Les Houches Lectures

A. Schekochihin, Aug. 2007

§1. The Fluctuation Dynamo Problem

Velocity: $\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\frac{\nabla p}{\rho} + \frac{\vec{B}\cdot\nabla\vec{B}}{4\pi\rho} + \nu\nabla^2\vec{u} + \vec{f}, \quad \nabla\cdot\vec{u} = 0$

M. field: $\frac{\partial \vec{B}}{\partial t} + \vec{u}\cdot\nabla\vec{B} = \underbrace{\vec{B}\cdot\nabla\vec{u}}_{\text{dynamo}} + \eta\nabla^2\vec{B}$

Consider isotropic, homogeneous, parity-invariant forced turbulence — the most idealised case.

Introduce an arbitrarily weak seed field.

Two questions

- Will the field energy grow? $\langle B^2\rangle \sim B_0^2 e^{\gamma t}, \gamma > 0$
  YES (Why?)
- How does it saturate and what is the saturated state?
  OPEN PROBLEM (more or less)

Understanding why the field grows may help figure out how it saturates.

Turbulence: energy injected at some scale L,

the injected power is $\epsilon \sim \frac{U^3}{L}$ ← (dimensionally)

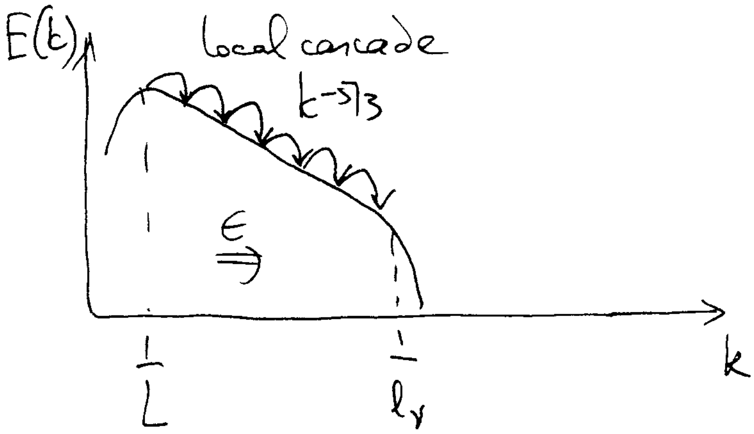
Energy is dissipated by viscosity

the viscous scale is $l_\nu \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \sim \frac{L}{Re^{3/4}}$,

where $Re \sim \frac{UL}{\nu}$ is the Reynolds #.

When $Re \gg 1$, $l_\nu \ll L$ and there is a scale-invariant range in between called the <u>inertial range</u>.

Assuming energy travels from scale to scale gradually, w/o jumping from v. large to v. small (<u>locality</u>), we have, dimensionally again:



$$\epsilon \sim \frac{\delta u_l^3}{l} = \text{const}$$
$$\Downarrow$$
$$\boxed{\delta u_l \sim (\epsilon l)^{1/3}}$$
$$\Updownarrow$$
$$E(k) \sim \epsilon^{2/3} k^{-5/3}$$

<u>Kolmogorov scaling</u>

Velocity gradients stretch the field and make it grow:

$$\frac{d}{dt}\frac{\langle B^2 \rangle}{2} = \langle \vec{B}\vec{B} : \nabla \vec{u} \rangle - \eta \langle |\nabla \vec{B}|^2 \rangle$$
$$\sim \gamma \langle B^2 \rangle \quad \text{if there is dynamo}$$

What is γ? $\qquad$ $\gamma \to$ const limit > 0 as $\eta \to +0$

Stretching rate due to motions of scale l:

$$\gamma_l \sim \frac{\delta u_l}{l} \sim \epsilon^{1/3} l^{-2/3},$$

So the smaller the eddy the faster it stretches.

What limits the scale of the stretching eddies?
Ohmic diffusion: let's estimate the scale l_η at which it becomes important:

$$\text{stretching} \sim \text{Ohmic diffusion}$$

$$\frac{\delta u_l}{l} \sim \frac{\eta}{l^2}$$

$$\delta u_l\, l \sim \eta$$

Inside the inertial range, $\delta u_l \sim (\epsilon l)^{1/3}$, so

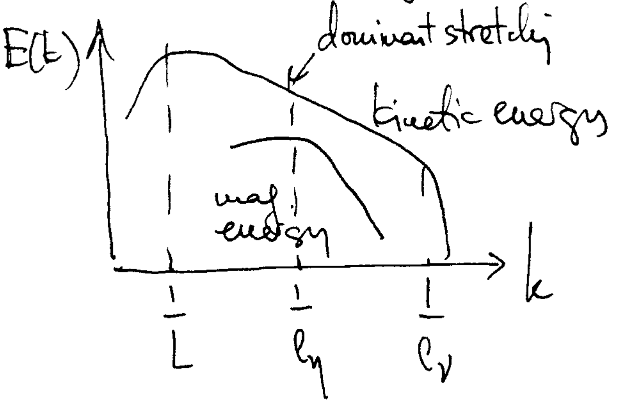
$$l_\eta \sim \left(\frac{\eta^3}{\epsilon}\right)^{1/4} \sim l_\nu\, Pm^{-3/4} \sim L\, Rm^{-3/4}$$

where $Pm = \frac{\nu}{\eta}$ is the magnetic Prandtl #.

$$Pm = \frac{Rm}{Re}$$

$$Rm = \frac{UL}{\eta} \text{ is the magnetic Reynolds \#}$$

Now we see that our $l_\eta \gg l_\nu$ (lies inside the inertial range, as we assumed) only if $\underline{Pm \ll 1}$!



The associated stretching rate is

$$\gamma \sim \frac{\delta u_{l_\eta}}{l_\eta} \sim \epsilon^{1/3} l_\eta^{-2/3} \sim \frac{U}{L} Rm^{1/2}$$

This is $\gg$ any time scale associated with the system-scale motions, so such a dynamo would be *much* faster than any mean-field dynamo. This is why fluctuation

dynamo is so important.
It is also universal: I did not use any assumptions on the global properties of my system.
Note that my estimates are good <u>if</u> the dynamo works — I have not, in fact, proven that it does. At the ~~xxxx~~ resistive scale, where the fastest stretching is, ~~xxx~~ I had

stretching ~ Ohmic diffusion

— there is no guarantee that stretching is bigger, i.e. that $\langle \vec{B}\vec{B} : \nabla\vec{u} \rangle > \eta \langle |\nabla \vec{B}|^2 \rangle$
Proving this mathematically is very hard even for simple, non-turbulent flows — usually impossible.
However, we are physicists, so what we'd really like is some intuitive model of how and why the dynamo works.
For $Pm \ll 1$, while we have numerical evidence that it does work (I'll show it in my last lecture), we do not have a physical model for it
— OPEN PROBLEM — and a challenge!
Now, is the limit $Pm \ll 1$ generic? Of course not.
For ionised plasmas,

$$Pm \sim 10^{-5} \frac{T^4}{n}$$

T ← temperature in K

n ← density in cm^{-3}

Liquid metals (planets, dynamo experiments) : $Pm \sim 10^{-5}$

Solar convective zone : $Pm \sim 10^{-4} .. 10^{-7}$
surface base

<u>but</u> ISM (warm phase) : $Pm \sim 10^{11}$

Galaxy clusters : $Pm \sim 10^{29}$

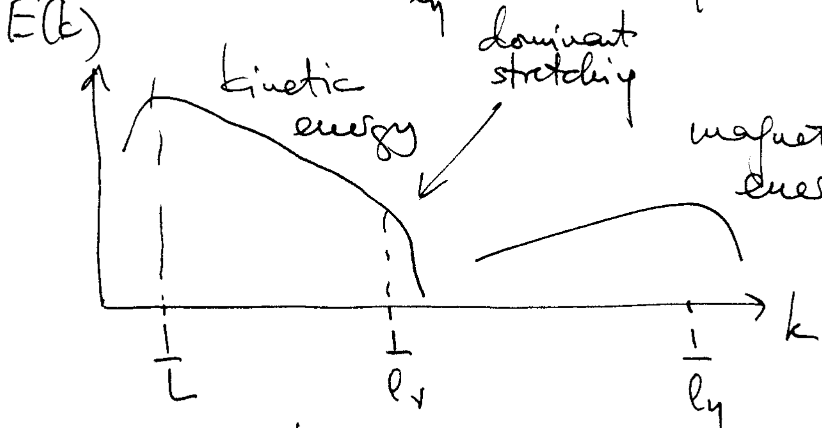
So what happens when <u>$Pm \gg 1$</u>? $\Rightarrow$ $l_\eta < l_\nu$, so the fastest stretching motions are the smaller-scale ones — the ones at the viscous scale:

$$\gamma \sim \frac{\delta u_{l_\nu}}{l_\nu} \sim \epsilon^{1/3} l_\nu^{-2/3} \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} \sim \frac{U}{L} Re^{1/2}$$

Estimate the <u>resistive scale</u>:

$$\gamma \sim \frac{\eta}{l_\eta^2} \Rightarrow l_\eta \sim \left(\frac{\eta}{\gamma}\right)^{1/2} \sim Pm^{-1/2} l_\nu \ll l_\nu$$

as expected.



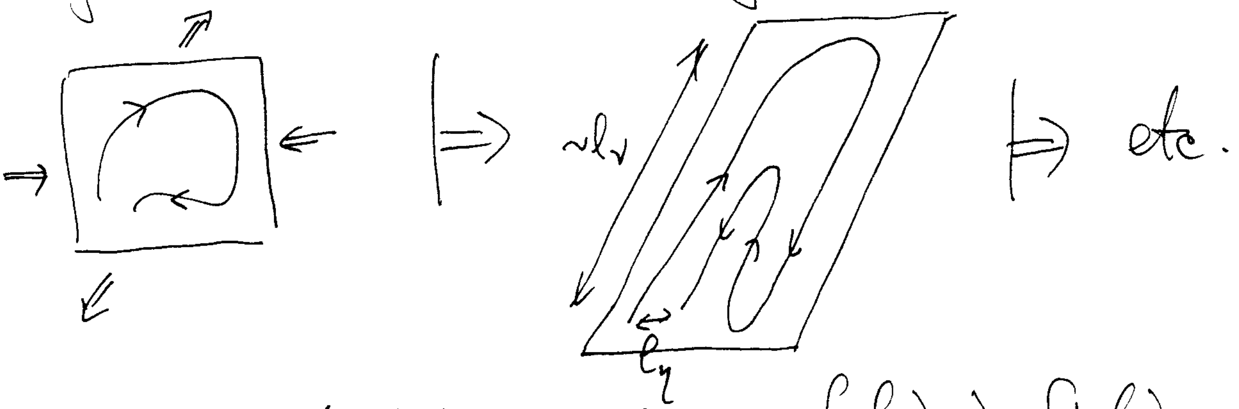
What are these viscous-scale motions like?

For $l \sim l_\nu$, we have $\epsilon \sim \frac{\nu}{l^2} \delta u_l^2$,

energy arriving from inertial range — energy dissipated

so $\boxed{\delta u_l \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} l}$ — <u>smooth</u> motions (Taylor-expandable)

So, the velocity field locally looks like a random (in time) linear shear/stretching.
This is much simpler and much more intuitive than turbulence in the inertial range.
Flows like ABC and other chaotic deterministic flows are of this type.
For such a velocity field, we ~~can~~ can easily figure out the way in which it amplifies magnetic field. Basically,



So, random stretching produces _folded_ fields.
Their direction-reversal scale is only limited by l_η – the Ohmic diffusion scale.
But _why does this work_?
Why are these fields not destroyed by diffusion?
Can one stretch and fold in a constructive way, so that magnetic energy grows?
It is not at all obvious that this can work.

For example, in 2D, it most definitely does not work! (Zeldovich theorem, proof based on
Sov. Phys. JETP 4, 460 (1957)
magnetic field in 2D being $\vec{B} = \nabla \times A\hat{z}$,
where $\partial_t A + \vec{u}\cdot\nabla A = \eta \nabla^2 A$

so $\frac{d}{dt}\langle A^2\rangle = -\eta\langle|\nabla A|^2\rangle$ decays eventually)

I will now use a simple mathematical argument to ~~[illegible]~~ illustrate how in 3D diffusion can be overcome (due to Zeldovich et al. 1984)
JFM 144, 1 (1984)

Note: If you want to read something, see a short review

Schekochihin & Cowley, astro-ph/0507686 (§3)

- on $Pm \gg 1$. This also contains all or most of ~~[illegible]~~ the relevant references.

On $Pm \ll 1$, see

Schekochihin et al., arXiv: 0704.2002
NJP 9, 300 (2007)

for the latest report on numerics and discussion of gen issues.

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§. Fluctuation Dynamo in a Linear Velocity Field.

Since the velocity field at viscous scales locally looks linear, let us write

$$u^i(t,\vec{x}) = u_0^i(t,0) + \sigma^i_m(t)\, x^m + \ldots$$

We are expanding about some reference point $\vec{x}=0$. We can always go to the reference frame that moves with velocity $\vec{u}_0(t,0)$, so set $\vec{u}_0(t,0)=0$. Then the induction equation is

$$\partial_t B^i + \sigma^l_m(t)\, x^m \frac{\partial B^i}{\partial x^l} = B^m \sigma^i_m + \eta \nabla^2 B^i \qquad (1)$$

Let us seek the solution to this equation as a sum of plane waves with time dependent wavevectors:

$$B^i(t,\vec{x}) = \int \frac{d^3 \vec{k}_0}{(2\pi)^3}\, \tilde{B}^i(t,\vec{k}_0)\, e^{i\tilde{k}_n(t,\vec{k}_0)\, x^n} \qquad (2)$$

where $\tilde{k}_n(t,\vec{k}_0) = k_{0n}$, so $\tilde{B}^i(0,\vec{k}_0) \equiv B_0^i(\vec{k}_0)$ is the Fourier transform of the initial field.

Since eq. (1) is linear, it's sufficient for each plane wave to be a solution — then (2) is a solution. Substitute into (1):

$$\partial_t \tilde{B}^i e^{i\tilde{k}_n x^n} = e^{i\tilde{k}_n x^n}\left(\partial_t \tilde{B}^i + \tilde{B}^i\, i x^m \partial_t \tilde{k}_m\right) =$$

$$= e^{i\tilde{k}_n x^n}\left(-\sigma^l_m x^m\, i\tilde{k}_l \tilde{B}^i + \tilde{B}^m \sigma^i_m - \eta \tilde{k}^2 \tilde{B}^i\right)$$

This must be satisfied $\forall \vec{x}$, so

$$\begin{cases} \partial_t \tilde{B}^i = \tilde{B}^m \sigma^i_m - \eta \tilde{k}^2 \tilde{B}^i & (3) \\ \partial_t \tilde{k}_m = - \sigma^l_m \tilde{k}_l & (4) \end{cases}$$

subject to initial conditions $\tilde{B}^i(0, \vec{k}_0) = B_0^i(\vec{k}_0)$, $\tilde{k}_l(0, \vec{k}_0) = k_{0l}$.

Now we could proceed to solve eqns (3-4) for general (even random) matrices $\sigma^i_m(t)$. This does lead to a solution, but spawns a lot of formalism on the way. I will skip all that (it will be in the printed notes) and consider an extremely simple case:

$$\hat{\sigma} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}$$

By incompressibility, $\lambda_1 + \lambda_2 + \lambda_3 = 0$ ($\nabla \cdot \vec{u}$ means $\mathrm{tr}\,\hat{\sigma} = 0$)

We will arrange our coord. system so that

$$\underset{\text{stretching}}{\lambda_1} \geq \underset{\text{"null"}}{\lambda_2} \geq \underset{\text{compression}}{\lambda_3} \quad \text{and} \quad \lambda_1 > 0, \ \lambda_3 < 0$$

Then the solution of eqs (3-4) is

$$\begin{cases} \tilde{B}^i(t, \vec{k}_0) = B_0^i(\vec{k}_0)\, e^{\lambda_i t - \eta \int_0^t dt' \tilde{k}^2(t')} & (5) \\ \tilde{k}_m(t, \vec{k}_0) = k_{0m}\, e^{-\lambda_m t} & (6) \end{cases}$$

Let us calculate the energy of this field: using (2),

$$\langle B^2\rangle = \int d^3\vec{x}\, |\vec{B}(t,\vec{x})|^2 =$$

$$= \int d^3\vec{x} \int \frac{d^3\vec{k}_0}{(2\pi)^3} \int \frac{d^3\vec{k}_0'}{(2\pi)^3} \tilde{B}^i(t,\vec{k}_0)\tilde{B}^i(t,\vec{k}_0')\, e^{i[\tilde{k}_n(t,\vec{k}_0)+\tilde{k}_n(t,\vec{k}_0')]x^n}$$

$$= \int \frac{d^3\vec{k}_0}{(2\pi)^3} \int \frac{d^3\vec{k}_0'}{(2\pi)^3} \tilde{B}^i(t,\vec{k}_0)\tilde{B}^i(t,\vec{k}_0')\,(2\pi)^3 \delta\Big(\underbrace{\tilde{k}_n(t,\vec{k}_0)+\tilde{k}_n(t,\vec{k}_0')}_{=\,(k_{0n}+k_{0n}')\,e^{-\lambda_n t}}\Big) =$$

$$= \delta(\vec{k}_0+\vec{k}_0') \quad \text{because } e^{-\lambda_1 t-\lambda_2 t-\lambda_3 t} = 1$$

$$= \int \frac{d^3\vec{k}_0}{(2\pi)^3} \tilde{B}^i(t,\vec{k}_0)\tilde{B}^i(t,-\vec{k}_0)$$

$$= \int \frac{d^3\vec{k}_0}{(2\pi)^3} |\tilde{\vec{B}}(t,\vec{k}_0)|^2 \qquad (7)$$

so the Parseval theorem holds and the energy of the field is the sum of energies of the plane waves.

Calculate the energy of each mode:

$$|\tilde{B}|^2(t,\vec{k}_0) = \Big(\underbrace{|B_{01}|^2 e^{2\lambda_1 t}}_{\text{exponentially larger than the rest}} + |B_{02}|^2 e^{2\lambda_2 t} + |B_{03}|^2 e^{2\lambda_3 t}\Big)\, e^{-2\eta\int_0^t dt'\,\tilde{k}^2(t')}$$

$$\approx |B_{01}|^2 e^{2\lambda_1 t - 2\eta\int_0^t dt'\,\tilde{k}^2(t')} =$$

$$\tilde{k}^2(t') = k_{01}^2 e^{-2\lambda_1 t'} + k_{02}^2 e^{-2\lambda_2 t'} + k_{03}^2 e^{-2\lambda_3 t'}$$

$$= |B_{01}|^2 e^{2\lambda_1 t - 2\eta\left[\frac{k_{01}^2}{2\lambda_1}(1-e^{-2\lambda_1 t}) + \frac{k_{02}^2}{2\lambda_2}(1-e^{-2\lambda_2 t}) + \frac{k_{03}^2}{2\lambda_3}(1-e^{-2\lambda_3 t})\right]}$$

small — small, assuming $\lambda_2 > 0$ — small

$$\approx |B_{01}|^2 e^{2\lambda_1 t - 2\eta\left[\frac{k_{01}^2}{2\lambda_1} + \frac{k_{02}^2}{2\lambda_2} + \frac{k_{03}^2}{2|\lambda_3|}e^{2|\lambda_3|t}\right]}$$

NB $\lambda_3 < 0$

We see that for most k_0, the corresponding modes decay superexponentially fast with time!
The domain in the k_0 space (over which we will integrate in eq.(7)) containing modes that are not exponentially small at any given time t, is:

$$2\lambda_1 t - 2\eta\left[\frac{k_{01}^2}{2\lambda_1} + \frac{k_{02}^2}{2\lambda_2} + \frac{k_{03}^2}{2|\lambda_3|}e^{2|\lambda_3|t}\right] > \text{const}$$

or $$\frac{k_{01}^2}{\lambda_1^2 t/\eta} + \frac{k_{02}^2}{\lambda_1\lambda_2 t/\eta} + \frac{k_{03}^2}{\lambda_1|\lambda_3|t\,e^{-2|\lambda_3|t}/\eta} < \text{const}$$

This is ~~[illegible]~~ the interior of an ellispsoid.
Its volume in k_0 space is

$$\sim \lambda_1^2(\lambda_2|\lambda_3|)^{1/2}\left(\frac{t}{\eta}\right)^{3/2} e^{-|\lambda_3|t}$$

Within this volume, $|\tilde{\vec{B}}|^2 \sim |B_{01}|^2 e^{2\lambda_1 t}$, so the energy is (from eq.(7))

$$\langle B^2\rangle(t) \sim |B_{01}|^2 e^{(2\lambda_1 - |\lambda_3|)t}\lambda_1^2(\lambda_2|\lambda_3|)^{1/2}\left(\frac{t}{\eta}\right)^{3/2}$$

$$\propto e^{(\lambda_1-\lambda_2)t}$$ – grows because $\lambda_1 > \lambda_2$

Exercises. 1) What happens if $\lambda_2 = 0$? $\lambda_2 < 0$?
2) Show that in 2D, a similar argument shows that the magnetic energy decays (no dynamo)
NB: You will need to use solenoidality of the

magnetic field, $B_{01} k_{01} + B_{02} k_{02} = 0$

3) In our 3D calculation, in view of the solenoidality of the field, $B_{01} k_{01} + B_{02} k_{02} + B_{03} k_{03}$, was it OK to neglect B_{02} and B_{03}?

Now let us discuss the meaning of this result.

Magnetic field is aligned with the stretching direction:

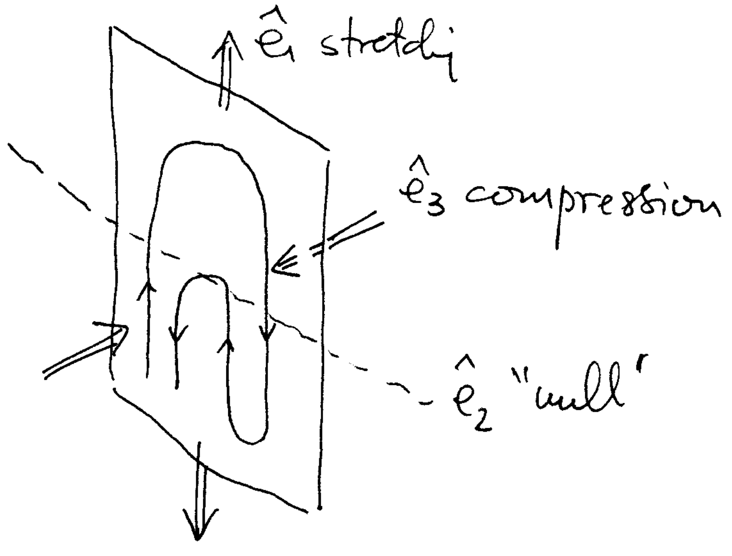
$$\vec{B} \sim \hat{e}_1 B_{01} e^{\lambda_1 t}$$

Its wave vector aligns with the compression direction:

$$\vec{k} \sim \hat{e}_3 k_{03} e^{|\lambda_3| t}$$

– but this makes modes decay superexponentially. The only ones that survive are those for which $\vec{k}_0$ was nearly $\perp \hat{e}_3$, with the permitted angular deviation from 90° decaying as $\sim e^{-|\lambda_3| t}$

Since $\vec{B}_0 \perp \vec{k}_0$ ($\nabla \cdot \vec{B}_0 = 0$), the modes that get stretched the most are $\vec{B}_0 \parallel \hat{e}_1$ and $\vec{k}_0 \parallel \hat{e}_2$



NB: this is only possible in 3D!
(in 2D, stretching is always accompanied by compression and annihilation of antiparallel fields)

In a more general case, when $\sigma^i_m(t)$ is not diagonal, time-dependent and/or random, the basic ideas outlined above survive.

One now calculate the growth of magnetic energy in terms of the finite-time Lyapunov exponents of the flow. Roughly speaking, one diagonalises the matrix

$$\hat{M} = e^{\int_0^t dt'\hat{\sigma}(t')} \cdot \left[e^{\int_0^t dt'\hat{\sigma}(t')}\right]^T \qquad \text{"metric"}$$

$$= \hat{R}^T \cdot \hat{L} \cdot \hat{R} \leftarrow \text{rotation}$$

$$\hookrightarrow \hat{L} = \begin{pmatrix} e^{\zeta_1(t)} & & 0 \\ & e^{\zeta_2(t)} & \\ 0 & & e^{\zeta_3(t)} \end{pmatrix}$$

Here $\frac{\zeta_i(t)}{2t}$ are the finite-time Lyapunov exponents (FLTEs).

It is possible to prove that

$$\hat{R}(t) \underset{\text{expo'tlly fast}}{\rightarrow} \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \quad \text{Lyapunov basis}$$

and $\frac{\zeta_i(t)}{2t} \rightarrow \lambda_i$ Lyapunov exponents.

For a random flow, $\zeta_i(t)$ are random functions.

One can show $\overline{} \leftarrow$ average over the distribution of ζ_i's.

$$\overline{\langle B^2 \rangle}(t) \propto \overline{e^{\frac{\zeta_1 - \zeta_2}{2}}}$$

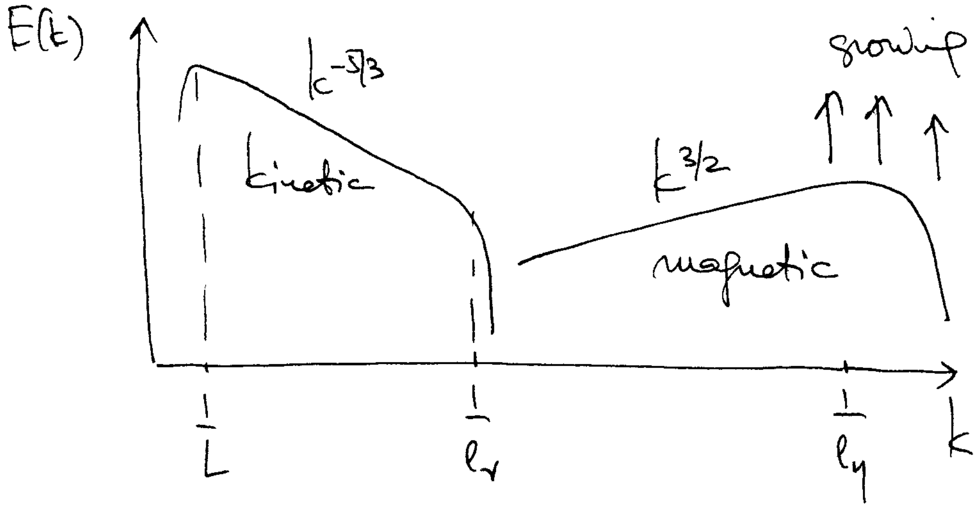
This generalises our result $\langle B^2 \rangle \propto e^{(\lambda_1 - \lambda_2)t}$

[See Ott, Phys. Plasmas 5, 1636 (1998) Chertkov et al., PRL 83, 4065 (1999)]

The distribution of FLTEs for any particular flow is quite hard to calculate.
One solvable case the Kazantsev Model - random Gaussian white-in-time velocity field. For this field the distribution of FLTEs is Gaussian in the long-time limit and one can compute everything. It is very useful to know how to calculate things for the Kazantsev field - a solvable model is always good! I probably won't have time to go through it in my lectures, but I will include a tutorial on the Kazantsev model in my typed notes.

Sov. Phys. JETP 26, 1031 (1968)

There is one important result that is only derivable for the Kazantsev model, and checks out in numerical simulations: the spectrum of the folded fields is $\propto k^{3/2}$ (spectrum of direction reversals)



$$\langle \overline{B^2} \rangle \propto e^{\frac{5}{4}\lambda_1 t}$$

where

$$\lambda_1 = \frac{\overline{J_1}}{2t}$$

[see Kulsrud & Anderson, ApJ 396, 606 (1992)]
2 reviews on Kazantsev approach: Zeldovich, Ruzmaikin, Sokoloff. The Almighty Chance (1990)
Falkovich et al. RMP 73, 913 (2001)

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§3. Saturation of the Fluctuation Dynamo.

If magnetic fields are small-scale, how can they coherently back react on velocity (large-scale) and effect saturation?

The Lorentz tension force $\vec{B}\cdot\nabla\vec{B} \sim \frac{B^2}{\ell_\parallel}$ does not know about direction reversals!

Antiparallel fields have large-scale coherence on the scale $\ell_\parallel \sim \ell_\nu$.

Nonlinearity is important when

$$\vec{B}\cdot\nabla\vec{B} \sim \vec{u}\cdot\nabla\vec{u} \sim \frac{\delta u_{\ell_\nu}^2}{\ell_\nu} \quad \Rightarrow B^2 \sim \delta u_{\ell_\nu}^2$$

$$\frac{B^2}{\ell_\parallel} \sim \frac{B^2}{\ell_\nu}$$

(magnetic energy ~ energy of viscous motions)

We don't know exactly how back reaction stops dynamo. An idea can be fruitfully (and, within the Kazantsev model, quantitatively) explored that it does so by making the velocity gradients anisotropic wrt the local field direction, suppressing $\hat{b}\hat{b}:\nabla\vec{u}$.

[PRL 92, 084504 (2004), APJ 612, 276 (2004)]

Suppose the dynamo action of the viscous motions is suppressed in some way. Then larger-scale motions come into play, — they are slower, but more energetic, so they can still do dynamo (albeit at smaller rate). Then they are suppressed and it's the turn of even larger scales — etc.

At time t, define stretching scale

$$l_s(t) \text{ s.t. } \delta u_{l_s}^2 \sim B^2(t)$$

Then $$\frac{d}{dt}B^2 \sim \frac{\delta u_{l_s}}{l_s}B^2 \sim \frac{\delta u_{l_s}^3}{l_s} \sim \epsilon = \text{const}$$

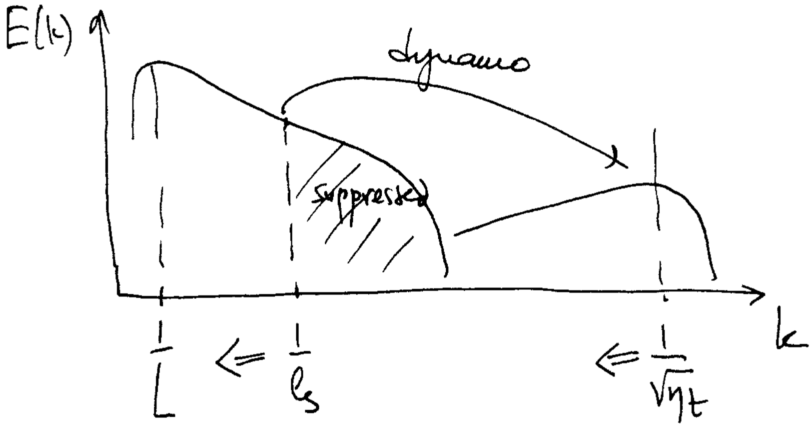
So $B^2 \sim \epsilon t$ — secular "nonlinear" dynamo

The folded structures get longer:

$$l_{\parallel} \sim l_s \sim \frac{\delta u_{l_s}^3}{\epsilon} \sim \frac{(\epsilon t)^{3/2}}{\epsilon} \sim \sqrt{\epsilon}\, t^{3/2}$$

The resistive scale gets larger:

$$l_\eta \sim \left[\frac{\eta}{\delta u_{l_s}/l_s}\right]^{1/2} \sim \sqrt{\eta t}$$



This goes on until

$$\left\{\begin{array}{l} l_s \sim L, \quad B^2 \sim U^2 \\ t \sim \left(\frac{L}{\sqrt{\epsilon}}\right)^{2/3} \sim \frac{L}{U} \\ l_{\parallel} \sim L \\ l_\eta \sim \frac{L}{\sqrt{Rm}} \end{array}\right.$$

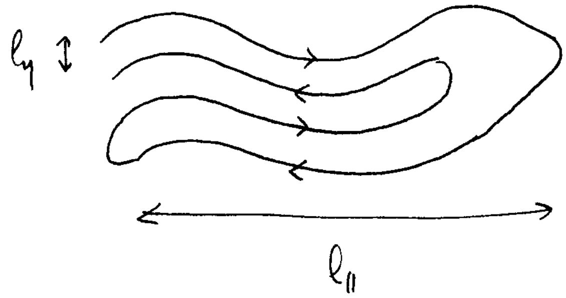
[NJP 4, 84 (2002), ApJ 612, 276 (2004)]

So, this argument predicts saturation at $B^2 \sim U^2$.

What happens in the inertial range?

Not all motions need to be suppressed.

For example, we can have a type of Alfvén waves that propagate along folds:



$$\frac{1}{\ell_\parallel} < k_\parallel < \frac{1}{\ell_\nu}$$

but these would be very hard to observe numerically.

$$\omega^2 = |\mathbf{k}\cdot\hat{\mathbf{b}}|^2 \langle B^2\rangle^{1/2}$$

[NJP 4, 84 (2002)]

Note that if we assume still $\ell_\nu \sim Re^{-3/4} L$,

Then $\dfrac{\ell_\eta}{\ell_\nu} \sim \dfrac{Rm^{-1/2}}{Re^{-3/4}} \sim \dfrac{Re^{1/4}}{Pm^{1/2}} \ll 0$ for $Pm \gg \sqrt{Re}$,

so not a regime easily captured in simulations with $Pm \sim 1$.

NB: Standard theory of Alfvénic turbulence, which predicts a $k^{-5/3}$ spectrum etc. assumes large-scale mean field and locality of interactions. This does not seem to be true for dynamo. So saturated spectrum probably still dominated by resistive scales (direction reversals)