## §2. Instabilities at Long Wavelengths. ($k\rho_i \ll 1$)

KMHD is a wonderful set of equations for gaining physical insight into the dynamics of magnetized plasma, but don't try to solve them on the computer – they will blow up! The reason for this is physical, not numerical: once you have pressure anisotropies (which will develop inevitably, once the plasma starts to move and heat to flow), everything goes wildly unstable. There is an entire zoo of plasma instabilities feeding on pressure anisotropy and heat fluxes – we can't do them all, but the simplest and most ~~[illegible]~~ illuminating thing to do is to start with linear ~~[illegible]~~ analysis of the stability of the KMHD eqns and see what dangers lurk therein.

### 2.1 Firehose Instability

The simplest case to do is to consider ~~[illegible]~~ a situation where we have some solution of KMHD (low frequency, long wavelength) and we then allow perturbations of this solution such that

$$\omega \gg |\nabla\vec{u}|, \quad k_\parallel \ell \gg 1 \quad \text{AND} \quad k_\perp = 0$$

only parallel wave #'s allowed.

time scale of the macroscopic solution

spatial scale of the macroscopic solution

The restriction to $k_\perp = 0$ is of course completely artificial, but we will see that it is self-consistent and certainly the easiest case to treat!

i.e. such perturbations do not engender any with $k_\perp \neq 0$

Start with the momentum eqn (13):

$$m_i n_i \frac{d\vec{u}}{dt} = -\nabla\left(p_\perp + \frac{B^2}{8\pi}\right) + \nabla\cdot\left[\hat{b}\hat{b}\left(p_\perp - p_\parallel + \frac{B^2}{4\pi}\right)\right]$$

$$-i m_i n_i \omega\,\delta\vec{u} = -i\vec{k}\left(\delta p_\perp + \frac{B\delta B_\parallel}{4\pi}\right) + i\vec{k}\cdot\left[\delta\hat{b}\,\hat{b}\left(p_\perp - p_\parallel + \frac{B^2}{4\pi}\right) + \right.$$
$$\left. + \hat{b}\,\delta\hat{b}\left(p_\perp - p_\parallel + \frac{B^2}{4\pi}\right) + \hat{b}\hat{b}\left(\delta p_\perp - \delta p_\parallel + \frac{B\delta B_\parallel}{2\pi}\right)\right]$$

NB: $\nabla\cdot\hat{b} = -\frac{\hat{b}\cdot\nabla B}{B}$ so $i\vec{k}\cdot\delta\hat{b} \approx -ik_\parallel \frac{\delta B_\parallel}{B}$

$$= -i\vec{k}_\perp\left(\delta p_\perp + \frac{B\delta B_\parallel}{4\pi}\right) \leftarrow \text{zero because } \vec{k}_\perp = 0$$
$$-i\hat{b}\,k_\parallel\left[\cancel{\delta p_\perp} + \cancel{\frac{B\delta B_\parallel}{4\pi}} + \frac{\delta B_\parallel}{B}\left(p_\perp - p_\parallel + \cancel{\frac{B^2}{4\pi}}\right) - \cancel{\delta p_\perp} + \delta p_\parallel - \cancel{\frac{B\delta B_\parallel}{2\pi}}\right]$$
$$+ i\delta\hat{b}\,k_\parallel\left(p_\perp - p_\parallel + \frac{B^2}{4\pi}\right) \qquad (28)$$

Induction equation (11):

$$\frac{d\vec{B}}{dt} = \vec{B}\cdot\nabla\vec{u} - \vec{B}\,\nabla\cdot\vec{u}$$

$$-i\omega\,\delta\vec{B} = i\vec{k}\cdot\vec{B}\,\delta\vec{u} - \vec{B}\,i\vec{k}\cdot\delta\vec{u} =$$
$$= +ik_\parallel\,\delta\vec{u}_\perp B - i\vec{k}_\perp\cdot\delta\vec{u}_\perp\,\vec{B} \qquad (29)$$

↑ zero because $\vec{k}_\perp = 0$

Setting $\vec{k}_\perp = 0$, we find

$$\frac{\delta\vec{B}_\perp}{B} = \delta\hat{b} = -\frac{k_\parallel}{\omega}\,\delta\vec{u}_\perp \quad \text{and} \quad \frac{\delta B_\parallel}{B} = 0 \qquad (30)$$

Then, from (28),

$$\delta\vec{u}_\perp = -\frac{k_\parallel}{m_i n_i \omega}\left(p_\perp - p_\parallel + \frac{B^2}{4\pi}\right)\delta\hat{b} \qquad (31)$$

$$\delta u_\parallel = \frac{k_\parallel}{m_i n_i \omega}\,\delta p_\parallel \qquad (32)$$

We see that $\delta\hat{b}$ and $\delta\vec{u}_\perp$ are completely decoupled from anything else — these are "Alfvénic" perturbations and in this approximation they don't talk to the "compressive" ones. So,


$$\omega^2 = k_\parallel^2 \, \frac{p_\perp - p_\parallel + B^2/4\pi}{m_i n_i}$$

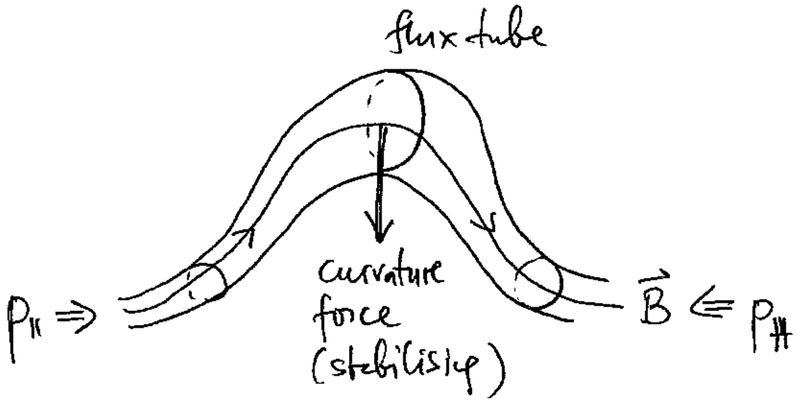
$$\boxed{\omega = \pm k_\parallel \sqrt{\frac{p_\perp - p_\parallel}{m_i n_i} + v_A^2}} \qquad (33)$$

Alfvén speed $v_A = \dfrac{B}{\sqrt{4\pi m_i n_i}}$

So, we have got Alfvén waves with phase speed modified by the presence of pressure anisotropy — makes sense since mathematically $p_\perp - p_\parallel$ looked like a modification of the tension force.

Note. Actually, I did not need to set $\vec{E}_\perp = 0$ to get this, I could have separated $\delta\vec{u}_\perp$ and $\delta\hat{b}$ anyway — Ex. Show this!

If $p_\perp - p_\parallel < 0$, the effect is to weaken the tension force.




If there's not enough tension, the field line cannot spring back $\Rightarrow$ instability if

$$\frac{p_\parallel - p_\perp}{m_i n_i} > v_A^2$$

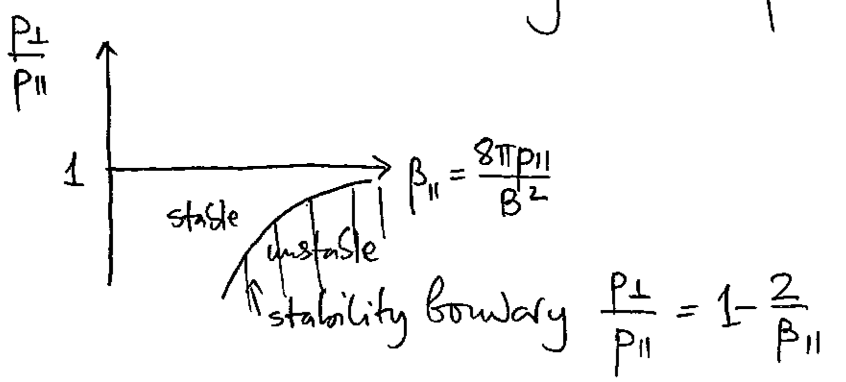
Equivalently, $$\boxed{\frac{p_\parallel - p_\perp}{p_{\parallel i}} > \frac{B^2}{4\pi p_{\parallel i}} = \frac{2}{\beta_{\parallel i}}} \qquad (34)$$

growth rate: $$\boxed{\gamma = |k_\parallel| \sqrt{\frac{p_\parallel - p_\perp}{m_i n_i} - v_A^2}} \qquad (35)$$

Rosenbluth 1956
Chandrasekhar et al. 1958
Parker 1958
Vedenov & Sagdeev 1958

UV catastrophe – within KMHD, no peak, $\gamma$ grows with $k_\parallel$. To find peak, we must include $k_\parallel \rho_i$ terms ("FLR") – this is quite a lot of work, I'll do it later!

Thus, negative pressure anisotropies give rise to a violent instability unless $\beta$ is low enough.




Measurements in the solar wind (Kasper et al. 2002, Hellinger et al. 2006, Matteini et al. 2007, Bale et al. 2009) show the wind is indeed loath to cross this boundary!

[for precise references here as elsewhere see my MNRAS-2010 paper v.405, 291]

## 2.2 Nonlinear (Quasilinear) Theory for the Firehose

So how do these firehose perturbations saturate and how does the pressure anisotropy manage to stay marginal?

Formally speaking, to construct any kind of nonlinear theory, we must include the FLR physics that sets the peak growth rate and the corresponding $k_\parallel$. We will not do this just yet but rather simply anticipate the result:

$$\text{for } k_\perp = 0 \quad \left\{ \begin{array}{l} \gamma_{peak} \sim \Omega_i \left| \Delta + \dfrac{2}{\beta} \right| \\ \text{and } k_{\parallel peak}\rho_i \sim \sqrt{\left| \Delta + \dfrac{2}{\beta} \right|} \end{array} \right. \qquad \Delta = \frac{p_\perp - p_\parallel}{p_{\parallel i}} < 0 \qquad \beta = \frac{p_{\parallel i}}{B^2/8\pi} \tag{36}$$

This means that

1) The stability parameter is correctly captured by the long-wavelength theory
2) Even though the perturbations may be fast growing and small-scale compared to the slow macroscopic solutions of KMHD that serves as "equilibrium", they are, at least when they are close to marginal stability, still sufficiently low-frequency and large-scale to conserve $\mu$ — which means that whatever the nonlinearity does to drive pressure anisotropy to marginal stability, it is __not__ going to be

an enhancement of particle scattering leading to higher effective collision rate and consequent relaxation of pressure anisotropy.

[I am raising this straw man because it is just such ~~an~~ natural — and often considered inevitable, obvious — assumption! ]

Ok, so what happens then?

Let us recall our eq. (23) for the pressure anisotropy and let us further assume that $p_\perp - p_\parallel \ll p_\perp, p_\parallel \approx p$ (which then means $|\Delta| \sim \frac{1}{\beta} \ll 1$, not an essential assumption, but one that will simplify our arguments)

We will also only be concerned with the <u>ion</u> nonlinearity because, as we argued on p. 13, the electron one will be smaller by $\sqrt{m_e/m_i}$.

$$\frac{d}{dt}(p_\perp - p_\parallel) = 3p\,\frac{1}{B}\frac{dB}{dt} - 2p\,\frac{1}{n}\frac{dn}{dt} - 3q_\perp \nabla\cdot\hat{b} - \nabla\cdot(\vec{q}_\perp - \vec{q}_\parallel) - 3\nu(p_\perp - p_\parallel) \tag{37}$$

Let us treat this equation as one that is <u>averaged</u> over small scales (i.e. scales at which the firehose perturbations are growing).

In fact, since $k_\perp = 0$ and the perturbations are purely Alfvénic ($\delta n = 0$, $\delta p_\perp = 0$, $\delta p_\parallel = 0$, $\delta B_\parallel = 0$, $\delta q_\perp = 0$, $\delta q_\parallel = 0$) the only term in (37) that can be affected by them is the magnetic-field one.

Namely, consider $\vec{B} = \underbrace{\vec{B}_0}_{\text{slowly decreasing background field}} + \underbrace{\delta\vec{B}_\perp}_{\text{fast growing but small firehose perturbation}}$

Then

$$\overline{\frac{1}{B}\frac{dB}{dt}} = \overline{\frac{1}{2}\frac{1}{B^2}\frac{d}{dt}B^2} \approx \overline{\frac{1}{2}\frac{1}{B_0^2}\frac{d}{dt}\left(B_0^2 + \underbrace{2\vec{B}_0\cdot\delta\vec{B}_\perp}_{0 \text{ under averaging}} + |\delta\vec{B}_\perp|^2\right)} =$$

$$\approx \underbrace{\frac{1}{B_0}\frac{dB_0}{dt}}_{\text{slowly decreasing field}} + \underbrace{\frac{1}{2}\frac{\partial}{\partial t}\frac{\overline{|\delta\vec{B}_\perp|^2}}{B_0^2}}_{\text{fast growing total energy of the firehose perturbations}} \tag{38}$$

Rigorous derivation: Rosin et al. 2011 MNRAS 413, 7 and refs therein

The general idea is already clear: fast growing small fluctuations can on average compensate for slowly decaying mean field and cancel the anisotropy! Let's do this a little more quantitatively:

$$\frac{d}{dt}(p_\perp - p_\parallel) = p\left(\underbrace{3\gamma_0}_{\text{this is all the slow anisotropy-driving terms in eq. (37)}} + \frac{3}{2}\frac{\partial}{\partial t}\frac{\overline{|\delta\vec{B}_\perp|^2}}{B_0^2}\right) - 3\nu(p_\perp - p_\parallel) \tag{39}$$

Solve this equation:

$$\frac{p_\perp - p_\parallel}{p} \equiv \Delta(t) = \underbrace{\Delta_0}_{\text{initial anisotropy}} e^{-3\nu t} + \int_0^t dt'\, e^{-3\nu(t-t')}\left[3\gamma_0(t') + \frac{3}{2}\frac{\partial}{\partial t'}\frac{\overline{|\delta\vec{B}_\perp|^2}}{B_0^2}\right] \tag{40}$$

What we do next depends on how we set up the problem we wish to solve.

D) Initial anisotropy, no drive

This means we assume that at some starting point there is a $\Delta_0$ and it is not replenished by anything, i.e. $\gamma_0 = 0$. This can only produce an interesting answer on time scales such that $\nu t \ll 1$ (collisionless plasma). Then

$$\Delta(t) = \underset{\substack{\uparrow \\ \text{negative}}}{\Delta_0} + \frac{3}{2}\underset{\substack{\uparrow \\ \text{positive}}}{\frac{|\delta \vec{B}_\perp(t)|^2}{B_0^2}} \to \underset{\substack{\uparrow \\ \text{marginal level}}}{-\frac{2}{\beta}} \tag{41}$$

Saturation :
$$\boxed{\frac{|\delta \vec{B}_\perp(t)|^2}{B_0^2} \to \frac{2}{3}\left|\Delta_0 + \frac{2}{\beta}\right|} \tag{42}$$

If we assume that $\Delta_0 + \frac{2}{\beta}$ was small, this gives a small saturated level, as it ought to be in order for various approximations that were needed to justify eq. (38) and (39) to hold.

Eq. (42) is the oldest and most standard result in this business: see Shapiro & Shevchenko 1964 (classic paper!)

This set up does not, however, represent real situations very well – in fact, in reality, pressure anisotropy is usually driven by the underlying macroscopic dynamics: e.g. solar wind keeps expanding!

Note: Most numerical literature also treats this case except recently Matteini et al. 2006

2) Driven anisotropy, collisional

Let us now ignore $\Delta_0$ and allow for the drive $\gamma_0$, which changes slowly compared to the coll. rate.
Then eq. (40) becomes

$$\Delta(t) \approx \frac{\gamma_0}{\nu} + \int_0^t dt' \, e^{-3\nu(t-t')} \, \frac{3}{2} \frac{\partial}{\partial t'} \frac{|\delta \vec{B}_\perp|^2}{B_0^2} \to -\frac{2}{\beta} \quad (43)$$

driven stationary anisotropy, as ~~given~~ in eq. (24) <u>negative</u>!

positive

marginal

$$\int_0^t dt' \, e^{-3\nu(t-t')} \frac{3}{2} \frac{\partial}{\partial t'} \frac{|\delta \vec{B}_\perp|^2}{B_0^2} \to \left| \frac{\gamma_0}{\nu} + \frac{2}{\beta} \right|$$

Solution:
$$\boxed{\frac{|\delta \vec{B}_\perp|^2}{B_0} \approx 2 \left| \gamma_0 + \frac{2}{\beta} \nu \right| t} \quad (44)$$

Secularly growing fluctuation level, which gives a constant $\frac{\partial}{\partial t} \frac{1}{2} \frac{|\delta \vec{B}_\perp|^2}{B_0} = \left| \gamma_0 + \frac{2}{\beta} \nu \right|$

This cancels most of the driven anisotropy and keeps the total anisotropy <u>marginal</u>.
Obviously, this solution only works for a time:

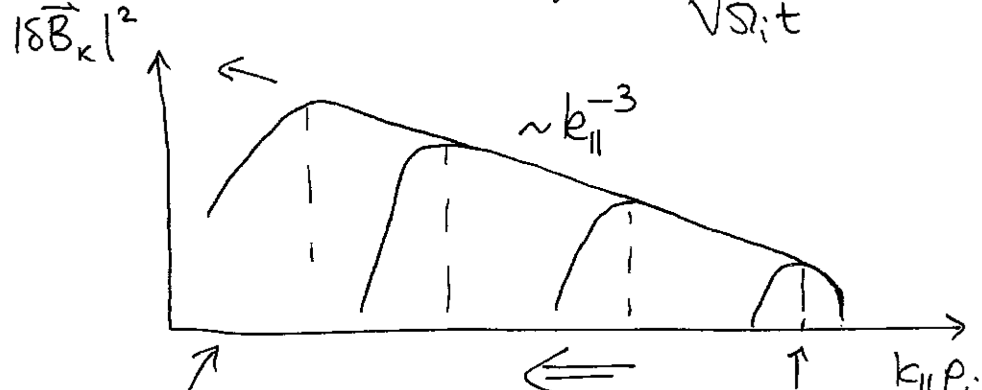
$$\left| \gamma_0 + \frac{2}{\beta} \nu \right| t \ll 1 \quad \text{(fluctuations stay small)}$$

We do not know how to handle the situation where $\frac{|\delta \vec{B}_\perp|^2}{B_0^2} \sim 1$.

Note that it is possible to show that during this secular regime,

$$\left|\Delta(t) + \frac{2}{\beta}\right| \sim \frac{1}{\Omega_i t} \to 0$$

$$\text{and} \quad k_{\parallel \text{peak}}\rho_i \sim \frac{1}{\sqrt{\Omega_i t}} \to 0 \qquad (45)$$




"Spectrum of firehose turbulence"

not clear what happens when this hits large scales (same time as $\frac{\delta B_\perp}{B_0} \sim 1$)

initial $\sim \sqrt{|\Delta + \frac{2}{\beta}|}$

This appears to be consistent with numerics [Quest & Shapiro '96, Matteini et al. 2006]

3) Driven anisotropy, collisionless

If you don't believe in collisions at all, set $\nu = 0$ in eq. (40):

$$\Delta(t) = \underbrace{\Delta_0 + 3\int_0^t dt' \gamma_0(t')}_{\text{negative}} + \underbrace{\frac{3}{2}\frac{|\delta\vec{B}_\perp(t)|^2}{B_0^2}}_{\text{positive}} \to \underbrace{-\frac{2}{\beta}}_{\text{marginal}} \qquad (46)$$

Then

$$\boxed{\frac{|\delta\vec{B}_\perp(t)|^2}{B_0^2} \approx \frac{2}{3}\left|\Delta_0 + 3\int_0^t dt'\gamma_0(t') + \frac{2}{\beta}\right|} \qquad (47)$$

secular growth again

So a very similar situation to the collisional case.

## 2.3 Mirror Instability

Ok, we are back to linear theory and now we want to look at what happens to compressive perturbations:

$\delta B_\parallel, \delta u_\parallel, \ldots$

These will also turn out to be unstable. Unlike the firehose, which is basically a fluid phenomenon, the mirror instability is essentially kinetic, involving particle resonances — so plasma physics par excellence!

[one can get an instability out of CGL eqns with no heat fluxes - but even the threshold will be wrong! ]
Ex. Try that.

Recall the perturbed momentum and induction eqns:

$$m_i n_i \omega \delta\vec{u} = \vec{k}_\perp \left(\delta p_\perp + \frac{B\delta B_\parallel}{4\pi}\right) + \hat{b} k_\parallel \left[\delta p_\parallel + (p_\perp - p_\parallel)\frac{\delta B_\parallel}{B}\right] - \delta\hat{b}\, k_\parallel \left(p_\perp - p_\parallel + \frac{B^2}{4\pi}\right) \quad (28)$$

$$\omega \frac{\delta\vec{B}}{B} = -k_\parallel \delta\vec{u}_\perp + \hat{b}\, \vec{k}_\perp \cdot \delta\vec{u}_\perp \quad (29)$$

We are interested in $\delta B_\parallel$ (having worked out $\delta\vec{B}_\perp$ before)

$$\omega \frac{\delta B_\parallel}{B} = \vec{k}_\perp \cdot \delta\vec{u}_\perp \leftarrow \text{get this from (28):}$$

$$m_i n_i \omega\, \delta\vec{u}_\perp \cdot \vec{k}_\perp = k_\perp^2 \left(\delta p_\perp + \frac{B\delta B_\parallel}{4\pi}\right) \underbrace{- \vec{k}_\perp \cdot \delta\hat{b}\, k_\parallel}_{= +k_\parallel^2 \frac{\delta B_\parallel}{B}} \left(p_\perp - p_\parallel + \frac{B^2}{4\pi}\right)$$

because $\nabla \cdot \vec{B} = 0$

$$\boxed{\begin{aligned} m_i n_i \omega^2 \frac{\delta B_\parallel}{B} &= k_\perp^2 \left(\delta p_\perp + \frac{B\delta B_\parallel}{4\pi}\right) + k_\parallel^2 \left(p_\perp - p_\parallel + \frac{B^2}{4\pi}\right)\frac{\delta B_\parallel}{B} \quad (48) \\ \text{and} \quad m_i n_i \omega \delta u_\parallel &= k_\parallel \left[\delta p_\parallel + (p_\perp - p_\parallel)\frac{\delta B_\parallel}{B}\right] \quad (49) \end{aligned}}$$

To get $\delta p_\perp$ and $\delta p_\parallel$, perturb the kinetic equation (16)
[ignore collisions: $\nu \ll \omega$]

$$\underbrace{-i\omega\delta f_s + i k_\parallel w_\parallel \delta f_s}_{= -i(\omega - k_\parallel w_\parallel)\delta f_s} + i\underbrace{\vec{k}_\perp\cdot\delta\hat{b}}_{= -k_\parallel \frac{\delta B_\parallel}{B}}\,\frac{w_\perp^2}{2}\frac{\partial f_s}{\partial w_\parallel} + \left(\frac{e_s}{m_s}\delta E_\parallel + \underline{i\omega\delta u_\parallel}\right)\left(\frac{w_\parallel}{w}\frac{\partial f}{\partial w} + \frac{\partial f}{\partial w_\parallel}\right)$$

$$- i\underbrace{\vec{k}\cdot\delta\vec{u}}_{= k_\parallel\delta u_\parallel + \vec{k}_\perp\cdot\delta\vec{u}_\perp = \cancel{k_\parallel\delta u_\parallel} + \omega\frac{\delta B_\parallel}{B}}\,\frac{w_\perp^2}{2}\frac{1}{w}\frac{\partial f}{\partial w} + \underline{i k_\parallel\delta u_\parallel}\left[\underbrace{\left(\cancel{\frac{w_\perp^2}{2}} - w_\parallel^2\right)\frac{1}{w}\frac{\partial f}{\partial w} - w_\parallel\frac{\partial f}{\partial w_\parallel}}_{= -w_\parallel\left(\frac{w_\parallel}{w}\frac{\partial f}{\partial w} + \frac{\partial f}{\partial w_\parallel}\right)}\right] = 0$$

$$\delta f_s = \delta u_\parallel\left(\frac{w_\parallel}{w}\frac{\partial f_s}{\partial w} + \frac{\partial f_s}{\partial w_\parallel}\right) -$$

$$- \frac{1}{\omega - k_\parallel w_\parallel}\left[i\frac{e_s}{m_s}\delta E_\parallel\left(\frac{w_\parallel}{w}\frac{\partial f_s}{\partial w} + \frac{\partial f_s}{\partial w_\parallel}\right) + \right.$$

$$\left. + \frac{\delta B_\parallel}{B}\frac{w_\perp^2}{2}\left(k_\parallel\frac{\partial f_s}{\partial w_\parallel} + \underbrace{\omega}_{= \omega - k_\parallel w_\parallel + k_\parallel w_\parallel}\frac{1}{w}\frac{\partial f_s}{\partial w}\right)\right]$$

this bit does not contribute to $\delta p_\perp$, $\delta p_\parallel$ same way it didn't in the derivation of CGL eqns

$$= \cancel{\delta u_\parallel\left(\frac{w_\parallel}{w}\frac{\partial f_s}{\partial w} + \frac{\partial f_s}{\partial w_\parallel}\right)} - \frac{\delta B_\parallel}{B}\frac{w_\perp^2}{2}\frac{1}{w}\frac{\partial f_s}{\partial w} -$$

$$- \frac{1}{\omega - k_\parallel w_\parallel}\left(i\frac{e_s}{m_s}\delta E_\parallel + k_\parallel\frac{\delta B_\parallel}{B}\frac{w_\perp^2}{2}\right)\left(\frac{w_\parallel}{w}\frac{\partial f_s}{\partial w} + \frac{\partial f_s}{\partial w_\parallel}\right) \qquad (50)$$

$$\delta p_{\perp s} = \int d^3\vec{w}\, m_s\frac{w_\perp^2}{2}\delta f_s = -\frac{\delta B_\parallel}{B}\underbrace{\int d^3\vec{w}\, m_s\frac{w_\perp^4}{4}\frac{1}{w}\frac{\partial f_s}{\partial w}}_{= -\int d^3\vec{w}\, m_s(w^2 - w_\parallel^2) = -2p_{\perp s}} -$$

$$- \int\frac{d^3\vec{w}\, m_s}{\omega - k_\parallel w_\parallel}\frac{w_\perp^2}{2}\left(i\frac{e_s}{m_s}\delta E_\parallel + k_\parallel\frac{\delta B_\parallel}{B}\frac{w_\perp^2}{2}\right)\left(\frac{w_\parallel}{w}\frac{\partial f_s}{\partial w} + \frac{\partial f_s}{\partial w_\parallel}\right)$$

For the purposes of this calculation, it is convenient to change variables $(w, w_\parallel) \to (w_\perp, w_\parallel)$. Then

$$\frac{w_\parallel}{w}\left(\frac{\partial f_s}{\partial w}\right)_{w_\parallel} + \left(\frac{\partial f_s}{\partial w_\parallel}\right)_w = \left(\frac{\partial f_s}{\partial w_\parallel}\right)_{w_\perp} \qquad \underline{\underline{\text{Ex.}}}\ \text{Check this!}$$

So we can carry out all the $w_\perp$ integrations and be left just with $w_\parallel$ ones.

Let $F_{0s}(w_\parallel) = 2\pi \int_0^\infty dw_\perp w_\perp f_s(w_\perp, w_\parallel)$

$$F_{2s}(w_\parallel) = 2\pi \int_0^\infty dw_\perp w_\perp \frac{w_\perp^2}{2} f_s(w_\perp, w_\parallel) \Big/ \frac{p_{\perp s}}{m_s n_s}$$

$$F_{4s}(w_\parallel) = 2\pi \int_0^\infty dw_\perp w_\perp \frac{w_\perp^4}{4} f_s(w_\perp, w_\parallel) \Big/ \frac{2p_{\perp s}^2}{(m_s n_s)^2}$$

The normalisations have been chosen so that all three of these functions turn into a 1D maxwellian in the special case of a bi-Maxwellian distribution (nothing special about it but a convenient parametrisation of a typical background distribution)

$$f_s(w_\perp, w_\parallel) = \frac{n_s}{\pi^{3/2} v_{ths\perp}^2 v_{ths\parallel}^2}\, e^{-\frac{w_\perp^2}{v_{ths\perp}^2} - \frac{w_\parallel^2}{v_{ths\parallel}^2}}$$

just the $\parallel$ bit

where $v_{ths\perp}^2 = \dfrac{2T_{\perp s}}{m_s} = \dfrac{2p_{\perp s}}{n_s m_s}$, $\quad v_{ths\parallel}^2 = \dfrac{2p_{\parallel s}}{n_s m_s}$

So, we get

$$\delta p_{\perp s} = 2p_{\perp s}\frac{\delta B_\parallel}{B} - \int \frac{dw_\parallel}{\omega - k_\parallel w_\parallel}\left[ i\frac{e_s}{m_s}\delta E_\parallel \cancel{m_s} \cdot \frac{p_{\perp s}}{\cancel{m_s} n_s}\left(\frac{\partial F_{2s}}{\partial w_\parallel}\right) + k_\parallel \frac{\delta B_\parallel}{B}\cancel{m_s}\frac{2p_{\perp s}^2}{m_s^{\cancel{2}} n_s^2}\left(\frac{\partial F_{4s}}{\partial w_\parallel}\right)\right] \qquad (51)$$

Landau integrals:

$k_\parallel \to |k_\parallel|$ because if $k_\parallel < 0$, can change $w_\parallel \to -w_\parallel$

assump $F_{2s}(w_\parallel)$ to be even!

$$\int_{-\infty}^{+\infty} \frac{dw_\parallel}{\omega - k_\parallel w_\parallel}\,\frac{\partial F_{2s}}{\partial w_\parallel} = -\frac{1}{k_\parallel}\int \frac{dw_\parallel}{w_\parallel - \frac{\omega}{|k_\parallel|}}\,\frac{\partial F_{2s}}{\partial w_\parallel} =$$

~~poles~~ ~~regular functions~~

$$= -\frac{2}{k_\parallel}\int \frac{dw_\parallel}{w_\parallel - \frac{\omega}{|k_\parallel|}}\, w_\parallel\, \frac{\partial F_{2s}}{\partial w_\parallel^2} = -\frac{2}{k_\parallel}\int dw_\parallel\, \frac{\partial F_{2s}}{\partial w_\parallel^2} -$$

$+\frac{\omega}{|k_\parallel|} - \frac{\omega}{|k_\parallel|}$

this is handier because simply turns into $-\frac{1}{v_{ths\parallel}^2}$ if $f_s$ was a bi-Maxwellian

$$-\frac{2}{k_\parallel}\,\frac{\omega}{|k_\parallel|}\int \frac{dw_\parallel}{w_\parallel - \frac{\omega}{|k_\parallel|}}\,\frac{\partial F_{2s}}{\partial w_\parallel^2}$$ ← nice regular function

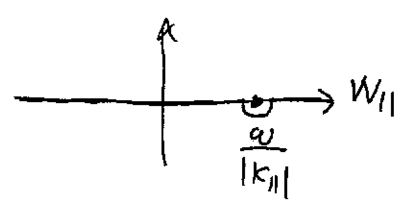
pole

The integration happens along the "Landau contour" — Landau's calculation is a beautiful piece of complex analysis applied to a real plasma problem — it's worth understanding it and the best way is to read Landau's original paper. Here I will simply contract the result into a hand mnemonic called the Plemelj formula:

$$\frac{1}{w_\parallel - \frac{\omega}{|k_\parallel|}} \to \mathcal{P}\,\frac{1}{w_\parallel - \frac{\omega}{|k_\parallel|}} + i\pi\,\delta\!\left(w_\parallel - \frac{\omega}{|k_\parallel|}\right)$$

principal value integral

pole



OK, so this gives

$$\int_{-\infty}^{+\infty}\frac{dw_\parallel\, F_{2s}'(w_\parallel)}{\omega-k_\parallel w_\parallel}=-\frac{2}{k_\parallel}\Bigg[\int dw_\parallel \frac{\partial F_2}{\partial w_\parallel^2}+\underbrace{\cancel{\frac{\omega}{|k_\parallel|}\,\mathcal{P}\underbrace{\int\frac{dw_\parallel}{w_\parallel-\frac{\omega}{|k_\parallel|}}\frac{\partial F_2}{\partial w_\parallel^2}}_{\text{nice regular function}}}}_{\text{this whole term can be neglected if we anticipate that } \frac{\omega}{|k_\parallel|}\ll v_{ths\parallel}}+\underbrace{i\pi\frac{\omega}{|k_\parallel|}\frac{\partial F_2}{\partial w_\parallel^2}\bigg|_{w_\parallel=\frac{\omega}{|k_\parallel|}}}_{}\Bigg]$$

If $f_s$ was a bi-Maxwellian (also $F_2(w_\parallel)$ is a Maxwellian) this is $-\frac{n_s}{v_{ths\parallel}}=-\frac{n_s^2 m_s}{2p_{\parallel s}}$

this one, however, must be kept because it is the lowest-order imaginary bit and will give us the instability!

Finally, give things names to simplify further calculations:

$$2\int dw_\parallel\frac{\partial F_2}{\partial w_\parallel^2}=-\frac{n_s^2 m_s}{p_{\parallel s}}\cdot C_{2s}\qquad (C_{2s}=1 \text{ for a bi-Max})$$

$$2\frac{\partial F_2}{\partial w_\parallel^2}=-\frac{n_s^2 m_s}{p_{\parallel s}}\cdot G_{2s}(w_\parallel)\left(G_{2s}=\frac{e^{-w_\parallel^2/v_{ths\parallel}^2}}{\sqrt{\pi}\,v_{ths\parallel}} \text{ for a bi-Max}\right)$$

Then
$$\int_{-\infty}^{+\infty}\frac{dw_\parallel}{\omega-k_\parallel w_\parallel}\frac{\partial F_{2s}}{\partial w_\parallel}\approx\frac{n_s^2 m_s}{k_\parallel p_{\parallel s}}\left[C_{2s}+i\pi\frac{\omega}{|k_\parallel|}G_{2s}\left(\frac{\omega}{|k_\parallel|}\right)\right]$$

and similarly for the $F_{4s}$ integral:

$$\int_{-\infty}^{+\infty}\frac{dw_\parallel}{\omega-k_\parallel w_\parallel}\frac{\partial F_{4s}}{\partial w_\parallel}\approx\frac{n_s^2 m_s}{k_\parallel p_{\parallel s}}\left[C_{4s}+i\pi\frac{\omega}{|k_\parallel|}G_{4s}\left(\frac{\omega}{|k_\parallel|}\right)\right]$$

<u>Note</u>: Since $\frac{\omega}{|k_\parallel|}\ll v_{ths\parallel}$, we may approximate

$$G_{2s}\left(\frac{\omega}{|k_\parallel|}\right)\approx G_{2s}(0)=\frac{g_{2s}}{\sqrt{\pi}\,v_{ths\parallel}}\qquad [g_{2s}=1 \text{ for a Max.}]$$

Assemble all this into (51):

$$\delta p_{\perp s} = 2p_{\perp s}\frac{\delta B_\parallel}{B} - i\frac{e_s n_s \delta E_\parallel}{k_\parallel}\frac{p_{\perp s}}{p_{\parallel s}}\left[C_{2s} + i\sqrt{\pi}\frac{\omega}{|k_\parallel|v_{ths\parallel}}g_{2s}\right]$$

$$- 2\frac{\delta B_\parallel}{B}\frac{p_{\perp s}^2}{p_{\parallel s}}\left[C_{4s} + i\sqrt{\pi}\frac{\omega}{|k_\parallel|v_{ths\parallel}}g_{4s}\right] \qquad (52)$$

This will go into eq. (48), but to complete the calculation, we need to calculate $\delta E_\parallel$. This is done using the quasineutrality condition:

$$0 = \sum_s e_s \delta n_s \doteq \sum_s e_s \int d^3\vec{w}\,\delta f_s \overset{\text{eq. (50)}}{\doteq}$$

$$= \sum_s e_s \Bigg[\delta u_\parallel \int d^3\vec{w}\,\underbrace{\cancel{\left(\frac{w_\parallel}{w}\frac{\partial f_s}{\partial w} + \frac{\partial f_s}{\partial w_\parallel}\right)}}_{=(\partial f_s/\partial w_\parallel)_{w_\perp}} - \frac{\delta B_\parallel}{B}\underbrace{\int d^3\vec{w}\,\frac{w_\perp^2}{2}\frac{1}{w}\frac{\partial f_s}{\partial w}}_{= -\int d^3\vec{w} f_s = -n_s}$$

$$0$$

$$- \int \frac{d^3\vec{w}}{\omega - k_\parallel w_\parallel}\left(i\frac{e_s}{m_s}\delta E_\parallel + k_\parallel\frac{\delta B_\parallel}{B}\frac{w_\perp^2}{2}\right)\left(\frac{\partial f_s}{\partial w_\parallel}\right)_{w_\perp}\Bigg] =$$

$$= \sum_s e_s \left[\cancel{n_s}\frac{\delta B_\parallel}{B} - \int\frac{dw_\parallel}{\omega - k_\parallel w_\parallel}\left(i\frac{e_s}{m_s}\delta E_\parallel\frac{\partial F_{0s}}{\partial w_\parallel} + k_\parallel\frac{\delta B_\parallel}{B}\frac{p_{\perp s}}{m_s n_s}\frac{\partial F_{2s}}{\partial w_\parallel}\right)\right] =$$

0 because $\sum_s e_s n_s = 0$

$$= -\sum_s e_s \left\{ i\frac{e_s}{\cancel{m_s}}\delta E_\parallel \frac{n_s^2 \cancel{m_s}}{k_\parallel p_{\parallel s}}\left[C_{0s} + i\sqrt{\pi}\frac{\omega}{|k_\parallel|v_{ths\parallel}}g_{0s}\right] + \right.$$

$$\left. + \cancel{k_\parallel}\frac{\delta B_\parallel}{B}\frac{p_{\perp s}}{\cancel{m_s n_s}}\frac{n_s^{\cancel{2}} \cancel{m_s}}{\cancel{k_\parallel} p_{\parallel s}}\left[C_{2s} + i\sqrt{\pi}\frac{\omega}{|k_\parallel|v_{ths\parallel}}g_{2s}\right]\right\}, \quad \text{so}$$

$$\frac{i\,\delta E_\parallel}{k_\parallel} = -\frac{\delta B_\parallel}{B}\,\frac{\sum_s e_s n_s \frac{p_{\perp s}}{p_{\parallel s}}\left[C_{2s} + i\sqrt{\pi}\frac{\omega}{|k_\parallel|v_{ths\parallel}}g_{2s}\right]}{\sum_s \frac{e_s^2 n_s^2}{p_{\parallel s}}\left[C_{0s} + i\sqrt{\pi}\frac{\omega}{|k_\parallel|v_{ths\parallel}}g_{0s}\right]} \equiv -\frac{\delta B_\parallel}{B}\chi \qquad (53)$$

OK, now from (52) we get

$$\delta p_\perp = \sum_s \delta p_{\perp s} = \frac{\delta B_\parallel}{B}\left\{ \sum_s 2p_{\perp s} - \sum_s 2\frac{p_{\perp s}^2}{p_{\parallel s}}\left[C_{4s} + i\sqrt{\pi}\frac{\omega}{|k_\parallel| v_{ths\parallel}} g_{4s}\right] + \frac{\left(\sum_s e_s n_s \frac{p_{\perp s}}{p_{\parallel s}}\left[C_{2s} + i\sqrt{\pi}\frac{\omega}{|k_\parallel| v_{ths\parallel}} g_{2s}\right]\right)^2}{\sum_s \frac{e_s^2 n_s^2}{p_{\parallel s}}\left[C_{0s} + i\sqrt{\pi}\frac{\omega}{|k_\parallel| v_{ths\parallel}} g_{0s}\right]} \right\} \qquad (54)$$

and this is to be subbed into (48):

$$\cancel{m_i n_i \omega^2 \frac{\delta B_\parallel}{B}} = k_\perp^2\left(\delta p_\perp + \frac{B^2}{4\pi}\frac{\delta B_\parallel}{B}\right) + k_\parallel^2\left(p_\perp - p_\parallel + \frac{B^2}{4\pi}\right)\frac{\delta B_\parallel}{B}$$

neglected because $\omega \ll k_\parallel v_{th\parallel s}$

$$\frac{\{\dots\}}{B^2/4\pi} + 1 + \frac{k_\parallel^2}{k_\perp^2}\left(\underbrace{\frac{p_\perp - p_\parallel}{B^2/4\pi}}_{=\sum_s \left(\frac{p_{\perp s}}{p_{\parallel s}} - 1\right)\frac{\beta_{\parallel s}}{2}} + 1\right) = 0 \qquad (55)$$

$$\frac{\{\dots\}}{B^2/4\pi} = \sum_s \beta_{\perp s} - \sum_s \frac{p_{\perp s}}{p_{\parallel s}}\beta_{\perp s}\left[C_{4s} + i\sqrt{\pi}\frac{\omega}{|k_\parallel| v_{ths\parallel}} g_{4s}\right] + \frac{\left(\sum_s e_s n_s \frac{p_{\perp s}}{p_{\parallel s}}\left[C_{2s} + i\sqrt{\pi}\frac{\omega}{|k_\parallel| v_{ths\parallel}} g_{2s}\right]\right)^2}{\sum_s 2\frac{e_s^2 n_s^2}{\beta_{\parallel s}}\left[C_{0s} + i\sqrt{\pi}\frac{\omega}{|k_\parallel| v_{ths\parallel}} g_{0s}\right]}$$

where $\beta_{\perp s} = \dfrac{p_{\perp s}}{B^2/8\pi}$, $\beta_{\parallel s} = \dfrac{p_{\parallel s}}{B^2/8\pi}$

This dispersion relation has the general form

$$A + B\gamma + \frac{(C - D\gamma)^2}{E - F\gamma} = 0 \quad \text{where } \gamma = -i\omega \;\left(\begin{array}{l}\text{real positive}\\ \text{means instability}\end{array}\right)$$

The instability threshold is

$$A + \frac{C^2}{E} = 0 \quad \text{(instability when this is } < 0\text{)}$$

The most favourable (to instability) situation is $k_\parallel \ll k_\perp$ and so we get

$$\boxed{\sum_s \left(\frac{p_{\perp s}}{p_{\parallel s}} C_{1s} - 1\right)\beta_{\perp s} > 1 + \frac{\left(\sum_s e_s n_s \frac{p_{\perp s}}{p_{\parallel s}} C_{2s}\right)^2}{2\sum_s \frac{e_s^2 n_s^2}{\beta_{\parallel s}} C_{0s}}} \qquad (56)$$

(cf. Hellinger 2007)

This represents a condition on the distribution functions, which is not terribly illuminating in the general case. We see that it clearly says that slow-mode polarised perturbations will be unstable provided $p_{\perp s}/p_{\parallel s}$ is sufficiently above unity.

The simplest version of this condition obtains for the (somewhat unphysical) case of "cold electrons" ($p_{\perp e} = p_{\parallel e} = 0$). In this case the electric field can be ignored ($\varphi = 0$ in eq. (53)) and we get, also assuming bi-Maxwellian ions,

$$\boxed{\frac{p_{\perp i} - p_{\parallel i}}{p_{\parallel i}} > \frac{1}{\beta_{\perp i}}} \qquad (57)$$

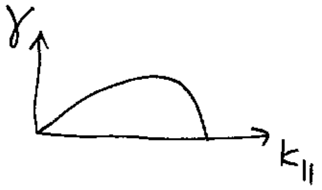
The dispersion relation in this case is

$$\beta_{\perp i}\left(1 - \frac{p_{\perp i}}{p_{\parallel i}}\right) + \sqrt{\pi}\,\frac{\gamma}{|k_\parallel| v_{thi\parallel}}\,\frac{p_{\perp i}}{p_{\parallel i}}\beta_{\perp i} + 1 + \frac{k_\parallel^2}{k_\perp^2}\left(\frac{p_{\perp i} - p_{\parallel i}}{p_{\parallel i}}\,\frac{\beta_{\parallel i}}{2} + 1\right) = 0$$

Therefore

$$\gamma = \frac{|k_\parallel| v_{thi\parallel}}{\sqrt{\pi}}\left[\frac{p_{\perp i}-p_{\parallel i}}{p_{\parallel i}} - \frac{1}{\beta_{\perp i}} - \frac{k_\parallel^2}{2k_\perp^2}\left(\frac{p_{\perp i}-p_{\parallel i}}{p_{\perp i}} + \frac{2}{\beta_{\perp i}}\right)\right]\frac{p_{\parallel i}}{p_{\perp i}} \qquad (58)$$

For a fixed $k_\perp$, this looks like




The peak is at

$$\left(\frac{\partial\gamma}{\partial k_\parallel}\right)_{k_\perp} \propto \frac{p_{\perp i}-p_{\parallel i}}{p_{\parallel i}} - \frac{1}{\beta_{\perp i}} - \frac{3}{2}\frac{k_\parallel^2}{k_\perp^2}\left(\frac{p_{\perp i}-p_{\parallel i}}{p_{\perp i}} + \frac{2}{\beta_{\perp i}}\right) = 0$$

So
$$\frac{k_\parallel^2}{2k_\perp^2}\left(\frac{p_{\perp i}-p_{\parallel i}}{p_{\perp i}} + \frac{2}{\beta_{\perp i}}\right) = \frac{1}{3}\left(\frac{p_{\perp i}-p_{\parallel i}}{p_{\parallel i}} - \frac{1}{\beta_{\perp i}}\right)$$

and
$$\boxed{\gamma = \frac{|k_\parallel| v_{thi\parallel}}{\sqrt{\pi}}\,\frac{2}{3}\left(\frac{p_{\perp i}-p_{\parallel i}}{p_{\parallel i}} - \frac{1}{\beta_{\perp i}}\right)\frac{p_{\parallel i}}{p_{\perp i}}} \qquad (59)$$

So 1) $\gamma \propto |k_\parallel|$ so a <u>UV catastrophe</u> again! Need FLR to regularise – see., e.g., Hellinger 2007

2) We assumed (p.40) $\gamma \ll |k_\parallel| v_{thi\parallel}$ – this is indeed satisfied as long as we stay <u>close to marginal stability</u>.

3) If we do stay close to marginal, we have

$$\frac{k_\parallel}{k_\perp} \sim \sqrt{\Lambda} \ll 1 \quad \text{where } \Lambda = \frac{p_{\perp i}-p_{\parallel i}}{p_{\parallel i}} - \frac{1}{\beta_{\perp i}} \qquad (60)$$

is the stability parameter

Thus, mirror modes are <u>highly oblique</u> (near threshold)

[<u>Note</u>. SW observations by T. Horbury suggest they get fatter as they get older and so more nonlinear]

Ok, we can now construct (roughly) the combined <u>mirror–firehose stability diagram</u>.

<u>Firehose</u>: $$\boxed{\frac{p_{\perp i}}{p_{\parallel i}} - 1 < -\frac{2}{\beta_{\parallel i}}} \qquad (34)$$

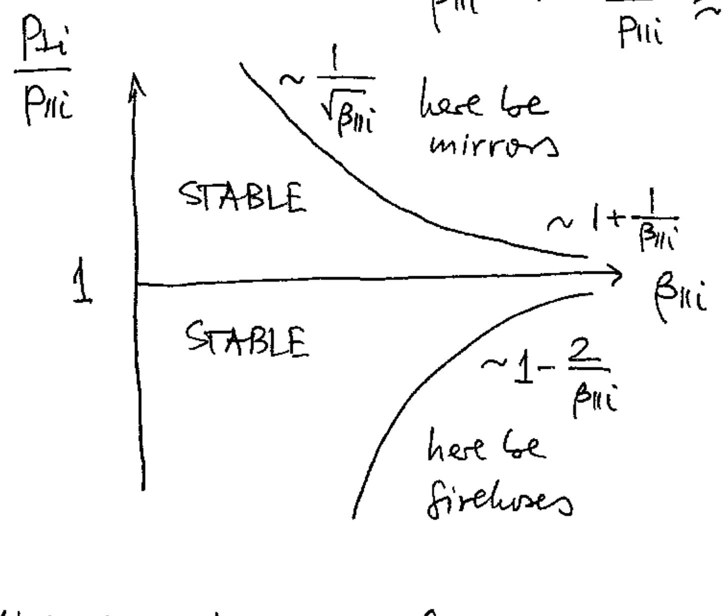
<u>Mirror</u>: $$\frac{p_{\perp i}}{p_{\parallel i}} - 1 > \frac{1}{\beta_{\perp i}} \qquad (57)$$ (for cold electrons, a little more complicated in the general case)

$$\beta_{\perp i}^{-1} = \frac{1}{\beta_{\parallel i}} \frac{p_{\parallel i}}{p_{\perp i}}$$

so, this is equivalent to

$$\boxed{\frac{p_{\perp i}}{p_{\parallel i}}\left(\frac{p_{\perp i}}{p_{\parallel i}} - 1\right) > \frac{1}{\beta_{\parallel i}}} \qquad (61)$$

$\hookrightarrow$ $\beta_{\parallel i} \ll 1$ $\quad \frac{p_{\perp i}}{p_{\parallel i}} \gtrsim \frac{1}{\sqrt{\beta_{\parallel i}}}$ ; $\beta_{\parallel i} \gg 1$ $\quad \frac{p_{\perp i}}{p_{\parallel i}} \gtrsim 1 + \frac{1}{\beta_{\parallel i}}$




The solar wind indeed seems to ~~stay~~ within these boundaries:
Hellinger et al. 2006
Matteini et al. 2007
Bale et al. 2009

they see predominantly $\delta B_\parallel$ fluctuations at the mirror threshold and $\delta B_\perp$ at the firehose – makes sense!

Note that there are other $(p_\perp - p_\parallel)$-driven instabilities that kick in if you include FLR (notable <u>ion-cyclotron</u> and some resonant versions of the firehose)

however, Hellinger et al. 2006 note that SW does not seem to care about this one, the mirror threshold overrides it!

## 2.4 Nonlinear Theory for Mirror Instability

Unlike for the (parallel) firehose, I don't really understand how the mirror saturates (although Califano possibly does – see Califano et al. 2008)

What seems clear – and important! – is that just like the firehose, the mirror will not produce saturated fluctuations that are small-scale enough to scatter particles efficiently.

If FLR regularisation is included, it turns out that $k_{\perp\mathrm{peak}}\rho_i \sim \sqrt{\Lambda}$

instability parameter p.44

so $$k_{\parallel\mathrm{peak}}\rho_i \sim \Lambda \quad [\text{see eq. (60)}] \qquad (62)$$

and $$\gamma \sim \Omega_i \Lambda^2 \quad [\text{see eq. (59)}]$$

Since $\Lambda \ll 1$ close to marginal stability, these are all small, so $\mu$ is happily conserved.

As far as I understand what will happen as fluctuations grow, the linear Landau regime will break down and particles will start getting trapped due to mirror forces:

particles trapped in regions of smaller B ($\delta B_\parallel < 0$)

if so, then statistically more particles will see lower average B, thus offsetting the effect of field growth.

Let us examine this idea: $\vec{B} = \vec{B}_0 + \vec{\delta B}$

$$\overline{\frac{1}{B}\frac{dB}{dt}} = \overline{\frac{1}{2B_0^2}\frac{d}{dt}\left(B_0^2 + 2B_0\delta B_{\parallel} + \delta B^2\right)} =$$

$$= \underbrace{\frac{1}{B_0}\frac{dB_0}{dt}} + \underbrace{\frac{d}{dt}\overline{\frac{\delta B_{\parallel}}{B_0}}}_{\sim\ -\frac{\partial}{\partial t}\overline{\frac{|\delta B_{\parallel}|}{B_0}}\cdot\text{ fraction of trapped particles}} + \dots \qquad (63)$$

$$\text{fraction of trapped particles} \sim \sqrt{\frac{|\delta B_{\parallel}|}{B_0}}$$

Pressure anisotropy is

$$\Delta \sim \frac{1}{\nu}\left[\gamma_0 - \frac{\partial}{\partial t}\left(\frac{|\delta B_{\parallel}|}{B_0}\right)^{3/2}\right] \to \frac{1}{\beta}$$

marginal stability (nevermind the electron terms)

$$\gamma_0 = \frac{1}{B_0}\frac{dB_0}{dt} + \text{the rest of the terms in eq. (24)} > 0 \quad \text{positive!} \qquad (64)$$

This gives

$$\boxed{\frac{|\delta B_{\parallel}|}{B_0} \sim \left[\left(\gamma_0 - \frac{1}{\beta}\nu\right)t\right]^{2/3}} \qquad (65)$$

AAS et al. PRL 100, 081301 (2008)

Note: Again we don't know what happens after $\delta B_{\parallel}/B \sim 1$ (as it can be, observations tell us)

<u>Secular growth</u> again (for the driven case)

I don't want to say anymore about this because, while this idea is appealing (to me), we have (as yet) no systematic theory that would produce this result in the same analytically convincing way as was done for eq. (44) in the Rosin et al. (2011) paper — and it is not necessarily clear that particle trapping is the dominant effect (e.g. Califano's simulations seem to be saying it is not). F. Rincon is currently working on this problem.