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§2. Turbulent Mixing

A. Turbulent Diffusion

Consider the motion of a particle in a turbulent fluid. The particle's coordinate (distance from the initial point of release) satisfies

$$\frac{d}{dt}\vec{x}(t) = \vec{u}(t, \vec{x}(t)) \equiv \vec{u}_L(t)$$

Lagrangian velocity

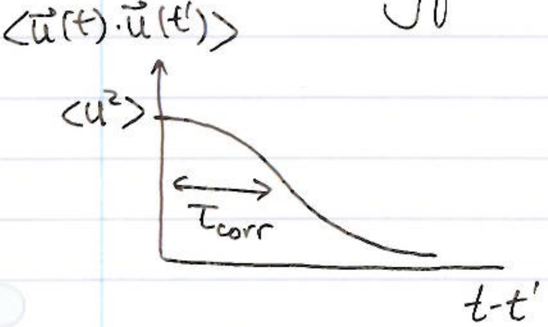
turbulent velocity field

Then $\frac{d}{dt}\langle x^2\rangle = 2\langle \vec{x}\cdot\vec{u}_L\rangle = 2\langle \vec{u}_L(t)\cdot\int_0^t dt'\,\vec{u}_L(t')\rangle$

ensemble average over many particles

$$= 2\int_0^t dt'\,\langle \vec{u}_L(t)\cdot\vec{u}_L(t')\rangle$$

Lagrangian time-correlation function of the velocity.



For $t \gg \tau_{corr}$, the integral is independent of time and

$$\int_0^\infty dt'\,\langle \vec{u}_L(t)\cdot\vec{u}_L(t')\rangle = D \sim U^2\tau_{corr}$$

$$\frac{d}{dt}\langle x^2\rangle = 2D \Rightarrow \langle x^2\rangle = 2Dt \qquad \text{diffusion}$$

Intuitively, $\tau_{corr} \sim \frac{L}{U}$ turnover of the largest eddy

so $D \sim UL$

(similar to molecular diffusion $D \sim v_{th}\cdot\lambda_{mfp}$)

Consider a tracer field (dye concentration, temperature...)

$$\frac{\partial \theta}{\partial t} + \vec{u}\cdot\nabla\theta = \varkappa \nabla^2 \theta$$

molecular diffusion.

If we ignore molecular diffusion and average this equation, we get

$$\frac{\partial}{\partial t}\langle\theta\rangle = -\nabla\cdot\langle\vec{u}\,\theta\rangle =$$

$$= \nabla\cdot\left\langle \vec{u}(t)\int_0^t dt'\,\vec{u}(t')\cdot\nabla\theta(t')\right\rangle$$

Nonrigorously, averages can be split because $\theta(t')$ can only depend on $\vec{u}$ at times before t'

$$= \nabla\cdot\int_0^t dt'\,\underbrace{\langle\vec{u}(t)\,\vec{u}(t')\rangle}_{=\,C_{ij}(t-t')}\cdot\nabla\langle\theta(t')\rangle$$

correlation tensor, only depends on $t-t'$ because of stationarity

$$= \nabla\cdot\int_0^t d\tau\,\underbrace{C_{ij}(\tau)}\cdot\nabla\underbrace{\langle\theta(t-\tau)\rangle}_{=\,\langle\theta(t)\rangle - \tau\frac{\partial}{\partial t}\langle\theta(t)\rangle + \dots} \approx$$

$$C_{ij}(\tau) = \frac{1}{3}\delta_{ij}\,C(\tau)$$

(isotropy)

$$\langle\theta(t)\rangle - \underbrace{\tau\frac{\partial}{\partial t}\langle\theta(t)\rangle}_{} + \dots$$

small (to be checked) as $t \gg \tau_{corr}$

As $t \gg \tau_{corr}$, $\int_0^t d\tau\, C(\tau) \to D = \text{const}$ (also space-independent because of homogeneity)

So we have

$$\boxed{\frac{\partial}{\partial t}\langle\theta\rangle = \frac{1}{3} D\,\nabla^2\langle\theta\rangle}$$

NB: $$\frac{\tau\frac{\partial}{\partial t}\langle\theta\rangle}{\langle\theta\rangle} \sim \frac{\tau_{corr} D}{L_\theta^2} \sim \frac{\frac{L}{U}\cdot UL}{L_\theta^2} \sim \frac{L^2}{L_\theta^2} \ll 1$$

Thus, all this works for the large-scale variation of the scalar field (>> scale of turbulence).

B) Particle Dispersion.

Now consider two ~~[illegible]~~ particles.
If one particle told us ~~about~~ how a spot of dye migrates in a turbulent fluid, two particles will tell us how it spreads.

Distance between particles: $\ell(t)$.

Now, roughly speaking,

$$\frac{d\,\ell(t)}{dt} \sim \delta u_{\ell(t)}$$ – velocity difference between two particles.

1) $\ell < \ell_\nu \Rightarrow \delta u_\ell \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} \ell$, so

$$\boxed{\ell(t) \sim \ell_0\, e^{(\epsilon/\nu)^{1/2} t}}$$

exponential separation (this is highly nonrigorous demonstration, but one can show, for smooth random velocity fields that trajectories do separate exponentially)

2) $l_\nu < l < L$ $\mapsto$ inertial range scales,
$\delta u_l \sim (\epsilon l)^{1/3}$

$$\frac{dl}{dt} \sim (\epsilon l)^{1/3}$$

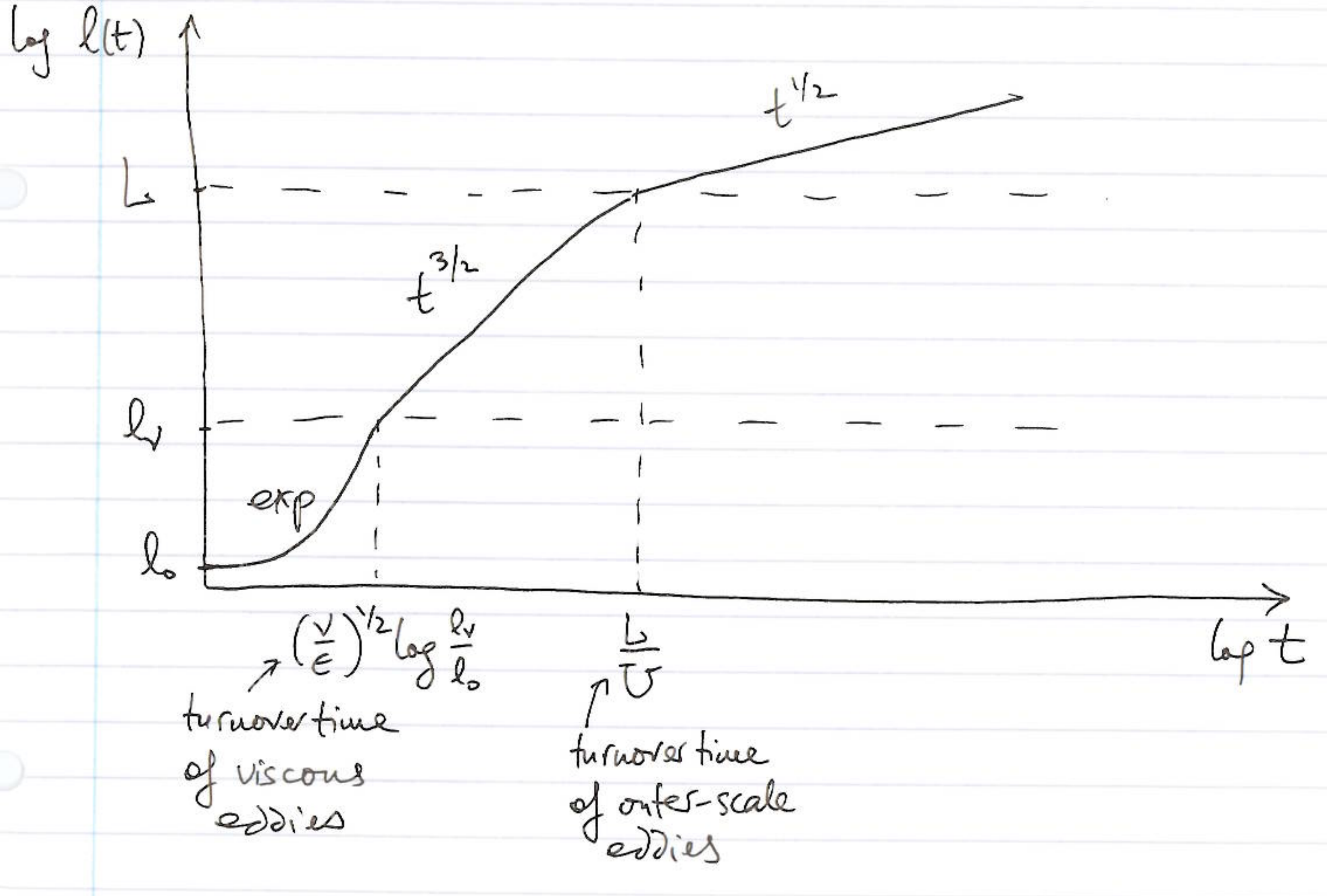
$$\frac{dl}{l^{1/3}} \sim \epsilon^{1/3} dt \mapsto l^{2/3} \sim \epsilon^{1/3} t$$

$$\boxed{l(t) \sim \sqrt{\epsilon}\, t^{3/2}}$$

- secular separation / Richardson law

3) $l > L$ - this is just like turbulent diffusion,

so $\boxed{l(t) \sim (Dt)^{1/2} \sim (ULt)^{1/2}}$



C. Spatial Structure of a Passive Scalar Field.

Let us go back to the advection-diffusion equation and ask what is the structure of the passive-scalar fluctuations.

$$\frac{\partial\theta}{\partial t} + \vec{u}\cdot\nabla\theta = \varkappa\nabla^2\theta + S$$

I have now added a source term, which I will assume to supply the scalar in a statistically stationary fashion and with a typical spatial scale L_θ.

[redacted] We now construct a theory of "scalar turbulence" similar in spirit to K41.

Conservation of scalar variance:

$$\frac{\partial}{\partial t}\frac{\langle\theta^2\rangle}{2} = \underbrace{\langle\theta S\rangle}_{\equiv\,\epsilon_\theta} - \varkappa\langle|\nabla\theta|^2\rangle = 0$$

again, this is a large scale quantity, indep. of $\varkappa$ [flux of variance into the system

real dissipation of [redacted] variance due to molecular diffusivity

stationary state

So by the same argument as for turbulence, the scalar field must develop small scales in order to dissipate.

NB: Turbulent diffusion is not real diffusion, it is simply the transfer of variance from large scales ($\gg L$) to small ones ($\lesssim L$).

At what scale does the molecular diffusion become important?

"turbulent diffusion" at scale l

$$\frac{\vec{u}\cdot\nabla\theta}{\varkappa\nabla^2\theta} \sim \frac{(\delta u_l/l)\delta\theta_l}{\varkappa\delta\theta_l/l^2} \sim \frac{\delta u_l\, l}{\varkappa} \sim \frac{\epsilon^{1/3} l^{4/3}}{\varkappa} \sim 1$$

for $l \sim l_\varkappa$

mol. diffusion

We may assume that scale fluctuations of size l are dominantly affected by ~~veloc~~ motions of scale l (locality again, for similar reasons)

$$l_\varkappa \sim \left(\frac{\varkappa^3}{\epsilon}\right)^{1/4} \sim \left(\frac{\varkappa}{\nu}\right)^{3/4} l_\nu \quad \text{diffusive scale}$$

visc. scale

Let $Sc = \frac{\nu}{\varkappa}$ Schmidt number

so $l_\varkappa \sim Sc^{-3/4}\, l_\nu$

Note that we used the inertial range scalings for velocity, so this is all valid only if $l_\varkappa \gg l_\nu$, i.e. $\underline{\underline{Sc \ll 1}}$

(will discuss the opposite limit shortly).

Note: If L_θ is the scale of the source, we have

$$l_\varkappa \sim \frac{\varkappa^{3/4}}{\epsilon^{1/4}} = \frac{\varkappa^{3/4}}{\left(\frac{\delta u_{L_\theta}^3}{L_\theta}\right)^{1/4}} = \left(\frac{\varkappa}{\delta u_{L_\theta} L_\theta}\right)^{3/4} L_\theta = Pe^{-3/4} L_\theta$$

where $Pe = \frac{\delta u_{L_\theta} L_\theta}{\varkappa}$ Péclet number

(analog of Re)

So, what is the scaling of $\delta\theta_\ell$ for $L_\theta \gg \ell \gg \ell_\varkappa$?
(this is called <u>inertial-convective range</u>)
Like in K41, we assume locality, isotropy, homogeneity...
Then the flux of scalar variance is

$$\epsilon_\theta \sim \frac{\delta\theta_\ell^2}{\tau_\ell} \sim \text{const}$$

cascade time the same as for turbulence:

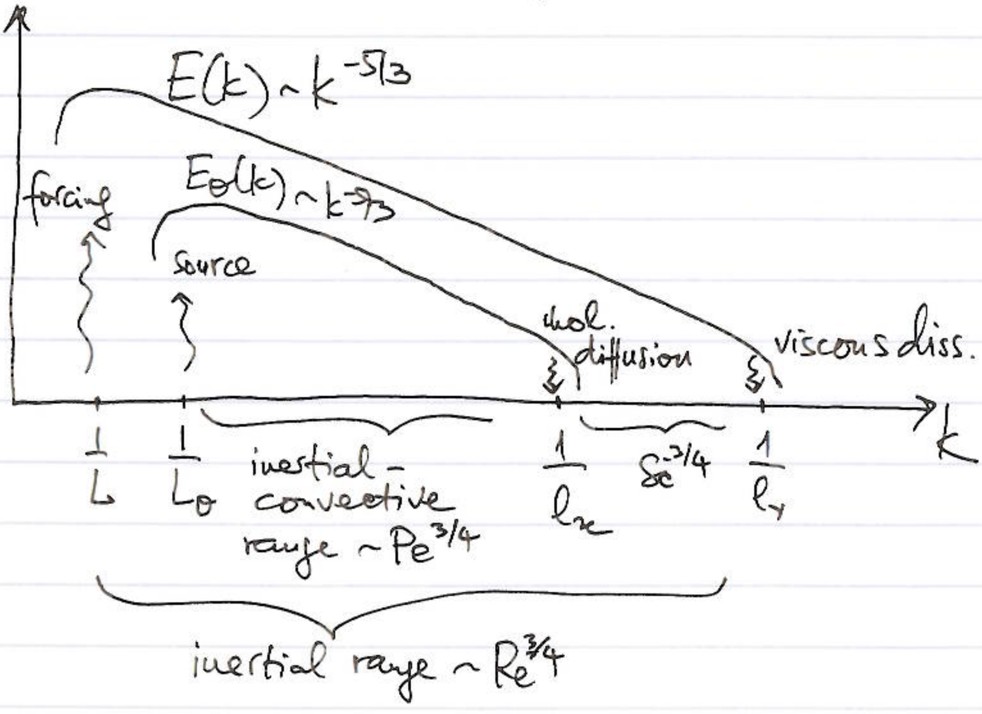
$$\frac{\delta u_\ell^2}{\tau_\ell} \sim \epsilon \Rightarrow \tau_\ell \sim \frac{\delta u_\ell^2}{\epsilon}$$, so we get

$$\boxed{\delta\theta_\ell \sim \left(\frac{\epsilon_\theta}{\epsilon}\right)^{1/2} \delta u_\ell \sim \frac{\epsilon_\theta^{1/2}}{\epsilon^{1/6}} \ell^{1/3}}$$

So the scalar traces the turbulence spectrum,

$$E_\theta(k) \sim \frac{\epsilon_\theta}{\epsilon^{1/3}} k^{-5/3}$$ <u>Obukhov-Corrsin spectrum</u>

$\boxed{Sc \ll 1}$:



Now let us consider the case $\underline{\underline{Sc \gg 1}}$.

Then $\varkappa \ll \nu$ and we expect that $l_\varkappa \ll l_\nu$.

To find it, again compare advective and diffusive terms, but this time use the viscous scaling for the velocity field:

$$\frac{\vec{u}\cdot\nabla\theta}{\varkappa\nabla^2\theta} \sim \frac{(\delta u_l/l)\,\delta\theta_l}{\varkappa\,\delta\theta_l/l^2} \sim \frac{\delta u_l\, l}{\varkappa} \sim \frac{(\epsilon/\nu)^{1/2} l^2}{\varkappa} \sim 1$$

for $l \sim l_\varkappa$

$$l_\varkappa \sim \varkappa^{1/2}\left(\frac{\nu}{\epsilon}\right)^{1/4} \sim \left(\frac{\varkappa}{\nu}\right)^{1/2}\frac{\nu^{3/4}}{\epsilon^{1/4}} \sim Sc^{-\frac{1}{2}}\, l_\nu \ll 1$$

for $Sc \gg 1$

We have a new range of scales: $l_\nu \gg l \gg l_\varkappa$ (called <u>viscous-convective range</u>).

The usual constant-flux argument gives

$$\epsilon_\theta \sim \frac{\delta\theta_l^2}{\tau_l} \sim \frac{\delta\theta_l^2}{(\nu/\epsilon)^{1/2}} \sim \text{const}$$

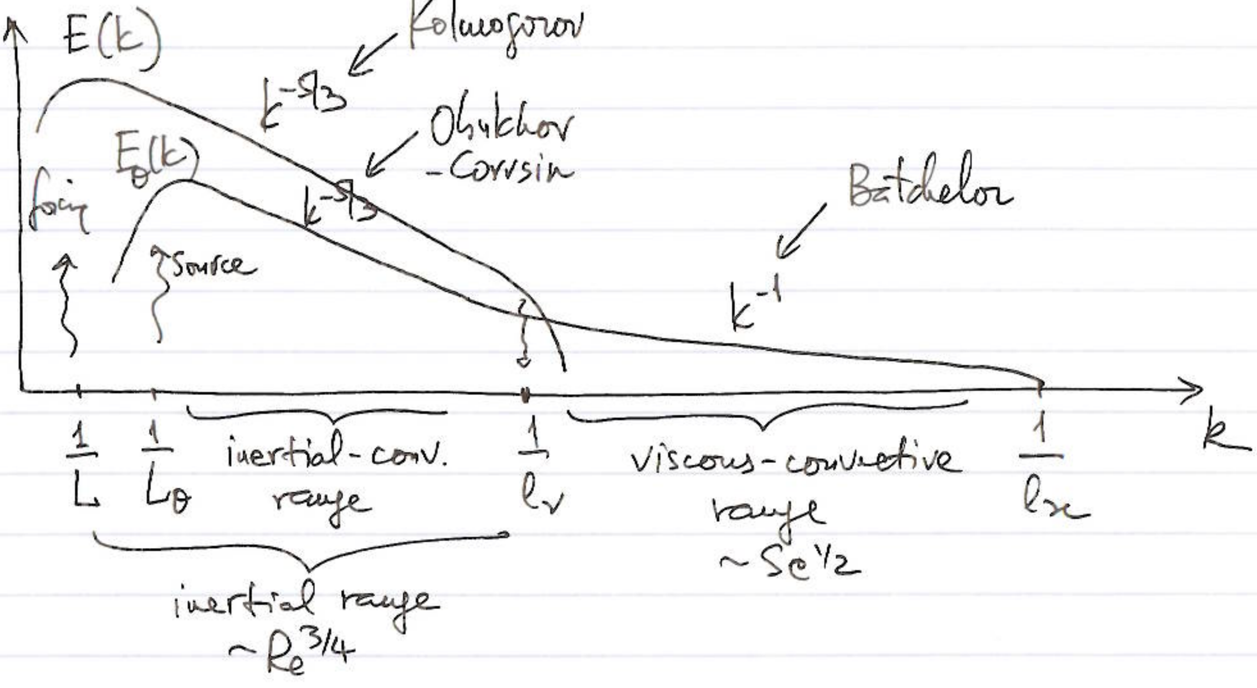
← the viscous eddy time

$$\delta\theta_l \sim \epsilon_\theta^{1/2}\left(\frac{\nu}{\epsilon}\right)^{1/4} \sim \text{const}$$

implying $\quad E_\theta(k) \sim \epsilon_\theta\left(\frac{\nu}{\epsilon}\right)^{1/2} k^{-1}\quad$ <u>Batchelor spectrum</u>

Note that in order for this argument to work, we have to <u>assume</u> constant flux (locality does not really work here because scalar fluctuations in the visc-conv. range are all interacting with the velocity field at the visc. scale — a "large" scale from their point of view)

$\boxed{Sc \gg 1}$



<u>Note</u>. Batchelor theory also describes the situation in flows that are chaotic / random, but not turbulent, i.e. there is no inertial range ($Re \sim 1$). Again, there is only one characteristic time scale $\sim L/U$, which is the mixing time for the scalar fluctuations, which should develop a k^{-1} spectrum. (same story in 2D turbulence, where the energy cascade is inverse and the forcing-scale motions dominate, as far as mixing is concerned)

D. Decay of Scalar Fluctuations.

Let us now consider a situation where there is no source and we want to know how quickly the scalar variance will decay

$$\langle \delta\theta^2 \rangle \to 0$$

Until quite recently, it was believed that the answer to this question is local in the sense that it is controlled by the small-scale local properties of the flow. The argument went roughly as follows.

For any initial ~~configuration~~ distribution of the scalar, small-scale structure quickly forms and then is dissipated by diffusion. The mixing rate must surely be determined by the fastest timescale of the flow, so, in the case of turbulence, by the turnover rate of the viscous eddies

$$\gamma \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} \sim \frac{U}{L} Re^{1/2}$$

and in the case of a single-scale chaotic/random flow, by the stretching/shearing rate associated with it, so

$$\gamma \sim \frac{U}{L}$$

In either case, trajectories separate exponentially, so one expects something like

$$\langle \delta\theta^2 \rangle \sim \langle \delta\theta_0^2 \rangle e^{-2\gamma t}$$

up to ~~a~~ some const of order unity.

This is called Lagrangian stretching theories.
They can be made quite mathematical — I will not go into details.
Two standard references are

Antonsen et al. Phys. Fluids 8, 3094 (1996)
Balkovsky & Fouxon Phys. Rev. E 60, 4164 (1999)
[review: Falkovich et al. Rev. Mod. Phys. 73, 913 (2001)]

However, experimental and numerical evidence emerged that suggested that in fact the decay is much slower than predicted by these theories and seems to be controlled by the size of the system — even though the scalar structure is very small-scale.
This was called the strange mode (by Pierrehumbert).
It is, in fact, not strange at all.
Consider what happens with a scalar field in a box such that $L_{box} \gg L_{flow}$.
The large-scale ($L_{box} \gtrsim \ell \gg L_{flow}$) component will be decaying by turbulent diffusion.
The slowest of all modes (Fourier modes, say) is the one with the smallest wave number:

$$\partial_t \theta_{k_{box}} = -\frac{1}{3} D k_{box}^2 \theta_{k_{box}} \qquad k_{box} \sim L_{box}^{-1}$$

All others decay faster.

The modes at scales comparable to or smaller than L_{flow} decay even faster ($\sim \frac{\delta u_\ell}{\ell}$).

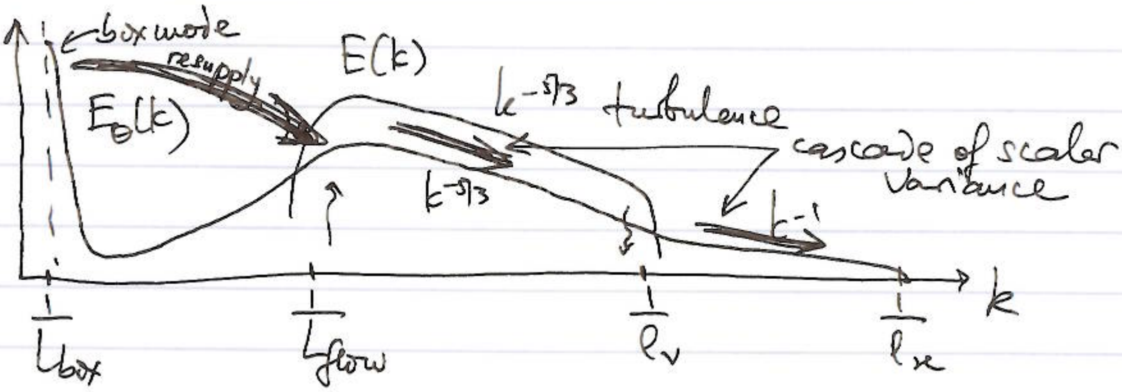
However, the slowly-decaying mode that is sitting at the box scale, will resupply scalar variance to smaller scales — that is indeed what turbulent diffusion is!

It is similar to having a scalar field with a very slowly decaying mean:

$$\theta = \langle\theta\rangle + \delta\theta \,, \text{ so}$$

$$\partial_t \delta\theta + \vec{u}\cdot\nabla\delta\theta = \varkappa\nabla^2\delta\theta \underbrace{- \vec{u}\cdot\nabla\langle\theta\rangle}_{\text{just like a source term.}}$$

So the scalar field at $\ell \lesssim L_{flow}$ is similar to the case with a source:



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Schekochihin, Haynes & Cowley Phys. Rev. E 70, 046304 (2004)
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