A. Schekochihin
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§3. Mean Flows and Simple Closure Models.

A. The Framework of Mean-Field Theory

NSEq: $\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \underbrace{\nu\nabla^2\vec{u}}_{\substack{\parallel \\ \nabla\cdot(2\nu\hat{S}) \\ \text{viscous stress}}}$ (1)

$\hat{S} = \frac{1}{2}\left[(\nabla\vec{u}) + (\nabla\vec{u})^T\right]$ rate-of-strain tensor

We'd like to split

$$\vec{u} = \vec{U} + \delta\vec{u}, \quad \text{where } \vec{U} = \langle\vec{u}\rangle$$

$\vec{U}$: "mean flow" — slow times, long scales

$\delta\vec{u}$: turbulence — fast times, short scales

$\langle\ \rangle$: averaging appropriately defined.

Similarly, $\langle p\rangle = P$, so $p = P + \delta p$
$\langle\hat{s}\rangle = \hat{S}$, so $\hat{s} = \hat{S} + \delta\hat{S}$ etc...

This can be organised rigorously in 2 cases

1) Turbulence is stirred "externally" via some fast-time, small-scale mechanism (forcing) — rather than coming from the destabilization of the mean flow. Then averages are over fast times/short scales.

2) Mean flow is <u>steady</u>, so averages are over time.

Average (1): $$\boxed{\frac{\partial\vec{U}}{\partial t} + \vec{U}\cdot\nabla\vec{U} = -\nabla P + \nabla\cdot(2\nu\hat{S} - \hat{R})} \quad (2)$$

where $-\hat{R} = -\langle\delta\vec{u}\,\delta\vec{u}\rangle$ is <u>Reynolds stress</u>

Thus, in order to compute the mean flow, we must know something about the turbulence.

Subtract (1) − (2) to get an equation for $\delta\vec{u}$:

$$\frac{\partial \delta\vec{u}}{\partial t} + \vec{U}\cdot\nabla\delta\vec{u} + \delta\vec{u}\cdot\nabla\vec{U} + \nabla\cdot\left(\delta\vec{u}\,\delta\vec{u} - \hat{R}\right) =$$

$$= -\nabla\delta p + \nu\nabla^2\delta\vec{u}$$

or, using tensor notation,

$$\partial_t \delta u_i + U_\ell\partial_\ell\delta u_i + \delta u_\ell\partial_\ell U_i + \partial_\ell\left(\delta u_\ell\delta u_i - R_{\ell i}\right) =$$

$$= -\partial_i\delta p + \nu\nabla^2\delta u_i$$

$$\partial_t(\delta u_i\delta u_j) + U_\ell\partial_\ell\,\delta u_i\delta u_j + \delta u_j\delta u_\ell\,\partial_\ell U_i + \delta u_i\delta u_\ell\,\partial_\ell U_j$$

$$+ \partial_\ell\left(\delta u_\ell\delta u_i\delta u_j\right) - \delta u_j\partial_\ell R_{\ell i} - \delta u_i\partial_\ell R_{\ell j} =$$

$$= -\partial_i(\delta p\,\delta u_j) - \partial_j(\delta p\,\delta u_i) + \delta p\,\underbrace{(\partial_i\delta u_j + \partial_j\delta u_i)}_{=\,2\delta S_{ij}}$$

$$+ \nu\underbrace{(\delta u_j\nabla^2\delta u_i + \delta u_i\nabla^2\delta u_j)}_{=\,\partial_\ell(\delta u_j\partial_\ell\delta u_i + \delta u_i\partial_\ell\delta u_j) - 2(\partial_\ell\delta u_j)(\partial_\ell\delta u_i) \,=\, \nabla^2(\delta u_i\delta u_j) - 2(\partial_\ell\delta u_j)(\partial_\ell\delta u_i)}$$

Average this:

$$\partial_t R_{ij} + \vec{U}\cdot\nabla R_{ij} = -R_{i\ell}\partial_\ell U_j - R_{j\ell}\partial_\ell U_i - \partial_\ell\langle\delta u_\ell\delta u_i\delta u_j\rangle$$

$$- \partial_i\langle\delta p\,\delta u_j\rangle - \partial_j\langle\delta p\,\delta u_i\rangle + 2\langle\delta p\,\delta S_{ij}\rangle +$$

$$+ \nu\nabla^2 R_{ij} - 2\nu\langle(\partial_\ell\delta u_i)(\partial_\ell\delta u_j)\rangle$$

In tensor form:

$$\begin{aligned}\partial_t \hat{R} + \vec{U}\cdot\nabla\hat{R} - \nu\nabla^2\hat{R} = &- \hat{R}\cdot\nabla\vec{U} - (\hat{R}\cdot\nabla\vec{U})^T - \nabla\cdot\langle\delta\vec{u}\,\delta\vec{u}\,\delta\vec{u}\rangle \\ &\underbrace{- \nabla\langle\delta p\,\delta\vec{u}\rangle - (\nabla\langle\delta p\,\delta\vec{u}\rangle)^T + 2\langle\delta p\,\delta\hat{s}\rangle - 2\nu\langle(\partial_\ell\delta\vec{u})(\partial_\ell\delta\vec{u})\rangle}_{\text{2nd-order averages of various other quantities.}}\end{aligned} \qquad (3)$$

3rd order correlations have appeared.

Thus, the equation for $\hat{R}$ is not closed.

In order to close it, we must do two things:

1) All averages we have written are 1-pt averages,

e.g. $\hat{R}_{ij} = \langle\delta\bar{u}_i(\vec{x})\,\delta\bar{u}_j(\vec{x})\rangle$

same point.

If instead of the Reynolds stress we consider a 2-pt correlation function,

$$\hat{C}_{ij}(\vec{x},\vec{x}') = \langle\delta\bar{u}_i(\vec{x})\,\delta\bar{u}_j(\vec{x}')\rangle$$

then some of 2nd order averages can be expressed:

e.g. $R_{ij}(\vec{x}) = C_{ij}(\vec{x},\vec{x})$,

$$\langle(\partial_\ell\delta u_i)(\partial_\ell\delta u_j)\rangle = \left.\frac{\partial^2 C_{ij}}{\partial x_\ell\,\partial x'_\ell}\right|_{\vec{x}=\vec{x}'} \quad \text{etc.}$$

Thus, 2-pt statistics should help us deal with the 2-order quantities.

2) In order to find 3-order quantities, we could try to write an equation for $\langle \delta u_i \delta u_j \delta u_k \rangle$.
However, this will involve 4-order quantities etc.
This is a classic closure problem.
Such things happen often in physics. They can only be dealt with rigorously if there is a small parameter in the problem. In our case, there is none, so we are in trouble.

multiplying the nonlinearity!

What we could (in our desperation) try to do is, instead of solving eq. (3), to come up with some model expression for R_{ij} in terms of other quantities that we already have. Such an expression is called a <u>closure model</u> (1-pt closure in this case) and the only constraints on our fantasy here is that no internal contradictions should arise and we must not violate any fundamental (conservation) laws or symmetries. For example,

$$(4) \qquad \boxed{-\hat{R} = 2\nu_T \hat{S} - \frac{2}{3} \mathbb{1} K}, \quad K = \frac{1}{2}\langle \delta u^2 \rangle \text{ kinetic energy}$$

$$\text{or} \quad -\langle \delta u_i \delta u_j \rangle = \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{1}{3} \delta_{ij} \langle \delta u^2 \rangle$$

Boussinesq <u>eqn.</u>

some coefficient

term to ensure $R_{ii} = \langle \delta u^2 \rangle$

Substituting (4) → (2), we get

$$\frac{\partial \vec{U}}{\partial t} + \vec{U}\cdot\nabla\vec{U} = -\nabla\left(P+\frac{2}{3}K\right) + \nabla\cdot\left[2(\nu+\nu_T)\,\hat{S}\right] \qquad (5)$$

"turbulent pressure" can be absorbed into P and found from $\nabla\cdot\vec{U}=0$

"turbulent viscosity" [models energy transfer from mean flow to turbulence]

Now the simplest way to proceed is to estimate

$$\nu_T = \underset{\langle \delta u^2\rangle^{1/2}}{\underset{\parallel}{\delta u_{rms}}}\, L$$

mixing length — can compute this from some set of simple experiments.

Unsurprisingly, this sometimes works passably well and sometimes not. For example, it is intuitively clear that if the turbulence is strongly anisotropic, an isotropic expression (4) cannot be a very good model for the Reynolds stress.

Generally, if we are determined to pursue the closure approach and parametrise the turbulence, we might try to make things more sophisticated and also increase the number of adjustable parameters. [Depending on your attitude, this is either a no-win or a no-lose game.]

because it's all wrong

because with a sufficient # of parameters, it's bound to "work".

B. The K-ε Model.

I was asked expressly to provide an intro ~~[illegible]~~ to this, so here it is:

There are many ways to estimate the turbulent viscosity in a way more sophisticated than just a number. In particular, when the turbulence is inhomogeneous [as it should be, strictly speaking, if it resulted from the destabilization of a spatially-dependent mean flow], we might want to have ν_T depend on space (and maybe also on time - slowly). Now we can write

$$\nu_T \sim \underset{\text{energy}}{K^{1/2}} \underset{\text{corr. length}}{L} \sim K \underset{\text{corr. time}}{\tau} \sim \frac{K}{\underset{\text{mean vorticity}}{\Omega}} \sim \frac{K^2}{\underset{\text{energy into turbulence, subsequently dissipated by viscosity}}{\epsilon}}$$

$$\epsilon \sim \frac{\delta u_L^3}{L}$$

The Kϵ mode takes

$$\nu_T = \underset{\text{adjustable constant}}{c_\mu} K^2/\epsilon \tag{6}$$

and we now need to calculate K and ϵ.

Now recall that $K = \frac{1}{2}\langle \delta u^2 \rangle = \frac{1}{2} R_{ii}$, so we could go back to eq. (3) to see what should be the structure of the equation for K.

$$\partial_t K + \vec{U}\cdot\nabla K - \nu\nabla^2 K = -\underbrace{R_{ij} S_{ij}}_{\substack{\text{energy transfer from mean flow,}\\ = 2\nu_T \hat{S}:\hat{S} \geq 0}} -$$

$$-\underbrace{\nabla\cdot\left[\tfrac{1}{2}\langle \delta\vec{u}\,\delta\vec{u}^2\rangle + \langle \delta p\,\delta\vec{u}\rangle\right]}_{\text{energy flux}} + \underbrace{\langle \delta p\,\delta s_{ij}\rangle}_{\to 0} - \underbrace{\nu\langle|\nabla\delta\vec{u}|^2\rangle}_{\epsilon}$$

energy flux

$\vec{T} = -\frac{1}{\sigma_K}\nu_T\nabla K$ || another closure assumption.

ϵ dissipation

σ_K: another adjustable constant

So,

$$\boxed{\partial_t K + \vec{U}\cdot\nabla K = \nabla\cdot\left[\left(\nu + \frac{1}{\sigma_k}\nu_T\right)\nabla K\right] + 2\nu_T\hat{S}:\hat{S} - \epsilon} \quad (7)$$

Finally, we need to find ϵ. The equation for it (fully invented) is

$$\boxed{\partial_t \epsilon + \vec{U}\cdot\nabla\epsilon = \nabla\cdot\left[\left(\nu + \frac{1}{\sigma_\epsilon}\nu_T\right)\nabla\epsilon\right] + c_1 2\nu_T\hat{S}:\hat{S}\frac{\epsilon}{K} - c_2\frac{\epsilon^2}{K}} \quad (8)$$

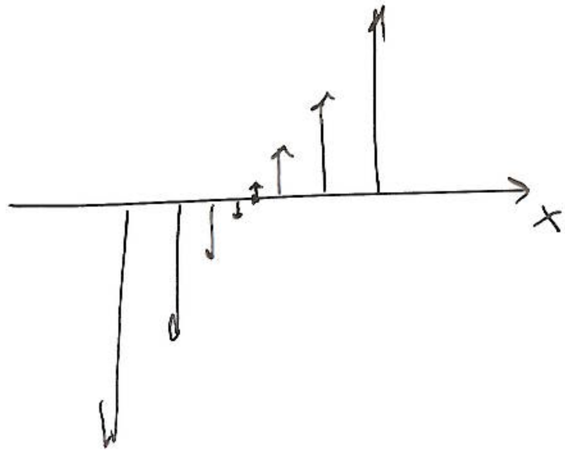
Equs (5)–(8) are the $K\epsilon$ model.
These are 5 adjustable constants:

$c_\mu = 0.09$ $\sigma_K = 1$ $\sigma_\epsilon = 1.3$ $c_1 = 1.44$ $c_2 = 1.92$

are the values that engineers like.
This tends to work better than it should and gives the flavour of this sort of closures.

c. A Simple Example: Homogeneous Shear Flow.

The simplest mean flow there is:



$$\vec{U} = Sx\,\hat{y}$$

$$\hat{S} = \frac{1}{2}\begin{pmatrix} 0 & S & 0 \\ S & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{S}:\hat{S} = \frac{1}{2} S^2$$

Energy equation: since turbulence should be homogeneous,

$$\partial_t K = -R_{xy} S - \epsilon \underset{\substack{\uparrow \\ K\epsilon\text{ model}}}{=} \nu_T S^2 - \epsilon = C_\mu \frac{K^2 S^2}{\epsilon} - \epsilon$$

$$\partial_t \epsilon = C_1 \nu_T S^2 \frac{\epsilon}{K} - C_2 \frac{\epsilon^2}{K} = C_1 C_\mu K S^2 - C_2 \frac{\epsilon^2}{K}$$

$$\partial_t \frac{K}{\epsilon} = \frac{1}{\epsilon}\partial_t K - \frac{K}{\epsilon^2}\partial_t \epsilon = C_\mu \frac{K^2 S^2}{\epsilon^2} - 1 - C_1 C_\mu \frac{K^2}{\epsilon^2} S^2 + C_2$$

$$= (C_2 - 1) - C_\mu (C_1 - 1)\frac{K^2 S^2}{\epsilon^2}$$

So we can have a self-similar solution with

$$\frac{K}{\epsilon} S = \sqrt{\frac{C_2 - 1}{C_\mu (C_1 - 1)}}$$

Generally, one finds for this type of flow that

$$\frac{KS}{\epsilon} \sim \text{const}, \qquad \frac{R_{xy}}{K} \sim \text{const} \quad \text{and} \quad \frac{R_{xy}S}{\epsilon} \sim \text{const}$$

Since $R_{xy} = -\nu_T S = -c_\mu \frac{K^2 S}{\epsilon}$ for our closure, this is fine.

P. Davidson quotes the following values from wind-tunnel measurements:

		$K\epsilon$ model
$\frac{KS}{\epsilon}$	6.3	4.8
$-\frac{R_{xy}}{K}$	0.28	0.43
$-\frac{R_{xy}S}{\epsilon}$	1.7	2.1

... so it's not as far <u>off</u> as it might have been...

... But I would not, if I were you, trust closure models quantitatively.