Introduction to Pressure-Anisotropy-Driven Instabilities.

Reading: MNRAS 405, 291 (2010) and refs therein

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§1. Kinetic Description of a Magnetised Plasma.

In order to start having an adequate theoretical understanding of the primary topic of these lectures, it is necessary first to gain a good mastery of the general kinetic framework in which dynamics of magnetised plasma is studied. I will spend quite a large fraction of time alloted for these lectures on this extended introduction because it involves material that is not generally part of standard plasma courses - although it provides a natural connection between fluid descriptions (MHD, etc.) and the fully kinetic ones.

Our goal is to develop a theoretical framework for plasmas that are strongly magnetised and weakly collisional in the sense that

$$\underbrace{\nu_{ie}, \nu_{ii}, \nu_{ei}, \nu_{ee}}_{\text{collision frequencies}} \ll \underbrace{\Omega_i, \Omega_e}_{\text{Larmor frequencies}}$$

or $$\underbrace{\rho_i, \rho_e}_{\text{Larmor radii}} \ll \underbrace{\lambda_{mfp}}_{\text{mean free path}}$$

We will also mostly consider low-frequency dynamics,

$$\omega \ll \Omega_i, \Omega_e$$

and (less generally) long wavelengths: $k\rho_i, k\rho_e \ll 1$.

1.1 Kinetic Equation

Let us start from "the beginning":
the Vlasov - Landau - Maxwell system of equations:
distribution function $f_s(t, \vec{r}, \vec{v})$ of species s $(=e, i)$
satisfies

$$\overbrace{\underbrace{\frac{\partial f_s}{\partial t} + \vec{v}\cdot\nabla f_s}_{\text{streaming}} + \underbrace{\frac{e_s}{m_s}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)}_{\text{Lorentz force}}\cdot\frac{\partial f_s}{\partial \vec{v}}}^{\text{Vlasov}} = \overbrace{\underbrace{\left(\frac{\partial f_s}{\partial t}\right)_c}_{\text{collisions}}}^{\text{Landau}} \qquad (1)$$

Maxwell

$$\cancel{\nabla\cdot\vec{E}} = \underbrace{4\pi \sum_s e_s n_s}_{\text{Poisson}}, \quad n_s = \int d^3\vec{v}\, f_s$$

$$\nabla\cdot\vec{B} = 0$$

$$\underbrace{\frac{\partial \vec{B}}{\partial t} = -c\nabla\times\vec{E}}_{\text{Faraday}}$$

$$\vec{J} = \sum_s e_s n_s \vec{u}_s, \quad \vec{u}_s = \frac{1}{n_s}\int d^3\vec{v}\,\vec{v} f_s$$

$$\underbrace{\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J}}_{\text{Ampère}} + \cancel{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}}$$

$\sim k^2\lambda_{De}^2 \ll 1$
neglected for wavelength
$\gg \lambda_{De}$ – quasineutrality
↓
because

$$\frac{kE}{4\pi e\delta n} \sim \frac{k^2\varphi}{4\pi e\delta n} \sim \frac{k^2 T}{4\pi e^2 n} \sim \frac{k^2 m_e v_{the}^2}{4\pi e^2 n} \sim \frac{k^2 v_{the}^2}{\omega_{pe}^2}$$

$\frac{\omega^2/k^2}{c^2} \ll 1$ } displacement current neglected when non-relat.

because

$$\frac{\frac{1}{c}\omega E}{kB} \sim \frac{\frac{1}{c^2}\omega^2\frac{1}{k}B}{kB} \sim \frac{\omega^2}{k^2c^2}$$

Our intuition likes imaging plasma as a fluid, with some density $n_s = \int d^3\vec{v}\, f_s$

and velocity $\vec{u}_s = \frac{1}{n_s}\int d^3\vec{v}\,\vec{v} f_s$

(and perhaps pressure, temperature or some generalisation thereof). This is rooted in the fact that gases we are used to (e.g. our atmosphere) are very collisional, so ~~usually~~ the $\left(\frac{\partial f}{\partial t}\right)_c$ term dominates ($\nu \gg \omega$) and so to lowest order the distribution function is a local Maxwellian:

$$F_s = \frac{n_s}{(\pi v_{ths}^2)^{3/2}} e^{-\frac{(\vec{v}-\vec{u}_s)}{v_{ths}^2}} , \quad v_{ths} = \sqrt{\frac{2T_s}{m_s}}$$

and then all we need to do is derive equations for n_s, $\vec{u}_s$, T_s and also sometimes for perturbations of the particle distribution function around F_s (to calculate transport coefficients: viscosity, thermal diffusivity etc.)

Here we will be concerned with a situation in which collisions are not quite so dominant ($\nu \sim \omega$ or $\ll \omega$). How do we generalise this fluid approach then? Let's first make a minor preliminary step, namely, change variables

$$\vec{v} \longrightarrow \vec{w} = \vec{v} - \vec{u}_s(t, \vec{r}) \quad \text{peculiar velocity}$$

where $\vec{u}_s = \frac{1}{n_s}\int d^3\vec{v}\, \vec{v} f_s$ the exact mean flow velocity.

So our particle kinetics will always be relative to the mean flow of the plasma.

$$\left(\frac{\partial}{\partial t}\right)_{\vec{v}} = \left(\frac{\partial}{\partial t}\right)_{\vec{w}} + \left(\frac{\partial \vec{w}}{\partial t}\right)_{\vec{v}} \cdot \frac{\partial}{\partial \vec{w}} = \frac{\partial}{\partial t} - \frac{\partial \vec{u}_s}{\partial t} \cdot \frac{\partial}{\partial \vec{w}}$$

$$(\nabla)_{\vec{v}} = (\nabla)_{\vec{w}} + (\nabla \vec{w})_{\vec{v}} \cdot \frac{\partial}{\partial \vec{w}} = \nabla - (\nabla \vec{u}_s) \cdot \frac{\partial}{\partial \vec{w}}$$

so $\vec{v}\cdot\nabla \rightarrow \vec{u}_s\cdot\nabla - (\vec{u}_s\cdot\nabla\vec{u}_s)\cdot\frac{\partial}{\partial \vec{w}} + \vec{w}\cdot\nabla - (\vec{w}\cdot\nabla\vec{u}_s)\cdot\frac{\partial}{\partial \vec{w}}$

Let $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \vec{u}_s \cdot \nabla$ convective derivative for species s

Then the kinetic equation becomes

$$\boxed{\underset{(1)}{\frac{df_s}{dt}} + \underset{(2)}{\vec{w}\cdot\nabla f_s} + \left(\underset{(3)}{\frac{e_s}{m_s}\frac{\vec{w}\times\vec{B}}{c}} + \underset{(4)}{\vec{a}_s} - \underset{(5)}{\vec{w}\cdot\nabla\vec{u}_s}\right)\cdot\frac{\partial f_s}{\partial \vec{w}} = \underset{(6)}{\left(\frac{\partial f_s}{\partial t}\right)_c}} \qquad (2)$$

where $\vec{a}_s = \dfrac{e_s}{m_s}\left(\dfrac{\vec{u}_s\times\vec{B}}{c} + \vec{E}\right) - \dfrac{d\vec{u}_s}{dt}$ acceleration (independent of $\vec{w}$!)

To this we must now attach Maxwell's equations:

$$\sum_s e_s n_s = 0 \quad \text{quasineutrality} \qquad (3)$$

$$\frac{\partial\vec{B}}{\partial t} = -c\,\nabla\times\vec{E} \quad \text{Faraday} \qquad (4)$$

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} = \frac{4\pi}{c}\sum_s e_s n_s \vec{u}_s \quad \text{Ampère} \qquad (5)$$

and the constraint $\int d^3\vec{w}\,\vec{w}\,f_s = 0$, which can be thought of as implicitly determining $\vec{u}_s$.

We are now going to spin out a fluid-like description for our plasma with these equations as the starting point.

1.2 Moment Equations.

Take moments of (2): NB use $\int d^3\vec{w}\,\vec{w} f_s = 0$!

$\int d^3\vec{w}$ (2) : $\quad \dfrac{dn_s}{dt} + (\nabla\cdot\vec{u}_s)\, n_s = 0$

from ① $\qquad$ from ⑤ after integration by parts

NB: $\int d^3\vec{w}\left(\dfrac{\partial f_s}{\partial t}\right)_c = 0$ conservation of particles

This is the continuity equation:

$$\boxed{\frac{\partial n_s}{\partial t} + \nabla\cdot(\vec{u}_s n_s) = 0} \qquad (6)$$

$\int d^3\vec{w}\, m_s \vec{w}$ (2) : $\quad \underbrace{\nabla\cdot\int d^3\vec{w}\, m_s \vec{w}\vec{w} f_s}_{\equiv\ \hat{P}_s \text{ pressure tensor}} - \vec{a}_s n_s m_s = \vec{R}_s$ interspecies friction force

particle momentum (relative to mean flow)

from ② $\qquad$ from ④ $\qquad$ from ⑥

$\vec{R}_s \doteq \int d^3\vec{w}\, m_s \vec{w}\left(\dfrac{\partial f_s}{\partial t}\right)_c$

This is the momentum equation:

$$\boxed{m_s n_s \frac{d\vec{u}_s}{dt} = -\nabla\cdot\hat{P}_s + e_s n_s\left(\vec{E} + \frac{\vec{u}_s\times\vec{B}}{c}\right) + \vec{R}_s} \qquad (7)$$

We are interested in mass flow (momentum) ~~[illegible]~~, so we add $(7)_i + (7)_e$ and use

$m_e\vec{u}_e + m_i\vec{u}_i \approx m_i\vec{u}_i \equiv m_i\vec{u} \quad (m_i \gg m_e)$

$\hat{P} \doteq \sum_s \hat{P}_s \quad , \quad \sum_s \vec{R}_s = 0$ (momentum conservation by elastic collision)

$\sum_s e_s n_s = 0$ (quasineutrality)

and $\quad \sum_s e_s n_s \vec{u}_s = \vec{J} = \dfrac{c}{4\pi}\nabla\times\vec{B}$ (Ampère)

This gives

$$\boxed{m_i n_i \frac{d\vec{u}}{dt} = -\nabla\cdot\hat{P} + \underbrace{\frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi}}_{\text{Lorentz force}}} \qquad (8)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla\vec{u}$

The rhs can be written in a nice divergence form:

$$\boxed{m_i n_i \frac{d\vec{u}}{dt} = -\nabla\cdot\Big(\hat{P} + \underbrace{\hat{\mathbb{1}}\frac{B^2}{8\pi} - \frac{\vec{B}\vec{B}}{4\pi}}_{\text{Maxwell stress}}\Big)} \qquad (9)$$

all the kinetic physics is in this tensor!

So, we have equations for n_i and $\vec{u}$, but still need to calculate $\vec{B}$ and $\hat{P}$.
We'll deal with $\vec{B}$ first, which is easier, and then discuss $\hat{P}$ at great length as this is where all the interesting (for the purposes of these lectures) physics is contained.

1.3 Magnetic Field

Faraday: $$\boxed{\frac{\partial \vec{B}}{\partial t} = -c\nabla\times\vec{E}} \qquad (4)$$

So we need to calculate $\vec{E}$. Note that we do this not from Poisson's equation (where $\nabla\cdot\vec{E}$ is small as $k^2\lambda_{De}^2 \ll 1$ and so $\vec{E}$ is not explicitly present) or from Ampère-Maxwell eqn (where the displacement current is small), but from the electron momentum equation, eq. $(7)_e$, which is also known as the "generalised Ohm's law":

$$\underbrace{\vec{E} + \frac{\vec{u}_e\times\vec{B}}{c}}_{\text{el. field in the frame of el. fluid}} = \underbrace{\frac{\vec{R}_e}{en_e}}_{\text{friction}} - \underbrace{\frac{\nabla\cdot\hat{P}_e}{en_e}}_{\text{"thermoelectric term"}} - \underbrace{\frac{m_e}{e}\frac{d\vec{u}_e}{dt}}_{\text{electron inertia}}$$

friction $\downarrow$ resistivity $=$ $\frac{1}{\sigma}\vec{J}$ (the original Ohmic term)

because $\vec{R}_e = -\nu_{ei}m_e n_e(\vec{u}_e - \vec{u}_i) = \frac{\nu_{ei}m_e}{e}\vec{J}$ So $\frac{1}{\sigma} = \frac{\nu_{ei}m_e}{e^2 n_e}$

Since $en_e(\vec{u}_i - \vec{u}_e) = \vec{J}$, this is $\frac{\vec{u}\times\vec{B}}{c} - \frac{\vec{J}\times\vec{B}}{cen_e}$ "Hall term"

Thus, $$\boxed{\vec{E} = -\frac{\vec{u}\times\vec{B}}{c} + \frac{1}{\sigma}\vec{J} - \frac{\vec{J}\times\vec{B}}{cen_e} - \frac{\nabla\cdot\hat{P}_e}{en_e} - \frac{m_e}{e}\frac{d}{dt}\left(\vec{u} - \frac{\vec{J}}{en_e}\right)} \qquad (10)$$

Since $\vec{J} = \frac{c}{4\pi}\nabla\times\vec{B}$ (5), everything here is expressed in terms of $\vec{u}$, $\vec{B}$ or $n_e \left(= \frac{e_i}{e}n_i = Zn_i\right)$, except $\hat{P}_e$, which we have not yet discussed. So, sub. (10) into (4):

$$\boxed{\underbrace{\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B})}_{\text{the usual induction equation}} + \eta\Delta\vec{B} + c\nabla\times\left[\frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi en_e} + \frac{\nabla\cdot\hat{P}_e}{en_e} - \frac{m_e}{e}\frac{d}{dt}\left(\vec{u} - \frac{c\nabla\times\vec{B}}{4\pi en_e}\right)\right]} \qquad (11)$$

$\eta = \frac{c^2}{4\pi\sigma} = \frac{c^2\nu_{ei}m_e}{4\pi e^2 n_e} = \nu_{ei}d_e^2$ magnetic diffusivity

All the terms except $\nabla\times(\vec{u}\times\vec{B})$ are small:

<u>Resistive</u>: $\dfrac{\eta k^2 B}{\omega B} \sim \dfrac{\nu_{ei}}{\omega} k^2 d_e^2 \ll 1$ if $\nu_{ei} \ll \omega$ and/or $k^2 d_e \ll 1$

<u>Hall</u>: $\dfrac{c k^2 B^2}{4\pi e n_e \omega B} \sim \dfrac{c^2 m_i}{4\pi e^2 n_i} \dfrac{k^2}{\omega} \dfrac{eB}{m_i c} \sim k^2 d_i^2 \dfrac{\Omega_i}{\omega}$

If $\omega \sim k v_A \sim \dfrac{kB}{\sqrt{4\pi n_i m_i}}$, then $\dfrac{\omega}{\Omega_i} \sim \dfrac{k\cancel{B} m_i c}{\sqrt{4\pi n_i m_i}\, e\cancel{B}} \sim k d_i$

so Hall term $\sim O(k d_i) \ll 1$

Note that $d_i \sim \dfrac{\rho_i}{\sqrt{\beta_i}}$ so Hall term is an FLR effect except at very low $\beta_i = \dfrac{n_i T_i}{B^2/8\pi}$ $\left(T_i = \dfrac{m_i v_{thi}^2}{2}\right)$.

<u>Thermoelectric</u>: $\dfrac{c k^2 p_e}{e n_e \omega B} \sim \dfrac{c k^2 p_i}{e n_e \omega B} \sim \dfrac{c k^2 m_i v_{thi}^2 \cancel{n_i}}{e \cancel{n_i} \omega B}$

$\sim k^2 \rho_i^2 \dfrac{\Omega_i}{\omega} \sim k\rho_i \dfrac{\rho_i}{d_i} \sim \dfrac{k d_i}{\sqrt{\beta_i}} \sim k\rho_i \sqrt{\beta_i}$

so this again is an FLR effect

<u>Electron inertia</u>: $\dfrac{c k m_e \cancel{\omega} u}{e \cancel{\omega} B} \sim \dfrac{k u}{\Omega_e} \sim \dfrac{\omega}{\Omega_e} \ll 1$

Thus, if we stay with $k\rho_i \ll 1$ and $\omega \ll \Omega_i$ (as we will, mostly), we can ignore all these terms.

<u>Note</u>, however, that this means that at long scales we <u>cannot</u> break magnetic flux conservation.

For any magnetic diffusion and/or reconnection to be possible, we must bring in small scale effects: resistivity, el. inertia, or some electron FLR bits of $\hat{P}_e$ (Hall and dominant-order (in $k\rho_e$) bit of the thermoelectric term are flux conserving)

1.4 Gyrotropic Plasmas.

Now let us tackle the pressure tensor

$$\hat{P}_s = \int d^3\vec{w}\, m_s \vec{w}\vec{w}\, f_s$$

In general, in order to know what it is, we still need to solve the kinetic equation (2), so despite all the work we have done so far, no real simplificati has yet been achieved.

Note that term ③ in eq. (2) can be written so:

$$\frac{e_s}{m_s}\frac{\vec{w}\times\vec{B}}{c}\cdot\frac{\partial f_s}{\partial\vec{w}} = -\Omega_s\left(\frac{\partial f_s}{\partial\vartheta}\right)_{w_\perp, w_\parallel}$$

where $\Omega_s = \frac{e_s B}{m_s c}$ is the Larmor frequency and ϑ is the gyroangle – angle at which the particle orbits the magnetic field.

Ex. Prove this! (just let $w_x = w_\perp \cos\vartheta$, $w_y = w_\perp \sin\vartheta$ etc.
– cylindrical coordinates locally in velocity space)
↳ wrt $\vec{B}$

So, eq. (2) can be written so:

$$\underset{③}{\Omega_s\left(\frac{\partial f_s}{\partial\vartheta}\right)_{w_\perp, w_\parallel}} = \underset{①}{\frac{d f_s}{dt}} + \underset{②}{\vec{w}\cdot\nabla f_s} + \underset{④}{(\vec{a}_s - \vec{w}\cdot\nabla\vec{u}_s)}\cdot\underset{⑤}{\frac{\partial f_s}{\partial\vec{w}}} - \underset{⑥}{\left(\frac{\partial f_s}{\partial t}\right)_c} \quad (12)$$

Let us now consider a situation when the lhs is ≫ the rhs. Then, to lowest order,

$$\frac{\partial f_s}{\partial \vartheta} = 0 \quad \text{and so} \quad f_s = f_s(t, \vec{r}, w_\perp, w_\parallel)$$

– The distribution function is gyrotropic, i.e., independent of the gyroangle.

To lowest order in what?

$$\frac{①}{③} \sim \frac{\omega}{\Omega_s} \ll 1 \quad \text{low frequency}$$

$$\sim \frac{k u_s}{\Omega_s} \sim k\rho_s \frac{u_s}{v_{ths}} \ll 1 \quad \text{long wavelength}$$

$$\frac{②}{③} \sim \frac{k v_{ths}}{\Omega_s} \sim k\rho_s \ll 1 \quad \text{long wavelength}$$

from gen'd Ohm's law

use $\nabla \cdot \hat{P}_e / e n_e$ (biggest term)

$$\frac{④}{③} \sim \frac{a_s}{\Omega_s v_{ths}} \sim \frac{\vec{E} + \frac{\vec{u}_s \times \vec{B}}{c}}{\frac{m_s}{e_s}\Omega_s v_{ths}} \sim k\rho_s \ll 1$$

or smaller

$$\frac{⑤}{③} \sim \frac{k v_{ths} u_s}{\Omega_s v_{ths}} \sim k\rho_s Ma \ll 1$$

$$\frac{⑥}{③} \sim \frac{\nu_s}{\Omega_s} \ll 1 \quad \text{weakly collisional}$$

(= magnetised)

[otherwise coll's will dominate and f_s will become isotropic]

Thus, we are safe in this approximation if

$$\omega \ll \Omega_i \quad \text{and} \quad k\rho_i \ll 1 \quad \text{and} \quad \nu_i \ll \Omega_i$$

magnetised weakly collisional plasma

For such a plasma, the pressure tensor is greatly simplified:

$$\hat{P}_s = \int d^3\vec{w}\, m_s \underbrace{\langle \vec{w}\vec{w} \rangle_\theta}_{\frac{w_\perp^2}{2}(\mathbb{1}-\hat{b}\hat{b}) + w_\parallel^2 \hat{b}\hat{b} \quad \text{where } \hat{b} = \frac{\vec{B}}{B}} f_s(t, \vec{r}, w_\perp, w_\parallel) =$$

$$= (\mathbb{1}-\hat{b}\hat{b}) \underbrace{\int d^3\vec{w}\, m_s \frac{w_\perp^2}{2} f_s}_{\equiv\, p_{\perp s}} + \hat{b}\hat{b} \underbrace{\int d^3\vec{w}\, m_s w_\parallel^2 f_s}_{\equiv\, p_{\parallel s}}$$

$$= \begin{pmatrix} p_{\perp s} & & 0 \\ & p_{\perp s} & \\ 0 & & p_{\parallel s} \end{pmatrix}$$

so we simply have two scalar pressures, perp. and parallel to the local direction of $\vec{B}$.

Denoting $p_\perp = \sum_s p_{\perp s}$

and $p_\parallel = \sum_s p_{\parallel s}$, we get

$$\nabla\cdot\hat{P} = \nabla\cdot\left[(\mathbb{1}-\hat{b}\hat{b})\, p_\perp + \hat{b}\hat{b}\, p_\parallel\right] = \nabla p_\perp - \nabla\cdot\left[\hat{b}\hat{b}\,(p_\perp - p_\parallel)\right]$$

and so the momentum equation is

$$\boxed{m_i n_i \frac{d\vec{u}}{dt} = -\nabla\left(p_\perp + \frac{B^2}{8\pi}\right) + \nabla\cdot\left[\hat{b}\hat{b}\left(p_\perp - p_\parallel + \frac{B^2}{4\pi}\right)\right]} \qquad (13)$$

usual scalar pressure, incl. magnetic

pressure anisotropy stress

Maxwell stress

1.5 Origin of Pressure Anisotropy.

Let us momentarily interrupt the formal flow and ask where pressure anisotropies might come from and how large they are likely to be. This discussion is qualitative and we will subsequently rederive everything more rigorously.

If the magnetic field in a plasma changes sufficiently slowly ($\omega \ll \Omega_i$) and particles rarely collide ($\nu_{ii} \ll \Omega_i$), then each particle has an adiabatic invariant (called first adiabatic invariant) $\mu = \dfrac{m w_\perp^2}{2B}$ (we'll prove that $\mu = \text{const}$ directly from eq. (12) later.)

Physically, this can be thought of as the magnetic moment of a current loop formed by a gyroorbit or angular momentum of the gyrating particle ($m w_\perp r = m w_\perp \cdot w_\perp / \Omega_i = m w_\perp^2 / (eB/mc) \propto m w_\perp^2 / B$).

Now the sum of all these μ's is

$$\int d^3\vec{w}\, \mu f = \frac{p_\perp}{B}$$

(in fact $N \dfrac{p_\perp}{nB}$, but let $n = \text{const}$ for now) [# particles; density]

Let us express this expectation that μ is conserved:

$$\frac{1}{p_\perp}\frac{dp_\perp}{dt} \sim \frac{1}{B}\frac{dB}{dt} - \nu\,\frac{p_\perp - p_\parallel}{p_\perp} \qquad (14)$$

non-rigorous at this stage, derivation later. — conservation — collisional tendency to isotropize pressure.

Thus, we expect that if ambient magnetic field changes in a plasma, this should cause $p_\perp$ to change, so as to preserve μ — a process possibly attenuated by collisions if they are large enough to compete.

If they indeed are large enough, we can make a simple estimate of the pressure anisotropy:

Example: solar wind is expanding, B is dropping, expect ~1 negative pressure anisotropy at 1AU

$$\Delta \equiv \frac{p_\perp - p_\parallel}{p_\perp} \sim \frac{1}{\nu}\,\frac{1}{B}\frac{dB}{dt} \sim \frac{\text{rate of change of } B}{\text{coll. rate}} \quad {}^{*)}$$

Let us recall the induction equation [eq. (11), dropping all the small-scale terms]:

$$\frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B}) = -\vec{u}\cdot\nabla\vec{B} + \vec{B}\cdot\nabla\vec{u} \qquad (\nabla\cdot\vec{u} = 0 \text{ for simplicity, incompressible})$$

$$\vec{B}\cdot \;\Big|\; \frac{d\vec{B}}{dt} = \vec{B}\cdot\nabla\vec{u}$$

$$\frac{1}{B}\frac{dB}{dt} = \hat{b}\cdot(\nabla\vec{u})\cdot\hat{b} \quad \leftarrow \text{rate of strain}$$

Thus,
$$\frac{p_\perp - p_\parallel}{p_\perp} \sim \frac{\hat{b}\cdot(\nabla\vec{u})\cdot\hat{b}}{\nu}$$

or
$$p_\perp - p_\parallel \sim \left(\frac{p_\perp}{\nu}\right)\hat{b}\hat{b} : \nabla\vec{u}$$

$$\frac{p_\perp}{\nu} \sim \frac{m_i n_i v_{thi}^2}{\nu_{ii}} \sim \text{collisional viscosity}$$

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*) Note that this means that the electron pressure anisotropy is usually smaller than the ion one:

$$\Delta_e/\Delta_i \sim \frac{\nu_i}{\nu_e} \sim \sqrt{\frac{m_e}{m_i}} \ll 1 \quad \text{(as indeed is the case in SW)}$$

In eq. (13) we have therefore

$$\nabla\cdot\left[\hat{b}\hat{b}(p_\perp - p_\parallel)\right] \sim \nabla\cdot\left[\frac{p_\perp}{\nu}\hat{b}\hat{b}\hat{b}\hat{b}:\nabla\vec{u}\right]$$

— precisely the familiar (to some, at least!) parallel (Braginskii) viscosity term [so, in the coll. limit, equations can be closed]

Thus, pressure anisotropy = parallel viscosity, although, as we are going to see shortly, while its effect on large scales is dissipative, at small scales it will be wildly destabilising.

Intuitively this is because $p_\perp \neq p_\parallel$ is a non-equilibrium situation and so is a source of free energy, the pressure-anisotropic system will want to relax towards isotropy. It can do so via collisions, of course, but it can (and will) also be impatient with their sluggishness and find ways of exciting instabilities, which will then push it towards equilibrium — a common phenomenon.

From eq. (13), we can also estimate under what conditions $p_\perp - p_\parallel$ is likely to prove an important effect: clearly we must compare it with $B^2/4\pi$

So $\quad \frac{p_\perp - p_\parallel}{p} \ll \frac{B^2}{4\pi p} \sim \frac{2}{\beta}$ pressure anisotropy irrelevant

$\frac{p_\perp - p_\parallel}{p} > \frac{2}{\beta}$ pressure anisotropy potentially important.

1.6 Kinetic MHD.

Let us now bring our quest for a fluid system of equations to a kind of completion by working out the evolution equations for $p_\perp$ and $p_\parallel$ (one of which will be a somewhat corrected form of eq. (14)).

Let us go back to eq. (12). We agreed that to lowest order, $\frac{\partial f_{0s}}{\partial \vartheta} = 0$, so $f_s = f_{0s} + \delta f_s$ and

$$\Omega_s \frac{\partial \delta f_s}{\partial \vartheta} = \frac{d f_{0s}}{dt} + \vec{w}\cdot\nabla f_{0s} + (\vec{a}_s - \vec{w}\cdot\nabla\vec{u}_s)\cdot\frac{\partial f_{0s}}{\partial \vec{w}} - \left(\frac{\partial f_{0s}}{\partial t}\right)_c \quad (15)$$

We can annihilate the lhs by averaging this equation over gyroangles. Since $f_{0s} = f_{0s}(t, \vec{r}, w_\perp, w_\parallel)$, the gyroaverage of the rhs will give us a closed equation for the distribution function.

It turns out that mathematically the least cumbersome calculation can be done if we use $(w, w_\parallel)$ instead of $(w_\perp, w_\parallel)$ as variables.

↳ $w = w_\perp^2 + w_\parallel^2$ [they also have the advantage that for isotropic distribution, $\partial f/\partial w_\parallel = 0$, so $f = f(t, \vec{r}, w)$]

The time and spatial derivatives in the rhs of (15) must be carefully transformed because our change of of variables mixes phase space:

$$(t, \vec{r}, \vec{w}) \to (t, \vec{r}, w, w_\parallel, \vartheta), \text{ where } w_\parallel = \vec{w}\cdot\hat{b}(t, \vec{r})$$

and $f_{0s} = f_{0s}(t, \vec{r}, w, w_\parallel)$ [drop subscripts in what follows]

NB: $\int d^3\vec{w} = 2\pi \int_0^\infty dw\, w \int_{-w}^{w} dw_\parallel$

Then $\left(\frac{df}{dt}\right)_{\vec{w}} = \left(\frac{df}{dt}\right)_{w,w_\parallel} + \left(\frac{dw_\parallel}{dt}\right)_{\vec{w}}\left(\frac{\partial f}{\partial w_\parallel}\right)_w$

$$= \frac{df}{dt} + \frac{d\hat{b}}{dt}\cdot\vec{w}\,\frac{\partial f}{\partial w_\parallel}$$

$$(\nabla f)_{\vec{w}} = (\nabla f)_{w,w_\parallel} + (\nabla w_\parallel)_{\vec{w}}\left(\frac{\partial f}{\partial w_\parallel}\right)_w$$

$$= \nabla f + (\nabla\hat{b})\cdot\vec{w}\,\frac{\partial f}{\partial w_\parallel}$$

Also

$$\frac{\partial f}{\partial \vec{w}} = \frac{\vec{w}}{w}\frac{\partial f}{\partial w} + \hat{b}\,\frac{\partial f}{\partial w_\parallel}$$

So, the gyroaveraged eqn (15) is

$$\frac{df}{dt} + \underbrace{\frac{d\hat{b}}{dt}\cdot\underbrace{\langle\vec{w}\rangle}_{=w_\parallel\hat{b}}}_{=0}\frac{\partial f}{\partial w_\parallel} + \underbrace{\langle\vec{w}\rangle}_{=w_\parallel\hat{b}}\cdot\nabla f + \underbrace{\langle\vec{w}\vec{w}\rangle}_{=\frac{w_\perp^2}{2}(\mathbb{1}-\cancel{\hat{b}\hat{b}})+w_\parallel^2\cancel{\hat{b}\hat{b}}}:(\nabla\hat{b})\frac{\partial f}{\partial w_\parallel} +$$

because $\hat{b}\cdot\frac{d\hat{b}}{dt} = \frac{1}{2}\frac{d\hat{b}^2}{dt} = 0$

NB: $(\nabla\hat{b})\cdot\hat{b} = \frac{1}{2}\nabla\hat{b}^2 = 0$

$$+\vec{a}\cdot\left(\underbrace{\frac{\langle\vec{w}\rangle}{w}}_{=\frac{w_\parallel}{w}\hat{b}}\frac{\partial f}{\partial w} + \hat{b}\frac{\partial f}{\partial w_\parallel}\right) - (\nabla\vec{u}):\underbrace{\frac{\langle\vec{w}\vec{w}\rangle}{w}}_{=\frac{w_\perp^2(\mathbb{1}-\hat{b}\hat{b})}{2w}+\frac{w_\parallel^2}{w}\hat{b}\hat{b}}\frac{\partial f}{\partial w} - \underbrace{\langle\vec{w}\rangle}_{=w_\parallel\hat{b}}\cdot(\nabla\vec{u})\cdot\hat{b}\frac{\partial f}{\partial w_\parallel}$$

$$= \left\langle\left(\frac{\partial f}{\partial t}\right)_c\right\rangle$$

$\nabla\cdot\hat{b} = -\frac{\hat{b}\cdot\nabla B}{B}$

$$\boxed{\begin{aligned}&\frac{df}{dt} + w_\parallel\hat{b}\cdot\nabla f + \underbrace{\frac{w_\perp^2}{2}(\nabla\cdot\hat{b})\frac{\partial f}{\partial w_\parallel}}_{\text{mirrorforce}} + \vec{a}\cdot\hat{b}\left(\frac{w_\parallel}{w}\frac{\partial f}{\partial w} + \frac{\partial f}{\partial w_\parallel}\right)\\ &\quad -(\nabla\cdot\vec{u})\frac{w_\perp^2}{2w}\frac{\partial f}{\partial w} + (\hat{b}\hat{b}:\nabla\vec{u})\left[\left(\frac{w_\perp^2}{2}-w_\parallel^2\right)\frac{1}{w}\frac{\partial f}{\partial w} - w_\parallel\frac{\partial f}{\partial w_\parallel}\right]\\ &\quad = \left\langle\left(\frac{\partial f}{\partial t}\right)_c\right\rangle\end{aligned}} \quad (16)$$

This equation, coupled with the definitions of $p_\perp$ & $p_\parallel$, the induction equation (11) (without the small-scale terms) & the momentum equation (13) constitute a closed system, known as "Kinetic MHD".

Note that the continuity eqn (6) is redundant (formally) as it can be obtained from (16) by integrating over velocities. The same is true about the parallel projection of the momentum equation, (13)$\cdot\hat{b}$, because $\vec{a}$ contains the $\frac{d\vec{u}}{dt}$ term.

In standard treatments (e.g. Kulsrud's review in the Handbook of Plasma Physics - 1983 or his earlier, more detailed, Varenna lecture notes), eq. (16) is written in $(w_\perp, w_\parallel)$ [or sometimes $(w_\perp, v_\parallel)$] variables – or $(\mu, w_\parallel)$ where $\mu = \frac{m w_\perp^2}{2B}$.

In this last form, μ conservation becomes manifest in the sense that the kinetic equation for $f(t, \vec{r}, \mu, w_\parallel)$ contains no μ derivatives:

$$\frac{df}{dt} + w_\parallel \hat{b}\cdot\nabla f - \Big(\hat{b}\cdot\frac{d\vec{u}}{dt} + w_\parallel \hat{b}\hat{b}:\nabla\vec{u} + \underbrace{\frac{\mu}{m}\hat{b}\cdot\nabla B}_{\text{mirror force}} - \frac{e}{m}E_\parallel\Big)\frac{\partial f}{\partial w_\parallel} = \left(\frac{\partial f}{\partial t}\right)_c \qquad (17)$$

(Ex. Derive this by changing variables from eq. (16))

This is nice and compact, but mixing B into velocity variables introduces some unwelcome complications into practical calculations, so I recommend eq. (16) over this.

Let us summarise the equations:

Our kinetic equation now is

$$\frac{df_s}{dt} + w_\parallel \hat{b}\cdot\nabla f_s + \frac{w_\perp^2}{2}(\nabla\cdot\hat{b})\frac{\partial f_s}{\partial w_\parallel} + \left(\frac{e_s}{m_s}E_\parallel - \hat{b}\cdot\frac{d\vec{u}_s}{dt}\right)\left(\frac{w_\parallel}{w}\frac{\partial f_s}{\partial w} + \frac{\partial f_s}{\partial w_\parallel}\right)$$

$$-(\nabla\cdot\vec{u}_s)\frac{w_\perp^2}{2w}\frac{\partial f_s}{\partial w} + (\hat{b}\hat{b}:\nabla\vec{u}_s)\left[\left(\frac{w_\perp^2}{2} - w_\parallel^2\right)\frac{1}{w}\frac{\partial f_s}{\partial w} - w_\parallel\frac{\partial f_s}{\partial w_\parallel}\right] =$$

$$= \left\langle\left(\frac{\partial f_s}{\partial t}\right)_c\right\rangle, \text{ where } w_\perp^2 = w^2 - w_\parallel^2 \qquad (16)$$

This contains three fields: $\vec{u}$, $\vec{B}$ (via $\hat{b}$) and $E_\parallel$

For $\vec{u}$, we have the momentum equation:

NB: $\vec{u}_i = \vec{u}$, $\vec{u}_e = \vec{u}_i - \vec{j}/en_e \approx \vec{u}_i$ as long as $kd_i \ll 1$

$$m_i n_i \frac{d\vec{u}}{dt} = -\nabla\left(p_\perp + \frac{B^2}{8\pi}\right) + \nabla\cdot\left[\hat{b}\hat{b}\left(p_\perp - p_\parallel + \frac{B^2}{4\pi}\right)\right] \qquad (13)$$

and for $\vec{B}$ the induction equation:

$$\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B}) \Leftrightarrow \frac{d\vec{B}}{dt} = \vec{B}\cdot\nabla\vec{u} - \vec{B}\,\nabla\cdot\vec{u} \qquad (11)$$

The pressures in eq. (13) are calculated from f_s:

$$p_\perp = \sum_s p_{\perp s}, \qquad p_{\perp s} = \int d^3\vec{w}\, m_s \frac{w_\perp^2}{2} f_s$$

$$p_\parallel = \sum_s p_{\parallel s}, \qquad p_{\parallel s} = \int d^3\vec{w}\, m_s w_\parallel^2 f_s$$

Note that the $\hat{b}\cdot$(13) projection is redundant: it can be obtained from eq. (16) by taking the $\int d^3\vec{w}\, w_\parallel$ (16) moment.

Density n_i in eq. (13) can be found either as $n_i = \int d^3\vec{w} f_i$ or by taking the moment of eq. (16):

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\vec{u}) = 0 \Leftrightarrow \frac{dn_i}{dt} = -n_i\nabla\cdot\vec{u} \qquad (6)$$

(this equation is again redundant wrt eq. (16))

Finally, the parallel electric field is determined implicitly, via the quasi-neutrality condition (which is the only of ~~the~~ Maxwell's equations that we have not yet properly utilized):

$$\sum_s e_s n_s = \sum_s q \int d^3\vec{w}\, f_s = 0 \qquad (3)$$

Note that there are some subtleties involved in the question of determining $E_\parallel$, which its worth discussing in some detail. On p. 7, I said that $\vec{E}$ must be determined from the electron momentum equation (10). Then we have: $\hat{b}\cdot(10)$:

$$E_\parallel = \frac{1}{\sigma} j_\parallel - \underbrace{\frac{(\nabla\cdot\hat{P}_e)\cdot\hat{b}}{en_e}}_{=} - \frac{m_e}{e}\left[\frac{d}{dt}\left(\vec{u} - \frac{\vec{j}}{n_e e}\right)\right]\cdot\hat{b}$$

small (exercise) $\sim \frac{\nu_{ei}}{\Omega_e}\frac{d_e^2}{\rho_e^2}$ compared to thermoelectric

small (exercise) $\leftarrow \sim \frac{u}{v_{thi}}\frac{m_e}{m_i}\frac{1}{\sqrt{\beta_i}}$ compared to thermoelectric

$$\frac{\left\{\nabla\cdot\left[p_{\perp e}(\mathbb{1}-\hat{b}\hat{b}) + p_{\parallel e}\hat{b}\hat{b}\right]\right\}\cdot\hat{b}}{en_e}$$

$$= \frac{\hat{b}\cdot\cancel{\nabla p_{\perp e}} - \nabla\cdot\hat{b}\,(p_{\perp e}-p_{\parallel e}) - \hat{b}\cdot\nabla(\cancel{p_{\perp e}}-p_{\parallel e})}{en_e}$$

$$\boxed{E_\parallel = -\frac{\hat{b}\cdot\nabla p_{\parallel e}}{en_e} + (p_{\perp e}-p_{\parallel e})\frac{\nabla\cdot\hat{b}}{en_e}} \qquad (18)$$

This, however, turns out to insufficient to calculate $E_\parallel$ from eq. (16) — to get it right one must keep the el. inertia term (because of some lowest-order cancellations) — so it's better to use eq. (3)

This concludes the discussion of KMHD — however, when we come to the treatment of firehose and mirror, we'll conclude that these equations, while useful for theoretical calculations, are ill posed — contain perturbations that grow at arbitrarily large k. So KMHD requires inclusion of non-gyrotropic FLR terms to ~~[illegible]~~ regularise solutions at small scales.

1.7 CGL Equations

↑ Chew-Goldberger-Low.

I promised evolution equations for $p_\perp$ and $p_\parallel$, but so far have only delivered a kinetic equation for the gyrotropic distribution function from which pressures can be calculated.

I am now going to make a further step by taking the $m\frac{w_\perp^2}{2}$ and $m w_\parallel^2$ moments of eq. (16) and hence deriving the desired eqns for $p_\perp$ and $p_\parallel$. So, $\int d^3\vec{w}\, m\frac{w_\perp^2}{2}$ (16):

$$\frac{dp_\perp}{dt} + \hat{b}\cdot\nabla \underbrace{\int d^3\vec{w}\, m\frac{w_\perp^2}{2} w_\parallel f}_{=\, q_\perp} + (\nabla\cdot\hat{b}) \int d^3\vec{w}\, m\frac{w_\perp^4}{4}\frac{\partial f}{\partial w_\parallel} +$$

$q_\perp$ ← parallel flux of "perp energy"

$$\int d^3\vec{w}\, m\frac{(w^2-w_\parallel^2)^2}{4}\frac{\partial f}{\partial w_\parallel} \overset{\text{by parts}}{=} \int d^3\vec{w}\, m\,(w^2-w_\parallel^2)\, w_\parallel f = 2q_\perp$$

$$+\left(\frac{e_s}{m_s}E_\parallel - \hat{b}\cdot\frac{d\vec{u}}{dt}\right)\int d^3\vec{w}\, m\frac{w^2-w_\parallel^2}{2}\left(\frac{w_\parallel}{w}\frac{\partial f}{\partial w}+\frac{\partial f}{\partial w_\parallel}\right)$$

$$= \int d^3\vec{w}\, m(w^2-w_\parallel^2)\left(w_\parallel\frac{\partial f}{\partial w^2}+\frac{1}{2}\frac{\partial f}{\partial w_\parallel}\right) \overset{\text{by parts}}{=} -\int d^3\vec{w}\, m\left(w_\parallel f - w_\parallel f\right) = 0$$

$$-(\nabla\cdot\vec{u})\int d^3\vec{w}\, m\frac{(w^2-w_\parallel^2)^2}{2}\frac{\partial f}{\partial w^2} +$$

$$= -\int d^3\vec{w}\, m\,(w^2-w_\parallel^2) f = -2p_\perp$$

$$+(\hat{b}\hat{b}:\nabla\vec{u})\int d^3\vec{w}\, m\,\frac{w^2-w_\parallel^2}{2}\left[(w^2-3w_\parallel^2)\frac{\partial f}{\partial w^2} - w_\parallel\frac{\partial f}{\partial w_\parallel}\right] =$$

$$= -\int d^3\vec{w}\, m\left(\cancel{\frac{w^2-3w_\parallel^2}{2}} + \frac{w^2-w_\parallel^2}{2} - \cancel{\frac{w^2-3w_\parallel^2}{2}}\right) f = -p_\perp$$

$$= \int d^3\vec{w}\, m\frac{w_\perp^2}{2}\left\langle\left(\frac{\partial f}{\partial t}\right)_c\right\rangle = -\nu(p_\perp - p_\parallel)$$

← this can be calculated using a model Coll. operator

see Note on p.22

e.g. Lorentz $\nu\frac{\partial}{\partial\xi}\frac{1-\xi^2}{2}\frac{\partial f}{\partial\xi}$ where $\xi = w_\parallel/w$

Assemble:

$$\frac{dp_\perp}{dt} = -\hat{b}\cdot\nabla q_\perp - 2q_\perp\nabla\cdot\hat{b} - 2p_\perp\nabla\cdot\vec{u} + p_\perp\hat{b}\hat{b}:\nabla\vec{u} - \nu(p_\perp - p_\parallel)$$

$$= -\nabla\cdot(\overbrace{\hat{b}q_\perp}^{\equiv\vec{q}_\perp}) - q_\perp\nabla\cdot\hat{b} - \nu(p_\perp - p_\parallel) + p_\perp\underbrace{(\hat{b}\hat{b}:\nabla\vec{u} - \nabla\cdot\vec{u})}_{=\frac{1}{B}\frac{dB}{dt}} - p_\perp\underbrace{\nabla\cdot\vec{u}}_{=-\frac{1}{n}\frac{dn}{dt}} \qquad (19)$$

$$\boxed{p_\perp\frac{d}{dt}\ln\frac{p_\perp}{nB} = -\nabla\cdot\vec{q}_\perp - q_\perp\nabla\cdot\hat{b} - \nu(p_\perp - p_\parallel)} \qquad (20)$$

This is the more rigorous version of eq. (14), incorporating now the effect of compressibility and heat fluxes. It is worth noting the size of the heat flux terms:

$$\frac{q_\perp}{u p_\perp} \sim \frac{\cancel{m} v_{th}^3\,\delta f/f}{u\,\cancel{m}v_{th}^2} \sim \frac{v_{th}}{u}\frac{\delta f}{f}$$

asymmetric part of the distribution function, usually small

This is small if flows are sonic $u \sim v_{th}$ ("Braginskii ordering")

and order unity if $\frac{u}{v_{th}} \sim \frac{\delta f}{f} \ll 1$ ("drift ordering", in the context of pressure anisotropies these are called "Mikhailovskii terms")

NB: obviously, for electrons, heat fluxes are always important!

So pressure anisotropies are caused both by changes in n and B ($\Leftarrow$ caused by flows) and by heat fluxes.

Let us complete the derivation and get $p_\parallel$:

$\int d^3\vec{w}\, m w_\parallel^2$ (16):

$$\frac{dp_\parallel}{dt} + \hat{b}\cdot\nabla \underbrace{\int d^3\vec{w}\, m w_\parallel^3 f}_{\substack{= \\ q_\parallel \,\leftarrow\, \text{parallel flux of "par. energy"}}} + (\nabla\cdot\hat{b}) \underbrace{\int d^3\vec{w}\, m w_\parallel^2 \frac{w^2-w_\parallel^2}{2}\frac{\partial f}{\partial w_\parallel}}_{\substack{= \\ -\int d^3\vec{w}\, m (w_\parallel w^2 - 2w_\parallel^3) f \\ = -\int d^3\vec{w}\, m (w_\perp^2 w_\parallel - w_\parallel^3) f \\ = -2q_\perp + q_\parallel}}$$

$$+ \left(\frac{e_s}{m_s}E_\parallel - \hat{b}\cdot\frac{d\vec{u}}{dt}\right) \underbrace{\int d^3\vec{w}\, m w_\parallel^2 \left(2w_\parallel \frac{\partial f}{\partial w^2} + \frac{\partial f}{\partial w_\parallel}\right)}_{\substack{= \\ 0}}$$

$$-(\nabla\cdot\vec{u}) \underbrace{\int d^3\vec{w}\, m w_\parallel^2 (w^2 - w_\parallel^2)\frac{\partial f}{\partial w^2}}_{\substack{= \\ -\int d^3\vec{w}\, m w_\parallel^2 f = -p_\parallel}} \quad +$$

$$+ (\hat{b}\hat{b}:\nabla\vec{u}) \underbrace{\int d^3\vec{w}\, m w_\parallel^2 \left[(w^2 - 3w_\parallel^2)\frac{\partial f}{\partial w^2} - w_\parallel \frac{\partial f}{\partial w_\parallel}\right]}_{\substack{= \\ -\int d^3\vec{w}\, m (w_\parallel^2 - 3w_\parallel^2) f = 2p_\parallel}} =$$

$$= \left.\int d^3\vec{w}\, m w_\parallel^2 \left\langle \left(\frac{\partial f}{\partial t}\right)_c \right\rangle \right\} = -2\nu(p_\parallel - p_\perp)$$

see Note

again from model, but can also be inferred from the requirement the energy $2p_\perp + p_\parallel$ is conserved by collisions

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Note: Lorentz operator, the simplest model for collisions:

$$\left\langle \left(\frac{\partial f}{\partial t}\right)_c \right\rangle = \nu \frac{\partial}{\partial \xi}\frac{1-\xi^2}{2}\frac{\partial f}{\partial \xi} = \nu\frac{\partial}{\partial w_\parallel}\frac{(w^2-w_\parallel^2)}{2}\frac{\partial f}{\partial w_\parallel}$$

So
$$\int d^3\vec{w}\, m w_\parallel^2\, \nu \frac{\partial}{\partial w_\parallel}\frac{(w^2-w_\parallel^2)}{2}\frac{\partial f}{\partial w_\parallel} = -\nu \int d^3\vec{w}\, m w_\parallel (w^2 - w_\parallel^2)\frac{\partial f}{\partial w_\parallel} =$$

$$= \nu \int d^3\vec{w}\, m (w^2 - 3w_\parallel^2) f = 2\nu(p_\perp - p_\parallel)$$

$$\int d^3\vec{w}\, m \frac{w^2-w_\parallel^2}{2}\nu\frac{\partial}{\partial w_\parallel}\frac{w^2-w_\parallel^2}{2}\frac{\partial f}{\partial w_\parallel} = +\nu\int d^3\vec{w}\, m w_\parallel \frac{w^2-w_\parallel^2}{2}\frac{\partial f}{\partial w_\parallel} = -\nu(p_\perp - p_\parallel)$$

Assemble:

$$\frac{dp_\parallel}{dt} = -\hat{b}\cdot\nabla q_\parallel - (\nabla\cdot\hat{b})(q_\parallel - 2q_\perp) - p_\parallel(\nabla\cdot\vec{u}) - 2p_\parallel\hat{b}\hat{b}:\nabla\vec{u} - 2\nu(p_\parallel - p_\perp)$$

$$= -\nabla\cdot(\overbrace{\hat{b}q_\parallel}^{=\vec{q}_\parallel}) + 2q_\perp\nabla\cdot\hat{b} - 2\nu(p_\parallel - p_\perp)$$

$$-2p_\parallel\underbrace{(\hat{b}\hat{b}:\nabla\vec{u} - \nabla\cdot\vec{u})}_{=\frac{1}{B}\frac{dB}{dt}} - 3p_\parallel\underbrace{\nabla\cdot\vec{u}}_{=-\frac{1}{n}\frac{dn}{dt}} \qquad (21)$$

$$\boxed{p_\parallel\frac{d}{dt}\ln\frac{p_\parallel B^2}{n^3} = -\nabla\cdot\vec{q}_\parallel + 2q_\perp\nabla\cdot\hat{b} - 2\nu(p_\parallel - p_\perp)} \qquad (22)$$

this is to do with the so-called "second adiabatic invariant" or bounce invariant

Equations (20), (22) are known as CGL equations — often in the version without the heat fluxes, in which case they are referred to as "double-adiabatic" equations (but that approximation tends to be very wrong because heat fluxes are not small, especially for electrons). If we must keep the heat fluxes, then obviously we still need the kinetic equation (16) to calculate them — this is the usual story with kinetic theory, moment equations do not close.

You might wonder why we bothered to derive these equations then. Well,

1) we learned some physics: what sets $p_\perp$ and $p_\parallel$

2) we have identified the key quantities that we might want to invent closures for: $p_\perp, p_\parallel, q_\parallel, q_\perp$.

Both pressure anisotropies and heat fluxes cause instabilities and one might take the view that instead of solving the kinetic eqn (which is ill posed anyway) we should set them to marginal values!

1.8 Pressure Anisotropy.

Since this is going to be the key quantity, let's work out what it is. Using eqs (~~19~~) and (21), we get

$$\frac{d}{dt}(p_\perp - p_\parallel) = p_\perp\left(\frac{1}{B}\frac{dB}{dt} + \frac{1}{n}\frac{dn}{dt}\right) - \nabla\cdot\vec{q}_\perp - q_\perp\nabla\cdot\hat{b} - \nu(p_\perp - p_\parallel)$$
$$- \left\{p_\parallel\left(-2\frac{1}{B}\frac{dB}{dt} + 3\frac{1}{n}\frac{dn}{dt}\right) - \nabla\cdot\vec{q}_\parallel + 2q_\perp\nabla\cdot\hat{b} - 2\nu(p_\parallel - p_\perp)\right\}$$

$$= (p_\perp + 2p_\parallel)\frac{1}{B}\frac{dB}{dt} + (p_\perp - 3p_\parallel)\frac{1}{n}\frac{dn}{dt} - 3q_\perp\nabla\cdot\hat{b} - \nabla\cdot(\vec{q}_\perp - \vec{q}_\parallel) - 3\nu(p_\perp - p_\parallel) \tag{23}$$

If we choose to express things in terms of total pressure $p = \frac{2}{3}p_\perp + \frac{1}{3}p_\parallel$, then

$$p_\perp = p + \frac{1}{3}(p_\perp - p_\parallel)$$
$$p_\parallel = p - \frac{2}{3}(p_\perp - p_\parallel)$$

Things simplify a bit when collisions are dominant, pressure anisotropy small and can be found by setting the rhs of (23) to 0:

$$\boxed{\frac{p_\perp - p_\parallel}{p} \approx \frac{1}{\nu}\left[\frac{1}{B}\frac{dB}{dt} - \frac{2}{3}\frac{1}{n}\frac{dn}{dt} - \frac{\nabla\cdot(\vec{q}_\perp - \vec{q}_\parallel) + 3q_\perp\nabla\cdot\hat{b}}{3p}\right]} \tag{24}$$

Note. Under the same assumption of high collisionality,

$$q_\perp = \frac{1}{3}q_\parallel = -\frac{1}{2}n\frac{v_{th}^2}{\nu}\hat{b}\cdot\nabla T \quad \text{where } T = \frac{p}{n}$$

this prefactor depends on the coll. operator used.

This follows from expanding around a Maxwellian equilibrium. [Note this again: when $\nu \gg \omega$, equations are closed: Braginskii theory]

1.9 Heating

Finally, it is sometimes useful – and revealing – to know the equation for total energy

$$\int d^3\vec{w}\, \frac{mw^2}{2} f = p_\perp + \frac{1}{2} p_\parallel = \frac{3}{2} p = \frac{3}{2} n \underbrace{T}_{\text{by definition}}$$

Again using (19) and (21),

$$\underbrace{\frac{3}{2}\frac{d}{dt} nT}_{\displaystyle = \frac{3}{2} n \frac{dT}{dt} + \underbrace{\frac{3}{2} p}_{= \cancel{p_\perp} + \frac{1}{2}p_\parallel} \frac{1}{n}\frac{dn}{dt}} = -\nabla\cdot \underbrace{\left(\vec{q}_\perp + \frac{1}{2}\vec{q}_\parallel\right)}_{\equiv\, \vec{q} \;\leftarrow\; \text{heat flux}} + \underbrace{(p_\perp - p_\parallel)\frac{1}{B}\frac{dB}{dt}}_{\text{viscous heating}} + \underbrace{\left(\cancel{p_\perp} + \frac{3}{2} p_\parallel\right)\frac{1}{n}\frac{dn}{dt}}_{\text{compressional heating}} \qquad (25)$$

$$\boxed{\frac{3}{2} n \frac{dT}{dt} = -\nabla\cdot\vec{q} + (p_\perp - p_\parallel)\frac{1}{B}\frac{dB}{dt} + p_\parallel \frac{1}{n}\frac{dn}{dt}} \qquad (26)$$

Note that eqs. (23) and (26) can be used instead of (20), (22).

In a simple incompressible (collisional, heatfluxless) situation, the viscous heating term is explicitly positive:

$$(p_\perp - p_\parallel)\frac{1}{B}\frac{dB}{dt} = \frac{p}{\nu}\left(\frac{1}{B}\frac{dB}{dt}\right)^2 = \frac{p}{\nu}\left(\hat{b}\hat{b}:\nabla\vec{u}\right)^2$$

<u>Ex.</u> Work out the total energy conservation law for

$$\mathcal{E} = \underbrace{\frac{m_i n_i u^2}{2}}_{\text{kinetic}} + \underbrace{\frac{B^2}{8\pi}}_{\text{magnetic}} + \underbrace{\sum_s \frac{3}{2} n_s T_s}_{\text{thermal}} \qquad (27)$$