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§5. Intermittency

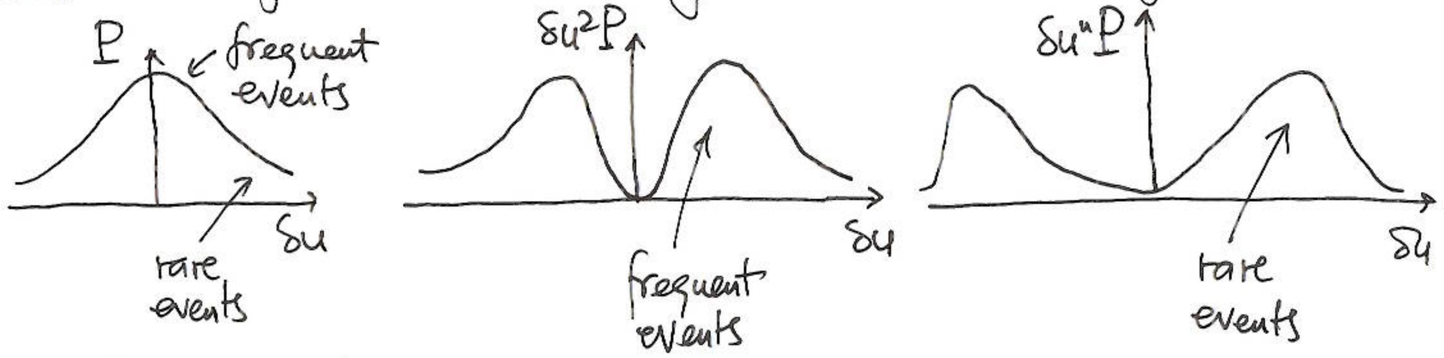
So far, we have an exact result for 3-d order statistics, $S_{LLL}(r) = -\frac{4}{5}\epsilon r$. (the $\frac{4}{5}$ law)

What about all other orders?

$$S_n(r) = \langle |\delta\vec{u}\cdot\hat{r}|^n \rangle \sim r^{\zeta_n} \text{ for } \ell_\nu \ll r \ll L$$

Can find ζ_n? (we know $\zeta_3 = 1$).

• Why do we care? High-order stats describe probabilities of rare events (large fluctuations) and come from the tails of the distribution function:



NB: These are hard to measure: it takes a long time for time averages and high resolution for volume averages
[rare in time] [sparse in space]

Note. It is not obvious that $S_n(r)$ exist – e.g. if $P(\delta u)$ was a power law, they would only exist up to some n. However, experiments suggest that (probably)

$$P(\delta u) \sim e^{-\delta u^\alpha}, \quad \alpha < 1 \quad \text{stretched exponential.}$$

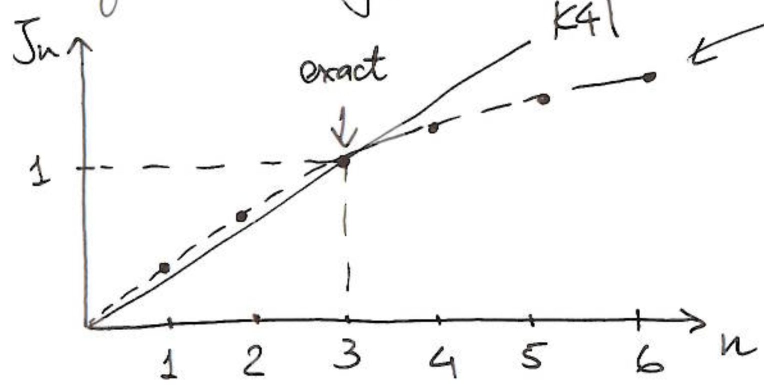
- What does the K41 argument predict?

K41 assumes self-similarity in the inertial range:

$$\delta u_{\lambda \ell} = \lambda^h \delta u_\ell \Rightarrow S_n(\lambda \ell) = \lambda^{hn} S_n(\ell) \text{ for all } n.$$

Since $S_3(\ell) = -\frac{4}{5}\epsilon \ell$ exactly, $h = \frac{1}{3}$ and we must have $\boxed{\zeta_n = \frac{n}{3}}$ in K41

- Experimentally, one sees



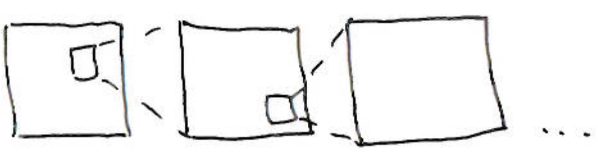
deviation from K41 grows with n.

Note: Mathematically, one can prove that
1) ζ_n vs. n is concave
2) ζ_n vs. n is non-decreasing
[see Frisch §8.4]

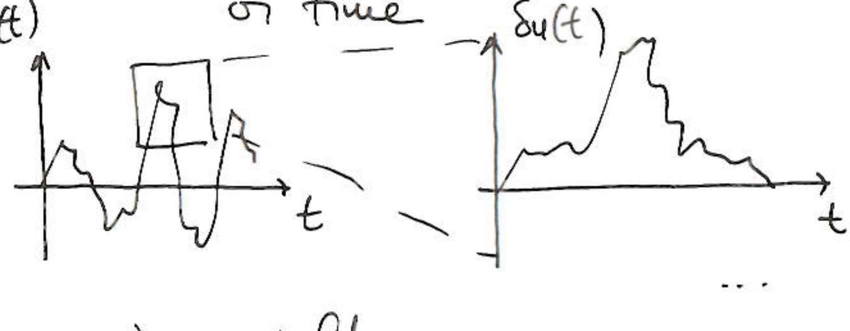
- So, something must be wrong with the assumption of self-similarity.

Self-similarity means that if we blow up small regions of the system, we see the same structure:

in space

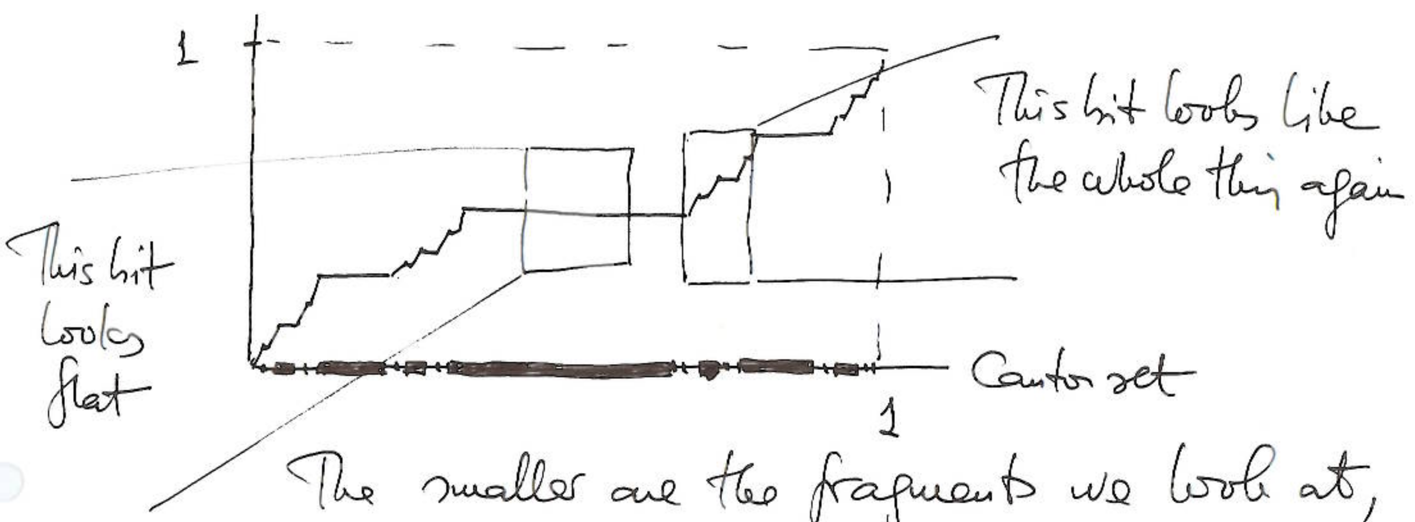


or time



This is true, e.g., for a random walk.

But there are examples of fluctuating behaviour, for which this is <u>not</u> the case, e.g. <u>devil's staircase</u>:



The smaller are the fragments we look at, the smaller is the probability that it won't just be flat!

Such non-self-similar behaviour is called <u>intermittency</u>

Intermittency is related to the <u>volume (or time)-filling property of the field</u>: crudely, let's imagine that the n'th order structure function only comes from velocity differences concentrated in a fraction $\chi_\ell^{(n)}$ of the system volume, where $\chi_\ell^{(n)} \sim \left(\frac{\ell}{L}\right)^{\mu_{n/3}}$ ← some exponent

Then

$$S_n(\ell) \sim \langle |\delta u_\ell|^n \rangle \sim \delta u_\ell^n \cdot \chi_\ell^{(n)} \sim (\epsilon \ell)^{n/3} \left(\frac{\ell}{L}\right)^{\mu_{n/3}}$$

$$\sim \epsilon^{n/3} L^{-\mu_{n/3}} \ell^{\frac{n}{3}+\mu_{n/3}}$$

So $\boxed{\zeta_n = \frac{n}{3} + \mu_{n/3}}$

NB: $\chi_\ell^{(n)}$ must depend on n, otherwise $\mu_{n/3} = 0$ because $\zeta_3 = 1$

What is the physical origin of this lack of volume filling?

Obukhov & Kolmogorov (1962) proposed to consider

$$\epsilon_\ell(\vec{x}) = \frac{1}{\frac{4}{3}\pi \ell^3} \int_{|\vec{x}'-\vec{x}|<\ell} d^3x' \, \nu \, |\nabla \vec{u}|^2(\vec{x}')$$

average over a ball of radius ℓ

Clearly, $\langle \epsilon_\ell(\vec{x}) \rangle = \epsilon$ Kolmogorov flux.

Batchelor & Townsend (1947): $\epsilon_\ell(\vec{x})$ is very intermittent (dissipation occurs in vortex filaments, ie a small fraction of the volume)

Perhaps, at scale ℓ, we have, generalizing K41:

$$\delta u_\ell \sim (\epsilon_\ell \ell)^{1/3}$$

Now $\langle \epsilon_\ell^n \rangle = \epsilon^n \left(\frac{\ell}{L}\right)^{\mu_n}$ and $\mu_1 = 0$, so $\langle \epsilon_\ell \rangle = \epsilon$

Then

$$S_n(\ell) \sim \langle \delta u_\ell^n \rangle \sim \langle (\epsilon_\ell \ell)^{n/3} \rangle =$$

so $S_3 \propto \ell$

$$= \langle \epsilon_\ell^{n/3} \rangle \ell^{n/3} = \epsilon^{n/3} L^{-\mu_{n/3}} \ell^{\frac{n}{3}+\mu_{n/3}}$$

So we have $\boxed{\zeta_n = \frac{n}{3} + \mu_{n/3}}$ where $\mu_{n/3}$ are scaling exponents of ϵ_ℓ

This is called the refined similarity hypothesis.

Can we construct the statistics of ϵ_ℓ from some plausible physical reasoning?

- Take a box the size of the system L.
  The mean ~~dissipation rate is~~ dissipated power is
  $$\langle \epsilon_L \rangle = \epsilon$$
- Divide the box into smaller boxes of size αL, $\alpha < 1$
  The mean dissipated power in each of these boxes is different (intermittency!)
  Model this variation by writing
  $$\epsilon_{\alpha L} = \epsilon W_1$$
  where W_1 is a random variable such that
  $$W_1 \geq 0 \text{ and } \langle W_1 \rangle = 1 \text{ (so that } \langle \epsilon_{\alpha L} \rangle = \epsilon)$$
- Divide these boxes again : $\alpha^2 L$ and let
  $$\epsilon_{\alpha^2 L} = \epsilon_{\alpha L} W_2 = \epsilon W_1 W_2 ,$$
  where W_2 is independent of W_1 and distributed the same way, so $\langle \epsilon_{\alpha^2 L} \rangle = \epsilon$ again
- Iterate this procedure, so for boxes of size $\ell = \alpha^k L$
  $$\epsilon_\ell = \epsilon W_1 W_2 \cdots W_k \quad , \quad \langle \epsilon_\ell \rangle = \epsilon \quad , \quad k = \log_\alpha \frac{\ell}{L}$$

Then $\langle \epsilon_\ell^m \rangle = \epsilon^m \langle W^m \rangle^k = \epsilon^m \alpha^{k \log_\alpha \langle W^m \rangle} = \epsilon^m \left(\frac{\ell}{L}\right)^{\log_\alpha \langle W^m \rangle}$

So we have $\langle \epsilon_\ell^m \rangle = \epsilon^m \left(\frac{\ell}{L}\right)^{\mu_m}$

where
$$\mu_m = \log_\alpha \langle W^m \rangle = \frac{\ln \langle W^m \rangle}{\ln \alpha}$$

Note that for K41, $W = 1$ (non random), $\mu_m = 0$

To model intermittency, we need to think about how W can be distributed.

- First of all, the way we divide our box must not influence the result, so replacing α by α^2 or α^r should not change the result. This means that the products $W_1 W_2$ ... or $W_1 \dots W_r$ must have the same distribution as W_1.

If we let $w_i = \ln W_i$, this means that for arbitrary r, $\sum_{i=1}^{r} w_i$ must have the same distribution as w_i, so that $W_1 \dots W_r = e^{\sum_{i=1}^{r} w_i}$ has the same distr. as W_i.

- The lognormal model. An obvious candidate is a gaussian distribution:
$$P(w_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(w-\bar{w})^2}{2\sigma^2}}$$
$\bar{w}, \sigma$ are parameters.

Then $\sum_{i=1}^{r} w_i$ is also gaussian with mean $= r\bar{w}$, Variance $= r\sigma^2$

[**NB**: This is not an application of the Central Limit Theorem, which says that

$$\frac{1}{\sqrt{r}}\left[\sum_{i=1}^{r} W_i - r\overline{W}\right] \quad \text{is Gaussian as } r\to\infty \quad \forall W_i$$

but not necessarily $\sum_{i=1}^{r} W_i$!]

Thus, ϵ_ℓ is lognormal (proposed by Obukhov 1962)

$$\langle W^m\rangle = \langle e^{mw}\rangle = \int dw \, \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{mw - \frac{(w-\overline{w})^2}{2\sigma^2}} = e^{m\overline{w} + \frac{\sigma^2 m^2}{2}}$$

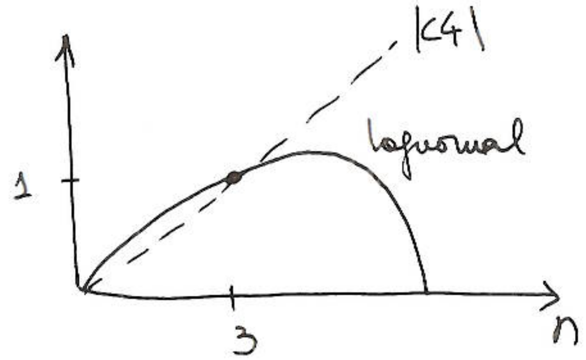
We must have $\langle W\rangle = 1 \Rightarrow \overline{w} + \frac{\sigma^2}{2} = 0$.

This leaves only one parameter: $\overline{w}$ and

$$\langle W^m\rangle = e^{\overline{w}\, m(1-m)} \Rightarrow \mu_m = \frac{\ln\langle W^m\rangle}{\ln\alpha} = \frac{\overline{w}}{\ln\alpha}\, m(1-m)$$

$$\frac{\overline{w}}{\ln\alpha} = \frac{\mu}{2} \quad \text{free parameter.}$$

$$\boxed{\zeta_n = \frac{n}{3} + \mu_{n/3} = \frac{n}{3} + \frac{\mu}{18}\, n(3-n)}$$



This is no good because ζ_n must be monotonic (otherwise δu_ℓ blows up as $\ell\to 0$ – see Frisch §8.4)

- The She-Lévêque Model (1994)

Another possibility for an infinitely divisible distribution is Poisson:

$$\left.\begin{aligned} w &= q \ln\beta + w_0 \\ \text{or } W &= e^{w_0}\beta^q \end{aligned}\right\} \quad \beta \text{ and } w_0 \text{ parameters and } q \text{ is a Poisson integer.}$$

$$P(q) = \frac{\lambda^q}{q!}e^{-\lambda}, \quad \langle q \rangle = \lambda \leftarrow \text{another parameter.}$$

Thus, ϵ_ℓ is log-Poisson (this interpretation of the SL Model was proposed by Dubrulle 1994)

$$\langle W^m \rangle = \sum_{q=0}^{\infty} e^{m(q\ln\beta + w_0) - \lambda}\frac{\lambda^q}{q!} = e^{mw_0-\lambda}\sum_{q=0}^{\infty}\frac{(\lambda\beta^m)^q}{q!} =$$

$$= e^{mw_0 - \lambda + \lambda\beta^m} = e^{mw_0 - \lambda(1-\beta^m)}$$

$$\langle W \rangle = 1 \implies w_0 - \lambda(1-\beta) = 0 \implies \lambda = \frac{w_0}{1-\beta}$$

so $\langle W^m \rangle = e^{w_0\left(m - \frac{1-\beta^m}{1-\beta}\right)}$, whence

$$\left.\mu_m = \frac{\ln\langle W^m\rangle}{\ln\alpha} = \underbrace{\frac{w_0}{\ln\alpha}}_{-x}\left(m - \frac{1-\beta^m}{1-\beta}\right)\right\} \quad \text{2 parameters: } x \text{ and } \beta$$

$$\boxed{\zeta_n = \frac{n}{3} + \mu_{n/3} = (1-x)\frac{n}{3} + x\,\frac{1-\beta^{n/3}}{1-\beta}}$$

monotonic is $\beta < 1$

The challenge now is to <u>find x and β</u>.

We have
$$\langle \epsilon_\ell^m \rangle = \epsilon^m \left(\frac{\ell}{L}\right)^{-x\left(m - \frac{1-\beta^m}{1-\beta}\right)}$$

Consider $m \to \infty$:
$$\langle \epsilon_\ell^m \rangle \sim \epsilon^m \left(\frac{\ell}{L}\right)^{-xm + \frac{x}{1-\beta}} \qquad (*)$$

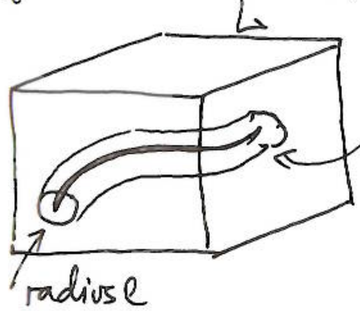
These large moments correspond to the most singular dissipative structures:

$$\langle \epsilon_\ell^m \rangle \underset{m\to\infty}{\sim} \underbrace{[\epsilon_\ell^{(\infty)}]^m}_{\text{diss. power in these structures}} \; \underbrace{\chi_\ell}_{\text{volume filling fraction of these structures}}$$

Here $\chi_\ell \sim \left(\frac{\ell}{L}\right)^{d-D}$, where $d\,(=3)$ is the dimension of space and D is the dimension of the most singular dissipative structures.

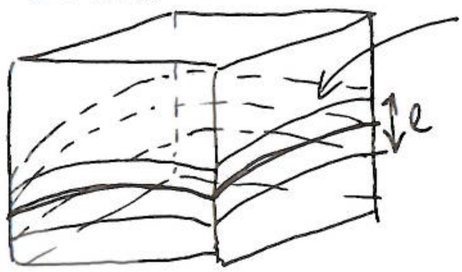
Indeed, χ_ℓ is the probability for a ball of radius ℓ to intersect with a diss. structure:

$D = 1$
$d = 3$
filaments



volume $\ell^2 L$ $\Rightarrow \chi_\ell \sim \frac{\ell^2 L}{L^3} \sim \left(\frac{\ell}{L}\right)^{2 = 3-1}$

$D = 2$
$d = 3$
sheets



volume ℓL^2 $\Rightarrow \chi_\ell \sim \frac{\ell L^2}{L^3} \sim \left(\frac{\ell}{L}\right)^{1 = 3-2}$

Thus, $\langle \epsilon_\ell^m \rangle \sim [\epsilon_\ell^{(\infty)}]^m \left(\frac{\ell}{L}\right)^{(d-D) \equiv C}$ codimension of the structures

Comparing this with (*), we get

$$\epsilon_\ell^{(\infty)} \sim \epsilon \left(\frac{\ell}{L}\right)^{-x} \quad \text{and} \quad \frac{x}{1-\beta} = C \Rightarrow \boxed{\beta = 1 - \frac{x}{C}}$$

$$\boxed{\zeta_n = (1-x)\frac{n}{3} + C\left[1 - \left(1 - \frac{x}{C}\right)^{\frac{n}{3}}\right]}$$

Now we need x, i.e. the scaling of the dissipated power in the most singular structures.
SL assumed that most of the energy is dissipated in them and that the cascade time has the dimensional K41 scaling:

$$\epsilon_\ell^{(\infty)} \sim \frac{U^2}{\tau_\ell} \sim U^2 \epsilon^{1/3} \ell^{-2/3} \sim U^2 \left[\frac{U^3}{L}\right]^{1/3} \ell^{-2/3} \sim \epsilon \left(\frac{\ell}{L}\right)^{-2/3}$$

whence $\boxed{x = \frac{2}{3}}$.

Finally, empirically we know that the diss. structures are vortex filaments, so $\boxed{C = 2}$ $(D=1)$.
This gives

$$\boxed{\zeta_n = \frac{n}{9} + 2\left[1 - \left(\frac{2}{3}\right)^{n/3}\right]}$$

This fits the experimental/observational data rather well.

Extended Self-Similarity (Benzi et al. 1993)

When measurements/simulations are done with only moderately large Re, it is rather hard to get good scalings. But there is a nifty trick:

plot $S_n(\ell)$ <u>not</u> vs. ℓ but vs. $S_3(\ell)$.

Since $S_3(\ell) \sim \ell$, we should have

$$S_n(\ell) \sim \ell^{\zeta_n} \sim S_3(\ell)^{\zeta_n}$$

so

$$\boxed{\zeta_n = \frac{\partial \ln S_n(\ell)}{\partial \ln S_3(\ell)}}$$

It turns out that this formula is much more robust that $\zeta_n = \frac{\partial \ln S_n(\ell)}{\partial \ln \ell}$ at moderate Re, i.e. the deviations from scaling arising from lack of asymptocity are correlated for all n and cancel when "relative" scaling exponents are computed. There is, however, no theory of this.