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§4. Small-Scale Structure of Turbulence: Correlation Functions and the $\frac{4}{5}$ Law.

For this lecture, I will use a set of notes that I wrote for a course on turbulence @ DAMTP in 2006 (pp. 4–12)
Apologies for discontinuous pagination.

§2. Correlation Functions (Stat. Descr. of Turbulence)

What does it mean "to solve turbulence"?

$\rho = 1$

$$\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \nu\Delta\vec{u} + \vec{f}$$
$$\nabla\cdot\vec{u} = 0$$

$\uparrow$ forcing – model. random

$\vec{u}$ is a random field: in space, time, and, if we use a random forcing function and an ensemble of random initial conditions, also in the strict statistical sense.

Predicting a particular realization $u(t,\vec{x})$ is probably impossible and also physically useless (non-universal). Our ambition should be to predict some average quantities / stat. distributions.

$$\langle\ldots\rangle_{\text{volume}} = \langle\ldots\rangle_{\text{time}} = \langle\ldots\rangle_{\text{ensemble}}$$
(for stationary st-lly processes)

Ergodic hypothesis is proven only for a few simple systems but is probably a safe assumption.

Formally, the full description would be

 $P_1[\vec{u}_1;\vec{x}_1]$ – 1-pt prob. to have vel. $\vec{u}_1$ at point $\vec{x}_1$

 $P_2[\vec{u}_1,\vec{u}_2;\vec{x}_1,\vec{x}_2]$ – 2-pt prob to have vel. $\vec{u}_1$ @ $\vec{x}_1$ and vel. $\vec{u}_2$ @ $\vec{x}_2$

 $P_3[\vec{u}_1,\vec{u}_2,\vec{u}_3;\vec{x}_1,\vec{x}_2,\vec{x}_3]$ – 3-pt etc...

So we need n-pt distributions for all n.

- At small scales (below forcing/injection scale), may assume turbulence is homogeneous, i.e. all points in space are statistically equal.

Then $P_1[\vec{u}_1 ; \vec{x}_1] = P_1[\vec{u}_1]$ indep. of position

$P_2 = P_2[\vec{u}_1, \vec{u}_2 ; \vec{x}_2 - \vec{x}_1]$ depends only on distance between pts.

$P_3 = P_3[\vec{u}_1, \vec{u}_2, \vec{u}_3 ; \vec{x}_2 - \vec{x}_1, \vec{x}_3 - \vec{x}_1]$ etc...

- Galilean invariance:

const velocity

$$\vec{u}'(t, \vec{x}) = \vec{u}(t, \vec{x} - \vec{U}t) + \vec{U}$$

If $\vec{u}$ is a sln, so is $\vec{u}'$ (without $\vec{f}$).

(See Frisch §6.2.6 on random gal. inv.)

So one measures velocity increments:

$$\delta\vec{u} = \vec{u}(\vec{x}_2) - \vec{u}(\vec{x}_1) \quad \text{for 2-pt stats.}$$

and $P[\delta\vec{u} ; \vec{x}_2 - \vec{x}_1]$

It is often hard to measure the full pdf and one looks at moments:

2-order:

$\langle \delta u_i \, \delta u_j \rangle = S_{ij}(\vec{r})$, $\vec{r} = \vec{x}_2 - \vec{x}_1$ structure function

$\langle u_i(\vec{x}_1) \, u_j(\vec{x}_2) \rangle = C_{ij}(\vec{r})$ correlation function

3-order: $\langle \delta u_i \, \delta u_j \, \delta u_k \rangle$ etc...

In general, we may talk about n-pt m-order stats – corr. fn that has m powers of u_i taken in n different points.

Note that energy $\frac{1}{2}\langle |\vec{u}(\vec{x})|^2 \rangle = \frac{1}{2} C_{ii}(0)$ is a 2-order 1-pt quantity independent of position.

Formally, $\langle u_i(\vec{x}_1) \, u_j(\vec{x}_2) \rangle = \int D\vec{u}_1 D\vec{u}_2 \, P[\vec{u}_1, \vec{u}_2 ; \vec{x}_2 - \vec{x}_1]$

Let's study properties of 2-order 2-pt correlation functions.

$$S_{ij}(\vec{r}) = \langle \delta u_i \delta u_j \rangle = \langle (u_{2i} - u_{1i})(u_{2j} - u_{1j}) \rangle$$

$$= \underbrace{\langle u_{2i} u_{2j} \rangle}_{C_{ij}(0)} + \underbrace{\langle u_{1i} u_{1j} \rangle}_{C_{ij}(0)} - \underbrace{\langle u_{1i} u_{2j} \rangle}_{C_{ij}(\vec{r})} - \underbrace{\langle u_{1j} u_{2i} \rangle}_{C_{ji}(\vec{r})}$$

- Symmetries can help reduce the number of unknown scalar functions

- Homogeneity gave us $C_{ij}(\vec{x}_1, \vec{x}_2) = C_{ij}(\underbrace{\vec{x}_2 - \vec{x}_1}_{\vec{r}})$
- Isotropy : no special directions

$\hat{r}_i = \frac{r_i}{r}$

$r = |\vec{r}|$

$$C_{ij}(\vec{r}) = C_1(r)\delta_{ij} + C_2(r)\hat{r}_i\hat{r}_j + \cancel{C_3(r)\epsilon_{ijk}\hat{r}_k}$$

- Parity (mirror symmetry) : $C_{ij}(-\vec{r}) = C_{ij}(\vec{r})$

(Note: trivially, from permutation symmetry, $C_{ij}(-\vec{r}) = -C_{ji}(\vec{r})$)

So, 2 scalar fns left: can rewrite

$$C_{ij}(\vec{r}) = C_{TT}(r)(\delta_{ij} - \hat{r}_i\hat{r}_j) + C_{LL}(r)\hat{r}_i\hat{r}_j$$

$C_{LL} = \hat{r}_i\hat{r}_j C_{ij}(\vec{r})$

transverse $\qquad$ longitudinal

Any 2-rank tensor depending on nothing is $\propto \delta_{ij}$, so

$$C_{ij}(0) = \text{const}\,\delta_{ij} \Rightarrow \text{const} = \frac{C_{ii}(0)}{3} = \frac{\langle u^2 \rangle}{3} = \frac{2}{3}\mathcal{E}$$

$\delta_{ii} = 3$ $\qquad$ $\mathcal{E} = \frac{1}{2}\langle u^2 \rangle$ energy.

$$C_{ij}(0) = \frac{1}{3}\langle u^2 \rangle \delta_{ij} = \frac{2}{3}\mathcal{E}\delta_{ij}$$

Thus, $S_{ij}(\vec{r}) = \frac{2}{3}\langle u^2\rangle \delta_{ij} - 2C_{ij}(\vec{r}) =$

$$= S_{TT}(r)(\delta_{ij} - \hat{r}_i\hat{r}_j) + S_{LL}(r)\hat{r}_i\hat{r}_j$$

$S_{TT} = \frac{2}{3}\langle u^2\rangle - 2C_{TT}$ $\qquad$ $S_{ii}(0) = 0$

$S_{LL} = \frac{2}{3}\langle u^2\rangle - 2C_{LL}$ $\qquad$ $C_{ii}(0) = 2C_{TT}(0) + C_{LL}(0)$

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• Incompressibility imposes a further constraint

$$\frac{\partial u_i}{\partial x_i} = 0, \text{ so } \quad \frac{\partial C_{ij}}{\partial x_{2j}} = \frac{\partial C_{ij}}{\partial r_j} = 0 =$$

$$= C'_{TT}(r)\,\hat{r}_j\underbrace{\cancel{(\delta_{ij} - \hat{r}_i\hat{r}_j)}}_{0} + C'_{LL}(r)\,\hat{r}_j\hat{r}_i\hat{r}_j +$$

$$\frac{\partial}{\partial r_j} = \frac{r_j}{r}\frac{\partial}{\partial r} = \hat{r}_j\frac{\partial}{\partial r}$$

$$+ (C_{LL} - C_{TT})\Big(\underbrace{\frac{\delta_{ij} r_j}{r^2} + \frac{r_i \cdot 3}{r^2} - 2\frac{r_i r_j}{r^3}\hat{r}_j}_{\frac{\partial}{\partial r_j}\frac{r_i r_j}{r^2}}\Big)$$

$$= \hat{r}_i\Big[\underbrace{C'_{LL} + \frac{2}{r}(C_{LL} - C_{TT})}\Big]$$

$$\downarrow$$

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 C_{LL} = \frac{2}{r} C_{TT}$$

<u>Von Kármán relation</u>

$$\boxed{C_{TT}(r) = \frac{1}{2r}\frac{\partial}{\partial r} r^2 C_{LL} = \left(1 + \frac{1}{2} r \frac{\partial}{\partial r}\right) C_{LL}}$$

Analogous relation holds for S_{TT} and S_{LL}

Thus, we only need <u>one</u> scalar function: say, $S_{LL}(r)$

There is an <u>equivalent description in k space</u>:

$$u_i(\vec{k}) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} u_i(\vec{x})$$

discrete (periodic system) case

$$u_i(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} u_i(\vec{k}) \Rightarrow \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} u_i(\vec{k}) \frac{1}{V}$$

$$\langle u_i(\vec{k}) u_j(\vec{k}')\rangle = \int d^3x \int d^3x' e^{-i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{x}'} \underbrace{\langle u_i(\vec{x}) u_j(\vec{x}')\rangle}_{C_{ij}(\vec{r}),\ \vec{r}=\vec{x}-\vec{x}'} =$$

$$= \underbrace{\int d^3r\, e^{-i\vec{k}\cdot\vec{r}} C_{ij}(\vec{r})}_{= C_{ij}(\vec{k})} \underbrace{\int d^3x' e^{-i\vec{x}'\cdot(\vec{k}+\vec{k}')}}_{= (2\pi)^3 \delta(\vec{k}+\vec{k}') \Rightarrow V \delta_{\vec{k},-\vec{k}'} \text{ (discr.)}}$$

So, $\langle u_i(\vec{k}) u_j(\vec{k}')\rangle = C_{ij}(\vec{k}) \underbrace{(2\pi)^3 \delta(\vec{k}+\vec{k}')}_{\text{this happens because of spatial homogeneity}}$

- <u>Isotropy</u> & <u>parity</u> give, as before, $C_{ij}(\vec{k}) = C_1(k)\delta_{ij} + C_2(k)\hat{k}_i\hat{k}_j$, $\hat{k}_i = \frac{k_i}{k}$
- <u>Incompressibility</u>: $k_i C_{ij} = 0 \Rightarrow C_2 = -C_1$, so

$$\boxed{C_{ij}(\vec{k}) = C(k)(\delta_{ij} - \hat{k}_i\hat{k}_j)}$$

It is conventional (and convenient) to define <u>spectrum</u> in such a way that

$$\frac{1}{2}\langle u^2\rangle = \mathcal{E} = \int_0^\infty dk\, E(k)$$

Well,

$$\frac{1}{2}\langle u^2\rangle = \frac{1}{2}\left\langle\left|\int\frac{d^3k}{(2\pi)^3}\vec{u}(\vec{k})e^{i\vec{k}\cdot\vec{x}}\right|\right\rangle = \quad \text{vol. average}$$

$$= \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}\underbrace{\langle u_i(\vec{k})u_i(\vec{k}')\rangle}_{=\underbrace{C_{ii}}_{=2C(k)}(\vec{k})(2\pi)^3\delta(\vec{k}+\vec{k}')} =$$

$$= \int\frac{d^3k}{(2\pi)^3}C(k) = \frac{1}{(2\pi)^3}\int_0^\infty dk\,k^2\underbrace{\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\varphi}_{=4\pi}\,C(k) =$$

$$= \int_0^\infty dk\,\frac{k^2C(k)}{2\pi^2}$$

So, we get $\boxed{E(k) = \frac{1}{2\pi^2}k^2C(k)}$

Now let us relate spectrum to corr. function:

$$C(k) = \frac{1}{2}C_{ii}(\vec{k}) = \frac{1}{2}\int d^3r\,e^{-i\vec{k}\cdot\vec{r}}C_{ii}(\vec{r}) =$$

$$= \frac{1}{2}\int_0^\infty dr\,r^2\Big[\underbrace{C_{TT}(r)}_{}\cdot 2 + C_{LL}(r)\Big]\underbrace{\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\varphi\,e^{-i\vec{k}\cdot\vec{r}}}_{}$$

$$C_{LL} + \frac{1}{2}rC_{LL}' \;=\; \qquad 2\pi\int_{-1}^{1}d\cos\theta\,e^{-ikr\cos\theta} = 2\pi\frac{e^{-ikr}-e^{ikr}}{-ikr} = 4\pi\frac{\sin kr}{kr}$$

$$= \frac{1}{2}\int_0^\infty dr\,r^2\Big[\underbrace{3C_{LL}+rC_{LL}'}_{=\frac{1}{r^2}\frac{\partial}{\partial r}r^3C_{LL}}\Big]4\pi\frac{\sin kr}{kr}$$

$$\text{Thus,}\quad C(k) = 2\pi \int_0^\infty dr\, \frac{\sin kr}{kr}\frac{\partial}{\partial r} r^3\, \overbrace{C_{LL}(r)}^{\frac{1}{3}\langle u^2\rangle - \frac{1}{2}S_{LL}(r)} =$$

$$= \frac{2\pi}{\cancel{3}}\langle u^2\rangle \underbrace{\int_0^\infty dr\, \frac{\sin kr}{kr}\cancel{3}r^2}_{\frac{1}{k}\int_0^\infty dr\, r\sin kr = -\frac{1}{k}\frac{\partial}{\partial k}\int_0^\infty dr\cos kr = 0} - \pi\int_0^\infty dr\, \frac{\sin kr}{kr}\frac{\partial}{\partial r} r^3 S_{LL}(r)$$

So, we have

NB: If $C_{LL}(\lambda r) = \lambda^\alpha C_{LL}(r) \Rightarrow E(\lambda k) = \lambda^{-1-\alpha}E(k)$

$$\boxed{E(k) = \frac{k^2}{2\pi^2}C(k) = \frac{1}{\pi}k^2\int_0^\infty dr\, \frac{\sin kr}{kr}\frac{\partial}{\partial r} r^3 C_{LL}(r)}$$

Note that for small k this gives

$$E(k) = \frac{1}{\pi}k^2\left[\underbrace{\int_0^\infty dr\, \frac{\partial}{\partial r} r^3 C_{LL}(r)}_{=0} - \frac{1}{6}k^2 \underbrace{\int_0^\infty dr\, r^2 \frac{\partial}{\partial r} r^3 C_{LL}(r)}_{= -2\int_0^\infty dr\, r^4 C_{LL}(r)} + \dots\right]$$

provided $C_{LL}(r\to\infty)\to 0$ faster than $\frac{1}{r^3}$

provided $C_{LL}(r)\to 0$ faster than $\frac{1}{r^5}$

$$= \frac{1}{3\pi}k^4 \underbrace{\int_0^\infty dr\, r^4 C_{LL}(r)}_{=\Lambda} + \dots$$

infrared asymptotic

Λ Loytsyanskii invariant

See Davidson
Landau & Lifshitz
for further discussion of this issue.

$$\partial_t u_i = - u_l \frac{\partial u_i}{\partial x_l} - \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + f_i$$

Can we compute correlation functions from the N-S equation? Let's try:

$$\partial_t \langle u_{1i} u_{2j} \rangle = \langle u_{1i} \partial_t u_{2j} \rangle + \langle u_{2j} \partial_t u_{1i} \rangle$$

$$= \Big\langle - u_{1i} u_{2l} \frac{\partial u_{2j}}{\partial x_{2l}} - u_{1i} \frac{\partial p_2}{\partial x_{2j}} + \nu u_{1i} \nabla_2^2 u_{2j} + u_{1i} f_{2j}$$

$$- u_{2j} u_{1l} \frac{\partial u_{1i}}{\partial x_{1l}} - u_{2j} \frac{\partial p_1}{\partial x_{1i}} + \nu u_{2j} \nabla_1^2 u_{1i} + u_{2j} f_{1i} \Big\rangle$$

$$= \underbrace{- \frac{\partial}{\partial x_{1l}} \langle u_{1i} u_{1l} u_{2j} \rangle - \frac{\partial}{\partial x_{2l}} \langle u_{1i} u_{2l} u_{2j} \rangle}_{\text{3-order correlators!}}$$

$$- \frac{\partial}{\partial x_{1i}} \langle p_1 u_{2j} \rangle - \frac{\partial}{\partial x_{2j}} \langle p_2 u_{1i} \rangle +$$

$$+ \underbrace{\nu \nabla_1^2 \langle u_{1i} u_{2j} \rangle + \nu \nabla_2^2 \langle u_{1i} u_{2j} \rangle}_{\text{diss. terms}} +$$

$$+ \underbrace{\langle u_{1i} f_{2j} \rangle + \langle u_{2j} f_{1i} \rangle}_{\text{energy injection terms}}.$$

If we now attempt to write an equation for 3-order corr. fns, we'll see that we need 4-order ones, etc.

$$\partial_t \langle uu \rangle \sim \langle uuu \rangle$$
$$\partial_t \langle uuu \rangle \sim \langle uuuu \rangle \quad \ldots$$

<u>Closure problem</u>: moments (corr. fns) satisfy an infinite hierarchy of equations.

This is a generic problem ~~inevitable~~ when one starts averaging <u>nonlinear</u> equations.

It can only be resolved rigourously if ~~[illegible]~~ there is a small parameter in the system that allows some sort of asymptotic expansion. No such luck in turbulence.

Some people look for <u>closure schemes</u> – heuristic ways of closing the hierarchy so moments become compatible.

Two basic approaches:

Millionshchikov-Chandrasekhar hypothesis

EITHER $\partial_t \langle uu \rangle \sim \langle uuu \rangle$

$\partial_t \langle uuu \rangle \sim \langle uuuu \rangle \sim \langle uu \rangle \langle uu \rangle$

as if u were Gaussian

See work of Kraichnan (DIA)
Orszag (EDQNM)

– McComb's book has full review.

OR $\partial_t \langle uu \rangle \sim \langle uuu \rangle \sim -\frac{1}{\tau} \langle uu \rangle$

τ-approximation

suitably chosen corr. time.

I will not cover closure theories in this course.

Kolmogorov's $\frac{4}{5}$ Law

It turns out that we can make exact analytical progress with the equation for the velocity correlation function (p. 11).

Because of homogeneity,

- $\langle u_{1i} u_{2j} \rangle = C_{ij}(\vec{r})$, $\vec{r} = \vec{x}_2 - \vec{x}_1$
- $\langle u_{1i} u_{1l} u_{2j} \rangle \equiv C_{il,j}(\vec{r})$
- $\langle u_{1i} u_{2l} u_{2j} \rangle = C_{lj,i}(-\vec{r}) = -C_{lj,i}(\vec{r})$
- $\dfrac{\partial}{\partial x_{1l}} = -\dfrac{\partial}{\partial r_l} \qquad \dfrac{\partial}{\partial x_{2l}} = \dfrac{\partial}{\partial r_l}$
- Because of isotropy, pressure terms vanish:

Pf. $\langle p_1 u_{2j} \rangle = f(r)\hat{r}_j$

incompressibility $\Rightarrow \dfrac{\partial}{\partial x_{2j}} \langle p_1 u_{2j} \rangle = \langle p_1 \dfrac{\partial u_{2j}}{\partial x_{2j}} \rangle = 0 =$

$$= \frac{\partial}{\partial r_j} f(r) \frac{r_j}{r} = f'(r) + f(r)\frac{3}{r} - f(r)\frac{r_j}{r^2}\frac{r_j}{r}$$

$$f'(r) + \frac{2}{r} f(r) = 0 \Rightarrow f(r) = \text{const}\,\frac{1}{r^2}$$

$$f(r \to 0) < \infty \Rightarrow \text{const} = 0 \Rightarrow f = 0 \quad \text{q.e.d.}$$

- Dissipation terms $= 2\nu \nabla^2 C_{ij}(\vec{r})$
- Injection terms $\equiv 2\epsilon_{ij}(\vec{r})$ some smooth large-scale function.

So, with these simplifications, we get

$$\partial_t C_{ij} = \frac{\partial}{\partial r_l}\left(C_{il,j} + C_{jl,i}\right) + 2\nu\nabla^2 C_{ij} + 2\epsilon_{ij}(\vec{r})$$

Since $C_{ij}(\vec{r})$ in fact only depends on one scalar function, we can take the trace of this equation without losing any information:

$$\boxed{\frac{1}{2}\partial_t C_{ii}(r) = \frac{\partial}{\partial r_l} C_{il,i} + \nu\nabla^2 C_{ii} + \epsilon_{ii}(r)} \qquad (1)$$

This is basically the 2-point version of the energy equation.

From isotropy etc., we know that

$$C_{ii}(r) = 2C_{TT}(r) + C_{LL}(r) =$$

$$= 2C_{LL} + r\frac{\partial}{\partial r}C_{LL} + C_{LL} = \frac{1}{r^2}\frac{\partial}{\partial r} r^3 \overset{-\frac{1}{2}S_{LL}+\frac{1}{3}\langle u^2\rangle}{\overset{\parallel}{C_{LL}(r)}}$$

$$= \langle u^2\rangle - \frac{1}{2r^2}\frac{\partial}{\partial r} r^3 S_{LL}(r)$$

It is also possible to prove that the triple correlator $C_{il,j}$ only depends on one scalar function, e.g. $S_{LLL}(r) = \hat{r}_i\hat{r}_j\hat{r}_k\langle \delta u_i \delta u_j \delta u_k\rangle$:

$$C_{il,j}(\vec{r}) = -\frac{1}{12}\Big[S_{LLL}\,\delta_{il}\hat{r}_j - \frac{1}{2}\left(rS'_{LLL} + 2S_{LLL}\right)\left(\delta_{ij}\hat{r}_l + \delta_{lj}\hat{r}_i\right)$$
$$+ \left(rS'_{LLL} - S_{LLL}\right)\hat{r}_i\hat{r}_l\hat{r}_j\Big]$$

(see, e.g. Landau & Lifshitz vol 6 §34)

Therefore

$$C_{i\ell,i} = -\frac{1}{12}\hat{r}_\ell\left[\cancel{S_{LLL}} - 2rS'_{LLL} - 4S_{LLL} + rS'_{LLL} - \cancel{S_{LLL}}\right]$$

$$= \frac{1}{12}\hat{r}_\ell\left(rS'_{LLL} + 4S_{LLL}\right) = \frac{1}{12}\frac{\hat{r}_\ell}{r^3}\frac{\partial}{\partial r}r^4 S_{LLL}$$

If we substitute all this into (1), we can get an equation relating S_{LL} and S_{LLL}, called the <u>von-Kármán-Howarth equation</u>.

It is not closed, of course, so not much use.

However, if we assume <u>stationarity</u>, i.e. $\partial_t C_{ii} = 0$ and <u>large-scale energy injection</u>, i.e.

$$\epsilon_{ii}(r) \approx \epsilon + \ldots \quad \text{(Taylor-expand)}$$

Then (1) simplifies to

$$\frac{\partial}{\partial r_\ell}\left[C_{i\ell,i} + \nu\frac{\partial}{\partial r_\ell}C_{ii}\right] = -\epsilon \qquad (2)$$

because $\frac{\partial r_\ell}{\partial r_\ell} = 3$

Integrate this:

$$\underbrace{C_{i\ell,i} + \nu \frac{\partial}{\partial r_\ell} C_{ii}}_{\parallel} = -\frac{1}{3}\epsilon r_\ell + \cancel{\text{const}}$$

$\cancel{\text{const}} \to 0$: So no singularity @ $r \to 0$

$$\frac{1}{12}\frac{\hat{r}_\ell}{r^3}\frac{\partial}{\partial r} r^4 S_{LLL} - \nu \hat{r}_\ell \frac{\partial}{\partial r}\frac{1}{2r^2}\frac{\partial}{\partial r} r^3 S_{LL}$$
$$= \frac{1}{12}\hat{r}_\ell \frac{1}{r^3}\frac{\partial}{\partial r} r^4 \left[S_{LLL} - 6\nu S'_{LL}\right]$$

$$\frac{1}{r^3}\frac{\partial}{\partial r} r^4 \left[S_{LLL} - 6\nu S'_{LL}\right] = -4\epsilon r$$

Integrating again, $\boxed{S_{LLL} = -\frac{4}{5}\epsilon r + 6\nu S'_{LL}}$ (3)

($6\nu S'_{LL}$) small for $r \gg \ell_\nu$

This is Kolmogorov's $\frac{4}{5}$ law

Note For decaying turbulence, $\epsilon_{ii}(r) = 0$ (no forcing) but we expect self-similar decay:

$$\frac{1}{2}\partial_t C_{ii}(r) = \frac{d\xi}{dt} - \cancel{\partial_t \frac{1}{2r^2}\frac{\partial}{\partial r} r^3 S_{LL}(r)} \to 0$$

$\frac{d\xi}{dt} = -\epsilon$ (structure function const)

So we again get eq. (2)

Eq. (3) is important because it is the only exact result available. It is also consistent with K41: $\langle \delta u_\ell^3 \rangle \propto \epsilon \ell \iff \delta u_\ell \sim (\epsilon\ell)^{1/3}$

Note. There is a similar result for passive scalar:

$$\hat{r}\cdot\langle \delta\vec{u}\,\delta\theta^2\rangle = -\frac{4}{3}\epsilon_\theta r + 2\varkappa\frac{\partial}{\partial r}\langle\delta\theta^2\rangle$$

— Yaglom's $\frac{4}{3}$ law.

It is easier to derive because the equation is scalar — exercise.

Note. These exact results depend on the assumptions of homogeneity and isotropy (although these can be weakened) but not on locality — so they do not constitute proof of locality or of the existence of a Kolmogorov-style cascade!