5 LECTURES ON TURBULENCE

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Cambridge 25–29 June 2007

§1. Introduction to turbulence. Kolmogorov's 1941 theory

Motion of incompressible fluid: Navier-Stokes Eqn

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \nu\nabla^2\vec{u} + \vec{f}}$$

external forcing.

pressure determined from incompressibility

viscosity (from material properties of the fluid)

Set $\rho = 1$ (incompressible in practice, $u \ll c_s$ speed of sound)

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0 \quad \underset{\rho = \text{const}}{\Rightarrow} \quad \nabla\cdot\vec{u} = 0$$

External forcing here is a stand in for whatever makes the fluid move: {energy injection}. This can be more complicated than a simple body force: energy can come from background gradients, shear etc. (i.e. converted from some external field: e.g. gravity) or from mechanical movement of parts of the system wrt fluid communicated to the latter via boundary conditions.

I will not go into the discussion of all these particular energy injection mechanisms. Instead, I will try to take a physicist's view and try to see what all such situations have in common.

The parameters of the system are:

- characteristic velocity, U
- characteristic scale, L

} these are det'd by the external driving

- viscosity ν — molecular properties of the fluid

There is a single dimensionless number one can cook up:

$$Re = \frac{UL}{\nu} \quad \text{Reynolds \#.}$$

• When $Re \lesssim 1$, we have a viscous flow, regular motion on system scales (except possibly boundary layers etc.)

Note that $Re = \dfrac{U/L}{\nu/L^2} = \dfrac{\text{fluid motion rate}}{\text{visc. diss. rate}}$

$$\sim \frac{|\vec{u}\cdot\nabla\vec{u}|}{|\nu\nabla^2\vec{u}|} = \frac{\text{nonlinear term}}{\text{visc. term.}}$$

- when Re is small, everything is linear.

• When $Re > Re_c$ ($\leftarrow$ depends on the system), the flow becomes destabilized and chaotic. There is a fairly complex process of transition to chaos (period doubling, strange attractors).

Instead of studying what happens near the transition point, we jump right away to the case of

$$Re \gg Re_c \quad (\gg 1)$$

— which is of most physical interest.

This is the regime of <u>fully developed turbulence</u>.

In this regime, $\vec{u}$ is very irregular in space and time - fluctuating at each point around its mean value $\vec{U}$ and varying rapidly in space.

So, $\vec{u}(t,\vec{x}) = \underbrace{\vec{U}}_{\text{mean}} + \underbrace{\delta\vec{u}}_{\text{fluctuating}}$

[This is the property Leonardo noticed]

The energy goes into fluctuations at the system scale L — it is ~~energy~~ also known as the <u>outer scale</u> or <u>energy-containing scale</u>.

At this scale, $\delta u_L \sim \delta U$ (fluctuations are the same order as the <u>change</u> in the mean flow — U itself does not, in fact, matter because of Galilean invariance).

So define $\boxed{Re = \frac{\delta u_L L}{\nu}}$

What happens to the energy: $\mathcal{E} = \frac{1}{2}\int d^3x\, |\vec{u}|^2$?

$$\frac{d\mathcal{E}}{dt} = -\nu \underbrace{\int d^3x\, |\nabla\vec{u}|^2}_{\text{dissipation}} + \underbrace{\int d^3x\, \vec{u}\cdot\vec{f}}_{\text{energy injection}}$$

Let us consider a stationary (statistically) situation: say the running time averages of all quantities become independent of time. (mathematically, this can be reformulated in terms of statistical ensemble averages over many realisations with different initial conditions, forcing histories etc.).

Then

$$\frac{d}{dt}\langle \mathcal{E} \rangle = 0 = -\nu \int d^3x \langle |\nabla \vec{u}|^2 \rangle + \underbrace{\int d^3x \langle \vec{u}\cdot\vec{f} \rangle}_{= V\epsilon}$$

Volume; injected power per unit vol.

So,

$$\underbrace{\nu \frac{1}{V} \int d^3x \langle |\nabla \vec{u}|^2 \rangle}\ = \epsilon$$

we will often include volume averaging into our definition of averages (justified for homogeneous systems)

Can we have an independent estimate of the total power going into the system?
Since the driving mechanism is associated with the outer scale, we might argue that this should be independent of viscosity:

$$\frac{|\nu \nabla^2 \vec{u}|}{|\vec{u}\cdot\nabla\vec{u}|} \sim \frac{\nu\, \delta u_L / L^2}{\delta u_L^2 / L} \sim \frac{1}{Re} \ll 1$$

So, dimensionally, $\boxed{\epsilon \sim \frac{\delta u_L^3}{L}}$

But we had

$$\epsilon = \nu \frac{1}{V} \int d^3x \langle |\nabla \vec{u}|^2 \rangle$$

finite small must be large!

The only way that ϵ can be finite and independent of viscosity is if $\langle |\nabla \vec{u}|^2 \rangle$ is dominated by large gradients (small scales).

Dimensionally, we can work out the viscous scale – the ~~scale~~ scale def'd by ϵ and ν:

$$l_\nu \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \sim \left(\frac{\nu^3 L^4}{\delta u_L^3 L^3}\right)^{1/4} = L\,Re^{-3/4} \ll L$$

This is called the <u>inner scale</u> (or <u>Kolmogorov scale</u>)

So we now have the following picture

Scale

L — energy injected

l — energy transported

l_ν $(\sim L Re^{-3/4})$ — energy dissipated

Turbulence is multiscale disorder

$L \gg l \gg l_\nu$

<u>inertial range</u>

Our ability to make further progress hinges on the (philosophical) assumption that the physics of the inertial range are <u>universal</u>

Key insight: Richardson (1922) conjectured that the energy transfer is local in scale space, i.e. occurs via a cascade:

$$L \to \frac{L}{2} \to \frac{L}{4} \to \text{etc. to the visc. scale}$$

(you might think of motion at each scale going unstable and breaking up into smaller scales like the mean flow did at the outer scale).

First quantitative theory: Kolmogorov 1941

Assumptions:
0) Universality

symmetries restored in the inertial range:
1) homogeneity: no special pts
2) isotropy: no special directions
3) scale invariance: no special scales

4) Locality of interactions (Richardson)

Then

the energy flux in and out of each scale is ϵ (energy cannot pile up anywhere in the inertial range because no scales are special)

Dimensionally, energy flux through scale ℓ is

$$\frac{\delta u_\ell^2}{\tau_\ell} \sim \epsilon = \text{const (indep. of } \ell\text{)}$$

energy

cascade time

But, dimensionally, we can only cook one time scale out of l and δu_l:

$$\tau_l \sim \frac{l}{\delta u_l}$$

[Obukhov's version of K41]

(think of "eddies" with velocity $\sim \delta u_l$, size $\sim l$, turnover time $\sim l/\delta u_l$)

So, $\frac{\delta u_l^3}{l} \sim \epsilon \Rightarrow \boxed{\delta u_l \sim (\epsilon l)^{1/3}}$

Kolmogorov-Obukhov law

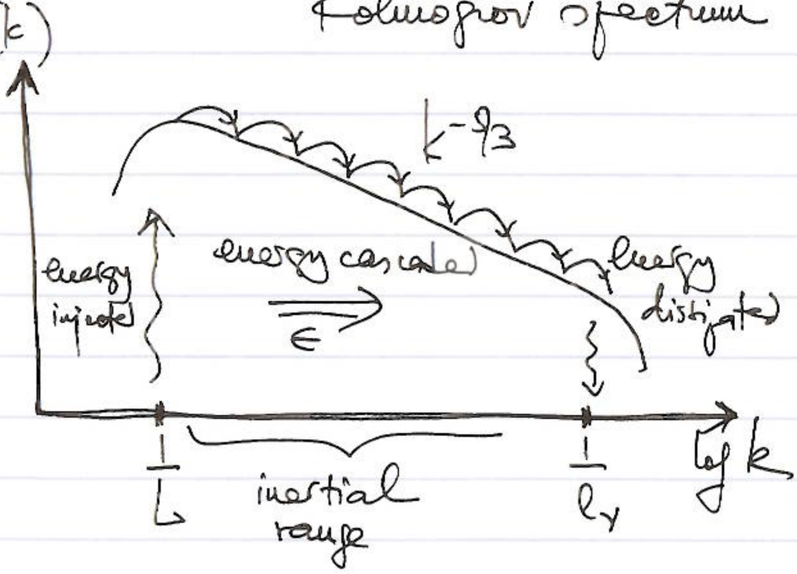
Spectral form of this law:

$E(k)\,dk$ - energy in $(k, k+dk)$

$$\delta u_l^2 \sim \int_{k=1/l}^{\infty} E(k)\,dk \sim \frac{\epsilon^{2/3}}{k^{2/3}} \Rightarrow \boxed{E(k) \sim \epsilon^{2/3} k^{-5/3}}$$

Kolmogorov spectrum

eddies of size $< l$ contribute to δu_l^2, those with size $> l$ do not because their velocity does not vary over l



- Energies: $\delta u_l^2 \sim \epsilon^{2/3} l^{2/3}$ dominated by large scales
- Gradients: $\frac{\delta u_l}{l} \sim \frac{\epsilon^{1/3}}{l^{2/3}}$ dominated by small scales (smaller eddies turn over faster)

Viscous cutoff: $\vec{u}\cdot\nabla\vec{u} \sim \nu\nabla^2\vec{u}$

$$\frac{\delta u_{l_\nu}^2}{l_\nu} \sim \nu\frac{\delta u_{l_\nu}}{l_\nu^2} \Rightarrow l_\nu \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$$ as estimated before.

What happens at $\ell \lesssim \ell_\nu$ (<u>dissipation range</u>)?

$$\epsilon \sim \nu |\nabla u|^2 \sim \nu \frac{\delta u_\ell^2}{\ell^2}$$

$$\delta u_\ell \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} \ell$$ — velocities are <u>smooth</u> (Taylor-expandable)
(unlike in the inertial range, where they are "rough")

<u>A note on locality of interactions</u>

We can now "confirm" (~~on~~ hand-wavingly) that the K41 results are consistent with the assumed locality of interactions.

Effect of motions at scale ℓ_1 on motions at scale ℓ_2:

$\ell_1 > \ell_2$: larger eddies sweep and shear the smaller eddies. Sweeping is uninteresting (galilean invariance), shearing happens at the rate

$$\sim \frac{u_{\ell_1}}{\ell_1} \sim \frac{\epsilon^{1/3}}{\ell_1^{2/3}}$$, so fastest shearing comes from eddies of similar size, $\ell_1 \sim \ell_2$

$\ell_1 < \ell_2$: smaller eddies "diffuse" ~~smaller~~ larger ones (see future lectures on turbulent diffusion)

The rate of this diffusion is

$$\sim \frac{u_{\ell_1}\ell_1}{\ell_2^2} \sim \frac{\epsilon^{1/3}\ell_1^{4/3}}{\ell_2^2}$$, so largest diffusion comes from eddies of similar size, $\ell_1 \sim \ell_2$

Some reading suggestions on §1

1. Landau & Lifshitz, vol. 6 (Fluid Dynamics) §33
— perhaps the shortest and most lucid intro to turbulence.

2. Original K41 papers in English: Proc Roy Soc A 434, 15 (1991)

3. U. Frisch. Turbulence.
— the book best loved by theoretical physicists

4. P. Davidson. Turbulence.
— a recent book intended for an engineering audience.

Some history:

5. Yaglom, Ann. Rev. Fluid Mech. 26, 1 (1994)
— on Kolmogorov and his school
Moffatt, Ann. Rev. Fluid Mech. 34, 19 (2002)
— on Batchelor and turbulence in Cambridge.

There are many other standard references, including books by

Batchelor
Monin & Yaglom (2 volumes)
Pope

etc...