_n}\right| = \left|\frac{\partial\Delta\varphi_n}{\partial\psi_{on+1}}\right|\left|\frac{\partial\psi_{on+1}}{\partial\varphi_n}\right| =$$

$$= \frac{\alpha}{\sqrt{K}}\frac{\Delta\psi_{os}}{|\psi_{on+1}-\psi_{os}|}|\cos\varphi_n| \approx \frac{\alpha}{\sqrt{K}}\frac{\Delta\psi_{os}}{|\psi_o-\psi_{os}|} \gtrsim 1$$

So the stochasticity criterion is

$$|\psi_0 - \psi_{0s}| \lesssim \frac{\alpha}{\sqrt{K}} \Delta\psi_{0s} = \frac{4\pi\epsilon\alpha^3}{\sqrt{K}} \frac{e^{\frac{\pi\alpha}{2\sqrt{K}}}}{\sinh\frac{\pi\alpha}{\sqrt{K}}}$$



This determines the thickness of the stochastic layer around the separatrix.

We can convert this into an estimate of Δx:

$$\psi_0 - \psi_{0s} \approx \Delta x \left.\frac{\partial\psi_0}{\partial x}\right|_s + \Delta y \underbrace{\left.\frac{\partial\psi_0}{\partial y}\right|_s}_{= K\sin y_s = 0} = \Delta x \underbrace{x_s}_{= 2\sqrt{K}}$$

take $(x_s, y_s) = (2\sqrt{K}, 0)$

So, $$\Delta x \approx \frac{2\pi\epsilon\alpha^3}{K} \frac{e^{\pi\alpha/2\sqrt{K}}}{\sinh(\pi\alpha/\sqrt{K})}$$

Recall that: x is normalised by $\left(\frac{B_{0y}'}{B_{0z}}\right)^{-1}$

$$K = \frac{\delta\psi_{10} B_{0y}'^2}{B_{0z}}, \quad \epsilon = \frac{\delta\psi_{11}}{\delta\psi_{10}}, \quad \alpha = \frac{1}{kR}$$

Consider "high-frequency" perturbations: $\frac{\alpha}{\sqrt{K}} \gg 1$

Then $$\boxed{\Delta x \approx \frac{4\pi\epsilon\alpha^3}{K} e^{-\frac{\pi\alpha}{2\sqrt{K}}}}$$

NB: Since $\Delta\psi_{0s} \approx 8\pi\epsilon\alpha^2 e^{-\frac{\pi\alpha}{2\sqrt{K}}}$ is exp-lly small, so is $|\psi_0 - \psi_{0s}|$, so, in fact ϵ does not need to be small for all of the above to be valid!

Let us look at what happens when $\epsilon = 1$

$$\psi = \underbrace{\frac{1}{2}x^2 - K\cos y}_{\text{this was treated as } \psi_0} - \underbrace{K\cos(y - \alpha z)}_{\text{this was the perturbation}}$$

Let's consider this term as zeroth-order and the second term as perturbation:

$$\psi_0 = \frac{1}{2}x^2 - K\cos(y - \alpha z)$$

Change variables: $y' = y - \alpha z$

$$\psi_0 = \frac{1}{2}x^2 - K\cos y'$$

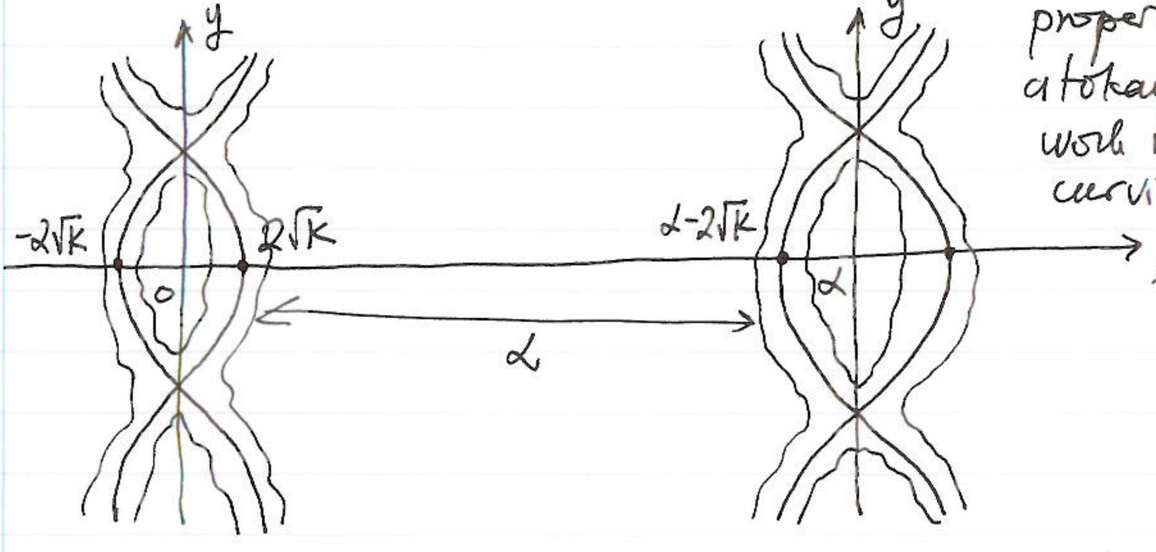
$$\frac{dx}{dz} = -K\sin y'$$

$$\frac{dy'}{dz} = \frac{dy}{dz} - \alpha = x - \alpha$$

$\Rightarrow$ Fixed points: $x = \alpha$

$y' = \pi l$

This basically point to the existence of another island chain at $x = \alpha$.



(To do this properly for a tokamak, work in curvilinear coords)

The condition $\frac{\alpha}{\sqrt{K}} \gg 1$ for a thin layer of stochasticity was equivalent to saying that the second (and all other) island chain was far away:

island width $\ll$ distance between islands

$$\sqrt{K}\,\frac{B_{0z}}{B'_{0y}} = \sqrt{\frac{\delta\psi_1}{B'_{0y}}} \ll \alpha\,\frac{B_{0z}}{B'_{0y}} = \frac{1}{kR}\,\frac{B_{0z}}{B'_{0y}} \sim \frac{\Delta q}{k} \sim a\,\Delta q$$

where $\Delta q = \frac{m_1}{n_1} - \frac{m_2}{n_2}$ is the distance between 2 neighbouring rational surfaces.

Since $\delta B_x \sim k\,\delta\psi_1$, we can rewrite this so:

$$\frac{\delta B_x}{B_{0z}} \ll \frac{1}{kR^2}\,\frac{B_{0z}}{B'_{0y}} \sim \frac{a}{R}\,\Delta q \quad \leftarrow \text{NB: gets smaller with } n!$$

When $\boxed{\frac{\delta B_x}{B_{0z}} \sim \frac{a}{R}\,\Delta q} \iff \boxed{\frac{\alpha}{\sqrt{K}} \sim 1}$,

island chains overlap and field lines become globally chaotic.

NB: We could get the same criterion if we asked when the width of the stochastic layer around the separatrix becomes comparable to the island width:

layer width $\sim$ island width

$$\Delta x \sim \sqrt{K}$$

or $\epsilon\left(\frac{\alpha}{\sqrt{K}}\right)^3 \dfrac{e^{\pi\alpha/2\sqrt{K}}}{\sinh\frac{\pi\alpha}{\sqrt{K}}} \sim 1$ happens for $\frac{\alpha}{\sqrt{K}} \sim 1$