THEORETICAL METHODS IN
PLASMA PHYSICS

by

N. G. VAN KAMPEN

and

B. U. FELDERHOF

Institute for theoretical physics, University of Utrecht

1967

NORTH-HOLLAND PUBLISHING COMPANY – AMSTERDAM

Library of Congress Catalog Card Number 67–20023

Publishers:

NORTH-HOLLAND PUBLISHING COMPANY – AMSTERDAM

Sole distributors for U.S.A. and Canada:

INTERSCIENCE PUBLISHERS, *a division of*

JOHN WILEY & SONS, INC. – NEW YORK

Printed in The Netherlands

Preface

Plasma physics is of interest not only because its results are important for many different branches of physics and astronomy, but also because it constitutes an interesting field for the application of various methods of theoretical physics. In the present book the latter aspect is emphasized more than practical applications. It is mainly intended as an introduction for the student of theoretical physics who wants to become acquainted with plasma theory, rather than as a handbook for those who specialize in plasmas.

Accordingly the derivation of the basic equations, their structure, and their general consequences are more fully discussed than are their detailed solutions for special cases. Yet a number of examples, and of course the various types of waves, are treated in some detail. The solution of the linearized Vlasov equation is carried out very explicitly in agreement with the authors' personal interest. On the other hand, several subjects, such as turbulence and emission of radiation, have not been included.

The numerous "Problems" in the text roughly fall into three classes. Some are simply exercises, others are derivations or proofs that are needed in the text but can be left to the reader, and the third class consists of problems that extend the text or apply it to special cases of interest. We hope that they will not frustrate the reader, but rather stimulate a more active participation in the material presented.

As to the literature references, it was felt that it was neither possible, nor particularly profitable to strive at completeness. Rather we have tried to make a selection which will be useful to the reader without confusing him. At the end of each chapter "General References" are listed pertaining to the subject material in the chapter, while more specific references are given in footnotes.

We are greatly indebted to Mr. J. W. Sanders for a careful perusal of the manuscript and to Miss E. Wesenhagen for her devoted secretarial work.

N. G. VAN KAMPEN
B. U. FELDERHOF

TABLE OF CONTENTS

PREFACE . . . v

I. INTRODUCTION . . . 1

II. DERIVATION OF THE MAGNETOHYDRODYNAMIC EQUATIONS . . . 8
   1. The equations of motion for a conducting fluid . . . 8
   2. The magnetohydrodynamic equations . . . 11
   3. Additional simplifications . . . 14

III. DISCUSSION OF THE MHD EQUATIONS . . . 17
   1. Structure of the equations . . . 17
   2. Balance equations . . . 18
   3. Magnetic pressure . . . 20
   4. Motion of the magnetic field . . . 21

IV. STATIC SOLUTIONS OF THE MHD EQUATIONS . . . 28
   1. The equations for the static case . . . 28
   2. Incompressible fluid . . . 29
   3. The pinched discharge . . . 31
   4. Static solution for compressible fluid with $\boldsymbol{g} \neq 0$ . . . 35

V. STATIONARY SOLUTIONS . . . 39
   1. The equations for the stationary case . . . 39
   2. Force-free fields . . . 40
   3. Planar flow of a conducting viscous fluid . . . 45
   4. Terrestrial magnetism . . . 49

VI. WAVES IN MAGNETOHYDRODYNAMICS . . . 54
   1. MHD waves in an incompressible plasma . . . 54
   2. MHD waves in the direction of $\boldsymbol{H}^0$ (Alfvén waves) . . . 57
   3. Compressible plasma: the linearized equations . . . 60
   4. Solution of the eigenvalue problem (6) . . . 62
   5. The remaining modes . . . 64

VII. Instability in magnetohydrodynamics . . . 66
1. Instabilities in classical mechanics . . . 67
2. Lagrange formulation of the MHD equations . . . 69
3. Equations of motion for neighboring solutions . . . 73
4. Applications . . . 76
VIII. Statistical mechanics of an ionized gas in equilibrium . . . 82
1. Debye shielding . . . 82
2. Statistical mechanics of a mixture of ionized gases . . . 86
3. Statistical mechanics of the electron gas . . . 87
4. Density fluctuations in the electron gas . . . 94
5. Field fluctuations in the electron gas . . . 95
6. Alternative calculation of the free energy of the electron gas . . . 97
IX. Two-component theory . . . 101
1. The equations of two-component theory . . . 101
2. Plasma waves in a cold plasma . . . 105
3. Discussion of the transverse waves in a cold plasma . . . 111
4. Plasma waves in a hot plasma . . . 113
5. Alternative derivation of waves in a cold plasma . . . 117
X. Kinetic foundations of the many-component theory . . . 119
1. Kinetic equations for the many-component plasma . . . 119
2. Conservation equations for macroscopic quantities . . . 121
3. The equations of many-component theory . . . 123
4. Discussion of the two-component theory . . . 125
XI. The Vlasov theory . . . 130
1. The Vlasov equation . . . 130
2. Properties of the Vlasov equation . . . 131
3. Vlasov equation for longitudinal waves . . . 134
XII. Correct treatment of longitudinal plasma waves . . . 139
1. Mathematical preliminaries . . . 139
2. Exact stationary solutions . . . 142
3. The initial value problem . . . 145
4. The evolution of the density . . . 150
5. Comments on the irreversible behavior of the density . . . 153
XIII. Transverse waves in the Vlasov theory . . . 159
1. The equations for transverse waves . . . 159
2. Stationary solutions of equation (4) . . . 161
3. The initial value problem for equation (4) . . . 163
4. The initial value problem for the linearized Vlasov equation . . . 165

5. Cut-off at the velocity of light . . . 166
6. The relativistic Vlasov equation . . . 170
7. Dispersion law and index of refraction . . . 172
XIV. Anisotropy and external magnetic field . . . 178
1. The linearized Vlasov equation with external field . . . 179
2. Isotropic equilibrium distribution in external field . . . 180
3. Anisotropic equilibrium distribution, no external field . . . 182
4. Anisotropic equilibrium distribution with external field . . . 184
XV. Collisions . . . 188
1. The Landau equation . . . 189
2. Discussion of the Landau equation . . . 195
3. Kinetic theory . . . 200
4. The method of Bogolyubov . . . 204
Author Index . . . 209
Subject Index . . . 212

CHAPTER I

INTRODUCTION

Definition. A plasma is a fluid (gas or liquid) in which the charged particles (ions and unbound electrons) are sufficiently numerous to influence the behavior of the fluid appreciably. The name "plasma" is due to Langmuir.

Occurrence. More than ninety-five percent* of all matter in the universe occurs in the form of plasma, the rest consists of dust clouds. The properties vary considerably depending on the temperature, density, composition, dimensions, strength of the magnetic field; see Table I. Plasma physics is used in the study of electrolytes and gas discharges, in astrophysics (corona, sun spots, nebulae), geophysics (ionosphere, the earth's magnetism), solid state physics (electrons in metals and semiconductors), and in astronautics (ionization of the air when re-entering). Yet the main stimulus came from the interest in thermonuclear fusion, which requires a very high temperature (of the order of 10^8 degrees) at which all matter is ionized.

Principal features. Plasmas are very good conductors owing to the freely moving charged particles. As a consequence strong electric currents occur so that *magnetic interaction is important*. Moreover there are often magnetic fields due to external sources, for instance the earth's magnetic field in the case of the ionosphere. Slow motions of the plasma can therefore be de scribed by the so-called *magnetohydrodynamic equations* (chapter II), which are the hydrodynamic equations with additional terms describing the inter action with the magnetic field. It is often a good approximation to treat the plasma as an ideal conductor, that is, to put the electrical resistivity equa to zero.

As a consequence of the high conductivity, magnetic fields do not die ou rapidly, because the only way their energy dissipates is through the relativel

* Astrophysical data such as this figure should not be taken too literally.

small electrical resistance. (The fact that the magnetic field does not die out when there is no electrical resistance is vellknown in the case of superconductors.)

For plasmas of astronomical dimensions the interaction through the magnetic field is predominant even when the conductivity is not very good by laboratory standards (chapter II).*

Waves and instabilities. Electrical interactions are much less important, since any electrical charge density disappears rapidly owing to the good conduction. When studying slow phenomena the plasma may therefore be treated as neutral ("quasi-neutrality approximation"). For rapid phenomena, however, the electrostatic interaction is important and gives rise to high frequency oscillations of the electrons and ions with respect to each other (*longitudinal plasma waves*, chapters IX and XII).

In a moving plasma the magnetic field will, in general, vary with time; hence there will be a coupling with the electric field according to Faraday's induction law. Vice versa, for very rapid motion the variation of the electric field gives rise to an additional magnetic field, due to Maxwell's displacement current. This interaction between both fields gives rise to *transverse electromagnetic waves*. They are nothing but the familiar electromagnetic waves modified by the presence of the charged plasma particles.

In addition to the longitudinal and transverse waves there exist mixed forms when the plasma is not isotropic, for example when there is an overall magnetic field due to external sources. The many different types of waves are important in the study of plasmas, because on the one hand they can be easily observed experimentally, and on the other hand they can readily be treated theoretically.

A plane wave whose frequency is found to be complex may have a growing amplitude and then constitutes an *instability*. The large variety of instabilities existing in a plasma is the main reason why it is impossible so far to maintain in a laboratory experiment a hot plasma longer than microseconds. The investigation of instabilities is therefore indispensable for thermonuclear research. However, we shall not go into the details, but only treat the basic ideas of the theory of instabilities (chapters VII and XIV).

Experimental observations. The typical properties of a plasma are consequences of the high degree of ionization and the resulting high conductivity. Experiments have been done with conducting liquids, viz. mercury and molten sodium, but their conductivity is relatively low. For normal gas discharges the degree of ionization is only of the order of one per cent, so that collisions with neutral particles still predominate; moreover there are the complications of ionizing collisions and recombination, while the influence of cathode and anode is not negligible.*

* A striking, but not quite relevant, example is the fact that for a proton on the earth the 10^{-7} gauss of the sun's magnetic field exerts a force that is 10^4 times the sun's gravitational pull.

Much higher ionization is reached in specially built machines (stellarator, OGRA, tokomak). In these experiments, however, difficulties with confinement and stabilization make it hard to obtain the clear-cut information that is desired by the theoreticians. Astronomical plasmas are often highly ionized, but they have the drawback that the circumstances are usually very complicated and not very well known. Some information is obtained from experiments with the electron gas in metals and semiconductors.**

In general, the agreement between theory and experiment is not bad, but the amount of information obtainable from observation leaves much to be desired.

Kinetic aspects. In the ordinary kinetic theory of neutral gases it is essential that the interaction between any two molecules has a short range, the range being much less than the average distance between neighboring molecules. This enables one to regard the molecules as free particles, whose velocities undergo sudden changes at random moments. Collisions of three or more molecules are rare and enter only into the higher order approximations (higher powers of the density).

In plasmas, on the other hand, the interaction consists mainly of the *long range Coulomb potential*, so that each particle continually interacts with a whole region around it. Actually the range is even infinite, inasmuch as the force exerted on one particular electron by all other electrons in a given solid angle does not decrease with increasing distance. It will be shown in chapter VIII, however, that the polarization of the plasma gives rise to a screening, which reduces the effective range to the "Debye length"; this reduced range is finite but still much larger than the interparticle distance.

In order to deal with the long range interaction, different methods have to be used than in kinetic gas theory. It appears reasonable to define an

* Special arrangements using cesium vapor, however, have reached an ionization of 90 per cent [J. Y. Wada and R. C. Knechtli, Proc. I.R.E. **49**, 1926 (1961)] and even 99% [N. Rynn in: *Engineering Aspects of Magnetohydrodynamics*, N. W. Mather and G. W. Sutton eds. (Gordon and Breach, New York 1964)].

** See for example: *Plasma Effects in Solids*, Paris Conference 1964 (Dunod, Paris 1965) average, or "smeared out" electric field, being the electric field averaged over a volume that is macroscopically small* but still contains many charged particles. Similarly, the magnetic interaction will also be described by an average, or smeared out magnetic field. As mentioned before, this field plays a dominant role in plasmas.

After the average field has been taken into account in this way, there still remain

(i) the force arising from the actual local field, after subtraction of the average field;

(ii) the usual short range forces between particles, such as the Van der Waals force.**

The effect of these remaining forces is regarded as *collisions*. These collisions become negligible if the density is low, the temperature high, and the phenomena considered very rapid.

Theoretical classification. It is easy to write down the (classical, nonrelativistic) equations of motion of the combined system of all particles and the electromagnetic field. In principle they contain all properties of the plasma, but in practice one has to make drastic approximations to derive useful results. Different choices of approximation lead to the following four models, each of which applies in different circumstances.

A. Interaction between particles is entirely negligible, so that they move as *independent particles* in the electromagnetic field of external sources. Examples: cathode ray tube, high energy accelerators, cosmic rays, Van Allen belts.*** The theory amounts to solving the equations of motion of a particle in a given field. This is not really plasma theory, but rather the mechanics of a charged particle in a given field. This subject will not be treated in this book.†

B. The interaction can be fully described by the smeared out electromagnetic field. This is called the *collisionless* or *Vlasov plasma*.†† Collisions

* That is: small compared to the distance over which macroscopic quantities (density, flow velocity, current, etc.) vary appreciably.

** The completely ionized *hydrogen plasma* ($T \gtrsim 30\,000$ °K) consists of protons and electrons, so that these short range forces are negligible.

*** Review articles: R. S. White, Phys. Today **19**, 25 (Oct. 1966). H. Elliot, Rept. Progr. Phys. **26**, 145 (1963); J. A. Van Allen in: *Space Science*, D. P. Le Galley ed. (Wiley, New York 1963).

† T. G. Northrop, *The Adiabatic Motion of Charged Particles* (Interscience, New York 1963); B. Lehnert, *Dynamics of Charged Particles* (North-Holland, Amsterdam 1964).

†† First studied by A. Vlasov, J. Phys. USSR 9, 25 (1945).

are neglected, or introduced as higher order corrections. This is correct when the average time between collisions is large compared to the time interval that characterizes the phenomenon one is interested in, e.g., the period of a plasma wave. The Vlasov plasma will be studied in chapters XI through XIV and the corrections due to collisions are the subject of chapter XV.

C. The collisions between ions are very frequent, so that they are able to maintain a local Maxwell velocity distribution. The ions may then be treated as a fluid, described by a local density, local velocity and local temperature. The same is supposed to apply to the electrons and possible neutral atoms or molecules. Thus the plasma is treated as a mixture of two or more interpenetrating gases; this is called the *two-component* or *many-component theory*. Admittedly this physical model is somewhat doubtful, but the two-component theory is often used as a shortcut for obtaining qualitatively correct information concerning plasma waves. The two-component plasma is studied in chapters IX and X.

D. Collisions between ions and neutral particles are also frequent, so that they are to be treated as *a single fluid* with one local velocity and one local temperature. The motion of the electrons with respect to this fluid gives rise to an electric current. This current is supposed to obey Ohm's law; in other words, the electrons collide so frequently with the heavy particles that an electric field acting on the electrons will give rise to a constant drift velocity, proportional to that field. One is thus led to the picture of a conducting fluid, applicable to high density, low temperature plasmas, when the phenomena are not very rapid. Accordingly one also neglects the Maxwell term $\partial \boldsymbol{D}/\partial t$ in the equations for the field. The resulting theory is called *magnetohydrodynamics*. It applies to the flow of conducting liquids in a magnetic field, to the earth's core, and to sufficiently slow phenomena in ionized gases. This case is treated in chapters II through VII.

Nomenclature. We have distinguished in the field of plasma physics two limiting cases: magnetohydrodynamics (very frequent collisions), Vlasov theory (negligible or almost negligible collisions); in addition there is the two-component theory as a hybrid mixture of both. Instead of Alfvén's term magnetohydrodynamics (MHD) one also uses "hydromagnetics". Sometimes these terms are used for the whole field of plasma physics; what we call MHD is then referred to as magneto-fluid dynamics or magneto-gas dynamics. Often the various cases are not clearly distinguished and all names are used interchangeably. Rather well established, however, is the distinction between "magnetohydrodynamic waves" (slow oscillations of

TABLE I

	n_e	n_{neutr}	T	$e^2/3kT$	$n_e^{-\frac{1}{3}}$	$\varkappa^{-1}$	$n_e\varkappa^{-3} = \Lambda/12\pi$	ω_p	ν_c	σ	H	ω_{ce}
	cm^{-3}	cm^{-3}	°K	cm	cm	cm		sec^{-1}	sec^{-1}	sec^{-1}	gauss	sec^{-1}
Interstellar gas												
H I-regions	10^{-3}	1	10^2	10^{-5}	10	10^3	10^6	10^3	10^{-4}	10^9	10^{-6}	10
H II-regions	1	0	10^4	10^{-7}	1	5×10^2	10^8	5×10^4	10^{-4}	10^{12}	10^{-6}	10
Sunspots	10^{11}	10^{14}	4×10^3	10^{-7}	10^{-4}	10^{-3}	10^2	10^{10}	10^6	10^{13}	3×10^3	5×10^{10}
Corona	10^8	0	10^6	10^{-9}	10^{-3}	0.5	10^6	5×10^8	10^2	10^{14}	10^{-4}	10^3
Interplanetary plasma	10^2	0	10^5	10^{-2}	10^{-1}	10^2	10^8	5×10^5	10^{-3}	10^{13}	10^{-4}	10^3
Ionosphere F-layer	10^6	10^{10}	10^3	10^{-6}	10^{-2}	10^{-1}	10^4	5×10^7	10^3	10^{11}	0.2	10^6
D-layer	10^2	10^{15}	10^2	10^{-5}	10^{-1}	5	10^4	5×10^5	10^7	10^3	0.2	10^6
Gas discharge	10^{14}	10^{16}	10^4	10^{-7}	10^{-5}	5×10^{-5}	10	5×10^{11}	10^8	10^{14}	10^3	10^{10}
Dilute hot plasma	10^{12}	0	10^6	10^{-9}	10^{-4}	5×10^{-3}	10^5	5×10^{10}	10^5	5×10^{15}	10^3	10^{10}
Dense hot plasma	10^{16}	0	10^5	10^{-9}	10^{-5}	5×10^{-5}	10^3	5×10^{12}	10^9	10^{16}	10^4	10^{11}
Thermonuclear plasma	10^{16}	0	10^8	10^{-11}	10^{-5}	5×10^{-4}	10^6	5×10^{12}	10^6	10^{19}	10^5	10^{12}
Mercury	10^{23}	0	10^2	10^{-5}	10^{-8}	10^{-10}	10^{-7}	10^{16}	10^{15}	10^{16}	10^3	10^{10}

the whole fluid, which can be treated by MHD; in particular Alfvén waves, see chapter VI) and "plasma waves" (rapid oscillations of the positive and negative charge carriers with respect to each other, which are found in the Vlasov theory and in the two-component theory). Equally well established is the distinction between "magnetohydrodynamic instabilities" and "microinstabilities"; the latter name comprises all instabilities that do not appear in the MHD approximation.

General references

G. H. A. Cole, *Fluid Dynamics* (Methuen, London 1962) ch. X.

L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford 1960) ch. VIII.

J. D. Jackson, *Classical Electrodynamics* (Wiley, New York 1962) ch. X.

T. G. Cowling, *Magnetohydrodynamics* (Interscience, New York 1957).

L. A. Arzimovich, *Elementary Plasma Physics* (Blaisdell, New York 1965).

C. L. Longmire, *Elementary Plasma Physics* (Interscience, New York 1963).

M. A. Uman, *Introduction to Plasma Physics* (McGraw-Hill, New York 1964).

E. Fünfer and G. Lehner in: *Ergebnisse der exakten Naturwissenschaften* **34** (Springer, Berlin 1962).

J. G. Linhart, *Plasma Physics* (North-Holland, Amsterdam 1960).

V. C. A. Ferraro and C. Plumpton, *An Introduction to Magneto-Fluid Mechanics* (Oxford University Press, Oxford 1961).

S. Chandrasekhar, *Plasma Physics* (University of Chicago Press, Chicago 1960).

CHAPTER II

DERIVATION OF THE MAGNETOHYDRODYNAMIC EQUATIONS

In this chapter we shall derive the basic equations for an electrically conducting, not necessarily incompressible, fluid in motion. The fluid is locally characterized by its density ρ, temperature T and velocity $\boldsymbol{V}$. Furthermore it has a charge density τ and electrical current density $\boldsymbol{j}$. Finally there is an electromagnetic field, described by $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{B}$. All these quantities are functions of the spatial coordinates (x, y, z) and the time t, and our task is to derive equations of motion which determine their behavior in time. We shall encounter additional quantities such as the pressure p and the conductivity σ, but these are supposed to be uniquely determined by ρ and T through equations of state or material equations.

1. The equations of motion for a conducting fluid

I. As for any continuous medium, conservation of mass is expressed by the *equation of continuity*

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \boldsymbol{V} = 0. \tag{1}$$

Alternatively one may write

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \boldsymbol{V} = 0,$$

where D/Dt is the time derivative *moving along with the fluid* or "substantial" derivative:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y} + V_z \frac{\partial}{\partial z} \equiv \frac{\partial}{\partial t} + (\boldsymbol{V} \cdot \nabla).$$

Problem. Let Q be an arbitrary quantity depending only on the local state of the fluid, and let Ω be a fixed volume enclosed by the surface S.

Prove that

$$\frac{d}{dt} \int_\Omega \rho Q \, d^3\boldsymbol{r} = \int_\Omega \rho \frac{DQ}{Dt} \, d^3\boldsymbol{r} - \oint_S \rho Q \boldsymbol{V} \cdot d\boldsymbol{S}.$$

Interpret this formula and show that if the volume moves with the fluid the same formula holds without the last term.

II. The *equation of motion for* $\boldsymbol{V}$ follows from conservation of momentum of a fluid element:

$$\rho \frac{D\boldsymbol{V}}{Dt} = \nabla \cdot \Pi + \boldsymbol{K}. \tag{2}$$

Here $\Pi \equiv (\Pi_{ij})$ is the (symmetrical) stress tensor and

$$(\nabla \cdot \Pi)_j = \sum_i (\partial/\partial x_i) \Pi_{ij}$$

is the force exerted by the surrounding medium. $\boldsymbol{K}$ is the sum of all other forces per unit volume; i.e.,

$$\boldsymbol{K} = \tau \boldsymbol{E} + \frac{1}{c} \boldsymbol{j} \wedge \boldsymbol{H} + \rho \boldsymbol{g}. \tag{3}$$

The velocity of light enters through our use of Gaussian units. $\boldsymbol{g}$ is the acceleration due to gravity and is therefore a given function of x, y, z, t.*

Equation (2) is exact, but useless as long as Π is unknown. In hydrodynamics one assumes that Π has the following form**

$$\Pi_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j} \right) - (\tfrac{2}{3}\mu - \mu') \left(\sum_k \frac{\partial V_k}{\partial x_k} \right) \delta_{ij}. \tag{4}$$

μ is the ordinary viscosity and μ' the bulk viscosity; we shall omit the latter because it usually is too small to observe. Then (2) becomes

$$\begin{aligned} \rho \frac{D\boldsymbol{V}}{Dt} = & -\nabla p + \mu \left\{ \nabla^2 \boldsymbol{V} + \tfrac{1}{3} \nabla (\nabla \cdot \boldsymbol{V}) \right\} \\ & + \tau \boldsymbol{E} + \frac{1}{c} \boldsymbol{j} \wedge \boldsymbol{H} + \rho \boldsymbol{g}. \end{aligned} \tag{5}$$

* This is no longer true if one takes into account the internal gravitational force of the plasma on itself. Furthermore, if one wants to include the Coriolis force (earth's core, chapte v) $\boldsymbol{g}$ also contains a term depending on $\boldsymbol{V}$.

** Because this is the only possible form satisfying the requirements: (i) it includes the elementary laws of hydrostatics and viscous flow; (ii) it has the correct transformation properties for an isotropic medium; (iii) it depends on $\boldsymbol{V}$ only through the first derivatives. By means of the hypothesis (4) one short-cuts the kinetic considerations.

The first line is the wellknown Navier–Stokes equation for viscous flow.

III. The electromagnetic field is of course governed by *Maxwell's equations*. In plasma physics the dielectric constant and the magnetic permeability are always put equal to one.* It is true that the ions and the neutral particles can be polarized, but for sufficiently high degree of ionization this effect is much smaller than that of the charges. We therefore have

$$\operatorname{curl} \boldsymbol{H} = \frac{4\pi}{c} \boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \quad (6) \qquad \operatorname{div} \boldsymbol{H} = 0 \quad (7)$$

$$\operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} \quad (8) \qquad \operatorname{div} \boldsymbol{E} = 4\pi\tau . \quad (9)$$

Conservation of charge follows from these equations:

$$\frac{\partial \tau}{\partial t} + \operatorname{div} \boldsymbol{j} = 0 . \quad (10)$$

IV. The relation between current density and electromagnetic field is given by Ohm's law. This law is a consequence of the interaction between electrons and heavy particles of the fluid and therefore only holds in its usual form for an observer moving locally with the fluid, so that

$$\boldsymbol{j}' = \sigma \boldsymbol{E}' ,$$

where $\boldsymbol{j}'$ and $\boldsymbol{E}'$ are the current and electric field strength measured by an observer moving with the local velocity $\boldsymbol{V}$. (In general σ should be a tensor, but we assume that the fluid is sufficiently isotropic** and that the Hall effect may be neglected.) Transforming back and supposing that $\boldsymbol{V}$ is a non-relativistic velocity, $|\boldsymbol{V}| \ll c$, one finds

$$\boldsymbol{j} - \tau \boldsymbol{V} = \sigma \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{V} \wedge \boldsymbol{H} \right). \quad (11)$$

Incidentally, the Joule heat is given by

$$\frac{1}{\sigma} (\boldsymbol{j}')^2 = \boldsymbol{j} \cdot \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{V} \wedge \boldsymbol{H} \right) - \tau \boldsymbol{V} \cdot \boldsymbol{E} . \quad (12)$$

* We are referring to the dielectric constant arising from the polarizability of molecules or ions. There exists an alternative description of plasmas, in which essential use is made of another kind of dielectric constant, see ch. IX, sec. 5.
** This is no longer true if the cyclotron radius of the particles in the magnetic field is not large in comparison with the mean free path, as in the corona.

The right-hand side represents the total work performed by the field on the particles, minus the work performed on the charge density moving with the fluid, which of course does not appear as heat.

V. Finally there is the *equation of state* of the fluid

$$p = f(\rho, T) . \quad (13)$$

In order to find the temperature T one must invoke the energy equation. For the internal energy per unit mass, u, one has according to the first law of thermodynamics

$$\frac{Du}{Dt} + p \frac{D}{Dt} \frac{1}{\rho} = \frac{q}{\rho} . \quad (14)$$

q is the heat per unit volume that is produced irreversibly as a result of heat conduction, viscosity and electrical conduction; for completeness we give the explicit expression* (in obvious notation):

$$q = \operatorname{div} (\lambda \operatorname{grad} T) + \mu \left\{ (\partial_i V_j)(\partial_j V_i) + (\partial_i V_j)(\partial_i V_j) - \tfrac{2}{3} (\partial_i V_i)^2 \right\} + \frac{1}{\sigma} (\boldsymbol{j}')^2 . \quad (15)$$

Of course there is also a relation between u and T of the form

$$u = g(\rho, T) . \quad (16)$$

Problem. Prove that

$$\frac{\partial g}{\partial T} = c_V , \qquad \rho \frac{\partial g}{\partial \rho} = \frac{p}{\rho} - \frac{T}{\rho} \frac{\partial f}{\partial T} .$$

Hence g is known if one knows the specific heat $c_V(\rho, T)$.

2. The magnetohydrodynamic equations

The equations (1), (5), (6), (7), (8), (9), (11), (13), (14), (15), (16) constitute a complete set of equations of motion for a compressible, viscous, electrically conducting fluid in interaction with the electromagnetic field. In magnetohydrodynamics one usually makes the following additional simplifications.

* H. Lamb, *Hydrodynamics* (Cambridge University Press, Cambridge 1932); S. R. de Groot and P. Mazur, *Non-Equilibrium Thermodynamics* (North-Holland, Amsterdam 1962).

(i) The viscosity term in (5) is neglected. An estimate of the importance of this term is obtained from the following dimensional analysis. Let V denote the order of magnitude of $\boldsymbol{V}$, and L the distance over which the flow pattern changes appreciably. Then the viscosity term is of order $\mu V/L^2$. On the other hand the convection term $\rho(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}$ is of order $\rho V^2/L$. Their ratio is the dimensionless number

$$R_e = \frac{VL}{\mu/\rho} \quad \text{(Reynolds' number).}$$

If $R_e \gg 1$ one may expect that viscosity is unimportant; compare table II.

(ii) One neglects the term $\tau\boldsymbol{E}$ in (5), but *not* $4\pi\tau$ in (9). The idea is that owing to the high conductivity small charges cause large currents, so that the magnetic force on the medium is much larger than the electric force. At the same time one neglects the convection term $\tau\boldsymbol{V}$ in (11) because it is of the same relative order as $\tau\boldsymbol{E}$ in (5). This is called the quasi-neutrality approximation.

In order to trace the orders of magnitude more closely we consider a charge density τ extending over a region of dimension L. Then (9) yields $E \sim L\tau$, hence $j \sim \sigma L\tau$. From (6) it then follows that $H \sim \sigma L^2\tau c^{-1}$, so that

$$\tau E \sim L\tau^2, \qquad j \wedge H/c \sim \sigma^2 L^3 \tau^2 c^{-2}.$$

Hence the ratio of magnetic and electric force is

$$\left(\frac{\sigma L}{c}\right)^2. \tag{17}$$

One finds the same order of magnitude for the ratio of the convection term $\tau\boldsymbol{V}$ in (11) and the term $(\sigma/c)\,\boldsymbol{V}\wedge\boldsymbol{H}$. See Table II. Note that even for small σ the magnetic field plays a predominant role provided that L is large. This is

TABLE II

	L	V	μ/ρ	R_e	σ	$\left(\frac{\sigma L}{c}\right)^2$	R_M
	cm	cm/sec	cm²/sec		sec⁻¹		
Earth's core	10^8	10^{-1}	10^{-3}	10^{10}	10^{15}	10^{25}	10^2
Ionosphere	10^7	10^4	10^7	10^4	10^8	10^9	10^{-1}
Solar atmosphere	10^8	10^5	10^4	10^9	10^{14}	10^{23}	10^7
Mercury	10	10^2	10^{-3}	10^6	10^{16}	10^{13}	10^{-1}
Gas discharge	10	10^7	10^2	10^6	10^{14}	10^9	10^2

the reason why astronomical plasmas are entirely dominated by magnetic forces. It is impossible to attain sufficiently high σ in the laboratory to equal these high values of $\sigma L/c$.

(iii) As remarked in chapter I, the Maxwell term $\partial E/\partial t$ in (6) is omitted because it is only important for rapid variations of the electromagnetic field, whereas magnetohydrodynamics only applies to rather slow phenomena anyway. This entails that (10) is replaced by

$$\operatorname{div} \boldsymbol{j} = 0.$$

One is not allowed to conclude that the charge density must be independent of t, but only that it changes too slowly to contribute noticeably to div $\boldsymbol{j}$. A simple dimensional analysis clarifies the meaning of "slow": the frequencies should be small compared to σ.

(iv) Finally one generally replaces equations (13), (14), (15), (16) by a relation between p and ρ alone,

$$p = f(\rho). \tag{18}$$

By calculating u from (14) and (15), subsequently solving T from (16) and inserting into (13) one obtains in principle p as a function of ρ. This does not yet permit to put $p = f(\rho)$, because the functional relation found here depends through (14) on the flow pattern. However, the last two terms in (15) are always neglected and usually also the first one. Then $q = 0$, and (14) expresses that the change of state of each fluid element separately is adiabatic. If one further assumes that the plasma was originally in equilibrium then the states of all fluid elements lie on the same adiabatic curve, so that there is a unique relation $p = f(\rho)$. For most plasmas one may take the ideal gas law

$$p = C\rho^\gamma, \text{ with } \gamma = c_p/c_V. \tag{19}$$

The assumptions that the changes of state are adiabatic and that the states of all fluid elements lie on the same adiabatic are both reasonable in the study of wave motions, but often questionable in other cases. Frequently they are made for reasons of simplicity rather than correctness. In the case of static gaseous plasmas one employs of course the isothermal equation of state

$$p = C\rho \text{ with } C = kT/m, \tag{20}$$

m being the mass of the atoms or ions.

With the approximations (i) through (iv) the equations of motion for our conducting fluid take the form of the usual magnetohydrodynamic equations, viz.,

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \boldsymbol{V} = 0 \tag{21a}$$

$$\rho \frac{D\boldsymbol{V}}{Dt} = - \operatorname{grad} p + \frac{1}{c} \boldsymbol{j} \wedge \boldsymbol{H} + \rho \boldsymbol{g} \tag{21b}$$

$$\operatorname{curl} \boldsymbol{H} = \frac{4\pi}{c} \boldsymbol{j} \qquad \operatorname{div} \boldsymbol{H} = 0 \tag{21c}$$

$$\operatorname{curl} \boldsymbol{E} = - \frac{1}{c} \frac{\partial \boldsymbol{H}}{\partial t} \tag{21d}$$

$$\boldsymbol{j} = \sigma \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{V} \wedge \boldsymbol{H} \right) \tag{21e}$$

$$p = f(\rho) . \tag{21f}$$

Problem. Why are (9) and (10) not included in these MHD equations?
Problem. Write the equations (21) in electromagnetic units and show that the only difference is that c disappears.
Problem. Write equations (1), (5), (6), (7), (8), (9), and (11) in dimensionless form and trace again the meaning of the approximations we have made.

3. Additional simplifications

(v) The conductivity σ is really a function of ρ (and possibly T), but it is usually treated as a constant, because that will not give rise to qualitative differences. As a simple, but important limiting case one often considers the *ideal conductor*: $\sigma = \infty$. Then (21e) becomes

$$\boldsymbol{E} = - \frac{1}{c} \boldsymbol{V} \wedge \boldsymbol{H} \tag{22}$$

so that $\boldsymbol{E}$ and $\boldsymbol{H}$ are perpendicular to each other.

This ideal conductor approximation is valid in the limiting case when (17) is very large. This is true for highly ionized plasmas, since they have a large σ. It is also true for low σ, provided that L is large, like in astronomical plasmas.

(vi) Another approximation consists in treating the plasma as an *incompressible fluid*, which amounts to replacing (21f) with the equation ρ = constant = ρ^0. Consequently (21a) reduces to div $\boldsymbol{V} = 0$, while in (21b) the pressure p is an independent quantity, no longer linked to the density. We shall investigate the applicability of this approximation.

First we rewrite (21a) in the form

$$\operatorname{div} \boldsymbol{V} = - \frac{1}{\rho} \frac{\partial \rho}{\partial t} - \frac{1}{\rho} \boldsymbol{V} \cdot \operatorname{grad} \rho .$$

The term on the left is of order $V/L \sim 1/t$, where t characterizes the time in which the state of the plasma changes appreciably. The first term on the right is of order $\delta\rho/\rho^0 t$, where $\delta\rho = \rho - \rho^0$ is the deviation of the actual density from the assumed constant density ρ^0. Similarly, the second term on the right is of order $(V/L)(\delta\rho/\rho^0)$. Clearly both these terms are of relative order $\delta\rho/\rho^0$ compared to the term on the left, and are therefore negligible when $\delta\rho/\rho^0 \ll 1$.

Next consider the effect of the approximation on (21b). By neglecting $\delta\rho$ one makes an error in the left-hand member of order $\delta\rho V/t$, which is again relatively small when $\delta\rho/\rho \ll 1$. In the right-hand member, however, one does not neglect the variation of p, that is, one does not neglect terms of order $f'(\rho)\delta\rho/L$. This is consistent if $f'(\rho)/L \gg V/t$. As the order of magnitude of $f'(\rho)$ will be the same as its value kT/m for the ideal gas, the condition becomes

$$\frac{kT}{m} \gg \frac{VL}{t} \sim V^2 .$$

This states that *the fluid motion should be slow compared to the heat motion of the molecules.* This is hardly an additional restriction, since MHD can only be applied to slow phenomena anyway.

Finally we have to investigate when $\delta\rho/\rho^0 \ll 1$. Of course, this will be true for conducting fluids like mercury or the earth's core (comp. ch. v, sec. 4). It is also true, however, for very hot plasmas owing to the fact that, according to (20), the pressure rises very steeply with increasing ρ. Indeed, the order of magnitude of the pressure variations is given by the "magnetic pressure" $H^2/8\pi$ (ch. III, sec. 3), where H denotes the order of magnitude of the magnetic field strength. According to (20) this corresponds to $\delta\rho \sim mH^2/8\pi kT$; hence we have $\delta\rho \ll \rho^0$ if

$$\beta \equiv \frac{8\pi n^0 kT}{H^2} \gg 1,$$

where n^0 is the particle density. Thus *for high β and slow variations the plasma may be treated as incompressible.* For $\beta \ll 1$ the gas pressure p may often be neglected.

(vii) It is hardly a restriction to assume a gravitational potential Φ,

$$\boldsymbol{g} = -\operatorname{grad} \Phi . \tag{23}$$

Moreover, in many cases the effect of gravity is negligible, so that one may then put $\boldsymbol{g} = 0$.

Problem. Prove that, if (23) holds,

$$\rho \frac{d\boldsymbol{V}}{dt} = -\nabla p - \rho\, \nabla(\Phi + \tfrac{1}{2}V^2) + \rho \boldsymbol{V} \wedge \operatorname{curl} \boldsymbol{V} + \frac{1}{c} \boldsymbol{j} \wedge \boldsymbol{H} .$$

Problem. Show that for an incompressible fluid the case with arbitrary Φ can be reduced to that with $\Phi = 0$.

General references

J. A. Shercliff, *A Textbook of Magnetohydrodynamics* (Pergamon, Oxford 1965) chs. II, III.
A. G. Kulikovskiy and G. A. Lyubimov, *Magnetohydrodynamics* (Addison-Wesley, Reading, Mass. 1965) ch. I.
W. B. Thompson, *An Introduction to Plasma Physics* (Pergamon, Oxford 1962) ch. IV.
– Rep. Progr. Phys. **16**, 363 (1961).
– in: *Advanced Plasma Theory*, M. N. Rosenbluth ed. (Academic Press, New York 1964).
Shih-I Pai, *Magnetogasdynamics and Plasma Dynamics* (Springer, Vienna 1962).
L. E. Kalikhman, *Elements of Magnetogasdynamics* (Saunders, Philadelphia 1967).
A. Jeffrey, *Magnetohydrodynamics* (Oliver and Boyd, Edinburgh 1966).

CHAPTER III

DISCUSSION OF THE MHD EQUATIONS

1. Structure of the equations

The seven equations of motion (II, 21) constitute a system of partial differential equations for the field quantities, ρ, $\boldsymbol{V}$, p, $\boldsymbol{j}$, $\boldsymbol{H}$, $\boldsymbol{E}$, considered as functions of x, y, z, t. One can distinguish two types of equations.

(i) Four "*condition equations*", which do not contain time derivatives, and hence are constraints on the possible values of the field quantities at every instant of time. They express the fact that more field quantities have been introduced than are needed to determine uniquely the momentary state of the system. For example, (II, 21e) can be used to eliminate p (unless, of course, the fluid is incompressible).

(ii) Three "*true equations of motion*", which do contain time derivatives and hence describe the evolution of the system.

Such a set of equations must satisfy two requirements.

(i) *Compatibility*: for arbitrary initial values of the field quantities, which obey the condition equations, it should be possible to find solutions for all subsequent times which again obey the condition equations.

(ii) *Completeness*: the solutions must be uniquely determined by the initial values.*

Problem. Verify that the MHD equations formally satisfy these requirements.

Remark. Formally it is true that for given initial situation ($t = 0$) the equations not only uniquely determine the states for $t > 0$, but also those for $t < 0$. Nevertheless there is an essential difference between past and future. This is apparent from the fact that a small change in the initial state also entails a small change in subsequent states; the dependence of the future

* Newton's equation of motion, strictly speaking, does not satisfy this requirement, because it is a second order differential equation. This can be trivially remedied, however, by introducing the velocity or the momentum as an additional variable.

on the initial state is *continuous*. On the other hand a small change in the state at $t = 0$ may correspond to an enormous modification in previous states; thus the dependence of the past on the presence is not continuous.* This is a general phenomenon of irreversible equations in physics. In our case irreversibility enters by Ohm's law (II, 21d). For ideal conductors, $\sigma = \infty$, the equations are reversible and there is no such difference between past and future.

Problem. Formulate and prove invariance for time reversal of the MHD equations in the case $\sigma = \infty$.

It is often useful to diminish the number of condition equations by eliminating $\boldsymbol{j}$ and $\boldsymbol{E}$. One then finds for (II, 21b) two alternative forms:

$$\rho \frac{DV}{Dt} = -\nabla p - \frac{1}{4\pi} \boldsymbol{H} \wedge \operatorname{curl} \boldsymbol{H} + \rho \boldsymbol{g} \tag{1a}$$

$$= -\nabla \left(p + \frac{H^2}{8\pi} \right) + \frac{1}{4\pi} (\boldsymbol{H} \cdot \nabla) \boldsymbol{H} + \rho \boldsymbol{g} . \tag{1b}$$

And by taking the curl of (II, 21d)

$$\frac{\partial \boldsymbol{H}}{\partial t} = \operatorname{curl} (\boldsymbol{V} \wedge \boldsymbol{H}) + \eta \nabla^2 \boldsymbol{H}, \qquad \eta = \frac{c^2}{4\pi\sigma}, \tag{2}$$

where σ has been assumed constant; η is a measure for the specific resistivity.** Equations (1), (2) together with $\operatorname{div} \boldsymbol{H} = 0$ are equivalent with (II,21b), (II, 21c), (II, 21d).

2. Balance equations

Equation (II, 21a) is the equation of mass balance or mass conservation, while (II, 21b) expresses the balance of momentum in the fluid. The latter equation can be given a more transparent form by means of the identity

$$\frac{1}{c} \boldsymbol{j} \wedge \boldsymbol{H} = \frac{1}{4\pi} (\operatorname{curl} \boldsymbol{H}) \wedge \boldsymbol{H} = \nabla \cdot \Pi^{\mathrm{M}},$$

* This distinction between continuous and discontinuous dependence of a solution on the initial or boundary data is a very important idea in mathematical physics, due to J. Hadamard; see: *Le Problème de Cauchy* (Hermann, Paris 1932), Livre premier, ch. II.
** η happens to be precisely the specific resistivity in rationalized electromagnetic units.

where Π^{M} is Maxwell's magnetic stress tensor

$$\Pi^{\mathrm{M}} = \frac{1}{4\pi} \begin{Bmatrix} H_x^2 - \frac{1}{2}H^2 & H_x H_y & H_x H_z \\ H_y H_x & H_y^2 - \frac{1}{2}H^2 & H_y H_z \\ H_z H_x & H_z H_y & H_z^2 - \frac{1}{2}H^2 \end{Bmatrix} . \tag{3}$$

In terms of this tensor (II, 21b) may be written

$$\rho \frac{DV}{Dt} = -\nabla p + \nabla \cdot \Pi^{\mathrm{M}} + \rho \boldsymbol{g} . \tag{4}$$

Integration over an arbitrary *fixed* volume Ω enclosed by a surface S yields the following *equation of momentum balance*

$$\frac{d}{dt} \int_{\Omega} \rho \boldsymbol{V} \, \mathrm{d}^3 \boldsymbol{r} = -\oint_S p \, d\boldsymbol{S} + \oint_S \Pi^{\mathrm{M}} \cdot d\boldsymbol{S} - \oint_S \rho \boldsymbol{V}(\boldsymbol{V} \cdot d\boldsymbol{S}) + \int_{\Omega} \rho \boldsymbol{g} \, d^3 \boldsymbol{r} . \tag{5}$$

The surface terms stand respectively for the force exerted on S by the fluid pressure, the force transmitted by the magnetic field, and the loss of momentum due to flow. If, on the other hand, Ω follows the motion of the fluid the last of these terms is to be omitted.

Note that the magnetic field transmits momentum but contains no momentum itself. This is due to the omission of Maxwell's term, which has the consequence that the electromagnetic field is no longer an autonomous physical system*, but serves as an interaction between the different parts of the fluid. Yet the field does contain energy, just like ordinary mechanical interactions, as shown by equation (6) below.

An *equation of energy balance* is obtained on multiplying (II, 21b) by $\boldsymbol{V}$ and integrating. We write for the compression energy per unit mass

$$\Psi(\rho) = -\int^{\rho} p \, d\frac{1}{\rho} = \int^{\rho} \frac{p}{\rho^2} \, d\rho$$

and assume for convenience that $\boldsymbol{g}$ can be derived from a potential Φ, in-

* This is exhibited by the fact that without the Maxwell term Maxwell's equations in vacuum do not have finite, non-vanishing solutions.

dependent of t.* Then one finds after some algebraic manipulations

$$\frac{d}{dt}\int_{\Omega}\left\{\tfrac{1}{2}\rho V^2+\rho\Psi(\rho)+\rho\Phi+\frac{1}{8\pi}H^2\right\}d^3r$$

$$=-\oint_S \rho\left\{\tfrac{1}{2}V^2+\Psi(\rho)+\Phi\right\}V\cdot dS$$

$$-\oint_S pV\cdot dS-\frac{c}{4\pi}\oint_S E\wedge H\cdot dS-\int_{\Omega}\frac{1}{\sigma}j^2\,d^3r. \qquad (6)$$

The surface terms are easily interpreted as energy flows. The last term is the energy dissipated in the fluid as Joule heat; the effect of the resulting temperature increase is neglected (assumption (iv) of ch. II, sec. 2).

Problem. Derive this formula. Note that the energy transferred per sec per cm^3 from the fluid to the field equals $(V\wedge j)\cdot H/c$.
Problem. Show that for adiabatic changes Ψ is the internal energy per unit mass, and that for isothermal changes Ψ is the Helmholtz free energy per unit mass.
Problem. Derive a balance equation for angular momentum.

3. MAGNETIC PRESSURE

If H is such that the *field lines are straight and parallel* the term $(H\cdot\nabla)H$ in (1b) is zero. The total force of the magnetic field on the fluid is then contained in the extra pressure $H^2/8\pi$. Because of its convenience this concept of magnetic pressure is frequently used to gain insight into the behavior of plasmas. However, $(H\cdot\nabla)H$ need not vanish, and in fact it need not be small compared to $\nabla(H^2)$. Thus one should use in the general case the complete tensor Π^{M}. The special form (1b) is obtained by splitting off a diagonal part from (4):

$$\Pi^{\mathrm{M}}_{ij}=-\frac{1}{8\pi}H^2\delta_{ij}+\frac{1}{4\pi}H_iH_j.$$

* See (II, 23). Otherwise there should be a term $\int\rho g\cdot V d^3r$ on the right-hand side of (6), as in equation (5).

On the other hand one can transform Π^{M} to principal axes by choosing the z-axis in the direction of H:

$$\Pi^{\mathrm{M}}=\frac{1}{8\pi}\begin{bmatrix}-H^2 & 0 & 0\\ 0 & -H^2 & 0\\ 0 & 0 & H^2\end{bmatrix}.$$

It appears that there is a *pressure $H^2/8\pi$ perpendicular to the field lines and a tension $H^2/8\pi$ along the field lines.* One can also say that there is *isotropic* pressure $H^2/8\pi$, plus a tension $H^2/4\pi$ along the field lines. In the case of straight parallel field lines this tension is immaterial because it does not contribute to $\nabla\cdot\Pi^{\mathrm{M}}$.

Hence in general the force exerted by the magnetic field on the fluid can not be described by a scalar pressure $H^2/8\pi$. In fact, there are "force-free" magnetic fields, i.e. fields for which $H\wedge\operatorname{curl}H=0$, so that they do not influence the fluid at all, although $\nabla(H^2/8\pi)$ may still be different from zero.*

Problem. Show that $(H\cdot\nabla)H=0$ for a field described by

$$H_x=0,\quad H_y=f(x,z-y\varphi(x)),\quad H_z=\varphi(x)f(x,z-y\varphi(x)), \qquad (7)$$

where f and φ are arbitrary functions. Sketch the field lines of this field. (It can also be shown that this is the most general field obeying $(H\cdot\nabla)H=0$, apart from rotations in space.)
Problem. Verify that for the field (7) the magnetic stress tensor (3) is equivalent to a scalar pressure equal to $H^2/8\pi$.

4. MOTION OF THE MAGNETIC FIELD

Equation (2) determines the evolution of H for given flow pattern $V(x,y,z,t)$. We shall investigate the significance of both terms on the right hand side separately.

First we derive a wellknown hydrodynamic formula for the change of flux through a surface S_t which follows the motion of the fluid.

* See chapter V. It may be remarked that for these fields the estimate $|\tau E|\ll|j\wedge H/c|$ no longer holds. Such exceptions are of course always possible in order of magnitude estimates. Strictly speaking it should be verified in each separate case whether indeed $j\wedge H/c$ is so small that τE can no longer be neglected. In practice however the term τE is always omitted unless there is reason to believe that it is important.

$$\int_{S_{t+\delta t}} \boldsymbol{H}_{t+\delta t} \cdot d\boldsymbol{S} - \int_{S_t} \boldsymbol{H}_t \cdot d\boldsymbol{S} = \delta t \int_{S_t} \frac{\partial \boldsymbol{H}}{\partial t} \cdot d\boldsymbol{S} + \int_{S_{t+\delta t}} \boldsymbol{H}_t \cdot d\boldsymbol{S} - \int_{S_t} \boldsymbol{H}_t \cdot d\boldsymbol{S} .$$

Since div $\boldsymbol{H} = 0$ the difference of the last two integrals equals the integral over the sidewall of the tube (Fig. 1), with a minus sign. This yields

$$-\oint \boldsymbol{H}_t \cdot (d\boldsymbol{l} \wedge \boldsymbol{V}\delta t) = -\delta t \oint (\boldsymbol{V} \wedge \boldsymbol{H}) \cdot d\boldsymbol{l} = -\delta t \int \text{curl}\,(\boldsymbol{V} \wedge \boldsymbol{H}) \cdot d\boldsymbol{S},$$

so that we find

$$\frac{d}{dt} \int_{S_t} \boldsymbol{H} \cdot d\boldsymbol{S} = \int_{S_t} \left\{ \frac{\partial \boldsymbol{H}}{\partial t} - \text{curl}\,(\boldsymbol{V} \wedge \boldsymbol{H}) \right\} \cdot d\boldsymbol{S} .$$

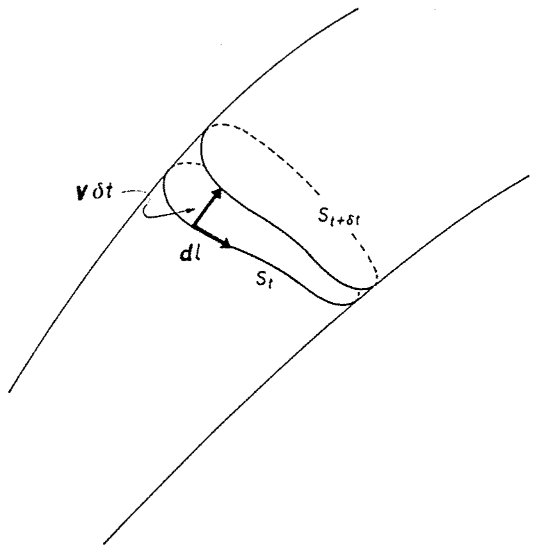

Fig. 1.

Problem. For the integrand { }, one may also write

$$\frac{D\boldsymbol{H}}{Dt} - (\boldsymbol{H} \cdot \nabla)\boldsymbol{V} + \boldsymbol{H}(\nabla \cdot \boldsymbol{V}) .$$

Interpret the separate terms in this expression for the case of a linear flow pattern described by:

$$V_i = \sum_j a_{ij} x_j$$

with constant a_{ij}.

a) *Ideal conductor*. In order to interpret the first term on the right-hand side of (2) we take $\eta = 0$:

$$\frac{\partial \boldsymbol{H}}{\partial t} = \text{curl}\,(\boldsymbol{V} \wedge \boldsymbol{H}) . \tag{8}$$

This equation implies that the field $\boldsymbol{H}$ at each point varies in such a way that the flux through any surface following the motion of the fluid remains constant:

$$\frac{d}{dt} \int_{S_t} \boldsymbol{H} \cdot d\boldsymbol{S} = 0 \quad \text{for every } S_t . \tag{9}$$

Usually this is expressed by saying: *the magnetic field lines are attached to the particles.* For a correct understanding it should be remarked that if one draws the field line pattern at every instant of time, there is no *a priori* method to identify the field lines of successive instants of time. The assertion is however: *if one agrees to let the field lines be carried along by the fluid particles, then equation (8) is satisfied at all times, so that one obtains the correct behavior of* $\boldsymbol{H}$ *as a function of t.*

This picture of field lines attached to the particles ("the field is frozen in") is very convenient but one can also describe the motion of the field lines in another fashion. In fact, (8) does not change when $\boldsymbol{V}$ is replaced by another velocity field $\boldsymbol{W} \equiv \boldsymbol{V} + \boldsymbol{U}$, provided that

$$\text{curl}\,(\boldsymbol{U} \wedge \boldsymbol{H}) = 0 .$$

Therefore it is also possible to let the field lines flow with velocity $\boldsymbol{W}$. The fact that the component of $\boldsymbol{U}$ along $\boldsymbol{H}$ is arbitrary simply means that the motion of a field line in its own direction does not correspond to a change in the field. This freedom can be used to choose $\boldsymbol{W}$ perpendicular to $\boldsymbol{H}$.

In order to investigate the remaining freedom in $\boldsymbol{U}$ we solve the above equation:

$$\boldsymbol{U} \wedge \boldsymbol{H} = \nabla\psi , \quad \psi \text{ arbitrary provided } \boldsymbol{H} \cdot \nabla\psi = 0 .$$

One therefore obtains all possible $\boldsymbol{U}$ by inserting for ψ all scalar functions which satisfy the requirement of being constant on every field line separately. The equation determines for each ψ the component of $\boldsymbol{U}$ perpendicular to $\boldsymbol{H}$. Apparently the sole effect of such a $\boldsymbol{U}$ is to interchange the field lines.

In the special case $\psi = 0$, the component of $\boldsymbol{U}$ parallel to $\boldsymbol{H}$ is still arbitrary. If one chooses it such that W is perpendicular to H the result is

$$W = V - H\frac{(V \cdot H)}{H^2} = \frac{H \wedge (V \wedge H)}{H^2}.$$

For an ideal conductor this becomes on account of (II, 22)

$$W = c\,\frac{E \wedge H}{H^2}. \tag{10}$$

In spite of this suggestive formula it should be kept in mind that the concept "motion of field lines" is not unique.

Problem. Solve the equations of motion for a single charged particle in constant, homogeneous fields E, H. Show that the particle exhibits a "drift velocity" equal to (10).

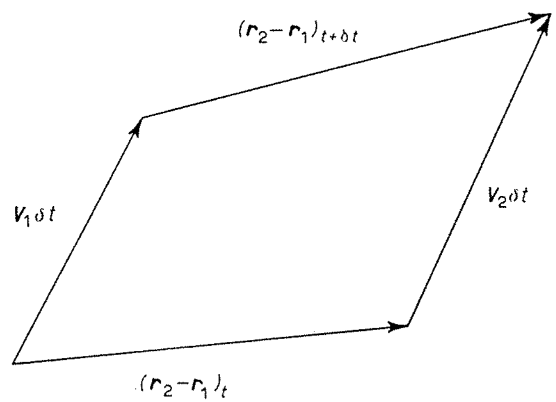

Fig. 2.

The picture of field lines attached to the fluid can still be further elaborated. From (II, 21a) and (8) one obtains (still for $\eta = 0$)

$$\frac{D}{Dt}\frac{H}{\rho} = \left\{\frac{H}{\rho}\cdot\nabla\right\}V. \tag{11}$$

This formula may be physically interpreted as follows. If two particles of the fluid are located at two neighboring points r_1 and r_2 at time t, then at time $t+\delta t$ the vector distance is (Fig. 2)

$$(r_2-r_1)_{t+\delta t} = (r_2-r_1)_t + (V_2-V_1)\,\delta t$$
$$= (r_2-r_1)_t + \delta t\{(r_2-r_1)\cdot\nabla\}V.$$

Hence the same formula holds for the vector distance r_2-r_1 and for H/ρ:

$$\frac{D}{Dt}(r_2-r_1) = \{(r_2-r_1)\cdot\nabla\}V.$$

Therefore: if at $t = 0$ two points lie on one field line and close together, then at time $t+\delta t$ they still lie on the same field line and the vector H/ρ varies proportionally to their distance.

Remark. Consider a segment of a tube of field lines (Fig. 3). Carried along by the motion of the fluid it will again be a segment of a field tube at a later

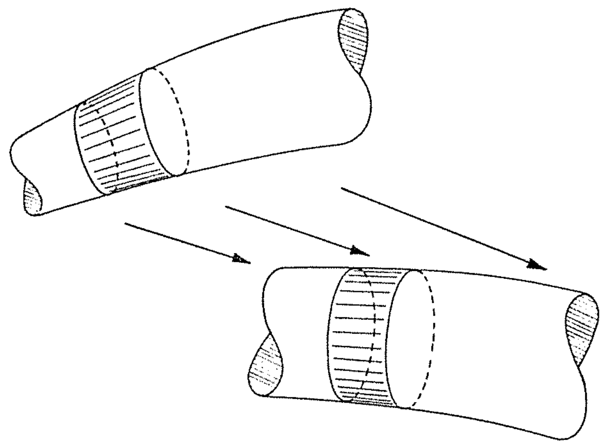

Fig. 3.

instant. Its length varies proportional to $|H|/\rho$, but its cross-section varies inversely proportional to $|H|$. Hence the volume of the segment multiplied by ρ is constant. This must in fact be so on account of mass conservation.

Problem. Conversely, derive (11) geometrically, using mass conservation and the fact that the field lines are attached to the fluid.

b) *Dissipation due to the resistivity* η. In order to interpret the second term in (2) we take $\eta > 0$, but $V = 0$. Then H satisfies

$$\frac{\partial H}{\partial t} = \eta\,\nabla^2 H. \tag{12}$$

This equation has the form of the equation of heat conduction or diffusion, so that it is clear that $\boldsymbol{H}$ will gradually decay. In fact, supposing $\boldsymbol{H}$ vanishes at infinity, it follows from (12) or (6) that

$$\frac{d}{dt}\int H^2\, d^3r = -\eta \int \sum_{i,j} \left(\frac{\partial H_i}{\partial x_j}\right)^2 d^3\boldsymbol{r}.$$

The right-hand side is negative (unless $\boldsymbol{H}$ is constant and therefore zero). This shows how the magnetic field energy dissipates. Because of the weakening of the field the distance between the field lines must increase; thus the field lines move away from each other and are no longer firmly attached to the fluid.

An estimate of the time in which $\boldsymbol{H}$ dissipates follows from (12) by dimensional arguments:

$$\text{dissipation time} \sim \frac{L^2}{\eta} \sim \frac{\sigma L^2}{c^2}; \qquad (13)$$

here L is the distance over which $\boldsymbol{H}$ changes noticeably. In laboratory experiments with mercury ($\sigma = 10^{16}\ \text{sec}^{-1}$, $L = 10$ cm) the dissipation time is $\sim 10^{-3}$ sec. For a copper sphere of 1 meter radius it is of the order of 10 sec, for the magnetic field of the sun 10^{10} years. For a fully ionized hydrogen plasma (13) is of the order $10^{-13} T^{\frac{3}{2}} L^2$. For the earth's core ($\sigma \approx 3 \times 10^{15}\ \text{sec}^{-1}$, $L \approx 3 \times 10^8$ cm) the dissipation time is approximately 10^4 years. This is too short to account for the earth's magnetic field as a small remnant of an originally much stronger field, trapped during the formation of the earth (comp. chapter V).

Problem. Show that for a sphere of radius L the dissipation time is $4\sigma L^2/\pi c^2$.

c) *Non-ideally conducting fluid in motion.* In a moving plasma both terms in (2) are present. Their ratio is given by the dimensionless number

$$R_{\mathrm{M}} = \frac{VL}{\eta} \quad \text{("magnetic Reynolds' number")}.$$

If $R_{\mathrm{M}} \gg 1$ the field lines may be visualized as attached to the fluid particles but slipping a little bit. If $R_{\mathrm{M}} \approx 1$ or less, the slip is so large that this picture is no longer useful. Again it is evident that large dimensions have the effect that the plasma behaves like a conductor. See Table II on page 12.

General references

D. J. Rose and M. Clark, *Plasmas and Controlled Fusion* (Wiley, New York 1961) chs. V, VI, VII.

Shih-I Pai, *Magnetogasdynamics and Plasma Dynamics* (Springer, Vienna 1962) chs. IV, V.

W. A. Newcomb, Ann. Phys. (N.Y.) **3**, 34 (1958).

CHAPTER IV

STATIC SOLUTIONS OF THE MHD EQUATIONS

1. The equations for the static case

We shall call static solutions of the equations (II, 21) those in which all quantities are independent of t, and moreover $V = 0$. In addition we shall put throughout this chapter $\boldsymbol{g} = -\nabla\Phi$. Then the MHD equations take the form

$$0 = -\operatorname{grad} p + \frac{1}{c}\boldsymbol{j} \wedge \boldsymbol{H} - \rho \operatorname{grad} \Phi \tag{1}$$

$$\operatorname{curl} \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j} \tag{2}$$

$$\operatorname{div} \boldsymbol{H} = 0 \tag{3}$$

$$\operatorname{curl} \boldsymbol{E} = 0 \tag{4}$$

$$\boldsymbol{j} = \sigma \boldsymbol{E} \tag{5}$$

$$p = f(\rho). \tag{6}$$

For an incompressible fluid (6) should be replaced by ρ = const.; for an ideal gas one has $f(\rho) = C\rho$ with $C = kT/m$. On eliminating $\boldsymbol{j}$ and $\boldsymbol{E}$ one obtains the two equations*

$$-4\pi(\nabla p + \rho\nabla\Phi) = \boldsymbol{H} \wedge \operatorname{curl} \boldsymbol{H} \tag{7a}$$

$$= \tfrac{1}{2}\nabla(H^2) - (\boldsymbol{H}\cdot\nabla)\boldsymbol{H} \tag{7b}$$

$$\nabla^2 \boldsymbol{H} = 0. \tag{8}$$

It is sufficient to find two functions $\boldsymbol{H}$ and ρ that obey (3), (7) and (8), p being regarded as related to ρ by (6). In fact it is easily checked that all other equations can then be satisfied.

* Of course these equations may also be obtained directly from (III, 1) and (III, 2).

As is wellknown, the Laplace equation (8) has no solutions that are everywhere finite and tend to zero at infinity (excepting the trivial solution $\boldsymbol{H} = 0$). It follows that no static solution is possible unless there are external sources for the magnetic field. Yet this conclusion has to be qualified in two respects.

Firstly it has been assumed that σ is constant everywhere. This is not always true; for instance when the plasma is confined to a certain region, the conductivity σ will be zero outside that region.

Secondly, the derivation of (8) leads at first to $\eta\nabla^2\boldsymbol{H} = 0$. Hence (8) need not hold for an ideal conductor, $\eta = 0$. For very good but not ideal conductors, η is very small. In this case, strictly speaking (8) should be valid, but in practice it cannot be trusted. For, as soon as there is a slight deviation from stationarity, so that $\partial\boldsymbol{H}/\partial t$ is not exactly zero in (III, 2), it is no longer possible to assert that $\nabla^2\boldsymbol{H}$ must be zero or small. For this reason the equations for an ideal conductor are often more realistic. They consist of the set (1), (2), (3), (6), or alternatively (7), (3), (6).

2. Incompressible fluid

For the static equilibrium of an incompressible fluid with negligible resistivity the problem amounts to finding solutions of

$$\boldsymbol{H} \wedge \operatorname{curl} \boldsymbol{H} = \operatorname{grad} \xi, \qquad \operatorname{div} \boldsymbol{H} = 0 \tag{9a}$$

with $p = -\rho\Phi - \xi/4\pi$. From (9a) follows that both $\boldsymbol{H}$ and curl $\boldsymbol{H}$ are perpendicular to grad ξ, and hence tangent to the surface ξ = const. Thus both the field lines of $\boldsymbol{H}$ and the flow lines of $\boldsymbol{j}$ lie on surfaces of constant ξ.

Instead of (9a) one may use the equivalent pair of equations

$$\operatorname{curl} \{(\boldsymbol{H}\cdot\nabla)\boldsymbol{H}\} = 0, \qquad \operatorname{div} \boldsymbol{H} = 0. \tag{9b}$$

Unfortunately, since (9a) and (9b) are nonlinear, there is no systematic way of solving these equations. It is possible, however, to construct special solutions, which may be of physical interest. We shall give three examples, but the first one is rather trivial.

Problem. Suppose one has a solution for an incompressible fluid obeying equations (1) to (5) in a potential field Φ. It is then possible to construct from it a solution for a compressible fluid obeying (1) to (6) with $\Phi = 0$. Prove this.

First example. Taking $\boldsymbol{H}$ = constant one has clearly satisfied (9b). It follows from (2) and (5) that $\boldsymbol{j} = \boldsymbol{E} = 0$, so that (4) is also satisfied. (1) reduces to

$$p + \rho\Phi = \text{constant}.$$

This is the familiar static solution for a fluid; the magnetic field does not exert any force on the fluid owing to $\boldsymbol{j} = 0$.

Second example. Take $\boldsymbol{H}$ in the direction of the z-axis. According to (3) one then has $\partial H_z/\partial z = 0$, so that $\boldsymbol{H}$ does not depend on z. This is the case of straight parallel field lines, so that (9b) is satisfied. $\boldsymbol{j}$ is perpendicular to the z-axis. Furthermore one finds from (7)

$$p + \rho\Phi + \frac{1}{8\pi} H_z^2 = \text{constant}. \qquad (10)$$

This equation tells that the hydrodynamical pressure p and the magnetic pressure $H_z^2/8\pi$ together compensate the force $\boldsymbol{g}$. Finally (8) takes the form

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = 0.$$

This solution has been employed by Alfvén to describe *sunspots*. For this purpose he neglected the barometric effect by putting Φ = constant. He also put H_z = constant (of the order of 3000 gauss) inside the sunspot. Outside it H_z is practically zero. Thus the equation $\nabla^2 H_z = 0$ cannot be satisfied, but that does not matter as he puts $\sigma = \infty$. In order to satisfy (10) it is necessary that p inside the sunspot is less than p outside it by the amount $H_z^2/8\pi = 0.36 \times 10^6$ dyne/cm^2 = 0.36 atm. This pressure difference must be due to different temperatures. In fact one measures experimentally 4600 °K inside and 5700 °K outside the sunspot. Since the density in the photosphere is roughly 10^{17} particles per cm^3, this observed temperature difference amounts to a pressure difference of 10^4 dyne/cm^2. This is regarded as a satisfactory agreement.

Note that in this solution $\boldsymbol{j}$ differs from zero only in a thin layer between the sunspot and its surroundings. The current runs in a circle around the sunspot. Because of the infinite conductivity it follows from (5) that $\boldsymbol{E} = 0$ everywhere. Finally the fluid is not regarded as incompressible, but rather as an ideal gas; hence (6) is taken to be $p = C\rho$, but with different values for C inside and outside the sunspot.

Third example. Cylindrical magnetic field:

$$\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{a} \wedge \boldsymbol{r} \quad \text{with constant } \boldsymbol{a}.$$

Equations (3) and (8) are satisfied. In addition, the identity

$$(\boldsymbol{H} \cdot \nabla)\boldsymbol{H} = \boldsymbol{a} \wedge (\boldsymbol{a} \wedge \boldsymbol{r}) = -\operatorname{grad} \tfrac{1}{2}(\boldsymbol{a} \wedge \boldsymbol{r})^2 = -\tfrac{1}{2}\nabla(H^2)$$

shows that (9b) is also satisfied. One finds $\boldsymbol{j} = (c/2\pi)\boldsymbol{a}$, so that the current density is constant in space. Furthermore

$$p + \rho\Phi + \frac{1}{4\pi} H^2 = \text{constant}. \qquad (11)$$

Note that the magnetic term is twice the usual $H^2/8\pi$, which demonstrates that one has to be careful with the concept of "magnetic pressure".

Problem. Show that the extra pressure in (11) may be regarded as a consequence of the stress along the circular magnetic field lines.

3. THE PINCHED DISCHARGE

In the above third example take $\Phi = 0$. (According to the problem on page 29 one may drop the condition that the fluid must be incompressible.) Furthermore take the z-axis along $\boldsymbol{a}$, and let $\boldsymbol{s}$ be the component of $\boldsymbol{r}$ perpendicular to it (Fig. 4). Then (11) takes the form

$$p + \frac{a^2}{4\pi} s^2 = p_0.$$

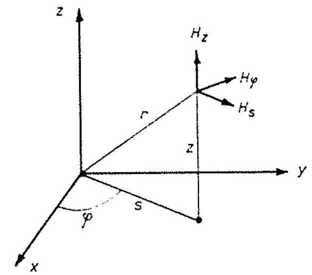

Fig. 4. Cylindrical coordinates.

Thus p as a function of s is a parabola, and p vanishes at $s = \sqrt{(4\pi p_0/a^2)} \equiv R$. For $s > R$ the solution breaks down, because p cannot be negative. However one may use a different solution for $s > R$, provided that certain conditions on the boundary $s = R$ are obeyed. We choose the trivial solution with zero density, so that $p = 0$ for $s > R$. The total current is $J =$

$= \pi R^2 j = 2\pi c p_0/a$. The magnetic field for $s > R$ obeys the equations in vacuum and has therefore the familiar form of the field of a given current J,

$$H_s = 0, \quad H_\varphi = \frac{2J}{c}\frac{1}{s}, \quad H_z = 0.$$

Clearly $\boldsymbol{H}$ satisfies (8) everywhere except on the boundary, hence curl $\boldsymbol{j} \neq 0$ on the boundary. Indeed, one finds on integrating around the dotted loop in Fig. 5

$$\oint \boldsymbol{j} \cdot d\boldsymbol{l} = j - 0 = \frac{ca}{2\pi}.$$

This is not in contradiction with the fact that curl $\boldsymbol{E} = 0$, since σ is no longer a constant in space, but drops to zero for $s > R$. From curl $\boldsymbol{E} = 0$ follows that E_z must have the same value outside and inside, namely $j/\sigma = ca/2\pi\sigma$. Thus the present solution represents a static plasma column in a constant electric field (due for instance to a fixed potential difference between

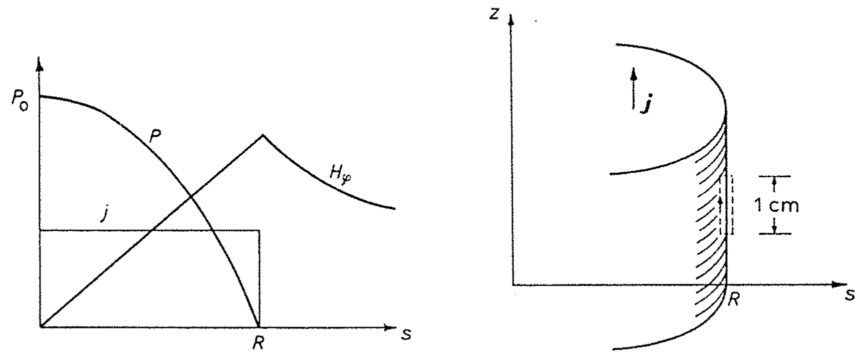

Fig. 5. The static plasma column.

two electrodes at both ends of the plasma column). A homogeneous current runs through the plasma, and the resulting magnetic field contains the plasma inside the cylinder with radius R. This effect of the magnetic field is called the *pinch effect*; it is important as a means to prevent a hot plasma from touching the walls.

Problem. Suppose that the potential difference between the electrodes and the total amount of fluid are given, and that σ and the temperature T are fixed. Show that J is uniquely determined by these data.

The contact with the electrodes can be circumvented by bending the cylinder into a toroidal shape, so that the plasma column becomes a ring. The electric field is then created by means of a changing magnetic flux through the torus. However, the main difficulty is that the plasma column – although a solution of the MHD equations – is not stable, as will be discussed in chapter VII. Hence it survives only for short periods (of the order of 10 microseconds), in spite of additional stabilizing devices. Accordingly, the actual discharges are not sufficiently static to guarantee the validity of (8); they are better described by the following solution.

Equations (9b), without (8), are obeyed by

$$\boldsymbol{H}(\boldsymbol{r}) = \mu(s)\,\boldsymbol{a} \wedge \boldsymbol{r} = \mu(s)\,\boldsymbol{a} \wedge \boldsymbol{s},$$

with constant vector $\boldsymbol{a}$ but arbitrary function $\mu(s)$. This is easily checked by means of the identity

$$(\boldsymbol{H}\cdot\nabla)\boldsymbol{H} = -a^2\mu^2(s)\boldsymbol{s} = -a^2\nabla M(s),$$

the function $M(s)$ being defined by

$$M(s) = \int_0^s s'\mu^2(s')ds'.$$

The corresponding current density is

$$\boldsymbol{j}(s) = \frac{c}{4\pi}\,\text{curl}\,\boldsymbol{H} = \frac{c}{2\pi}(\mu + \tfrac{1}{2}s\mu')\boldsymbol{a}, \tag{12}$$

while the pressure obeys according to (7)

$$p(s) + \frac{a^2}{4\pi}(M + \tfrac{1}{2}sM') = p_0.$$

Problem. Show that this solution contains the previously described static pinch as a special case, obtained by an appropriate choice of $\mu(s)$ for $s < R$ and $s > R$. What is the reason why this case now appears as one single solution rather than as two solutions matched at the boundary?
Problem. Find the total current J in terms of the undetermined function μ.

In a discharge in a highly conducting plasma the current is almost entirely concentrated on the surface,

$$\boldsymbol{j}(s) = \frac{c}{2\pi}\boldsymbol{a}R\delta(s - R).$$

(The factor R has been inserted for dimensional reasons.) In order to obtain this as a special case one must take, according to (12),

$$\mu(s) = 0 \qquad \text{for} \quad s < R,$$
$$\mu(s) = 2R^2/s^2 \quad \text{for} \quad s > R.$$

From this it is easy to deduce by means of the above formulas

$$\boldsymbol{H} = 0, \qquad p = p_0 = a^2R^2/2\pi \quad \text{for} \quad s < R,$$
$$H_\varphi = 2R^2a/s, \quad p = 0 \qquad \text{for} \quad s > R.$$

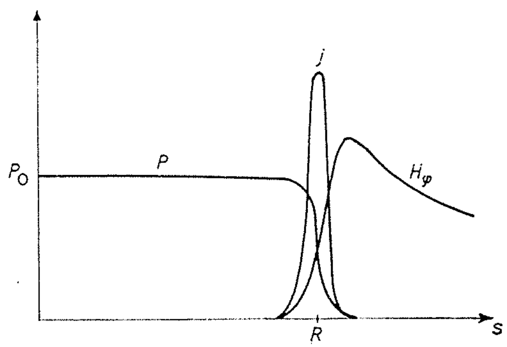

Fig. 6. The almost static plasma column.

Clearly one has in this case

$$p_0 = H_\varphi^2(R)/8\pi,$$

which states that the hydrostatic pressure p_0 inside the plasma column is balanced by the magnetic pressure of the field around it. In a gas discharge the ideal gas law is applicable, so that

$$p_0 = nkT = NkT/\pi R^2,$$

where N is the number of particles per unit length of the column. If one substitutes this and also the total current $J = caR^2$, the above condition for equal pressures takes the form

$$NkT = J^2/2c^2 \quad \text{(Bennett's relation)}.$$

Problem. Derive this relation for the general case described by (12); comp. Fünfer and Lehner, p. 26.

Remark. For experiments and pictures of pinch discharges see Fünfer and Lehner. Actually the MHD approximation is not applicable to real pinch discharges, because in reality the mean free path of the particles is not short compared to the dimensions of the plasma. Nevertheless it is often used, because the results are very similar to those of a more realistic treatment – as happens so often in plasma physics. A better treatment, taking into account the dynamics of the individual particles, is due to M. Rosenbluth.*

4. Static solution for compressible fluid with $g \neq 0$

The equations are

$$\operatorname{div} \boldsymbol{H} = 0 \qquad [\nabla^2 \boldsymbol{H} = 0] \tag{13}$$

$$p = f(\rho) \tag{14}$$

$$\nabla p + \rho \nabla \Phi = -\frac{1}{4\pi} \boldsymbol{H} \wedge \operatorname{curl} \boldsymbol{H}. \tag{15}$$

It is again impossible to give a complete solution, but it is possible to reduce the equations to an equation for $\boldsymbol{H}$ alone, that is, to eliminate p and ρ.

On taking the curl of (15) one obtains

$$\nabla \rho \wedge \nabla \Phi = \operatorname{curl} \boldsymbol{K}, \tag{16}$$

where

$$\boldsymbol{K} = -\frac{1}{4\pi} \boldsymbol{H} \wedge \operatorname{curl} \boldsymbol{H}. \tag{17}$$

Thus curl $\boldsymbol{K}$ is normal to $\nabla\rho$ and $\nabla\Phi$; and because of (14) also to ∇p. It therefore follows from (15) that

$$\boldsymbol{K} \cdot \operatorname{curl} \boldsymbol{K} = 0. \tag{18}$$

This identity is the condition for $\boldsymbol{K}$ to be representable in the form

$$\boldsymbol{K} = \lambda \operatorname{grad} X, \tag{19}$$

with two scalar fields λ and X.** Thus $\boldsymbol{K}$ is normal to the surfaces X = constant, so that curl $\boldsymbol{K}$ is tangent to them.

* M. Rosenbluth, *International Summer Course in Plasma Physics* (Risö 1960). p. 431; see also J. D. Jackson, *Classical Electrodynamics* (Wiley, New York 1962) ch. x, and Rose and Clark.

** To put it differently, (18) is the condition that the differential $K_x\,dx + K_y\,dy + K_z\,dz$ has an integrating factor $1/\lambda$. See e.g. I. N. Sneddon, *Elements of Partial Differential Equations* (McGraw-Hill, New York 1957) ch. I.

The situation is as follows. There are two sets of surfaces, Φ = constant and X = constant. The intersections are the field lines of curl $\boldsymbol{K}$. On account of (16) ρ, and hence also p, is constant along each field line, so that ρ and

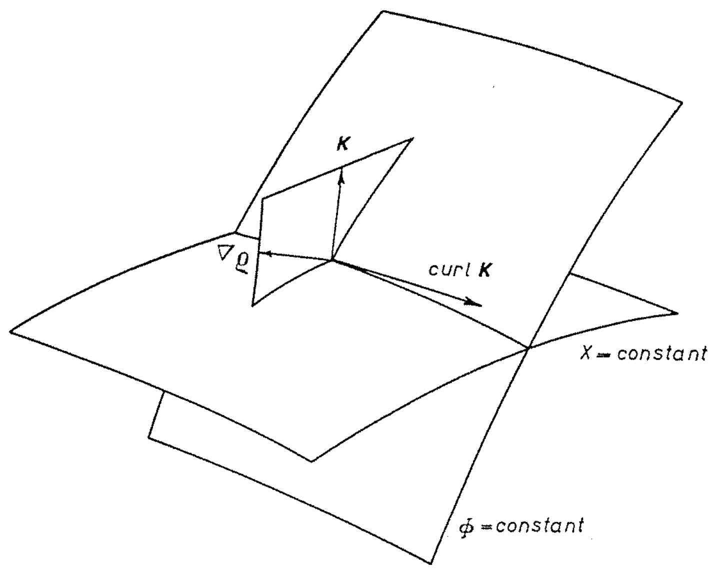

Fig. 7.

p may be written as functions of the two variables Φ and X. The same applies to λ, because it follows from (19) that grad λ is normal to curl $\boldsymbol{K}$. Hence (15) takes the form

$$\left(\frac{\partial p}{\partial \Phi} + \rho\right) \nabla\Phi + \frac{\partial p}{\partial X} \nabla X = \lambda \nabla X .$$

Unless $\nabla\Phi$ and ∇X happen to be parallel it follows that

$$\frac{\partial p}{\partial \Phi} = -\rho \tag{20a}$$

$$\frac{\partial p}{\partial X} = \lambda . \tag{20b}$$

The general solution of (20a) is

$$P(\rho) = -\Phi + \Omega(X) . \tag{21}$$

Here Ω denotes an arbitrary function, while*

$$P(\rho) = \int \frac{dp}{\rho} = \int^{\rho} \frac{f'(\rho)}{\rho}\, d\rho .$$

In order to satisfy (20b) one must have

$$\frac{d\Omega(X)}{dX} = \frac{1}{\rho}\frac{\partial p}{\partial X} = \frac{\lambda}{\rho} .$$

Thus (19) becomes

$$\boldsymbol{K} = \rho \nabla \Omega . \tag{22}$$

According to (21) it is possible to write ρ as a function of P,

$$\rho = F(P) = F(-\Phi + \Omega) .$$

Consequently we have found that $\boldsymbol{K}$ must have the form

$$\boldsymbol{K} = F(\Omega - \Phi) \text{ grad } \Omega , \tag{23}$$

where F is uniquely determined by f, whereas Ω is an arbitrary scalar field.

Problem. Show that any divergence-free field $\boldsymbol{H}$ for which (23) holds provides a solution of (15).
Problem. Show that (23) reduces to (9) if either $\Phi = 0$ or ρ = constant.
Problem. Show that (23) also reduces to (9) if $\nabla\Phi$ and ∇X are parallel.

In the case of an *ideal gas* the condition (23) simplifies considerably. One then has $p = C\rho$, so that $P(\rho) = C \log \rho$, and

$$\rho = \exp \frac{P}{C} = \exp \left[\frac{\Omega - \Phi}{C}\right] .$$

On putting $C \exp (\Omega/C) = \Xi$ one has for (23)

$$\boldsymbol{K} = e^{-\Phi/C}\, \nabla \Xi . \tag{24}$$

The corresponding density is

$$\rho = \frac{1}{C} e^{-\Phi/C}\, \Xi = \frac{m}{kT} e^{-m\Phi/kT}\, \Xi .$$

* In adiabatic variations P is the enthalpy per unit mass, in isothermal variations the chemical potential.

Thus the factor Ξ represents the correction to the barometric density due to the magnetic field. The condition for the possibility of such a solution is that $\boldsymbol{K}$ can be written in the form (24), that is, that

$$\operatorname{curl}\{e^{\Phi/C}\boldsymbol{K}\} \equiv \operatorname{curl}\{e^{\Phi/C}\boldsymbol{H}\wedge\operatorname{curl}\boldsymbol{H}\} = 0. \qquad (25)$$

Example. Let $\boldsymbol{H} = (Bz, 0, 0)$ and $\Phi = gz$. Then one has $\boldsymbol{K} = B^2(0, 0, -z/4\pi)$, and (25) is obeyed. One finds

$$\Xi = \frac{B^2}{4\pi}\frac{kT}{mg}\left(\frac{kT}{mg} - z\right)e^{mgz/kT}.$$

The corresponding density is

$$\rho = \frac{B^2}{4\pi}\frac{kT}{mg^2}\left(1 - \frac{mgz}{kT}\right).$$

Thus instead of the barometric formula one finds a linear decrease of ρ with increasing altitude, because the magnetic field pulls the plasma down. Of course this solution only applies for $z \leqslant kT/mg$; above this limit the density and hence the conductivity are zero, so that the MHD equations have to be replaced with the free field equations.

Problem. Show that for an ideally conducting plasma obeying the ideal gas law a solution exists with $\boldsymbol{H} = (\mu(z), 0, 0)$, where $\mu(z)$ is an arbitrary function. Verify that the density decreases more slowly with z than barometrically if $d\mu^2/dz < 0$. Study the special case $\mu(z) = B\,e^{-gz/2C}$.

General references

D. J. Rose and M. Clark, *Plasmas and Controlled Fusion* (Wiley, New York 1961) chs. VII, XIV.

J. A. Shercliff, *A Textbook of Magnetohydrodynamics* (Pergamon, Oxford 1965) ch. IV.

H. Alfvén, *Cosmic Electrodynamics* (Oxford University Press, Oxford 1950).

L. A. Artsimovitch, *Controlled Thermonuclear Reactions* (Oliver and Boyd, Edinburg 1964) ch. VI.

J. M. Greene and J. L. Johnson in: *Advances in Theoretical Physics*, Vol. 1, K. A. Brueckner ed. (Academic Press, New York 1965).

E. Fünfer and G. Lehner in: *Ergebnisse der exakten Naturwissenschaften* **34** (Springer, Berlin 1962).

CHAPTER V

STATIONARY SOLUTIONS

1. The equations for the stationary case

A solution of the MHD equations (II, 21) in which all quantities are independent of t, but in which $\boldsymbol{V}$ does not vanish, is called stationary. We again assume $\boldsymbol{g} = -\nabla\Phi$, and moreover ρ = constant, because the examples that we shall consider all involve incompressible fluids. The MHD equations may then be written in the following form

$$\operatorname{div}\boldsymbol{V} = 0 \qquad (1a)$$

$$\boldsymbol{V}\wedge\operatorname{curl}\boldsymbol{V} = \operatorname{grad}\left(\frac{p}{\rho} + \tfrac{1}{2}V^2 + \Phi\right) + \frac{1}{4\pi\rho}\boldsymbol{H}\wedge\operatorname{curl}\boldsymbol{H} \qquad (1b)$$

$$\operatorname{div}\boldsymbol{H} = 0 \qquad (1c)$$

$$\operatorname{curl}(\boldsymbol{V}\wedge\boldsymbol{H}) + \eta\nabla^2\boldsymbol{H} = 0. \qquad (1d)$$

The general solution of these nonlinear equations is of course unknown. Instead we shall study some special solutions.

Remark. As the term with η dissipates energy, there can only be stationary solutions when energy is supplied from outside. Although the equations contain no external force term, this energy supply is implicitly contained in the solution in one (or more) of the following ways.

(i) The field $\boldsymbol{H}$ or the velocity field $\boldsymbol{V}$ extends to infinity.

(ii) The equations are only valid in a certain region; they must therefore be supplemented with appropriate boundary conditions.

(iii) Rather than solving (1b) one adopts a certain flow pattern $\boldsymbol{V}(\boldsymbol{r})$ and solves (1c) and (1d); this device will presently be used in connection with terrestrial magnetism.

Problem. The fact that for $\eta > 0$ no solutions exist that vanish at infinity follows from the identity

$$\int_\Omega \eta\,(\operatorname{curl}\boldsymbol{H})^2\,d^3\boldsymbol{r} =$$

$$= \oint_S [\eta\,\boldsymbol{H} \wedge \operatorname{curl}\boldsymbol{H} - \boldsymbol{H} \wedge (\boldsymbol{V} \wedge \boldsymbol{H}) - 4\pi\boldsymbol{V}(p+\tfrac{1}{2}\rho V^2+\rho\Phi)] \cdot d\boldsymbol{S}.$$

Deduce this identity from equations (1). Show that this identity remains valid when η is a function of $\boldsymbol{r}$.

The "equipartition solution". Let $\eta = 0$. A solution is obtained by starting from an arbitrary divergence-free field $\boldsymbol{H}$, choosing $\boldsymbol{V}$ to be

$$\boldsymbol{V} = (4\pi\rho)^{-\frac{1}{2}}\,\boldsymbol{H},$$

and subsequently determining p from

$$p+\tfrac{1}{2}\rho V^2+\rho\Phi = \text{constant}.$$

For this solution the kinetic and magnetic energies in each volume element are equal:

$$\tfrac{1}{2}\rho V^2 = \frac{H^2}{8\pi}. \tag{2}$$

Of course, this is not the same as the equipartition law of statistical mechanics, which says that the irregular heat motion has the effect that ultimately the energy is equally distributed over the degrees of freedom. It has been suggested that similarly, irregular turbulent flow should lead to (2), but this has not been proved convincingly. Yet Chandrasekhar has proved that the present solution is *stable* (for incompressible fluids).

Problem. Show that the above solution is still valid for a compressible fluid with $\Phi = 0$. Note that then ρ must be constant along every field line.

2. Force-free fields

A magnetic field $\boldsymbol{H}$ satisfying

$$\boldsymbol{H} \wedge \operatorname{curl}\boldsymbol{H} = 0 \tag{3}$$

exerts no force on the plasma. Such fields have been extensively studied in astronomy.* Starting from such a force-free field one may construct a solution of the equations (1) by setting

$$\boldsymbol{V} = \beta\boldsymbol{H}, \qquad \beta = \text{constant},$$

$$p/\rho+\tfrac{1}{2}V^2+\Phi = \text{constant}.$$

For the special value $\beta = (4\pi\rho)^{-\frac{1}{2}}$ this reduces to the solution found above for arbitrary $\boldsymbol{H}$.

We are left with the problem of finding solutions of (3). This equation may also be written

$$\operatorname{curl}\boldsymbol{H} = \alpha\boldsymbol{H}, \tag{4}$$

where α is an arbitrary scalar function of $\boldsymbol{r}$. In spite of the efforts of astrophysicists the general solution has not been found.

Solutions with constant α. A special class of solutions obtains by taking α = constant.** Then the equation has become a set of three linear differential equations with constant coefficients and may be solved by standard methods. Inserting $\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{a}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ one finds a solution if the two constant vectors $\boldsymbol{a}$ and $\boldsymbol{k}$ satisfy

$$i\boldsymbol{k} \wedge \boldsymbol{a} = \alpha\boldsymbol{a},$$

which can easily be solved. Note that this equation implies that $\boldsymbol{a}$ is perpendicular to itself, which is only possible for complex $\boldsymbol{a}$.

Problem. Find all solutions of this type and sketch the magnetic field lines.

The general solution for given constant α consists of a superposition of the solutions of this type, but this is not always the most suitable form. For example, to find solutions that fit a cylindrical geometry (like that of a spiral arm of a galaxy), it is convenient to write (4) in cylindrical coordinates s, φ, z:

$$\frac{1}{s}\frac{\partial H_z}{\partial\varphi} - \frac{\partial H_\varphi}{\partial z} = \alpha H_s, \qquad \frac{\partial H_s}{\partial z} - \frac{\partial H_z}{\partial s} = \alpha H_\varphi,$$

$$\frac{1}{s}\frac{\partial}{\partial s}sH_\varphi - \frac{1}{s}\frac{\partial H_s}{\partial\varphi} = \alpha H_z.$$

* R. Lüst and A. Schlüter, Z. Astrophys. **34**, 263 (1954).
** S. Chandrasekhar, Proc. Natl. Acad. Sci. **42**, 1 (1956); S. Chandrasekhar and L. Woltjer, Proc. Natl. Acad. Sci. **44**, 285 (1958).

Assume that the components H_s, H_φ, H_z depend on φ and z through the common factor $e^{im\varphi+ikz}$, where m is an integer and k a constant.

$$\frac{im}{s}H_z - ikH_\varphi = \alpha H_s$$

$$ikH_s - \frac{\partial H_z}{\partial s} = \alpha H_\varphi$$

$$\frac{1}{s}\frac{\partial}{\partial s}sH_\varphi - \frac{im}{s}H_s = \alpha H_z .$$

Eliminating H_s and H_φ one finds for H_z

$$\frac{\partial^2 H_z}{\partial s^2} + \frac{1}{s}\frac{\partial H_z}{\partial s} + \left(\alpha^2 - k^2 - \frac{m^2}{s^2}\right)H_z = 0$$

Hence the solution is a Bessel function,

$$H_z(s, \varphi, z) = J_m\,[s\sqrt{(\alpha^2-k^2)}]\;e^{im\varphi+ikz} .$$

Problem. Show that the corresponding H_s and H_φ are

$$H_s = \left(\frac{im}{s(\alpha-k)}J_m - \frac{ik}{\sqrt{(\alpha^2-k^2)}}J_{m+1}\right)e^{im\varphi+ikz}$$

$$H_\varphi = \left(\frac{-m}{s(\alpha-k)}J_m + \frac{\alpha}{\sqrt{(\alpha^2-k^2)}}J_{m+1}\right)e^{im\varphi+ikz},$$

where all Bessel functions have the same argument $s\sqrt{(\alpha^2-k^2)}$.
Problem. In particular, find the only solution with cylindrical symmetry,*

$$H_s = 0, \quad H_\varphi = J_1(\alpha s), \quad H_z = J_0(\alpha s).$$

Problem. Find the same solution by noting that from (4) follows

$$(\nabla^2+\alpha^2)\,\boldsymbol{H} = 0, \qquad \operatorname{div}\boldsymbol{H} = 0 .$$

Problem. Write (4) as an equation for the spherical components H_r, H_ϑ, H_φ. Show that the general solution with $H_r = 0$ and constant α is given by

$$H_\varphi + iH_\vartheta = \frac{f(\tan\frac{1}{2}\vartheta\; e^{i\varphi})}{\sin\vartheta}\frac{e^{i\alpha r}}{r},$$

where f is an arbitrary complex function of its argument.

* S. Lundquist, Arkiv Fysik **2**, 361 (1950).

If α is not supposed constant, it is possible* to find the general solution with cylindrical symmetry, i.e., independent of φ and z. In this case (4) reduces to

$$H_s = 0, \quad -\frac{dH_z}{ds} = \alpha H_\varphi, \quad \frac{1}{s}\frac{d}{ds}sH_\varphi = \alpha H_z .$$

Elimination of the unknown function α yields

$$H_z\frac{dH_z}{ds} + \frac{1}{s}H_\varphi\frac{d}{ds}sH_\varphi = 0$$

or

$$\frac{d}{ds}\tfrac{1}{2}(H_z^2+H_\varphi^2) + \frac{1}{s}H_\varphi^2 = 0 .$$

Hence if one puts $\frac{1}{2}(H_z^2+H_\varphi^2) \equiv q(s)$, one has

$$H_\varphi^2 = -s\frac{dq}{ds}, \qquad H_z^2 = 2q + s\frac{dq}{ds} = \frac{1}{s}\frac{d}{ds}s^2 q .$$

Vice versa, if one takes for q an arbitrary function of s, the H_φ and H_z defined in this way, together with $H_s = 0$, constitute a force-free field with cylindrical symmetry.

Problem. Find the same result by writing (3) in cylindrical coordinates.
Problem. Find Lundquist's solution by choosing a suitable $q(s)$.

Another class of solutions is found by assuming $H_z = 0$. In that case the divergence condition tells that H_x and H_y derive from a scalar "potential" $\chi(x, y, z)$

$$H_x = \frac{\partial\chi}{\partial y}, \qquad H_y = -\frac{\partial\chi}{\partial x}.$$

The three components of (4) then lead to

$$\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} = 0, \qquad \frac{\partial}{\partial z}\left\{\left(\frac{\partial\chi}{\partial x}\right)^2 + \left(\frac{\partial\chi}{\partial y}\right)^2\right\} = 0 .$$

The first equation is the Laplace equation in two dimensions. Its general solution can be obtained by taking an analytic function F of $w \equiv x+iy$ and putting

$$\chi(x, y, z) = \operatorname{Re} F(x+iy, z) = \operatorname{Re} F(w, z) .$$

* A. Schlüter, Z. Naturf. **12a**, 855 (1957).

z enters only as a parameter. One then has

$$\frac{\partial F}{\partial w} = \frac{\partial \chi}{\partial x} - i\,\frac{\partial \chi}{\partial y} = -H_y - iH_x .$$

The second equation for χ states that $|\,\partial F/\partial w\,|^2$ is independent of z, so that

$$\frac{\partial F(w,z)}{\partial w} = \frac{\partial F(w,0)}{\partial w}\, e^{i\psi(z)}$$

with real $\psi(z)$ obeying $\psi(0) = 0$. If we write $-f(w)$ for $\partial F(w,0)/\partial w$ the result is

$$H_y + iH_x = f(x+iy)\, e^{i\psi(z)},$$

where f is an arbitrary analytic function and ψ an arbitrary real function.

Problem. Verify the solution obtained directly and show that $\alpha = d\psi/dz$.

Problem. Why is ψ a function of z alone rather than of w and z?

Problem. Find the force-free field which has both cylindrical symmetry and $H_z = 0$.

Problem. Find a force-free field for which H_z is a constant other than zero.

So far we have constructed solutions of (3) without paying attention to boundary conditions. It is not known which are the proper boundary conditions that make the solution possible and unique. However, the following calculation shows that a force-free field cannot vanish sufficiently rapidly at infinity to make the total field energy finite.*

Let Ω be an arbitrary volume, then

$$\int_\Omega \boldsymbol{r}\cdot(\boldsymbol{H}\wedge \operatorname{curl}\boldsymbol{H})d^3\boldsymbol{r} = \int_\Omega \boldsymbol{H}\cdot \operatorname{curl}(\boldsymbol{r}\wedge\boldsymbol{H})d^3\boldsymbol{r} + \oint_S \boldsymbol{H}\wedge(\boldsymbol{r}\wedge\boldsymbol{H})\cdot d\boldsymbol{S}$$

$$= -2\int_\Omega H^2\, d^3\boldsymbol{r} - \tfrac{1}{2}\int_\Omega (\boldsymbol{r}\cdot\nabla)H^2\, d^3\boldsymbol{r} + \oint_S H^2\boldsymbol{r}\cdot d\boldsymbol{S} - \oint_S (\boldsymbol{r}\cdot\boldsymbol{H})\boldsymbol{H}\cdot d\boldsymbol{S}$$

$$= -\tfrac{1}{2}\int_\Omega H^2\, d^3\boldsymbol{r} + \tfrac{1}{2}\oint_S H^2\boldsymbol{r}\cdot d\boldsymbol{S} - \oint_S (\boldsymbol{r}\cdot\boldsymbol{H})\boldsymbol{H}\cdot d\boldsymbol{S}.$$

For force-free fields the left-hand side vanishes. Since the term with the volume integral on the right-hand side is certainly negative, the surface

* See Chandrasekhar, p. 158.

integrals cannot both vanish. Hence there are no fields which are force-free everywhere in space and approach zero faster than $r^{-\frac{3}{2}}$ at infinity (which is necessary if the field energy is to be finite). If it is true that there is a force-free magnetic field in the Crab nebula*, then it must have a boundary layer, which is not force-free.

A slightly more general solution of the stationary MHD equations obtains by requiring for the magnetic field, instead of (3),

$$\boldsymbol{H}\wedge \operatorname{curl}\boldsymbol{H} = \operatorname{grad}\xi, \tag{5}$$

with arbitrary $\xi(\boldsymbol{r})$.** The equations (1) can again be satisfied by setting $\boldsymbol{V} = \beta\boldsymbol{H}$, provided that p is determined from

$$\frac{p}{\rho} + \tfrac{1}{2}V^2 + \Phi + \left(\frac{1}{4\pi\rho} - \beta^2\right)\xi = \text{constant}.$$

Problem. Show that (5) is satisfied if $\boldsymbol{H}$ is of the form

$$H_s = 0, \quad H_\varphi(s), \quad H_z(s).$$

3. Planar flow of a conducting viscous fluid

In order to account for experiments on flow of mercury in an external magnetic field*** ρ is again taken constant, but electrical resistance and viscosity cannot be neglected. The energy dissipated will have to be supplied by an externally applied pressure head. The relevant equations are for this case, compare (II, 5),

$$\operatorname{div}\boldsymbol{V} = 0 \tag{6a}$$

$$\rho(\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = -\operatorname{grad}(p+\rho\Phi) + \mu\nabla^2\boldsymbol{V} - \frac{1}{4\pi}\boldsymbol{H}\wedge\operatorname{curl}\boldsymbol{H} \tag{6b}$$

$$\operatorname{div}\boldsymbol{H} = 0 \tag{6c}$$

$$\operatorname{curl}(\boldsymbol{V}\wedge\boldsymbol{H}) + \eta\nabla^2\boldsymbol{H} = 0. \tag{6d}$$

Consider a fluid between two horizontal plane plates ($z = L$ and $z = -L$), which flows in the x-direction, so that all quantities are independent of y. Let there be a homogeneous external field $\boldsymbol{H}^0$ in the z-direction. The fluid

* L. Woltjer, Bull. Astron. Inst. Neth. **14**, 39 (1958).

** This is the same equation as in the case of static equilibrium, (IV, 9).

*** J. Hartmann and F. Lazarus, Kgl. Danske Videnskab. Selbskab, Mat.-Fys. Medd. **15**, nos. 6 and 7 (1937); W. Murgatroyd, Phil. Mag. **44**, 1348 (1953).

pulls at the field lines and bends them, so that a field component H^1 in the x-direction arises inside the fluid. Flow velocity, field and pressure *gradient* do not depend on x but only on z. Thus we seek a solution of the form

$$\boldsymbol{V} = (V(z), 0, 0), \qquad \boldsymbol{H} = (H^1(z), 0, H^0),$$

$$p + \rho\Phi = -xK_1(z) + K_2(z), \tag{7}$$

where K_1 and K_2 are functions to be determined. Clearly (6b) and (6c) are automatically satisfied, while (6d) reduces to

$$H^0 \frac{dV}{dz} + \eta \frac{d^2H^1}{dz^2} = 0, \tag{8}$$

and (6b) yields

$$0 = K_1 + \mu \frac{d^2V}{dz^2} + \frac{H^0}{4\pi}\frac{dH^1}{dz}, \tag{9}$$

$$0 = x\frac{dK_1}{dz} - \frac{dK_2}{dz} - \frac{1}{4\pi} H^1 \frac{dH^1}{dz}. \tag{10}$$

Since H^1 was assumed not to depend on x, one finds from (10) that dK_1/dz must be zero, hence K_1 = constant. Furthermore it follows from (10) that

$$K_2(z) + \{H^1(z)\}^2/8\pi = \text{constant}.$$

On integrating (8) one obtains

$$H^0 V + \eta \frac{dH^1}{dz} = A, \tag{11}$$

where A is a constant of integration. Hence (9) becomes

$$\mu \frac{d^2V}{dz^2} - \frac{(H^0)^2}{4\pi\eta} V = -K_1 - \frac{H^0 A}{4\pi\eta}.$$

This ordinary linear differential equation with constant coefficients can readily be solved to give

$$V(z) = B\cosh\left[\frac{H^0}{\sqrt{(4\pi\eta\mu)}} z\right] + C\sinh\left[\frac{H^0}{\sqrt{(4\pi\eta\mu)}} z\right] + \frac{4\pi\eta}{(H^0)^2} K_1 + \frac{A}{H^0}.$$

In order to determine the integration constants B and C we insert the boundary conditions $V(\pm L) = 0$. Moreover the dimensionless number

$$R_H = \frac{H^0 L}{\sqrt{(4\pi\eta\mu)}} \qquad \text{(Hartmann's number)}$$

is introduced. The result is

$$V(z) = \left[\frac{4\pi\eta}{(H^0)^2} K_1 + \frac{A}{H^0}\right]\left[1 - \frac{\cosh(R_H z/L)}{\cosh R_H}\right]. \tag{12}$$

This solution still contains the integration constant A, which is therefore not uniquely determined by the boundary conditions. It can be seen from (11) that A/c represents the strength of the electric field, which points in the y-direction. In fact, one is still free to apply an arbitrary electric field in this direction, thereby causing an electric current on which a force is exerted by the magnetic field. In order to remove this indeterminacy one has to know how the experiment is done. The space between the plates is of course not of infinite extent in the y-direction, but is closed by two vertical walls. If these are made of metal and interconnected, A must be zero.* In the actual experiment the walls were covered by an insulating layer so that the total current in the y-direction must vanish:

$$0 = \int_{-L}^{L} j_y\, dz = \frac{c}{4\pi}\int_{-L}^{L} \frac{dH^1}{dz} dz = \frac{c}{4\pi\eta}\int_{-L}^{L} (A - H^0 V) dz.$$

Solving for A and inserting the result into (12) one obtains finally

$$V = \frac{K_1 L^2}{\mu} \frac{\cosh R_H - \cosh R_H z/L}{R_H \sinh R_H}.$$

Problem. Show that for $R_H \ll 1$ this formula describes the plane laminar flow of ordinary hydrodynamics.

In Fig. 8 the velocity profile is sketched for several values of R_H and constant pressure gradient K_1. The larger H^0, the more difficult it is for the fluid layers to slide over each other, because the fluid particles tend to stick to the field lines. The magnetic field lends a certain stiffness to the fluid, which enhances the effect of viscosity.

This "magnetic stiffness" also becomes apparent in a different manner. It is known in ordinary hydrodynamics that for large velocity the laminar

* The walls influence the flow also in a different manner but such boundary effects may be neglected, provided the extent in the y-direction is large. Otherwise one has to solve the equations for a tube of rectangular cross section, see e.g. Shih-I Pai.

flow becomes unstable and *turbulence* occurs.* Viscosity improves stability, so that Reynolds' number can be used as a rough measure for the turbulence limit (R_e = 2000–100000 depending on shape and smoothness of the walls). Turbulence impedes the flow considerably. Now the pressure of a magnetic field helps viscosity in preventing turbulence and therefore raises the critical

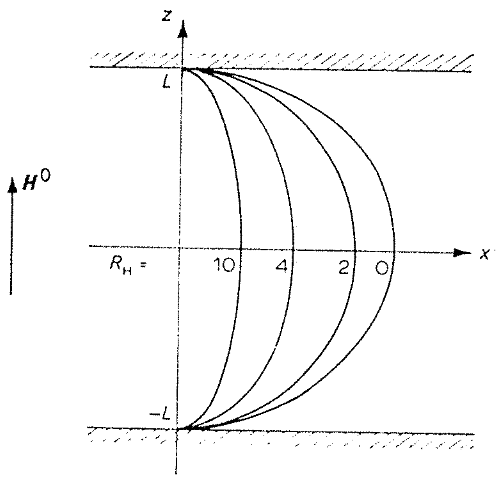

Fig. 8. Planar flow of a conducting fluid in a magnetic field $\boldsymbol{H}^0$.

Reynolds' number. Accordingly, for large velocity the flow will be increased by the presence of a magnetic field, although small velocity (laminar flow) is decreased by it. For further illustrations of magnetic stiffness, see Cowling and Chandrasekhar.

Problem. Consider a stationary, axially symmetric flow of an ideal plasma ($\sigma = \infty$), such that each fluid element moves in a horizontal circle whose center lies on the vertical z-axis:

$$V_s = 0, \quad V_\varphi = s\Omega(s, z), \quad V_z = 0.$$

$\Omega(s, z)$ is the angular velocity of the particles on the horizontal circle specified by the cylindrical coordinates ($s; 0 \leqslant \varphi < 2\pi; z$). The magnetic field is also assumed stationary and axially symmetric, but otherwise unspecified. Derive from equations (1c) and (1d) that Ω must be constant along a magnetic field line:

$$(\boldsymbol{H} \cdot \nabla)\Omega = 0.$$

* See e.g. L. Prandtl, *Essentials of Fluid Dynamics* (Blackie, London 1952); G. H. A. Cole, *Fluid Dynamics* (Methuen, London 1962).

(This is Ferraro's law of isorotation*; it also follows directly from the fact that the magnetic field lines are attached to the plasma, but it was discovered earlier.)

Problem. Show that the law of isorotation also applies to plasmas with finite conductivity, provided one imposes the additional restriction $H_\varphi = 0$.

Problem. Write the MHD equations for the case that
(i) the plasma is incompressible;
(ii) all quantities are independent of x and y;
(iii) the velocity lies in the x-direction,

$$\boldsymbol{V} = (V_x(z, t), 0, 0);$$

(iv) $H_y = 0$.
Show that in this case the time-dependent equations can be solved.**

4. Terrestrial Magnetism

The magnetic field of the earth is mainly a dipole field of the order of 0.5 gauss at the earth's surface (0.6 at the poles, 0.3 at the equator) corresponding to a dipole moment of 8×10^{25} gauss cm^3. Superimposed on this there are local variations and fluctuations in time with periods up to 30 years. These variations show a western drift of approximately 0.2° per year $\approx$ 1 mm/sec.

The earth consists of a *crust* (thickness ~35 km), the *mantle* (2900 km) and the *core* (radius 3500 km). This core contains liquid iron and nickel in which the heat production by radioactive decay of uranium and thorium causes convection currents. The core is regarded as the source of the earth's magnetism. The conductivity at the prevailing pressure (2 to 3 million atm) and temperature (estimated at 5000 °K) has been computed to be $\sigma = 3 \times 10^{-6}$ emu $= 3 \times 10^{15}$ sec^{-1}, that is one thirtieth of ordinary iron. The mantle has no influence on the magnetic field, as its conductivity is 10^4 times smaller and its magnetic permeability is practically unity. The magnetic field at the surface of the core is therefore $2^3 \times 0.5 = 4$ gauss, but inside the core field strengths of some tens of gauss are expected.***

A magnetic field in the core decays in 15000 years (see page 26). On the

* V. C. A. Ferraro, Monthly Notices Roy. Astron. Soc. **97**, 458 (1937).
** J. Szabo, Acta Phys. Hung. (to be published).
*** All such data are rather uncertain. For example, there are indications that there is also a nucleus of solid iron with a radius of 1300 km. See the review article by W. M. Elsasser, Rev. Mod. Phys. **22**, 1 (1950); also: C. W. Allen, *Astrophysical Quantities*, 2nd ed. (University of London, London 1963).

other hand a fluctuating field of frequency $\omega = 2\pi/(100 \text{ years})$ has a penetration depth of

$$\left(\frac{c^2}{2\pi\sigma\omega}\right)^{\frac{1}{2}} \approx 50 \text{ km}.$$

The fluctuating component is therefore only related to a thin surface layer of the core. The western drift corresponds to a velocity of 0.05 cm/sec in this layer which is a lower bound for the prevailing velocities. From this one finds $R_M \approx 600$, so that the earth's core behaves as a good conductor.

In 1919 Larmor advanced the idea that fluid flow in the core is responsible for maintaining the magnetic field of the earth. The idea is that the motion across the field lines generates a current, which is the source of the magnetic field ("*dynamo theory*").* This poses the problem of solving the complete set of equations of motion for $\boldsymbol{H}$ and V, taking into account the gravitational force, the Coriolis force and production and transport of heat (which have not been included in our MHD equations). The energy dissipated by electrical resistance and viscosity is supplied by the radioactive heating. However, since this problem is much too difficult it is simplified by disregarding the equation of motion for V; one tries instead to guess a reasonable flow pattern. On substituting this $V(r)$ in (1d) one is left with a linear equation for the magnetic field $H(r)$. The energy dissipated by the second term of (1d) is supplied by the first term; this is possible because $V(r)$ is regarded as given and therefore acts as a kind of external force on the magnetic field. As the flow is taken to be stationary, the treatment only refers to the dominant, time-independent part of the magnetic field. In addition to (1d) the following conditions must be obeyed.

(i) div $\boldsymbol{H} = 0$ throughout space,

(ii) div $V = 0$; $V_{\text{normal}} = 0$ at the surface,

(iii) $(\text{curl } H)_{\text{normal}} = (4\pi/c) j_{\text{normal}} = 0$ at the surface,

(iv) $\boldsymbol{H}_{\text{inside}} = \boldsymbol{H}_{\text{outside}}$ at the surface,

(v) curl $\boldsymbol{H} = 0$ outside the core and $\boldsymbol{H} \to 0$ at infinity.

Problem. Derive the energy balance equation for the "simplified problem", and show that the energy supplied by the material flow equals $-V \cdot (j \wedge H)/c$ per unit time per unit volume.

Problem. An infinitely long conducting cylinder rotates about its axis with constant angular velocity. Find all solutions $\boldsymbol{H}(r)$ that are finite and have cylindrical symmetry.

* W. M. Elsasser, Rev. Mod. Phys. **28**, 135 (1956).

Even this simplified approach has not yet led to a satisfactory understanding of the earth's magnetism. The reason is that no simple form for the magnetic field $\boldsymbol{H}(r)$ is possible. This has been demonstrated by Cowling in the following way. Try an axially symmetric field of the type one would expect from a (continuously distributed) ring current, see Fig. 9. Clearly

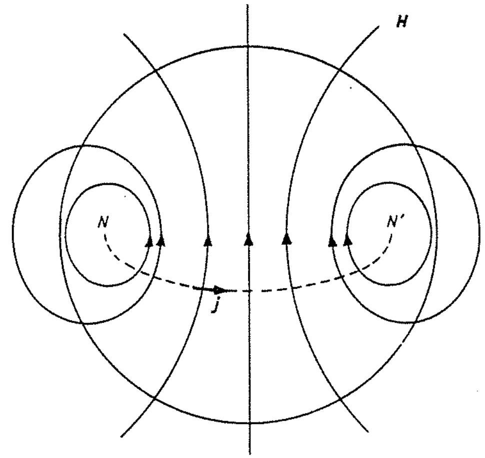

Fig. 9. Cowling's argument concerning the possibility of a dynamo.

$\boldsymbol{H} = 0$ at the points N and N', and therefore on the entire dotted circle. On the other hand curl $\boldsymbol{H} \neq 0$ on this circle, so that there must be a current running along the circle. This, however, is impossible because the line integral around the circle obeys the identity

$$\oint j \cdot dl = \sigma \oint E \cdot dl = 0,$$

since $\boldsymbol{H} = 0$ and curl $\boldsymbol{E} = 0$. This proves that no solution with a magnetic field of this simple type exists. The same conclusion applies to a number of other simple forms of $\boldsymbol{H}(r)$.

Problem. Prove that for any surface S that lies within the earth's core and is bounded by a closed field line one has

$$\int_S \nabla^2 \boldsymbol{H} \cdot d\boldsymbol{S} = 0.$$

Does this still hold when the surface has a bulge outside the core?

Problem. Prove for a cylindrically symmetric conductor rotating with constant angular velocity about its axis (the z-axis) that every cylindrically symmetric solution satisfies

$$\nabla^2 H_z = 0 \quad \text{and} \quad \nabla^2(\boldsymbol{r} \cdot \boldsymbol{H}) = 0 .$$

Conclude from this that a rotating rigid sphere cannot create a magnetic field that tends to a dipole field at large distance.

Problem. Take $\boldsymbol{V} = \mu \boldsymbol{a} \wedge \boldsymbol{r}$, where $\boldsymbol{a}$ is a constant vector in the z-direction, and μ an arbitrary function of the cylindrical coordinates s and z, but independent of φ. Show that the magnetic field obeys the equations

$$\nabla^2 H_z = \frac{a\mu(s, z)}{\eta} \frac{\partial H_z}{\partial \varphi}, \qquad \nabla^2(\boldsymbol{r} \cdot \boldsymbol{H}) = \frac{a\mu(s, z)}{\eta} \frac{\partial}{\partial \varphi}(\boldsymbol{r} \cdot \boldsymbol{H}) .$$

Bullard* makes the following flow pattern, due to radioactive heating, plausible (see Fig. 10). The material rises along one diameter in the equatorial

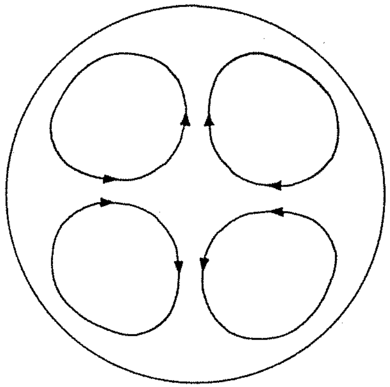

Fig. 10. Bullard's flow pattern in the earth's core.

plane, reaches the surface of the core at two opposite points, fans out along the equator and descends back towards the center along a diameter at right angles with the first one. The field lines, which are attached to the fluid, are stretched, which amounts to an increase of magnetic energy. This generates tens of gauss, of which only a small fraction extends beyond the earth's core and is observed as a dipole field. Bullard has supported this view with numerical calculations. However, it has not yet been shown in a definitive way that this flow pattern will actually maintain a magnetic field.

* E. C. Bullard, Proc. Roy. Soc. (London) **A197,** 433 (1949); Phil. Trans. Roy. Soc. London **A247,** 213 (1954); see also Cowling.

The question may be asked whether there exists at all a possible flow pattern that acts as a dynamo. A simple example of a self-exciting dynamo is shown in Fig. 11. The dynamics of such a model have been studied, and it was found that they are actually able to maintain a constant current and hence a constant magnetic field.* Moreover Bullard found oscillating solutions, in which the magnetic field may even reverse; this he related to the fact that the earth's magnetism reverses once every million years.

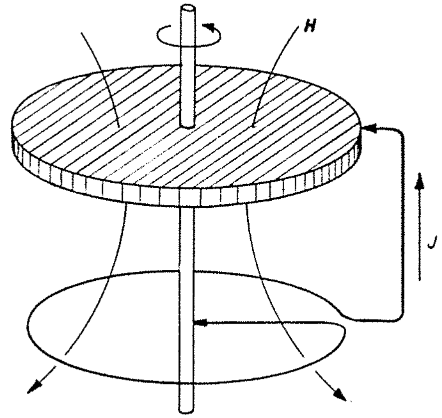

Fig. 11. Simple example of a self-exciting dynamo.

Yet this model is not entirely convincing, since it involves a conductor that is multiply connected (in the topological sense). The possibility of a self-exciting dynamo with a simply connected conductor has been thoroughly investigated by Backus**, and a simple working model has been constructed.***

General references

Shih-I Pai, *Magnetogasdynamics and Plasma Dynamics* (Springer, Vienna 1962) ch. VI.
J. A. Shercliff, *A Textbook of Magnetohydrodynamics* (Pergamon, Oxford 1965) ch. VI.
A. G. Kulikovskiy and G. A. Lyubimov, *Magnetohydrodynamics* (Addison-Wesley, Reading, Mass. 1965) chs. II, VI.
S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, Oxford 1961) ch. IV.
T. G. Cowling, *Magnetohydrodynamics* (Interscience, New York 1957) ch. V.

* E. C. Bullard, Proc. Cambridge Phil. Soc. **51,** 744 (1955); D. W. Allan, idem **58,** 671 (1962).
** G. Backus, Ann. Phys. (N.Y.) **4,** 372 (1958).
*** A. Herzenberg, Phil. Trans. Roy. Soc. London **A250,** 543 (1958); F. J. Lowes and I. Wilkinson, Nature **198,** 1158 (1963).

CHAPTER VI

WAVES IN MAGNETOHYDRODYNAMICS

To find time dependent solutions of the MHD equations is even more difficult than static or stationary solutions. The only practical method is to restrict oneself to solutions in the vicinity of a known (static or stationary) solution. The small deviations from the known solution are then treated to first order, higher orders being neglected. In other words, the MHD equations are linearized in the same way in which the hydrodynamic equations are linearized to obtain sound waves. In this way one obtains linear equations with constant coefficients, whose solutions have the form of waves. Owing to the greater complexity of the MHD equations, however, we shall find not only sound waves, but also a new kind of waves, called MHD waves.

For the known or "unperturbed" solution we take the static solution

$$\rho = \rho^0 = \text{constant}, \qquad p = p^0 = \text{constant}, \qquad V = 0, \qquad H = H^0 = \text{constant}. \tag{1}$$

We shall first study the case of an incompressible plasma, because it exhibits already the solution of most interest, the MHD wave. Subsequently we shall study the compressible plasma, in which not only MHD waves, but also sound waves occur.

1. MHD WAVES IN AN INCOMPRESSIBLE PLASMA

For an *ideally conducting*, *incompressible* plasma without external force ($g = 0$) the MHD equations reduce to

$$\nabla \cdot V = 0$$

$$\rho \frac{\partial V}{\partial t} + \rho(V \cdot \nabla)V = -\nabla p - \frac{1}{4\pi} H \wedge (\nabla \wedge H)$$

$$\frac{\partial H}{\partial t} = \nabla \wedge (V \wedge H) \qquad \nabla \cdot H = 0.$$

In order to seek a nonstationary solution in the vicinity of (1) we put

$$\rho = \rho^0, \quad p = p^0 + p^1, \quad V = V^1, \quad H = H^0 + H^1,$$

where p^1, V^1, H^1 are functions of r and t, but are small. If squares and products of these small quantities are neglected, the above set of equations becomes

$$\nabla \cdot V^1 = 0$$

$$\rho^0 \frac{\partial V^1}{\partial t} = -\nabla p^1 - \frac{1}{4\pi} H^0 \wedge (\nabla \wedge H^1)$$

$$\frac{\partial H^1}{\partial t} = \nabla \wedge (V^1 \wedge H^0)$$

$$\nabla \cdot H^1 = 0.$$

These equations are linear and homogeneous, and have coefficients that do not depend on r and t. Hence they must admit solutions of the form of plane waves:

$$p^1 = p_k e^{i(k \cdot r - \omega t)}, \qquad V^1 = V_k e^{i(k \cdot r - \omega t)}, \qquad H^1 = H_k e^{i(k \cdot r - \omega t)}.$$

Here k, ω, p_k, V_k, H_k are constants, which must be determined from the equations

$$k \cdot V_k = 0 \tag{2a}$$

$$-\omega \rho^0 V_k = -k p_k - \frac{1}{4\pi} H^0 \wedge (k \wedge H_k) \tag{2b}$$

$$-\omega H_k = k \wedge (V_k \wedge H^0) = V_k (k \cdot H^0) \tag{2c}$$

$$k \cdot H_k = 0. \tag{2d}$$

Clearly (2d) may be omitted since it is a consequence of (2c).* The remaining equations constitute for a given k a set of seven linear homogeneous equations for the seven unknowns p_k, V_k, H_k. In order that there is a non-trivial solution the determinant must vanish, which determines one or several values of ω for each k. However, rather than writing down a seven by seven determinant we shall solve the equations more directly.

(2c) states that H_k and V_k are parallel and

$$H_k / V_k = -(k \cdot H^0)/\omega.$$

* Unless $\omega = 0$, in which case (2d) must be taken into account separately.

Hence (2b) may be written

$$\left\{\omega\rho^0 - \frac{(\boldsymbol{k}\cdot\boldsymbol{H}^0)^2}{4\pi\omega}\right\}\boldsymbol{V}_k = \left\{p_k + \frac{1}{4\pi}\boldsymbol{H}^0\cdot\boldsymbol{H}_k\right\}\boldsymbol{k}. \tag{3}$$

According to (2a) the vectors $\boldsymbol{k}$ and $\boldsymbol{V}_k$ are perpendicular, so that (3) cannot be satisfied unless both members vanish. The vanishing of the left-hand side yields the *dispersion law**

$$\omega^2 = \frac{(\boldsymbol{k}\cdot\boldsymbol{H}^0)^2}{4\pi\rho^0}.$$

It is customary to use the abbreviation

$$\boldsymbol{u}_A = \frac{\boldsymbol{H}^0}{\sqrt{(4\pi\rho^0)}} \quad (\text{“Alfvén velocity”}). \tag{4}$$

The dispersion law then becomes $\omega^2 = (\boldsymbol{k}\cdot\boldsymbol{u}_A)^2$.

From the dispersion law follows the *phase velocity* of the wave $\omega/k = u_A \cos\vartheta$, where ϑ is the angle between the direction of propagation and $\boldsymbol{H}^0$. The fact that the phase velocity depends on the direction of the wave illustrates that the plasma is *anisotropic*, owing to the presence of the magnetic field $\boldsymbol{H}^0$. The fact that the phase velocity is independent of the wave length is expressed by saying that there is no dispersion. A consequence of this is that the *group velocity* $d\omega/dk$ is equal to the phase velocity. Hence one may speak without ambiguity about a “velocity of propagation” of these waves. The same remark applies to all waves that we shall find in magnetohydrodynamics.

In order to further investigate the properties of these MHD waves, we conclude from (2a) that the oscillation $\boldsymbol{V}_k$ is perpendicular to $\boldsymbol{k}$, so that they are *transverse* waves. In ordinary hydrodynamics transverse waves do not exist, because there is no elastic force to counteract the sliding of neighboring layers of the fluid with respect to each other. In magnetohydrodynamics, however, such a sliding stretches the magnetic field lines (which are attached to the fluid particles) and is therefore counteracted by the tension in the field lines. This is another illustration of the stiffness of a plasma provided by an external magnetic field.

* We use the name “dispersion law” for the equation that connects the frequency with the wave length. In plasma physics this is often called “dispersion relation”, but we shall avoid this expression, because in field theory a “dispersion relation” is something quite different, namely an equation of the type of a Kramers–Kronig relation.

Since the right-hand side of (3) must vanish,

$$p_k = -\frac{1}{4\pi}\boldsymbol{H}^0\cdot\boldsymbol{H}_k = \frac{1}{4\pi\omega}(\boldsymbol{H}^0\cdot\boldsymbol{k})(\boldsymbol{H}^0\cdot\boldsymbol{V}_k).$$

For waves propagating in the direction of the external field, $\boldsymbol{k}//\boldsymbol{H}^0$, one has $\boldsymbol{H}^0\cdot\boldsymbol{V}_k = 0$ so that the pressure variation p_k vanishes. For waves propagating in another direction, $\boldsymbol{k}$ not parallel to $\boldsymbol{H}^0$, there are the following two different modes, or polarizations. (a) $\boldsymbol{V}_k$ and $\boldsymbol{H}_k$ are perpendicular to the plane through $\boldsymbol{H}^0$ and $\boldsymbol{k}$, so that again the pressure variation vanishes. (b) $\boldsymbol{V}_k$ and $\boldsymbol{H}_k$ lie in this plane and $p_k \neq 0$. Of course any linear combination of these two different polarizations is also a solution of the equations (2).

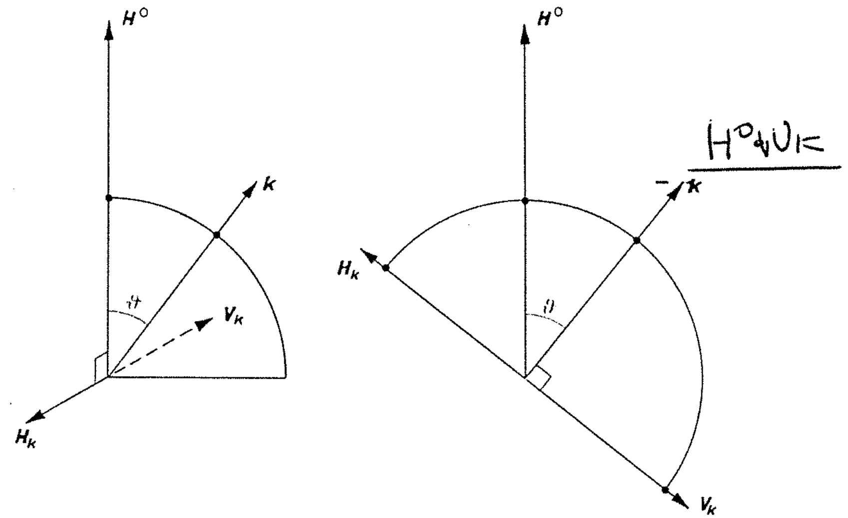

Fig. 12. Two modes of MHD waves propagating at an angle ϑ with the magnetic field.

Problem. Show that our solution does satisfy all equations (2).
Problem. Investigate the special case that $\boldsymbol{k}$ is perpendicular to $\boldsymbol{H}^0$.
Problem. Investigate the case $\omega = 0$, which has so far been excluded.

2. MHD WAVES IN THE DIRECTION OF $\boldsymbol{H}^0$ (ALFVÉN WAVES)

Consider the case $\boldsymbol{k}//\boldsymbol{H}^0$, so that the wave propagates along the unperturbed field lines of $\boldsymbol{H}^0$ with velocity $\boldsymbol{u}_A$. We shall use the name “Alfvén waves”

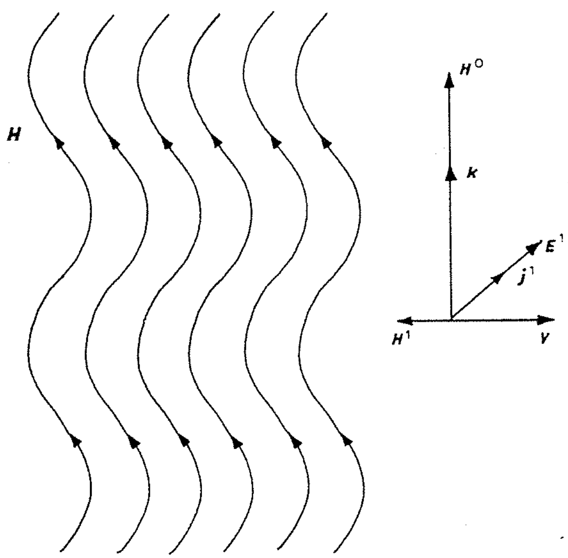

Fig. 13. Alfvén waves.

for this special case of MHD waves.* Take the z-axis in this direction and the x-axis in the direction of V_k. The explicit form of our solution is then found to be

$$V_x^1(z,t) = V_k \cos(kz-\omega t)$$

$$H_x^1(z,t) = H_k \cos(kz-\omega t) = -(kH^0/\omega)V_k \cos(kz-\omega t)$$

$$j_y^1(z,t) = \frac{c}{4\pi}\frac{k^2H^0}{\omega} V_k \sin(kz-\omega t)$$

$$E_y^1(z,t) = (H^0/c)V_k \cos(kz-\omega t).$$

Just like in the case of an electromagnetic wave in vacuum, $\boldsymbol{E}^1$ and $\boldsymbol{H}^1$ oscillate in phase with each other, and perpendicular to each other and to the direction of propagation (Fig. 13). The displacement of a fluid element is

$$\int V_x^1(z,t)\,dt = -\frac{V_k}{\omega}\sin(kz-\omega t).$$

On the other hand the angle between the perturbed field line and the z-axis is H_x^1/H^0, so that the displacement of the field line is

* This name is often used for all MHD waves, but it is convenient to make this distinction between both terms.

$$\int \frac{H_x(z,t)}{H^0}\,dz = -\frac{V_k}{\omega}\sin(kz-\omega t).$$

The fact that this is identical with the displacement of the fluid element demonstrates again that the field lines are attached to the fluid.

The *energy density* is according to (III, 6), apart from an additive constant,

$$\tfrac{1}{2}\rho V^2 + \frac{1}{8\pi}(H^1)^2 = \tfrac{1}{2}\left\{\rho V_k^2 + \frac{1}{4\pi}\left(\frac{kH^0V_k}{\omega}\right)^2\right\}\cos^2(kz-\omega t).$$

Since $k/\omega = 1/u_A = \sqrt{(4\pi\rho/H^0)}$, the two terms are equal in magnitude.* The energy density of the Alfvén wave, averaged over a period, is therefore $\frac{1}{2}\rho V_k^2$.

The *energy current density* in the z-direction is according to (III, 6)

$$-\frac{c}{4\pi}E_y^1H_x^1 = \frac{k}{4\pi\omega}(H^0V_k)^2\cos^2(kz-\omega t),$$

and averaged over a period

$$\frac{(H^0V_k)^2}{8\pi u_A} = (\tfrac{1}{2}\rho V_k^2)u_A.$$

As was to be expected this equals energy density times velocity of propagation.

Lundquist** has demonstrated MHD waves experimentally in mercury. In these experiments H^0 was 13000 gauss, so that u_A = 1000 cm/sec. In the actual experiment the oscillation was much more complicated than the plane wave calculated here, owing to the shape of the vessel and the strong damping caused by resistivity. Alfvén waves have also been observed in metals.*** Moreover, MHD waves are a source of astrophysical speculations.†

The fact that the sun rotates almost uniformly as if it were a solid body, can be ascribed to magnetic stiffness. Suppose that this uniform rotation is

* In this connection again the word "equipartition" is sometimes used, but actually this equality only expresses the fact that for an harmonic oscillator the kinetic and potential energies are equal on the average (virial theorem!).
** S. Lundquist, Nature **164**, 145 (1949).
*** See R. Bowers in: *Plasma Effects in Solids* (Dunod, Paris 1965). In discharges oscillations have been observed which were interpreted also as MHD waves; see e.g. S. Glasstone and R. H. Lovberg, *Controlled Thermonuclear Reactions* (Van Nostrand, Princeton 1960) p. 263.
† See e.g. J. W. Dungey, *Cosmic Electrodynamics* (Cambridge University Press, Cambridge 1958).

perturbed, for example by an outwards directed convection, which at the surface will rotate more slowly owing to the Coriolis force. This causes a bending of the magnetic field lines; as these try to straighten themselves out Alfvén waves are generated. The waves traverse the sun with a velocity of 0.2 cm/sec, that is, in 10000 years. Hence at least a time of this order is needed to restore uniformity; fortunately this time is indeed much shorter than the life-time of the sun. Walén has tried to explain the generation of sunspots from such perturbations in the magnetic field.

Problem. Complete the following derivation of the Alfvén waves. Take $\boldsymbol{H}^0$ in the z-direction and let $\xi(z)$ be the displacement in the x-direction of the fluid. Since the magnetic field is attached to the fluid it obtains an x-component $H^1(z) = H^0(d\xi/dz)$, so that the magnetic energy per unit volume is

$$\frac{H^2}{8\pi} = \frac{(H^0)^2}{8\pi} + \frac{(H^0)^2}{8\pi}\left(\frac{d\xi}{dz}\right)^2 .$$

The kinetic energy per unit volume is $\frac{1}{2}\rho^0\dot{\xi}(z)^2$. Form the Lagrange function and find from this the dispersion law for Alfvén waves.
Remark. The correct result is also obtained by regarding the field lines as strings of mass ρ^0 per unit length, provided one takes for their tension $(H^0)^2/4\pi$ rather than $(H^0)^2/8\pi$; compare page 21.

Problem. Supplement the equations of motion with dissipative terms with resistivity η and viscosity μ. Show that now the dispersion law for Alfvén waves is

$$k^2u_A^2 = (\omega+i\eta k^2)(\omega+i\mu k^2/\rho^0).$$

Note that the phase velocity ω/k is no longer independent of the wave length, in other words there is dispersion.

3. Compressible plasma: the linearized equations

For an *ideally conducting*, *compressible* plasma without external force $\boldsymbol{g}$ the MHD equations read

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho \boldsymbol{V} = 0$$

$$\rho\frac{\partial \boldsymbol{V}}{\partial t} + \rho(\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = -\nabla p - \frac{1}{4\pi}\boldsymbol{H}\wedge(\nabla\wedge\boldsymbol{H})$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = \nabla\wedge(\boldsymbol{V}\wedge\boldsymbol{H})$$

$$\nabla\cdot\boldsymbol{H} = 0$$

$$p = f(\rho).$$

Again (1) is a solution and we seek a neighboring nonstationary solution by putting

$$\rho = \rho^0+\rho^1, \qquad \boldsymbol{V} = \boldsymbol{V}^1, \qquad \boldsymbol{H} = \boldsymbol{H}^0+\boldsymbol{H}^1,$$

where ρ^1, $\boldsymbol{V}^1$, $\boldsymbol{H}^1$ depend on time, but are supposed so small that their squares and products may be neglected. They must satisfy the linearized equations

$$\frac{\partial\rho^1}{\partial t} + \rho^0\nabla\cdot\boldsymbol{V}^1 = 0$$

$$\rho^0\frac{\partial \boldsymbol{V}^1}{\partial t} = -f'(\rho^0)\nabla\rho^1 - \frac{1}{4\pi}\boldsymbol{H}^0\wedge(\nabla\wedge\boldsymbol{H}^1)$$

$$\frac{\partial \boldsymbol{H}^1}{\partial t} = \nabla\wedge(\boldsymbol{V}^1\wedge\boldsymbol{H}^0)$$

$$\nabla\cdot\boldsymbol{H}^1 = 0.$$

Again these equations admit solutions of the form

$$\rho^1 = \rho_k\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \qquad \boldsymbol{V}^1 = \boldsymbol{V}_k\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \qquad \boldsymbol{H}^1 = \boldsymbol{H}_k\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \tag{5}$$

where $\boldsymbol{k}$, ω, ρ_k, $\boldsymbol{V}_k$, $\boldsymbol{H}_k$ are constants to be determined from

$$-\omega\rho_k+\rho^0\boldsymbol{k}\cdot\boldsymbol{V}_k = 0 \tag{6a}$$

$$-\omega\rho^0\boldsymbol{V}_k = -f'(\rho^0)\boldsymbol{k}\rho_k - \frac{1}{4\pi}\boldsymbol{H}^0\wedge(\boldsymbol{k}\wedge\boldsymbol{H}_k) \tag{6b}$$

$$-\omega\boldsymbol{H}_k = \boldsymbol{k}\wedge(\boldsymbol{V}_k\wedge\boldsymbol{H}^0) \tag{6c}$$

$$\boldsymbol{k}\cdot\boldsymbol{H}_k = 0. \tag{7}$$

Again, since (7) is a consequence of (6c), this constitutes (for fixed $\boldsymbol{k}$ and ω) a set of seven linear homogeneous equations for the seven unknowns ρ_k, $\boldsymbol{V}_k$, $\boldsymbol{H}_k$. In order that there is a non-trivial solution the determinant must vanish, which yields a relation between $\boldsymbol{k}$ and ω. It is easily seen that for any given $\boldsymbol{k}$ there are in general seven possible eigenvalues of ω. Hence for every $\boldsymbol{k}$ there are seven solutions ρ_k, $\boldsymbol{V}_k$, $\boldsymbol{H}_k$, the eigenvectors (each with seven components). For a given direction of propagation there are thus seven modes of oscillation, each with its own frequency.

Problem. In the special case $\boldsymbol{H}^0 = 0$ the equations (6a) and (6b) together constitute a set of four equations for ρ_k and $\boldsymbol{V}_k$. Show that there are two eigenvalues $\omega = \pm k\sqrt{(f'(\rho^0))}$ corresponding to ordinary longitudinal sound waves, and two eigenvalues $\omega = 0$ corresponding to transverse motions of the fluid. In this connection we introduce an abbreviation for the sound velocity,

$$u_S = \sqrt{f'(\rho^0)}. \tag{8}$$

Problem. Let ρ^0, $\boldsymbol{V}^0$, $\boldsymbol{H}^0$ be an arbitrary time dependent solution of the MHD equations. Find the linearized equations for neighboring solutions. Show that in this case (5) does not lead to a solution.

4. Solution of the Eigenvalue Problem (6)

As the determinant of the equations (6) is too large to write down we reduce the problem by solving ρ_k and $\boldsymbol{H}_k$ from (6a) and (6c) and substituting into (6b). With the abbreviations (4) and (8),

$$-\omega^2 \boldsymbol{V}_k = -u_S^2(\boldsymbol{k}\cdot\boldsymbol{V}_k)\boldsymbol{k} + \boldsymbol{u}_A \wedge [\boldsymbol{k}\wedge\{\boldsymbol{k}\wedge(\boldsymbol{V}_k\wedge\boldsymbol{u}_A)\}].$$

Work out the vector product and put $(\boldsymbol{k}\cdot\boldsymbol{u}_A) = \kappa$,

$$(\omega^2-\kappa^2)\boldsymbol{V}_k = \{(u_S^2+u_A^2)(\boldsymbol{k}\cdot\boldsymbol{V}_k) - \kappa(\boldsymbol{u}_A\cdot\boldsymbol{V}_k)\}\,\boldsymbol{k} - \kappa(\boldsymbol{k}\cdot\boldsymbol{V}_k)\boldsymbol{u}_A. \tag{9}$$

These are three linear homogeneous equations for the three components of $\boldsymbol{V}_k$. Rather than computing the cartesian components, it is convenient to compute the components in the directions $\boldsymbol{k}$, $\boldsymbol{u}_A$, and $\boldsymbol{k}\wedge\boldsymbol{u}_A$. Strictly speaking the case that $\boldsymbol{k}$ is parallel to $\boldsymbol{u}_A$ should be considered separately, but it can also be treated as a limit of the general case.

Abbreviate

$$\boldsymbol{k}\cdot\boldsymbol{V}_k = \alpha, \quad \boldsymbol{u}_A\cdot\boldsymbol{V}_k = \beta, \quad (\boldsymbol{k}\wedge\boldsymbol{u}_A)\cdot\boldsymbol{V}_k = \gamma.$$

By taking the scalar product of (9) with $\boldsymbol{k}$, $\boldsymbol{u}_A$, $\boldsymbol{k}\wedge\boldsymbol{u}_A$ respectively one finds three linear homogeneous equations for α, β, γ:

$$\{\omega^2 - k^2(u_S^2+u_A^2)\}\alpha + k^2\kappa\beta = 0 \tag{10a}$$

$$-\kappa u_S^2\alpha + \omega^2\beta = 0 \tag{10b}$$

$$(\omega^2-\kappa^2)\gamma = 0. \tag{10c}$$

By solving this set of equations one obtains three different modes.

First mode. $\omega = \pm\kappa$, $\alpha = \beta = 0$. The velocity of propagation is again $\omega/k = u_A\cos\vartheta$ (ϑ is the angle between $\boldsymbol{H}^0$ and $\boldsymbol{k}$). The oscillation $\boldsymbol{V}_k$ is perpendicular to $\boldsymbol{H}^0$ and $\boldsymbol{k}$, so that according to (6a) $\rho_k = 0$. This solution is identical with one of the modes of the MHD wave in the incompressible plasma, viz. the one without pressure variation (Fig. 12a).

The other two modes contained in (10) are obtained by putting $\gamma = 0$ and solving α and β from (10a) and (10b). The eigenvalue equation of this problem is

$$\omega^4 - \omega^2k^2(u_S^2+u_A^2) + k^2\kappa^2u_S^2 = 0. \tag{11}$$

Remark. Since (10) has only six eigenvalues, one of the seven eigenvalues of (6) must have been lost. The reason is that (6) contains ρ_k, $\boldsymbol{V}_k$, $\boldsymbol{H}_k$ as unknowns, whereas (9) and (10) only contain the three components of $\boldsymbol{V}_k$ (ρ_k and $\boldsymbol{H}_k$ having been eliminated). A solution of (6) for which $\boldsymbol{V}_k = 0$, $\rho_k \neq 0$, $\boldsymbol{H}_k \neq 0$ corresponds to the trivial solution $\alpha = \beta = \gamma = 0$ of (10), which is not counted as a solution when the eigenvalues of (10) are determined. Hence the solutions of (6) with $\boldsymbol{V}_k = 0$ have to be studied separately. On inserting $\boldsymbol{V}_k = 0$ in (6) one readily finds $\omega = 0$, so that this solution describes a *static displacement* of the fluid. In addition one finds from (6)

$$\left\{f'(\rho^0)\rho_k + \frac{1}{4\pi}\boldsymbol{H}^0\cdot\boldsymbol{H}_k\right\}\boldsymbol{k} = \frac{1}{4\pi}(\boldsymbol{H}^0\cdot\boldsymbol{k})\boldsymbol{H}_k. \tag{12}$$

Moreover one must take (7) into account separately, because for $\omega = 0$ it is no longer a consequence of (6c). One then finds from (12) $\boldsymbol{H}^0\cdot\boldsymbol{k} = 0$, which states that the displacement is perpendicular to $\boldsymbol{H}^0$. Finally one finds from (12)

$$f'(\rho^0)\,\rho^1(\boldsymbol{r}) + \frac{1}{4\pi}\boldsymbol{H}^0\cdot\boldsymbol{H}^1(\boldsymbol{r}) = 0,$$

which states that the sum of hydrostatic and magnetic pressure is constant.

Problem. Analyze all possible solutions of (6) and (7) with $\omega = 0$, including those that have been dropped in the foregoing analysis by tacitly supposing $\boldsymbol{k} \neq 0$.

5. The remaining modes

It remains to study the modes corresponding to the eigenvalues determined by (11). It may be supposed that $\omega \neq 0$ and hence $\boldsymbol{k} \neq 0$, as the case $\omega = 0$ has been studied separately in the above "Problem". Moreover, for simplicity only the special case $\boldsymbol{k} \perp \boldsymbol{H}^0$ and $\boldsymbol{k} // \boldsymbol{H}^0$ will be investigated in detail.

Second mode. $\boldsymbol{k} \perp \boldsymbol{H}^0$, so that $\kappa = 0$. On account of (10b) and (10c) one has $\beta = \gamma = 0$. Consequently V_k must lie in the direction of $\boldsymbol{k}$: the wave is *longitudinal.* The velocity of propagation is

$$|\,\omega/k\,| = \sqrt{u_S^2 + u_A^2}\,.$$

For $\boldsymbol{H}^0 \to 0$ this wave reduces to an ordinary sound wave; the magnetic field simply adds an additional stiffness. Hence, unlike the MHD waves this mode is not purely a magnetohydrodynamic effect, and is therefore called *magnetoacoustic* or *magnetosonic* wave (Fig. 14).

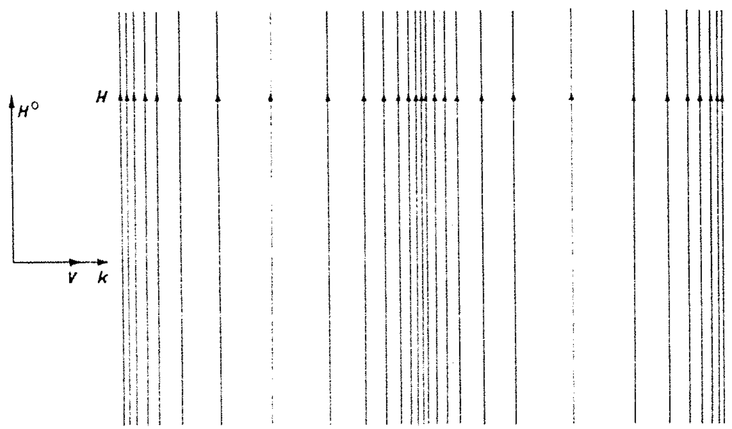

Fig. 14. Magneto-acoustic or magnetosonic wave.

Third mode. $\boldsymbol{k} // \boldsymbol{H}^0$. Then $\kappa = k u_A$ and the solutions of (11) are $\omega^2 = k^2 u_S^2$ and $\omega^2 = k^2 u_A^2$. However, as has been remarked before, in this case (10) is no longer a logical consequence of (9), so that we have to start from (9) itself. For $\boldsymbol{k} // \boldsymbol{H}^0$ this equation reduces to

$$(\omega^2 - k^2 u_A^2) V_k = (u_S^2 - u_A^2)\,(\boldsymbol{k} \cdot V_k)\boldsymbol{k}\,.$$

One solution is evidently $\omega^2 = k^2 u_A^2$, with $V_k \perp \boldsymbol{k}$; this is again the Alfvén wave treated before. If $\omega^2 \neq k^2 u_A^2$ one finds (on multiplying by $\boldsymbol{k}$)

$$\omega^2 = k^2 u_S^2\,, \qquad V_k // \boldsymbol{k}\,.$$

This is the third mode; it is simply an ordinary longitudinal sound wave propagating along the field lines of $\boldsymbol{H}^0$ (Fig. 15). It is not affected by the magnetic field, because the fluid particles move only parallel to the field; indeed, $\boldsymbol{H}^1 = 0$ according to (6c).

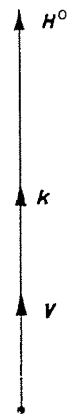

Fig. 15. Sound wave along $\boldsymbol{H}^0$.

Problem. It is clear that there is a solution in which the fluid moves along the field lines, with a velocity varying from one field line to the next. Where does this solution belong in our catalog?

Problem. Study the modes corresponding to the eigenvalues of (11) for an arbitrary angle between $\boldsymbol{H}^0$ and $\boldsymbol{k}$.

General references

H. Alfvén, *Cosmical Electrodynamics* (Oxford University Press, Oxford 1950) ch. IV.
H. Grad in: *The Magnetodynamics of Conducting Fluids*, D. Bershader ed. (Stanford University Press, Stanford 1959).
R. Lüst, Fortschr. Physik **7**, 503 (1959).

CHAPTER VII

INSTABILITY IN MAGNETOHYDRODYNAMICS

The static and stationary solutions found so far need not all be stable. A solution is called *unstable* if, by slightly modifying the initial values of ρ, V, $\boldsymbol{H}$ it is possible to obtain another solution which in the course of time moves farther and farther away from the original solution.* If this is the case the physical state described by the static or stationary solution is short-lived. For, in practice, it is impossible to realize the initial conditions so precisely and to avoid perturbations so completely that the plasma is exactly in the desired static or stationary state. In case of stability that does not matter, because any deviation remains small. In case of instability, however, the deviation grows indefinitely, so that in due time the state of the plasma is entirely different from the one described by the static or stationary solution.

Such instabilities occur also in ordinary hydrodynamics and mechanics. In plasma physics they are the rule rather than the exception. For practical applications it is naturally of great interest to know whether or not a given solution is stable. In addition one often wants to know whether a certain instability grows rapidly or perhaps so slowly that it is harmless. A great deal of research in plasma physics is therefore concerned with the investigation of the many kinds of instabilities. In this chapter we only give the general theory without going into the details of the specific instabilities. Moreover we restrict ourselves to MHD and to ideal conduction: $\eta = 0$. The theory of instabilities in classical mechanics is developed first and subsequently applied to plasmas.

* To distinguish it from other kinds of stability, this kind is more specifically called "stability in the sense of Lyapunov". We emphasize the difference with *continuous* dependence on the initial data, mentioned in ch. III, sec. 1. Continuous dependence means that the quantities at a *fixed* time $t > 0$ are continuous functions of the initial data; stable dependence means that this continuity is *uniform for all* $t > 0$. All our solutions of the MHD equations have the former property, but the question whether or not they are stable is the subject of this chapter.

1. Instabilities in Classical Mechanics

Consider a system with f generalized coordinates $\{q_k\}$, $(k = 1, 2, \ldots, f)$, whose motion is described by the Lagrange function $L(\dot{q}_1, \ldots, \dot{q}_f; q_1, \ldots, q_f)$. Let $\{q_k^0(t)\}$ be a certain solution of the equations of motion, such that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)^0 - \left(\frac{\partial L}{\partial q_k}\right)^0 = 0. \tag{1}$$

The upper index ^{0} indicates that after differentiation q_k^0 and $\dot{q}_k^0$ must be inserted. In order to describe neighboring solutions we transform from the $\{q_k\}$ to new coordinates $\{\xi_k\}$ defined by

$$q_k = q_k^0 + \xi_k .$$

In terms of these coordinates the Lagrange function becomes

$$L(\dot{q}^0+\dot{\xi};\, q^0+\xi) = L(\dot{q}^0; q^0) + \sum_k \left[\left(\frac{\partial L}{\partial \dot{q}_k}\right)^0 \dot{\xi}_k + \left(\frac{\partial L}{\partial q_k}\right)^0 \xi_k\right]$$
$$+\tfrac{1}{2}\sum_{k,l}\left[\left(\frac{\partial^2 L}{\partial \dot{q}_k \partial \dot{q}_l}\right)^0 \dot{\xi}_k\dot{\xi}_l + 2\left(\frac{\partial^2 L}{\partial \dot{q}_k \partial q_l}\right)^0 \dot{\xi}_k \xi_l + \left(\frac{\partial^2 L}{\partial q_k \partial q_l}\right)^0 \xi_k \xi_l\right] + \ldots \ .$$

This is a time dependent Lagrange function from which the Lagrange equations of motion for the ξ_k may be derived in the usual way. Since we are interested in solutions close to $\{q_k^0\}$ the ξ_k are small and we omit higher orders. The first term on the right-hand side does not involve $\dot{\xi}_k$ and ξ_k, and may therefore be omitted as it does not affect the equations of motion. The second term, which is linear in $\dot{\xi}_k$ and ξ_k may be written with the aid of (1) as a total derivative,

$$\sum_k \left[\left(\frac{\partial L}{\partial \dot{q}_k}\right)^0 \dot{\xi}_k + \left(\frac{\partial L}{\partial q_k}\right)^0 \xi_k\right] = \frac{d}{dt}\left[\sum_k \left(\frac{\partial L}{\partial \dot{q}_k}\right)^0 \xi_k\right].$$

Total derivatives in the Lagrange function do not affect the equations of motion (as is apparent from the variational form of the equations of motion), so that this term may also be omitted. Consequently the equations for the ξ_k are determined by the quadratic Lagrange function

$$\Lambda(\dot{\xi}, \xi) = \tfrac{1}{2}\sum_{k,l}\left[A_{kl}\dot{\xi}_k\dot{\xi}_l + 2B_{kl}\dot{\xi}_k\xi_l + C_{kl}\xi_k\xi_l\right]. \tag{2}$$

Here A_{kl}, B_{kl} and C_{kl} are square matrices which may depend on t through the chosen solution $\{q_k^0(t)\}$; the matrices A_{kl} and C_{kl} are symmetric, B_{kl} in general is not.

In most cases, and certainly in our future applications, L has the form

$$L(\dot{q}\,; q) = \tfrac{1}{2} \sum_{k,l} A_{kl}\dot{q}_k\dot{q}_l - W(q)\,. \tag{3}$$

Clearly this A_{kl} is the same matrix as in (2). For our purpose it suffices to take $A_{k.}$ constant (i.e., independent of the q_k); it follows that

$$B_{kl} = 0\,, \qquad C_{kl} = -\left(\frac{\partial^2 W}{\partial q_k \partial q_l}\right)^0 . \tag{4}$$

Hence the equations of motion for $\{\xi_k\}$ are

$$\sum_l A_{kl}\ddot{\xi}_l = \sum_l C_{kl}\xi_l\,. \tag{5}$$

This is a set of f linear homogeneous differential equations. So far it has not yet been assumed that the unperturbed solution $\{q_k^0(t)\}$ is static; this assumption will now be made, so that the q_k^0, and hence the coefficients C_{kl} do not depend on t.

Equation (5) may now be solved by the familiar method of the theory of small oscillations. Putting $\xi_k(t) = X_k e^{-i\omega t}$ one finds for $\{X_k\}$ the eigenvalue equation

$$-\omega^2 \sum_l A_{kl}X_l = \sum_l C_{kl}X_l\,. \tag{6}$$

There are f eigenvalues ω^2 (not necessarily all different), each with an eigenvector $\{X_k\}$. Three cases may be distinguished.

(i) All eigenvalues ω^2 are positive. In this case all frequencies ω are real so that all solutions $\{\xi_k(t)\}$ oscillate. The waves treated in the previous chapter are examples of such solutions. These deviations never become large: the static solution $\{q_k^0\}$ is *stable*.

(ii) One or more eigenvalues ω^2 are negative. The corresponding frequencies $\pm i\sqrt{(-\omega^2)}$ are imaginary. Hence there are $\{\xi_k(t)\}$ that increase exponentially with t, so that $\{q_k^0\}$ is *unstable*.*

(iii) There are no negative eigenvalues, but one or more are zero. For the corresponding $\{X_k\}$ the right-hand side of (6) vanishes; hence $\xi_k(t) = (C_1 + C_2 t)X_k$ (with two arbitrary constants C_1, C_2) is a solution of (5). This is called a state of *neutral equilibrium*.

Remark. For the validity of this theory it is essential that A_{kl} and C_{kl} are

* Of course, the ξ_k become rapidly so big that the linear approximation is no longer valid. In reality therefore the increase of ξ_k with t need not be exponential.

symmetric matrices.* A consequence of this property is that ω^2 is real, so that only real and purely imaginary ω can occur. A complex eigenvalue would mean that $\{\xi_k(t)\}$ oscillates with exponentially increasing amplitude. This form of instability is called "overstability", but it cannot occur in this case.

In order to decide whether or not instability occurs it is not always necessary to completely solve the eigenvalue problem (6). If $\{X_k\}$ is a solution, one also has

$$\omega^2 \sum_{k,l} A_{kl}X_kX_l = -\sum_{k,l} C_{kl}X_kX_l = \sum_{k,l}\left(\frac{\partial^2 W}{\partial q_k \partial q_l}\right)^0 X_kX_l\,. \tag{7}$$

Since A_{kl} is positive definite it follows that ω^2 is positive or negative according as the right-hand side is positive or negative. This shows: If the matrix formed by the second derivatives of the potential energy is positive definite the state is stable. Conversely, if all eigenvalues ω^2 are positive this matrix must be positive definite. Hence if one finds an arbitrary $\{X_k\}$ for which the right-hand side of (7) becomes negative, there is certainly a negative eigenvalue and instability occurs. In plasma physics this elementary property of matrices is called the *energy principle*.

2. Lagrange Formulation of the MHD Equations

In order to apply this theory to MHD it is necessary to derive the MHD equations from a Lagrange function. This is only possible if there is no dissipation; hence we are forced to restrict ourselves to the case of ideally conducting plasmas: $\eta = 0$. Moreover it is first necessary to cast the MHD in a different form.

In hydrodynamics one distinguishes two ways of describing the kinetics of a fluid: following Euler and following Lagrange. In the Euler description ρ and V are expressed as functions of space and time, as we have done so far. In the Lagrange description the position of a fluid element is expressed as a function of t and of the initial position of that element, as we shall do now.** To avoid confusion we emphasize that this "Lagrange description in hydrodynamics" is something else than the "Lagrange form of the equations of classical point mechanics". However, once hydrodynamics has been

* Moreover A_{kl} must be positive definite. According to (3) this is the case, since the kinetic energy is always positive.

** See e.g. A. Sommerfeld, *Mechanik der deformierbaren Medien* (*Vorlesungen* II) (Dieterich, Wiesbaden 1947) sec. 34; or: H. Lamb, *Hydrodynamics* (Cambridge University Press, Cambridge 1932) ch. I.

written in the Lagrange description, it is possible to cast the hydrodynamical equations of motion in a form that is similar to the Lagrange equations of the point mechanics.

Let the position of a fluid element, which was in $\boldsymbol{a}$ at time 0, be $\boldsymbol{x}(\boldsymbol{a}, t)$ at time t. The vector function $\boldsymbol{x}(\boldsymbol{a}, t)$ gives a complete description of the fluid flow. For example

$$V = \dot{x}(a, t), \qquad DV/Dt = \ddot{x}(a, t). \tag{8}$$

If one wants to know V as a function of x and t, then $\boldsymbol{a}$ should be expressed in $\boldsymbol{x}$ by solving $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a}, t)$ for $\boldsymbol{a}$, and the result should be substituted in $\dot{x}(a, t)$. The volume of the fluid element d^3x equals the volume of the corresponding fluid element $d^3\boldsymbol{a}$ multiplied by the Jacobi determinant $d(x)/d(\boldsymbol{a})$. Conservation of mass therefore yields an equation for the density,

$$\rho \frac{d(x)}{d(a)} = \rho_a, \tag{9}$$

where ρ is the density at time t at position $\boldsymbol{x}(\boldsymbol{a}, t)$ and ρ_a the density at time 0 at position $\boldsymbol{a}$. This also determines the pressure.

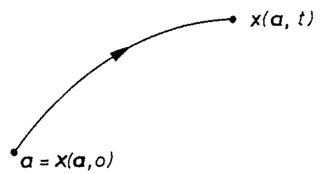

Fig. 16. The Lagrange description of hydrodynamics.

In order to find the magnetic field at time t, if it is given for $t = 0$, we employ the considerations on pages 24 and 25. At $t = 0$ take a point $\boldsymbol{a}$ and a neighboring point $\boldsymbol{a}+\Delta\boldsymbol{a}$ lying on the same field line at a distance $\varepsilon H/\rho_a$, i.e. $\Delta \boldsymbol{a} = \varepsilon \boldsymbol{H}(\boldsymbol{a}, 0)/\rho(\boldsymbol{a}, 0)$. Here ε is an arbitrary number but sufficiently small to treat the field line between $\boldsymbol{a}$ and $\boldsymbol{a}+\Delta\boldsymbol{a}$ as a straight line. At time t the corresponding fluid elements have moved to $\boldsymbol{x}(\boldsymbol{a}, t)$ and

$$x(a, t) + \left(\Delta a \cdot \frac{\partial}{\partial a}\right) x(a, t) = x(a, t) + \varepsilon \left(\frac{H(a, 0)}{\rho(a, 0)} \cdot \frac{\partial}{\partial a}\right) x(a, t)$$

respectively. Since the fluid has been assumed ideally conducting these two points still lie on the same field line at a mutual distance $\varepsilon H/\rho$; hence

$$\frac{H(x, t)}{\rho(x, t)} = \left(\frac{H(a, 0)}{\rho(a, 0)} \cdot \frac{\partial}{\partial a}\right) x(a, t). \tag{10}$$

Equations (8), (9), (10) express the values of V, ρ, $\boldsymbol{H}$ at time t in terms of their initial values once the function $\boldsymbol{x}(\boldsymbol{a}, t)$ that describes the flow is known. This is the Lagrange formulation of the kinematics of an ideally conducting plasma.

Problem. Let Q be an arbitrary quantity following the fluid flow, that is, $Q(\boldsymbol{x}, t) = Q(\boldsymbol{a}, 0)$. Prove that

$$\left(\frac{H}{\rho} \cdot \nabla\right) Q = \left(\frac{H_a}{\rho_a} \cdot \nabla_a\right) Q_a, \tag{11}$$

where $\boldsymbol{H}_a = \boldsymbol{H}(\boldsymbol{a}, 0)$ and $Q_a = Q(\boldsymbol{a}, 0)$.

The energy expression (III, 6) suggests that there is a Lagrange function of the form

$$L(\dot{x}; x) = \int \left\{\tfrac{1}{2}\rho_a \dot{x}^2 - \rho_a \Phi(x) - \rho_a \Psi(\rho) - \frac{1}{8\pi}\frac{\rho_a}{\rho} H^2\right\} d^3\boldsymbol{a}. \tag{12}$$

Here $\boldsymbol{x}(\boldsymbol{a}, t)$ plays the role of the $q_k(t)$, with $\boldsymbol{a}$ corresponding to the index k. Furthermore ρ and $\boldsymbol{H}$ are to be regarded as implicit functions of $\boldsymbol{x}(\boldsymbol{a}, t)$, given by (9) and (10) respectively. We shall presently justify this Lagrange function by verifying that it leads to the correct MHD equations. For this purpose it is necessary to calculate the first order variation of L when $\boldsymbol{x}(\boldsymbol{a}, t)$ is varied, but we shall calculate simultaneously the second order variation, because that will be required for stability considerations.

Let $\boldsymbol{x}(\boldsymbol{a}, t)$ be a given fluid flow, and $\boldsymbol{x}(\boldsymbol{a}, t)+\xi(\boldsymbol{a}, t)$ a neighboring one.* We again expand to second order in ξ,

$$\begin{aligned} L(\dot{x}+\dot{\xi}; x+\xi) = L(\dot{x}; x) &+ \int \rho_a(\dot{x} \cdot \dot{\xi} + \tfrac{1}{2}\dot{\xi}^2) d^3\boldsymbol{a} \\ &- \int \rho_a\{(\xi \cdot \nabla)\Phi(x) + \tfrac{1}{2}(\xi \cdot \nabla)^2 \, \Phi(x)\} \, d^3\boldsymbol{a} \\ &- \int \rho_a\{\delta\rho \Psi'(\rho) + \tfrac{1}{2}(\delta\rho)^2 \Psi''(\rho)\} \, d^3\boldsymbol{a} \\ &- \frac{1}{8\pi} \int \rho_a \left\{\delta\rho \left(\frac{H}{\rho}\right)^2 + \rho\delta\left(\frac{H}{\rho}\right)^2 + \delta\rho\delta\left(\frac{H}{\rho}\right)^2\right\} d^3\boldsymbol{a}. \end{aligned} \tag{13}$$

* It would be more consistent with the notation in sec. 1 to write x^0 rather than x, but that would make the formulas even more cumbersome.

Here $\delta\rho$ and $\delta(\boldsymbol{H}/\rho)^2$ are the variations in ρ and $(\boldsymbol{H}/\rho)^2$ induced by the variation $\boldsymbol{\xi}$ in $\boldsymbol{x}$, not yet restricted to lowest orders.

The first and second line of (13) are ready for use, but in the third line $\delta\rho$ must still be calculated to second order in ξ. One finds from (9)

$$\frac{\rho+\delta\rho}{\rho} = \left[\frac{d(x+\xi)}{d(x)}\right]^{-1} = \left[1+\nabla\cdot\xi + \sum_{\text{cycl}}\left\{(\partial_x\xi_x)(\partial_y\xi_y)-(\partial_x\xi_y)(\partial_y\xi_x)\right\}\right]^{-1}$$

$$= 1-\nabla\cdot\xi+\tfrac{1}{2}(\nabla\cdot\xi)^2+\tfrac{1}{2}\sum_{i,j}(\partial_i\xi_j)(\partial_j\xi_i). \tag{14}$$

This yields for the third line of (13)

$$\int \rho_a\left[\rho\left\{\nabla\cdot\xi-\tfrac{1}{2}(\nabla\cdot\xi)^2-\tfrac{1}{2}\sum_{i,j}(\partial_i\xi_j)(\partial_j\xi_i)\right\}\Psi'(\rho)-\tfrac{1}{2}\rho^2(\nabla\cdot\xi)^2\Psi''(\rho)\right]d^3\boldsymbol{a}.$$

Insert $\rho^2\Psi'(\rho) = p, \quad \rho^3\Psi''(\rho) = \rho(dp/d\rho)-2p,$

$$\int\frac{\rho_a}{\rho}p\left[\nabla\cdot\xi+\tfrac{1}{2}\sum_{i,j}\left\{(\partial_i\xi_i)(\partial_j\xi_j)-(\partial_i\xi_j)(\partial_j\xi_i)\right\}\right]d^3\boldsymbol{a}-\tfrac{1}{2}\int\rho_a\frac{dp}{d\rho}(\nabla\cdot\xi)^2\,d^3\boldsymbol{a}.$$

Finally, in order to calculate the fourth line of (13) we note that according to (10) and (11)

$$\delta\frac{\boldsymbol{H}}{\rho} = \left(\frac{\boldsymbol{H}_a}{\rho_a}\cdot\nabla_a\right)\xi = \left(\frac{\boldsymbol{H}}{\rho}\cdot\nabla\right)\xi. \tag{15}$$

(Note the similarity with (III, 11)!) On substituting this, one finds for the fourth line of (13)

$$\frac{1}{8\pi}\int\frac{\rho_a}{\rho}H^2(\nabla\cdot\xi)\,d^3\boldsymbol{a}-\frac{1}{4\pi}\int\frac{\rho_a}{\rho}\boldsymbol{H}\cdot(\boldsymbol{H}\cdot\nabla)\xi\,d^3\boldsymbol{a}$$

$$-\frac{1}{16\pi}\int\frac{\rho_a}{\rho}\left\{(\nabla\cdot\xi)^2+\sum_{i,j}(\partial_i\xi_j)(\partial_j\xi_i)\right\}H^2\,d^3\boldsymbol{a}$$

$$-\frac{1}{8\pi}\int\frac{\rho_a}{\rho}\{(\boldsymbol{H}\cdot\nabla)\xi\}^2\,d^3\boldsymbol{a}+\frac{1}{4\pi}\int\frac{\rho_a}{\rho}(\nabla\cdot\xi)\boldsymbol{H}\cdot(\boldsymbol{H}\cdot\nabla)\xi\,d^3\boldsymbol{a}.$$

These integrals can be simplified by transforming from the integration variable $\boldsymbol{a}$ to $\boldsymbol{x}$. The connection between both variables is given by $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a}, t)$, and according to (9) one has $\rho_a\,d^3\boldsymbol{a} = \rho\,d^3\boldsymbol{x}$. The quantities ρ, $\dot{\boldsymbol{x}}$, ξ, $\dot{\xi}$, $\boldsymbol{H}$ are to be regarded as functions of $\boldsymbol{x}$. It is then possible to perform partial integrations since ∇ and ∂_i denote differentiations with respect to $\boldsymbol{x}$. (We shall presently specialize to the case that $\boldsymbol{x}(\boldsymbol{a}, t)$ is a static solution, so that $\boldsymbol{x}(\boldsymbol{a}, t) \equiv \boldsymbol{a}$ and the transformation reduces to the identity.)

The terms of first order in ξ and $\dot{\xi}$ are

$$\int\rho\dot{\boldsymbol{x}}\cdot\dot{\xi}\,d^3\boldsymbol{x}-\int\rho(\xi\cdot\nabla)\Phi(\boldsymbol{x})\,d^3\boldsymbol{x}+\int p\nabla\cdot\xi\,d^3\boldsymbol{x}$$

$$+\frac{1}{8\pi}\int H^2\nabla\cdot\xi\,d^3\boldsymbol{x}-\frac{1}{4\pi}\int\boldsymbol{H}\cdot(\boldsymbol{H}\cdot\nabla)\xi\,d^3\boldsymbol{x}.$$

Apart from a total time derivative (which may be omitted in a Lagrange function) this is equal to

$$\int\xi\cdot\left[-\rho\ddot{\boldsymbol{x}}-\rho\nabla\Phi-\nabla p-\frac{1}{8\pi}\nabla(H^2)+\frac{1}{4\pi}(\boldsymbol{H}\cdot\nabla)\boldsymbol{H}\right]d^3\boldsymbol{x}$$

$$+\oint\left(p+\frac{1}{8\pi}H^2\right)\xi\cdot d\boldsymbol{S}-\frac{1}{4\pi}\oint(\boldsymbol{H}\cdot\xi)\boldsymbol{H}\cdot d\boldsymbol{S}.$$

In order that this first order variation vanishes for all ξ the expression [ ] in the volume integral must be zero. The fact that this condition is identical with the equation of motion (III, 1) proves that (12) is indeed the correct Lagrange function. The surface integrals account for the force exerted by the surroundings on the surface of the plasma. For reasons of simplicity we shall however assume that the plasma is of infinite extent and that the variation ξ vanishes at infinity so that all surface integrals may be omitted. In many applications the behavior of the surface must however also be considered.

3. Equations of Motion for Neighboring Solutions

The terms of second order in ξ in (13) constitute the Lagrange function for neighboring motions:

$$\Lambda(\dot{\xi};\xi) = \frac{1}{2}\int\rho\dot{\xi}^2\,d^3\boldsymbol{x}-\frac{1}{2}\int\rho\sum_{i,j}\xi_i\xi_j\partial_i\partial_j\Phi\,d^3\boldsymbol{x}$$

$$-\frac{1}{2}\int\left(\rho\frac{dp}{d\rho}-p+\frac{H^2}{8\pi}\right)(\nabla\cdot\xi)^2\,d^3\boldsymbol{x}-\frac{1}{2}\int\left(p+\frac{H^2}{8\pi}\right)\sum_{i,j}(\partial_i\xi_j)(\partial_j\xi_i)\,d^3\boldsymbol{x}$$

$$-\frac{1}{8\pi}\int\{(\boldsymbol{H}\cdot\nabla)\xi\}^2\,d^3\boldsymbol{x}+\frac{1}{4\pi}\int(\nabla\cdot\xi)\boldsymbol{H}\cdot(\boldsymbol{H}\cdot\nabla)\xi\,d^3\boldsymbol{x}. \tag{16}$$

In order to find the equations of motion for ξ the variation must be formed,

$$\Lambda(\dot{\xi}+\delta\dot{\xi};\, \xi+\delta\xi)-\Lambda(\dot{\xi};\, \xi) =$$

$$\int \rho\dot{\xi}\cdot\delta\dot{\xi}\, d^3x - \int \rho\delta\xi\cdot(\xi\cdot\nabla)\nabla\Phi\, d^3x$$

$$+ \int \delta\xi\cdot\nabla\left\{\left(\rho\frac{dp}{d\rho} - p + \frac{H^2}{8\pi}\right)\nabla \ \xi\right\} d^3x$$

$$+ \int \sum_{i,j}\delta\xi_i\partial_j\left\{\left(p+\frac{H^2}{8\pi}\right)\partial_i\xi_j\right\} d^3x$$

$$+ \frac{1}{4\pi}\int \delta\xi\cdot(H\cdot\nabla)(H\cdot\nabla)\xi\, d^3x$$

$$- \frac{1}{4\pi}\int \delta\xi\cdot\nabla\{H\cdot(H\cdot\nabla)\xi\}\, d^3x - \frac{1}{4\pi}\int \delta\xi\cdot\{(H\cdot\nabla)H(\nabla\cdot\xi)\}\, d^3x.$$

This yields the equations of motion

$$\rho\ddot{\xi}_i = -\rho(\xi\cdot\nabla)\partial_i\Phi + \left(\rho\frac{dp}{d\rho}+\frac{H^2}{4\pi}\right)\partial_i(\nabla\cdot\xi)$$

$$-(\nabla\cdot\xi)\partial_i\left(\rho\frac{dp}{d\rho} - p + \frac{H^2}{8\pi}\right) + \{(\partial_i\xi)\cdot\nabla\}\left(p+\frac{H^2}{8\pi}\right)$$

$$+ \frac{1}{4\pi}\Big[(H\cdot\nabla)(H\cdot\nabla)\xi_i - \partial_i\{H\cdot(H\cdot\nabla)\xi\} - (H\cdot\nabla)H_i(\nabla\cdot\xi)\Big]. \quad (17)$$

In this equation ρ, p, H refer to the unperturbed state. For simplicity we assume this to be a static state, so that ρ, p, H are independent of time. The derivative $dp/d\rho$ is determined by the equation of state. Since the deviations from the static state are likely to vary rapidly, it is reasonable to take this derivative along the adiabatic, so that one has

$$\rho(dp/d\rho) = \gamma p.$$

The equation of motion (17) has the form

$$\rho\ddot{\xi} = F\xi, \quad (18)$$

where F is a linear operator. F differentiates, combines and mixes the three components of ξ, but the equation is of the general form (5). The role of the subscript l is now taken by the label that distinguishes the components of ξ, together with the variables x, y, z of which ξ is a function. By means of rather lengthy calculations F may be transformed to the more commonly used form

$$F\xi = (\nabla\cdot\rho\xi)\nabla\Phi + \nabla\{\gamma p(\nabla\cdot\xi)+(\xi\cdot\nabla)p\}$$

$$-\frac{1}{4\pi}[H\wedge \text{curl curl}(\xi\wedge H)+\{\text{curl}(\xi\wedge H)\}\wedge \text{curl}\, H]. \quad (19)$$

ρ, p, H still refer to the static solution whose stability is under investigation.

The potential energy $W(\xi)$ associated with the deviation ξ follows directly from the Lagrange function (16); after rearranging some terms it is

$$W(\xi) = \frac{1}{2}\int \rho\sum_{i,j}\xi_i\xi_j\partial_i\partial_j\Phi\, d^3x + \frac{1}{2}\int\rho\frac{dp}{d\rho}(\nabla\cdot\xi)^2\, d^3x$$

$$+\frac{1}{8\pi}\int\{H(\nabla\cdot\xi)-(H\cdot\nabla)\xi\}^2\, d^3x$$

$$+\frac{1}{2}\int\left(p+\frac{H^2}{8\pi}\right)\sum_{i,j}\{(\partial_i\xi_j)(\partial_j\xi_i)-(\partial_i\xi_i)(\partial_j\xi_j)\}\, d^3x. \quad (20)$$

By lengthy calculations this may again be transformed to the more commonly used form

$$W(\xi) = \frac{1}{2}\int\Big[-(\nabla\cdot\rho\xi)(\xi\cdot\nabla)\Phi+\gamma p(\nabla\cdot\xi)^2+(\nabla\cdot\xi)(\xi\cdot\nabla)p$$

$$+\frac{1}{4\pi}\{\text{curl}(\xi\wedge H)\}^2 - \frac{1}{4\pi}(\xi\wedge\text{curl}\, H)\cdot\text{curl}(\xi\wedge H)\Big]\, d^3x. \quad (21)$$

Problem. Show that the last line of (20) is also equal to

$$\frac{1}{2}\int\left(p+\frac{H^2}{8\pi}\right)\left\{\sum_{i,j}\alpha_{ij}^2 - \tfrac{2}{3}(\text{div}\,\xi)^2 - \tfrac{1}{2}(\text{curl}\,\xi)^2\right\} d^3x,$$

where α_{ij} stands for

$$\alpha_{ij} = \tfrac{1}{2}(\partial_i\xi_j+\partial_j\xi_i) - \tfrac{1}{3}\delta_{ij}\,\text{div}\,\xi.$$

Problem. Verify that (20) is indeed equal to (21) apart from boundary terms. Show that the boundary terms that have to be added to (21) are

$$\frac{1}{2}\oint\rho\{(\xi\cdot\nabla)\Phi\}\xi\cdot dS + \oint\left(p+\frac{H^2}{8\pi}\right)\sum_{i,j}\{(\partial_i\xi_j)(\xi_i dS_j)-(\partial_i\xi_i)(\xi_j dS_j)\}$$

$$+\frac{1}{8\pi}\oint\{\xi\cdot(\xi\cdot\nabla)H\}(H\cdot dS) - \frac{1}{8\pi}\oint\{\xi\cdot(H\cdot\nabla)H\}(\xi\cdot dS).$$

(*Hint*. Partial integration of the first term of (20) gives rise to the first term of (21) plus a remaining term. The latter can be combined by means of (IV, 7b) with one of the terms obtained by partially integrating the term with $(\partial_i \xi_j)(\partial_j \xi_i)$ once. In this way all terms involving Φ are seen to be equal, and the same can be shown for the terms involving p. Do not use (IV, 7b) again! Finally the terms with ξ and $\boldsymbol{H}$ can be shown to be identical, for instance by writing each of them in such a way that the differentiations act on $\boldsymbol{H}$ alone.)

Problem. Derive (19) by differentiating (21) with respect to ξ; in other words prove that (omitting boundary terms)

$$W(\xi+\delta\xi)-W(\xi) = -\int \delta\xi \cdot F\xi \, d^3x + O(\delta\xi^2).$$

Conclude from this that F is a self-adjoint operator, in the sense that for any two vector fields ξ, η

$$\int \eta \cdot F\xi \, d^3x - \int \xi \cdot F\eta \, d^3x = \text{surface integral}.$$

Remark. The usual derivation of (19) starts from the equations of motion themselves.* One expresses the deviations of $\rho, p, \boldsymbol{H}$ from the static solution in terms of ξ, neglecting all second order terms. This derivation of (19) is more direct and less cumbersome, since only first order terms need be retained. However, in order to apply the energy principle for stability one now has to construct (21) from (19), which involves the same kind of calculations that we had to use.

4. Applications

As an example consider the static solution of the problem on page 38,

$$\Phi = gz, \quad \boldsymbol{H} = (\mu(z), 0, 0), \quad p = (kT/m)\rho.$$

Take a special variation ξ that does not depend on x and whose divergence vanishes; this can be expressed by a single scalar function χ:

$$\xi_x = 0, \quad \xi_y = \frac{\partial\chi}{\partial z}, \quad \xi_z = -\frac{\partial\chi}{\partial y}.$$

* I. B. Bernstein, E. A. Frieman, M. D. Kruskal and R. M. Kulsrud, Proc. Roy. Soc. (London) A **244**, 17 (1958); also in: Jeffrey and Taniuti, p. 39. See also: K. Hain, R. Lüst and A. Schlüter, Z. Naturf. **12a**, 833 (1957).

Substitution in (21) yields

$$W(\xi) = -\tfrac{1}{2}g \int \left(\frac{\partial\chi}{\partial y}\right)^2 \frac{d\rho}{dz} \, d^3x.$$

Hence the solution is certainly unstable if ρ increases with height. It is therefore not possible for a plasma to float in the gravitational field with the aid of a magnetic field of this form. The instability described by this ξ consists simply of a rotation of a fluid element around a field line, which amounts to a gain in gravitational energy (as denser material descends while less dense material rises), without change in magnetic energy. Such instabilities, based on the energy gained when two layers of the plasma with their field lines interchange position, are called "*interchange-instabilities*".

Remark. Actually ξ cannot be entirely independent of x, since it should vanish for $x \to \pm\infty$ in order that the integral for $W(\xi)$ converges. That means that the magnetic field lines are slightly stretched when two plasma layers are interchanged, but it is easy to see that this involves only a magnetic energy of arbitrarily small amount when the plasma is very large.

If the plasma is bounded by conducting surfaces, however, say at $x = x_0$ and $x = -x_0$, the end points of the magnetic field lines are fixed on the surface (owing to the general rule that magnetic field lines in a conductor are fixed to the material). In that case the interchange of plasma layers does increase the magnetic energy by a finite amount, which may be sufficient to suppress the interchange instability. Another way of suppressing the interchange instability is by replacing the parallel field lines with a sheared field as given by (III, 7).

Problem. Estimate the value of x_0 needed to make a plasma with given density and given field strength stable.

In many cases of practical interest instabilities occur for which also the plasma surface is displaced. In order to treat these one must take into consideration the surface terms in the equations for $W(\xi)$ and $F\xi$. Moreover the field must fit the perturbed magnetic field in the vacuum outside the plasma. This calculation would lead us too far, but we shall discuss an important application.

For the *static longitudinal pinch* (p. 33) with surface current (Fig. 6) one considers small displacements ξ of the interior and of the surface of the plasma column. This leads again to an equation of the form (18). On account of the cylindrical symmetry of the static solution it is appropriate to introduce cylindrical coordinates and to assume a periodic dependence on φ and z:

$$\xi = (\xi_s(s), \xi_\varphi(s), \xi_z(s)) \, e^{im\varphi + ikz - i\omega t} .$$

k is an arbitrary real number, m an arbitrary integer. The displacement of the surface is $\xi_s(R)$. Subsequently one also writes the equation for the magnetic field in the surrounding vacuum in cylindrical coordinates. At the surface the normal components of the magnetic field within and without the plasma must fit. If it is assumed that everything is surrounded by a conducting cylinder of radius $R_0 > R$, one has the further requirement $H_s(R_0) = 0$.

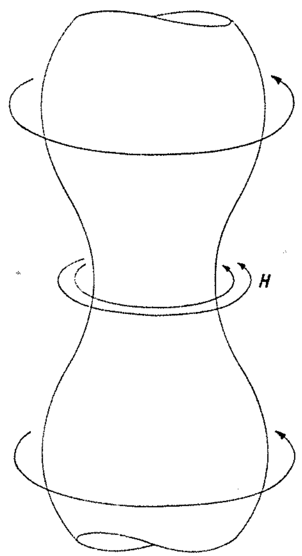

Fig. 17. Sausage instability of a plasma column.

All these equations together lead to an eigenvalue problem, from which ω can be solved numerically.* For each m and k one obtains a series of eigenvalues for ω. As it turns out that some of the eigenvalues are imaginary, the longitudinal pinch is unstable. The most important instabilities are the following.

$m = 0$: *Sausage instability*. Independent of φ, hence axially symmetric; see Fig. 17. If the discharge happens to be slightly constricted at one place,

* The explicit calculation is given in S. Glasstone and R. H. Lovberg, *Controlled Thermonuclear Reactions* (Van Nostrand, Princeton 1960) p. 263.

the external magnetic field increases at that place and hence also the magnetic pressure. Therefore the constriction continues to increase. This instability can be reduced by applying an additional axial magnetic field within the plasma. This causes a magnetic counter pressure, which is increased by an accidental constriction and therefore counteracts the increased pressure of the external azimuthal magnetic field. An applied magnetic field in the z-direction outside the plasma also helps to prevent the sausage instability, because its field lines are stretched by a constriction of the discharge, which amounts to an increased magnetic energy.

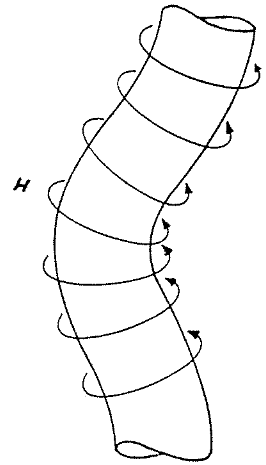

Fig. 18. Kink instability of a plasma column.

$m = 1$: *Kink instability* (more properly: screw-shaped instability); see Fig. 18. The magnetic field is clearly stronger on the inside than on the outside, so that the kink continues to grow. Again an axial magnetic field improves the stability, because the tension along the field lines tries to straighten the kink. But for small k, long drawn-out kink, this is clearly not very effective. To prevent kink instability with small k one surrounds the discharge with a conducting cylinder, so that the magnetic field is confined and is compressed at the convex side.

Photographs of these instabilities can be found in Fünfer and Lehner.* For higher m one also finds unstable eigenvalues but these are even more easily made stable by the applied magnetic field in the z-direction.

* E. Fünfer and G. Lehner in: *Ergebnisse der exakten Naturwissenschaften* **34** (Springer, Berlin 1962) pp. 108, 109.

All this concerns the static pinch with the current concentrated at the surface. One can also consider a static pinch where the current is distributed over a layer of finite thickness, or even over the entire plasma (Fig. 5). For that case it is possible to derive from the energy principle a necessary condition for stability, which is valid regardless of the special current distribution:

$$\frac{s}{4}\left(\frac{d}{ds}\log\frac{H_\varphi}{sH_z}\right)^2+\frac{8\pi}{H_z^2}\frac{dp}{ds}>0 \quad \text{(Suydam's criterion)}.$$

Here $H_\varphi(s)/sH_z(s)$ is a measure for the pitch of the screw-shaped field lines. As dp/ds is usually negative the condition states that the pitch must vary sufficiently strongly with the distance to the axis. It is clear that the discharge in Fig. 5 does not satisfy this criterion. Newcomb* has derived a necessary and sufficient condition for stability of the pinch, which is however more difficult to apply.

In another practical device the linear pinch contains a solid conductor at the center of the column which carries a current in opposite direction: "*hard core pinch*". This configuration has been shown to be theoretically stable. In practice the pinch discharge is still beset by other instabilities, which are ascribed to the fact that the conductivity is finite.

Problem. Consider a plasma in a force-free field with $\Phi = 0$ and $p =$ = constant. Verify that (21) reduces to

$$W(\xi) = \tfrac{1}{2}\gamma p \int (\operatorname{div}\xi)^2\, d^3\boldsymbol{x}$$
$$+\frac{1}{4\pi}\int \left[\{\operatorname{curl}(\xi\wedge \boldsymbol{H})\}^2-\alpha(\xi\wedge\boldsymbol{H})\cdot\operatorname{curl}(\xi\wedge\boldsymbol{H})\right]d^3\boldsymbol{x}.$$

Hence show that the minimum pressure for which the plasma is stable is found by minimizing the second line with the constraint

$$\int (\operatorname{div}\xi)^2\, d^3\boldsymbol{x} = 1\,.$$

Write the corresponding variational equation for ξ.**

* W. A. Newcomb, Ann. Phys. (N.Y.) **10**, 232 (1960), reprinted in: Jeffrey and Taniuti, p. 169.

** D. Voslamber and D. K. Callebaut, Phys. Rev. **128**, 2016 (1962).

General references

L. Spitzer, *Physics of Fully Ionized Gases*, 2nd ed. (Interscience, New York 1962) ch. IV.
D. J. Rose and M. Clark, *Plasmas and Controlled Fusion* (Wiley, New York 1961) ch. XII.
S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, Oxford 1961).
W. B. Thompson, *An Introduction to Plasma Physics* (Pergamon Oxford 1962) ch. VI.
R. Kulsrud in: *Advanced Plasma Theory*, M. N. Rosenbluth ed. (Academic Press, New York 1964).
J. M. Greene and J. L. Johnson in: *Advances in Theoretical Physics*, Vol. 1, K. A. Brueckner ed. (Academic Press, New York 1965).
A. A. Vedenov, E. P. Velikhov and R. Z. Sagdeev, Soviet Phys.-Usp. **4**, 332 (1961).
Magnetohydrodynamic Stability and Thermonuclear Containment, a collection of reprints with an introductory review by A. Jeffrey and T. Taniuti (Academic Press, New York 1966).

For the theory of instability in the presence of dissipative effects see:
H. P. Furth in: *Advanced Plasma Theory*, M. N. Rosenbluth ed. (Academic Press, New York 1964), and in: *Propagation and Instabilities in Plasmas*, W. I. Futterman ed. (Stanford University Press, Stanford 1963).
B. Coppi in: *Propagation and Instabilities in Plasmas*, W. I. Futterman ed. (Stanford University Press, Stanford 1963).

CHAPTER VIII

STATISTICAL MECHANICS OF AN IONIZED GAS IN EQUILIBRIUM

We now abandon the magnetohydrodynamic point of view and regard the plasma as a collection of charged particles rather than as a fluid. In this chapter only the equilibrium state of such a collection of particles is studied, so that the familiar methods of statistical mechanics apply. First we give the elementary approach due to Debye, and subsequently a more systematic treatment.

In general a plasma consists of several components, i.e., several species of particles with different masses and charges. It is often convenient, however, to study the simple case of one component first. It is then necessary to add a constant charge density of opposite sign, the so-called *neutralizing background*. This is not an unrealistic description of a plasma of electrons and ions, because the heavy ions are not much affected by the motion of the electrons, and the electrons move so fast that they only feel the averaged or smeared out field of the ions. This simplified picture of a plasma is often called the *electron gas*.

1. Debye shielding

For convenience we take the simple case of a gas of electrons, charge $-e$, and a neutralizing positive background with charge density $n_0 e$. In equilibrium there are n_0 electrons per unit volume. Now suppose there is an additional point charge e_0 at the origin. This changes the electron density in the vicinity into $n(\boldsymbol{r})$ say. The electrostatic potential $\varphi(\boldsymbol{r})$ will obey Poisson's equation

$$\nabla^2\varphi(\boldsymbol{r}) = 4\pi e n(\boldsymbol{r}) - 4\pi e n_0 - 4\pi e_0 \delta(\boldsymbol{r}). \tag{1}$$

On the other hand $n(\boldsymbol{r})$ is given by Boltzmann's formula

$$n(\boldsymbol{r}) = A \exp \frac{e\varphi(\boldsymbol{r})}{kT} \tag{2}$$

in which the constant A must be adjusted so as to satisfy the condition

$$\int_V n(\boldsymbol{r})\, d^3\boldsymbol{r} = N.$$

N is the total number of electrons in the volume V, and $N = n_0 V$ since the plasma is neutral. It is clear that if V is large*, (2) can only satisfy this condition when $\varphi(\boldsymbol{r})$ tends to a constant value $\varphi(\infty)$, which may be taken zero; it then follows that $A = n_0$.

Substitution of (2) into (1) yields a nonlinear differential equation for φ,

$$\nabla^2\varphi(\boldsymbol{r}) = 4\pi e n_0 \left\{\exp \frac{e\varphi(\boldsymbol{r})}{kT} - 1\right\} - 4\pi e_0 \delta(\boldsymbol{r}). \tag{3}$$

This equation is linearized by expanding the exponent to first order

$$\nabla^2\varphi(\boldsymbol{r}) - \kappa^2\varphi(\boldsymbol{r}) = -4\pi e_0\delta(\boldsymbol{r}), \qquad \kappa^2 = \frac{4\pi n_0 e^2}{kT}. \tag{4}$$

The solution we are interested in depends on $|\boldsymbol{r}|$ alone and is easily seen to be

$$\varphi(\boldsymbol{r}) = e_0 \frac{e^{-\kappa r}}{r}, \qquad n(\boldsymbol{r}) - n_0 = \frac{e_0}{e}\frac{\kappa^2}{4\pi}\frac{e^{-\kappa r}}{r}. \tag{5}$$

This shows that around the point charge e_0 a charge cloud is formed, which decreases exponentially with the distance. The result is a shielding of the electrostatic potential, so that the interaction is no longer a Coulomb potential, but has the form of a Yukawa potential. The range is finite, viz.,

$$\frac{1}{\kappa} = \sqrt{\frac{kT}{4\pi n_0 e^2}} = \text{"Debye length"}. \tag{6}$$

The spherical region in which the density is increased (or decreased, depending on the sign of e_0) and in which the potential differs appreciably from zero is called the *Debye sphere*.** Rigorously speaking it extends to infinity, but it may be visualized as a sphere whose radius is of the order of the Debye length $1/\kappa$.

* More precisely, we take the "thermodynamic limit": $V \to \infty$, $N \to \infty$, $N/V = n_0 =$ = constant.

** P. Debye and E. Hückel, Physik. Z. **24**, 185 (1923); reprinted in: *Collected Papers of P. Debye* (Interscience, New York 1954) p. 217.

Remark. This shielding effect is responsible for the fact that no large scale field can be maintained in a plasma. Static fields of external sources cannot penetrate deeper than roughly a Debye length. Rapidly oscillating fields, however, can exist; they are called plasma waves and are studied in chapters IX and XII.

Remark. The density $n(r)$ is defined as the average number of electrons in a volume element, divided by the volume of that element. This is only physically meaningful if the element is large enough to contain many electrons: otherwise the fluctuations in the number are of the same magnitude as the average itself. This restriction implies that $n(r)$ must be practically constant over ranges large compared to the interparticle distance ($d = n^{-\frac{1}{3}}$), or

$$n^{-\frac{1}{3}} \,|\, \mathrm{grad}\, n \,| \ll n .$$

Obviously, it makes no sense to talk about variations of n over distances in which only a few or no particles are contained. Thus $n(r)$ is a "coarse-grained" quantity.

Similarly the smeared out or average potential $\varphi(r)$ is also coarse-grained: it is only defined as long as it varies slowly over an interparticle distance. It follows that the result (5) is only meaningful if the Debye length turns out to be large compared to the interparticle distance d, or

$$kT \gg e^2/d , \quad \text{or} \quad kT \gg e^2 n_0^{\frac{1}{3}} . \tag{7}$$

This is the fundamental condition for the validity of the theory in this and following chapters. It states that the average kinetic energy of a particle must be much larger than the average potential energy of neighboring particles. Plasmas obeying this condition will be referred to as "*hot dilute plasmas*".

Problem. Prove the following properties of the Debye sphere (5).
(i) The total number of excess electrons is e_0/e.
(ii) The interaction energy between e_0 and the charge in the Debye sphere is $e_0^2\kappa$.
(iii) The energy of the charge density in the Debye sphere is $\frac{1}{4}e_0^2\kappa$.

Problem. By changing units, (3) can be reduced to

$$\frac{1}{r}\frac{d^2}{dr^2}\, r\varphi = e^{\varphi} - 1 , \qquad (r > 0)$$

with the condition at the origin

$$\varphi(r) = \frac{a}{r} + \text{finite function} , \qquad a = \frac{\kappa e_0 e}{kT} .$$

Prove that this equation has no solution for $a > 0$, that is, when e_0 is positive. (This is the familiar collapse of positive and negative charges, which is remedied by quantum mechanics.*)

Let us select one particular electron of the plasma. By a slight generalization of the result (5) one may expect that this electron is also surrounded by a Debye sphere, in which the average potential and the average density of electrons are given by (5), or rather in this case

$$\varphi(r) = -e\frac{e^{-\kappa r}}{r} , \qquad n(r) = n_0 - \frac{\kappa^2}{4\pi}\frac{e^{-\kappa r}}{r} . \tag{8}$$

Thus the density of the other electrons is slightly less than n_0, but if one takes the selected electron at the center into account, the total density is again n_0. Hence the overall density of the electron gas remains n_0, but the electrons have a tendency to stay farther apart from each other than if they would have no interaction.

Since the density of the other electrons is less than average, there is a net positive charge density due to the positive background; hence our selected electron is surrounded by a positive charge cloud, which is responsible for reducing its Coulomb potential to the "Debye potential" given by (8). The charge cloud is appreciable within a radius of order $1/\kappa$, its total charge is e. Hence it just compensates the $-e$ of the selected electron, so that the electrostatic force does not extend beyond the Debye sphere. For this reason one sometimes treats a plasma as a gas of particles whose interaction is a two-body "Debye force" instead of a Coulomb force. The range $1/\kappa$ of this Debye force is still much larger than the distance between particles, but it is no longer infinite.

Yet it should be borne in mind that *the Debye potential is a statistical concept* and is the result of the average behavior of the surrounding particles. This becomes particularly clear if one estimates the fluctuations about this average behavior. The number of electrons in the Debye sphere is large, because of the fundamental condition (7), in actual plasmas of the order 1000. Hence the fluctuations are of the order of 30 electrons. However, the decrease of the number of electrons in this neighborhood is 1, which is much less than the fluctuations of the number of electrons in the range $1/\kappa$. Thus the Debye sphere has a somewhat ghost-like existence.

Problem. Show that the average electrostatic energy of the Debye sphere is much less than the energy fluctuations in it.

* R. Dekeyser, Physica **31**, 1405 (1965).

Problem. If one knows that electron 1 is at $\boldsymbol{r}_1$, the probability of finding electron 2 in a volume element $d^3\boldsymbol{r}_2$ at $\boldsymbol{r}_2$ is denoted by $g(\boldsymbol{r}_1, \boldsymbol{r}_2)d^3\boldsymbol{r}_2/V$. Show that this "pair distribution function" $g(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is

$$g(\boldsymbol{r}_1, \boldsymbol{r}_2) = 1 - \frac{\kappa^2}{4\pi n_0} \frac{e^{-\kappa|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}. \tag{9}$$

Problem. Prove that

$$\int \{1-g(\boldsymbol{r}_1, \boldsymbol{r}_2)\}d^3\boldsymbol{r}_2 = \frac{1}{n_0},$$

and conclude from this again that the average number of electrons in the Debye sphere is 1 less than normal.

Problem. Generalize these considerations for a mixture of ionized gases, and show in particular that the Debye length is given by

$$\kappa^2 = \frac{4\pi}{kT} \sum_\nu e_\nu^2 n_\nu, \tag{10}$$

where ν labels the various kinds of ions, and e_ν and n_ν are their respective charges and densities.

2. Statistical Mechanics of a Mixture of Ionized Gases

The different species of particles (electrons and various ions) will be labeled by the index ν, with mass m_ν and charge e_ν. There are N_ν particles of each species, they are identified by a subscript s, their positions and momenta are $\boldsymbol{r}_{\nu s}$ and $\boldsymbol{p}_{\nu s}$. The total kinetic energy is

$$W_{\text{kin}} = \sum_\nu \frac{1}{2m_\nu} \sum_s \boldsymbol{p}_{\nu s}^2;$$

and the total potential energy is

$$W_{\text{pot}} = \tfrac{1}{2} \sum_{\nu,\nu'} e_\nu e_{\nu'} \sum_{\substack{s,s' \\ s\neq s'}} \frac{1}{|\boldsymbol{r}_{\nu s}-\boldsymbol{r}_{\nu' s'}|} + \sum_{\nu,s} \Phi_\nu(\boldsymbol{r}_{\nu s}),$$

where $\Phi_\nu(\boldsymbol{r})$ is the potential of the external forces acting on particles of the species ν. The total energy $W_{\text{tot}} = W_{\text{kin}} + W_{\text{pot}}$ is a function of all $\boldsymbol{p}_{\nu s}$ and $\boldsymbol{r}_{\nu s}$.

The probability for finding the plasma in a specified microscopic state,

i.e., with all variables $\boldsymbol{p}_{\nu s}$ and $\boldsymbol{r}_{\nu s}$ given within small ranges $d^3\boldsymbol{p}_{\nu s}$ and $d^3\boldsymbol{r}_{\nu s}$ is

$$\frac{1}{\prod_\nu N_\nu!} e^{\beta F - \beta W_{\text{tot}}(p,r)} \prod_{\nu,s} d^3\boldsymbol{p}_{\nu s}\, d^3\boldsymbol{r}_{\nu s}.$$

Here $\beta = 1/kT$, and F is the Helmholtz free energy, given by

$$e^{-\beta F(V,T)} = \frac{1}{\prod_\nu N_\nu!} \int \dots \int e^{-\beta W_{\text{tot}}(p,r)} \prod_{\nu,s} d^3\boldsymbol{p}_{\nu s}\, d^3\boldsymbol{r}_{\nu s}. \tag{11}$$

The particles are supposed to be enclosed in a volume V, which determines the range of the coordinates $\boldsymbol{r}_{\nu s}$. The factorials have been added to ensure that the free energy is additive; for our purpose they are not necessary (although convenient), because we only consider plasmas with fixed N_ν. All thermodynamic properties of the plasma can be derived when F as a function of V and T has been computed from (11).

The integration over the variables $\boldsymbol{p}_{\nu s}$ in (11) is trivial

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-\beta W_{\text{kin}}(p)} \prod_{\nu,s} d^3\boldsymbol{p}_{\nu s} = \prod_\nu \left(\frac{2\pi m_\nu}{\beta}\right)^{\frac{3}{2}N_\nu}.$$

Accordingly one has

$$F = -\tfrac{3}{2}kT \sum_\nu N_\nu \log(2\pi m_\nu kT) + F_{\text{pot}}$$

where F_{pot} is given by

$$e^{-\beta F_{\text{pot}}} = \frac{1}{\prod_\nu N_\nu!} \int_V \dots \int_V e^{-\beta W_{\text{pot}}(r)} \prod_{\nu,s} d^3\boldsymbol{r}_{\nu s}. \tag{12}$$

This integral cannot, however, be evaluated exactly, so that a suitable approximate method must be found. In order to do this we shall again confine ourselves to the simple case of an electron gas with neutralizing background.

3. Statistical Mechanics of the Electron Gas

The problem is to evaluate the $3N$-fold integral

$$e^{-\beta F_{\text{pot}}} = \frac{1}{N!} \int_V \dots \int_V e^{-\beta W_{\text{pot}}} d^3\boldsymbol{r}_1 \dots d^3\boldsymbol{r}_N, \tag{13}$$

$$W_{\text{pot}} = \tfrac{1}{2} \sum_{s,s'} \frac{e^2}{|\boldsymbol{r}_s - \boldsymbol{r}_{s'}|} + \sum_s \Phi(\boldsymbol{r}_s) + W_{\text{bg}}. \tag{14}$$

W_{bg} is the potential energy of the background; it is constant but has been included to cancel a divergence due to the infinite size of the plasma. The energy potential $\Phi(\boldsymbol{r})$ arises from external sources and from the neutralizing background,

$$\Phi = -e\varphi_{ex} - e\varphi_{bg}.$$

Problem. Show that if $\Phi = 0$ the physically significant part of the free energy must have the form

$$F(V, T, N, e) = NTf(VT^3/Ne^6), \tag{15}$$

where f is a function of the single argument VT^3/Ne^6.
(*Hint*: This result follows from three remarks: the partition function (13) contains β and e only in the combination βe^2; the scale transformation $\boldsymbol{r} = \lambda \boldsymbol{r}'$, $V = \lambda^3 V'$, $\beta = \lambda\beta'$ only multiplies the partition function by a constant; the free energy is an extensive quantity.)

We now subdivide the volume V in cells of volume Δ, large enough to contain many electrons, but so small that the density $n(\boldsymbol{r})$ and the electrostatic potential $\varphi(\boldsymbol{r})$ are practically constant in each cell. This is possible under the fundamental condition (7), provided that also the external field varies slowly. Let the position of the center of cell λ be $\boldsymbol{R}_\lambda$ and the number of electrons in it N_λ. Of course one has

$$\sum_\lambda N_\lambda = N. \tag{16}$$

(14) may now be written approximately in the form

$$W_{pot} = \frac{e^2}{2}\sum_{\lambda,\lambda'} \frac{N_\lambda N_{\lambda'}}{|\boldsymbol{R}_\lambda - \boldsymbol{R}_{\lambda'}|} + \sum_\lambda N_\lambda \Phi(\boldsymbol{R}_\lambda) + W_{bg}. \tag{17}$$

For $\lambda = \lambda'$ one should replace $|\boldsymbol{R}_\lambda - \boldsymbol{R}_{\lambda'}|$ by a number roughly equal to the diameter of a cell, but these terms will presently become harmless.

The integration in (13) can now be performed in three steps. First confine each electron to one cell and integrate its variables only over that cell. The result is* Δ^N. Next count the number of ways in which the particles can be assigned to cells when the occupation numbers N_λ are given; the answer is

$$\frac{N!}{N_1!\,N_2!\,N_3!\ldots}.$$

* The particles are considered as points without proper volume, which is quite reasonable for electrons. For ions the volume of each particle ought to be taken into account.

Finally multiply with the energy exponential and sum over all occupation numbers subject to the condition (16). The final expression is

$$e^{-\beta F_{pot}} = \sum_{\{N_\lambda\}}' \frac{\Delta^N}{N_1!\,N_2!\ldots} \exp -\beta\left[\frac{e^2}{2}\sum_{\lambda,\lambda'}\frac{N_\lambda N_{\lambda'}}{|\boldsymbol{R}_\lambda - \boldsymbol{R}_{\lambda'}|} + \sum_\lambda N_\lambda \Phi(\boldsymbol{R}_\lambda) + W_{bg}\right].$$

The dash attached to $\sum$ is a reminder of (16). Using Stirling's formula one may write

$$e^{-\beta F_{pot}} = \sum_{\{N_\lambda\}}' e^{\Psi\{N_\lambda\}}, \tag{18}$$

with

$$\Psi\{N_\lambda\} = -\beta\frac{e^2}{2}\sum_{\lambda,\lambda'}\frac{N_\lambda N_{\lambda'}}{|\boldsymbol{R}_\lambda - \boldsymbol{R}_{\lambda'}|} - \sum_\lambda N_\lambda\left\{\beta\Phi(\boldsymbol{R}_\lambda) + \log\frac{N_\lambda}{\Delta} - 1\right\} - \beta W_{bg}. \tag{19}$$

In order to evaluate (18) we first determine the largest term by maximizing $\Psi\{N_\lambda\}$ under the subsidiary condition (16).

$$\frac{\partial\Psi}{\partial N_\lambda} \equiv -\beta e^2 \sum_{\lambda'}\frac{N_{\lambda'}}{|\boldsymbol{R}_\lambda - \boldsymbol{R}_{\lambda'}|} - \beta\Phi(\boldsymbol{R}_\lambda) - \log\frac{N_\lambda}{\Delta} = \alpha, \tag{20}$$

α being a Lagrange multiplier. This equation is easily interpreted by noting that, if $\varphi_{el}(\boldsymbol{r})$ is the electrostatic potential due to the electrons,

$$\varphi_{el}(\boldsymbol{R}_\lambda) = -e\sum_{\lambda'}\frac{N_{\lambda'}}{|\boldsymbol{R}_\lambda - \boldsymbol{R}_{\lambda'}|}. \tag{21}$$

Hence (20) is equivalent with

$$N_\lambda/\Delta = e^{-\alpha} e^{\beta e\{\varphi_{el}(\boldsymbol{R}_\lambda) + \varphi_{ex}(\boldsymbol{R}_\lambda) + \varphi_{bg}(\boldsymbol{R}_\lambda)\}}. \tag{22}$$

Thus $\Psi\{N_\lambda\}$ is maximized by those N_λ that obey the Boltzmann distribution law.

Each set of occupation numbers N_λ may also be described by a density distribution $n(\boldsymbol{r})$ in space, such that $N_\lambda/\Delta = n(\boldsymbol{R}_\lambda)$. Unless the N_λ vary wildly from one cell to the next, $n(\boldsymbol{r})$ will be a smooth function. Then (22) becomes

$$n(\boldsymbol{r}) = e^{-\alpha} e^{\beta e\varphi(\boldsymbol{r})},$$

which is identical with (2). Also $\varphi = \varphi_{el} + \varphi_{ex} + \varphi_{bg}$ obeys Poisson's equation

$$\nabla^2\varphi = 4\pi e n(\boldsymbol{r}) - 4\pi e n_0 + \nabla^2\varphi_{ex},$$

which is identical with (1). Thus we have derived from the principles of statistical mechanics that the density distribution with maximum probability obeys the two equations on which Debye's theory is based. It follows that for a neutral plasma $e^{-\alpha} = n_0$.

Problem. Prove that for an infinite plasma with $\varphi_{ex} = 0$ these equations have no other solution than $n(r) = n_0$, φ = constant. (*Hint*: show that n cannot have a maximum or a minimum.)

Our next task is the evaluation of the free energy of the electron gas. By substituting (22) in (19) one finds for the maximum of Ψ

$$\Psi^{\max} = \beta\frac{e}{2}\int n(r)\varphi_{el}(r)\, d^3r + \beta e\int n(r)\,\{\varphi_{ex}(r)+\varphi_{bg}(r)\}\, d^3r$$

$$-\int n(r)\,\{\log n(r)-1\}\, d^3r - \beta W_{bg}.$$

We take $\varphi_{ex}(r) = 0$, so that $n(r) = n_0$. Then it is clear that the first and the last term together are exactly canceled by the interaction term

$$\beta e\int n\varphi_{bg}\, d^3r.$$

Hence

$$\Psi^{\max} = -\int n(r)\,\{\log n(r)-1\}\, d^3r = -\log\prod_\lambda N_\lambda!, \tag{23}$$

which is the same value as for an ideal gas. Thus the familiar procedure of identifying the sum in (18) with its maximum term turns out to be too crude an approximation to include the plasma correction to the free energy.

Problem. Show that (23) leads to

$$F_{pot} = -NkT\log(V/N) - NkT,$$

and that this agrees with the well known thermodynamic free energy for an ideal gas.

We shall now proceed to the next approximation. For this purpose we expand Ψ to second order

$$\Psi = \Psi^{\max} + \frac{1}{2}\sum_{\lambda,\lambda'}\left(\frac{\partial^2\Psi}{\partial N_\lambda \partial N_{\lambda'}}\right)^{\max}\delta N_\lambda \delta N_{\lambda'},$$

where $\delta N_\lambda = N_\lambda - N_\lambda^{\max}$. One readily finds from (20)

$$\left(\frac{\partial^2\Psi}{\partial N_\lambda \partial N_{\lambda'}}\right)^{\max} = -\frac{\beta e^2}{|\boldsymbol{R}_\lambda - \boldsymbol{R}_{\lambda'}|} - \frac{\delta_{\lambda\lambda'}}{N_\lambda^{\max}},$$

so that, omitting higher orders in δN_λ,

$$\Psi - \Psi^{\max} = -\frac{1}{2}\sum_\lambda \frac{(\delta N_\lambda)^2}{N_\lambda^{\max}} - \frac{\beta e^2}{2}\sum_{\lambda,\lambda'}\frac{\delta N_\lambda \delta N_{\lambda'}}{|\boldsymbol{R}_\lambda - \boldsymbol{R}_{\lambda'}|}. \tag{24}$$

Here $N_\lambda^{\max} = n_0\Delta$ is independent of λ since we consider the homogeneous case, $\varphi_{ex} = 0$. This quadratic form can then be transformed to principal axes by the Fourier transformation

$$\delta N_\lambda/\Delta = V^{-\frac{1}{2}}\sum_{k\neq 0} v_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_\lambda}, \qquad v_{-\boldsymbol{k}} = v_{\boldsymbol{k}}^*. \tag{25}$$

This constitutes a transformation from the variables $\{\delta N_\lambda\}$ to the variables $\{v_{\boldsymbol{k}}\}$. $\boldsymbol{k}$ takes all values which correspond to wave lengths that fit in the volume V, and are not shorter than the diameter of a cell Δ. Hence one has

$$\sum_\lambda e^{i\boldsymbol{k}\cdot\boldsymbol{R}_\lambda} e^{-i\boldsymbol{k}'\cdot\boldsymbol{R}_\lambda} = \delta_{\boldsymbol{k}\boldsymbol{k}'}\frac{V}{\Delta}.$$

In the summation (25) the value $\boldsymbol{k} = 0$ is to be omitted because of $\sum\delta N_\lambda = 0$. Substitution of (25) in (24) yields

$$\Psi - \Psi^{\max} = -\frac{1}{2}\sum_{\boldsymbol{k}\neq 0}\left(\frac{1}{n_0} + \frac{4\pi\beta e^2}{k^2}\right)|v_{\boldsymbol{k}}|^2.$$

By substituting this in (18) one finds the next approximation to the free energy. The summation over all δN_λ amounts to an integration over all $v_{\boldsymbol{k}}$. However, the $v_{\boldsymbol{k}}$ are complex, and not independent since $v_{-\boldsymbol{k}} = v_{\boldsymbol{k}}^*$. We therefore write

$$v_{\boldsymbol{k}} = v_{\boldsymbol{k}}' + iv_{\boldsymbol{k}}'', \qquad v_{-\boldsymbol{k}} = v_{\boldsymbol{k}}' - iv_{\boldsymbol{k}}'',$$

and

$$\Psi - \Psi^{\max} = -\sum_{\boldsymbol{k}>0}\left(\frac{1}{n_0} + \frac{4\pi\beta e^2}{k^2}\right)(v_{\boldsymbol{k}}'^2 + v_{\boldsymbol{k}}''^2). \tag{26}$$

The symbol $\boldsymbol{k} > 0$ indicates that $\boldsymbol{k} = 0$ is excluded, and that of each pair $\boldsymbol{k}$, $-\boldsymbol{k}$ only one is included in the sum. Substitution in (18) yields

$$e^{-\beta F_{pot}} = e^{\Psi^{max}} \prod_{k>0} \int\int_{-\infty}^{\infty} \exp\left[-\frac{1}{n_0}\left(1+\frac{\kappa^2}{k^2}\right)(v_k'^2+v_k''^2)\right] dv_k'\, dv_k''$$

$$= e^{\Psi^{max}} \prod_{k>0} \frac{\pi n_0}{1+\kappa^2/k^2}.$$

Taking the logarithm

$$F_{pot} = -kT\Psi^{max} - kT\sum_{k>0}\left[\log \pi n_0 - \log\left(1+\frac{\kappa^2}{k^2}\right)\right].$$

The term with $\log \pi n_0$ is a correction to the free energy F^{id} of the ideal gas, so that we find for the free energy of the electron gas

$$F = F^{id} + \frac{kT}{2}\sum_{k\neq 0} \log\left(1+\frac{\kappa^2}{k^2}\right).$$

Problem. Show that the correction to the ideal gas free energy is roughly equal to

$$-kT\log(\pi n_0)\sum_{k>0} 1 = -\frac{kT}{12\pi^2}\frac{V}{\Delta}\log\left(\pi\frac{N}{V}\right),$$

and that this correction is relatively small.

Problem. What is the divergence that has been cancelled by W_{bg}?

To evaluate the sum we replace it with an integral:

$$\sum_{k\neq 0}\log\left(1+\frac{\kappa^2}{k^2}\right) = \frac{V}{8\pi^3}\int \log\left(1+\frac{\kappa^2}{k^2}\right) d^3k = \frac{V}{2\pi^2}\int_0^K \log\left(1+\frac{\kappa^2}{k^2}\right) k^2\, dk.$$

The upper cut-off has been called K and is of order $\Delta^{-1/3}$. The restriction $k \neq 0$ is automatically satisfied since the integrand vanishes at $k = 0$. Straightforward integration yields

$$\tfrac{1}{3}K^3\log\left(1+\frac{\kappa^2}{K^2}\right) + \tfrac{2}{3}\kappa^2 K - \tfrac{2}{3}\kappa^3 \arctan\frac{K}{\kappa}.$$

As $K \gg \kappa$ this becomes $\kappa^2 K - \frac{1}{3}\pi\kappa^3$, so that finally

$$F = F^{id} + \frac{kTV}{4\pi^2}\kappa^2 K - \frac{kTV}{12\pi}\kappa^3 = F^{id} + Ne^2K/\pi - \tfrac{2}{3}\pi^{\frac{1}{2}}e^3 n_0^{\frac{3}{2}} V(kT)^{-\frac{1}{2}}. \quad (27)$$

The *first term* of the right-hand member of (27) is the ideal gas free energy. The *second term* is independent of V and T and is therefore simply a constant added to the energy. If K^{-1} would be of the order of an electron radius*, it would be possible to interpret this constant as the electrostatic self-energy of the electrons. Although K^{-1} is actually of the order of the diameter of a cell Δ, this interpretation is still valid, because we have cut off the electrostatic field strength at this distance. Finally the *third term* in (27) represents the correction to the free energy due to the electrostatic interaction between electrons. Note that it is of the form (15), as it should.

Problem. Show that the pressure of the electron gas is given by the equation of state

$$p = p^{id} - \tfrac{1}{3}\pi^{\frac{1}{2}}e^3 n_0^{\frac{3}{2}}(kT)^{-\frac{1}{2}}. \quad (28)$$

Problem. Omitting the ideal gas contribution and the constant term we write for the free energy of the electron gas

$$F^{el} = -\tfrac{2}{3}\pi^{\frac{1}{2}}e^3 n_0^{\frac{3}{2}} V(kT)^{-\frac{1}{2}}.$$

Derive from this the thermodynamic relations

$$p^{el}V = \tfrac{1}{3}U^{el} = TS^{el}.$$

Problem. Derive the relation $p^{el}V = \frac{1}{3}U^{el}$ directly from the partition function (13) in the following way.** Take Φ constant, so that

$$e^{-\beta F_{pot}} = C\int_V \ldots \int_V d^3\mathbf{r}_1 \ldots d^3\mathbf{r}_N \exp\left[-\tfrac{1}{2}\beta\sum_{s\neq s'}\frac{e^2}{|\mathbf{r}_s - \mathbf{r}_{s'}|}\right].$$

Let V be a cube of edge L, and introduce new variables by putting $\mathbf{r}_s = L\mathbf{r}_s'$. Then calculate $p = -\partial F/\partial V$ by varying L and using the fact that the energy is homogeneous in L of degree -1.

Problem. Derive the last term of (27) in the following elementary way. Owing to the surrounding electrons, the potential energy of each electron is lowered by

$$\left[e^2\frac{e^{-\kappa r}}{r} - \frac{e^2}{r}\right]_{r=0} = -e^2\kappa.$$

* The classical electron radius r_e is found by putting the electrostatic energy ($\approx e^2/r_e$) equal to the relativistic rest energy mc^2, so that $r_e \approx e^2/mc^2 = 2.8\times 10^{-13}$ cm. It is the only quantity of the dimension of a length that can be made with e, m, c. For example, the classical cross-section for scattering of light by a free electron is $\frac{8}{3}\pi r_e^2$ ("Thomson cross-section").

** D. ter Haar and H. Wergeland, Astrophys. Norvegica 9, 233 (1964).

(More correctly: this is the potential energy minus the smeared out potential energy, the latter being cancelled by the positive background.) Hence

$$U = U^{\text{id}} - \tfrac{1}{2}Ne^2\kappa,$$

from which F can be found thermodynamically.

Problem. Show that the self-energy term in (27) originates from the term with $\lambda = \lambda'$ in (17).

4. Density fluctuations in the electron gas

According to (18) the probability for finding a given set of occupation numbers $\{N_\lambda\}$ is proportional to $\exp \Psi\{N_\lambda\}$. On substituting (24) one finds the probability distribution of the density fluctuations near equilibrium. Of course $\langle \delta N_\lambda \rangle = 0$, but quadratic quantities do not vanish. One has

$$\langle \delta N_\rho \delta N_\sigma \rangle = \frac{\sum' \delta N_\rho \delta N_\sigma \exp\left[\Psi\{N_\lambda^{\max} + \delta N_\lambda\} - \Psi^{\max}\right]}{\sum' \exp\left[\Psi\{N_\lambda^{\max} + \delta N_\lambda\} - \Psi^{\max}\right]}. \tag{29}$$

For $\rho = \sigma$ this represents the mean square fluctuation $\langle \delta N_\rho^2 \rangle$; for $\rho \neq \sigma$ it represents the correlation between the fluctuations in two different cells.

In order to evaluate (29) it is convenient to compute first the correlation between two different ν_k. Their probability distribution is given by (26); this shows that each ν'_k and ν''_k is distributed according to a Gaussian law, with mean square

$$\langle \nu_k'^2 \rangle = \langle \nu_k''^2 \rangle = \tfrac{1}{2} n_0 (1 + \kappa^2/k^2)^{-1}.$$

Moreover, they are statistically independent apart from the relations $\nu'_{-k} = \nu'_k$, $\nu''_{-k} = -\nu''_k$. One easily verifies

$$\langle \nu_k \nu_{k'}^* \rangle = \delta_{kk'} n_0 (1 + \kappa^2/k^2)^{-1}.$$

By means of (25) one finds subsequently

$$\frac{\langle \delta N_\rho \delta N_\sigma \rangle}{\Delta^2} = V^{-1} \sum_{kk'} \langle \nu_k \nu_{k'}^* \rangle e^{i(k \cdot R_\rho - k' \cdot R_\sigma)}$$

$$= \frac{n_0}{8\pi^3} \int \frac{e^{ik \cdot (R_\rho - R_\sigma)}}{1 + \kappa^2/k^2} d^3k$$

$$= n_0 \left\{ \delta(R_\rho - R_\sigma) - \frac{\kappa^2}{4\pi} \frac{e^{-\kappa |R_\rho - R_\sigma|}}{|R_\rho - R_\sigma|} \right\}.$$

This is the formula for the correlation between the density fluctuations at two points R_ρ and R_σ in an electron gas.

Problem. Show that this result is identical with (9). (Note, however, that (9) was derived by applying the result that had been obtained for an external charge, to each electron of the plasma. No such logical jump is used in the present derivation.)

Problem. Consider a particular plane wave mode

$$\delta N_\lambda / \Delta = A \cos k \cdot R_\lambda.$$

Show that its average electrostatic energy is $2\pi n_0 e^2 (k^2 + \kappa^2)^{-1}$, and its average compression energy $\frac{1}{2} kTk^2 (k^2 + \kappa^2)^{-1}$, which add up to $\frac{1}{2} kT$.

5. Field fluctuations in the electron gas

Any fluctuation in the density gives rise to a fluctuation of the electrostatic potential,

$$\delta\varphi(r) = -e \sum_\lambda \frac{\delta N_\lambda}{|r - R_\lambda|}.$$

The correlation function of these fluctuations can be found by the following calculation.

$$\langle \delta\varphi(r)\, \delta\varphi(r') \rangle = e^2 \sum_{\lambda,\lambda'} \frac{\langle \delta N_\lambda\, \delta N_{\lambda'} \rangle}{|r - R_\lambda|\, |r' - R_{\lambda'}|}$$

$$= e^2 \frac{\Delta^2}{V} \sum_{k,k'} \langle \nu_k \nu_{k'}^* \rangle \sum_{\lambda,\lambda'} \frac{e^{i(k \cdot R_\lambda - k' \cdot R_{\lambda'})}}{|r - R_\lambda|\, |r' - R_{\lambda'}|}$$

$$= \frac{e^2}{V} \sum_k \frac{n_0}{1 + \kappa^2/k^2} e^{ik \cdot (r - r')} \left| \int \frac{e^{ik \cdot R}}{R} d^3R \right|^2$$

$$= \frac{e^2 n_0}{V} \int \frac{e^{ik \cdot (r - r')}}{1 + \kappa^2/k^2} \frac{V}{8\pi^3} d^3k \left(\frac{4\pi}{k^2} \right)^2$$

$$= 4e^2 n_0 \int_0^\infty \frac{dk}{k^2 + \kappa^2} \frac{e^{ik|r - r'|} - e^{-ik|r - r'|}}{ik|r - r'|}$$

$$= \frac{4e^2 n_0}{|r - r'|} \int_{-\infty}^{\infty} \frac{dk}{k^2 + \kappa^2} \frac{\sin k|r - r'|}{k}.$$

Since the integrand is an even function of k, we were entitled to extend the integration from $-\infty$ to $+\infty$ and divide by 2. The integral can now easily be evaluated by complex integration. First the integration path is bent slightly below the origin; the integral then is

$$\frac{1}{2i}\int \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{k}\frac{dk}{k^2+\kappa^2}-\frac{1}{2i}\int \frac{e^{-ik|\boldsymbol{r}-\boldsymbol{r}'|}}{k}\frac{dk}{k^2+\kappa^2}.$$

In the second term the integration path is shifted all the way down, so that this integral turns out to be equal to the residue in $k = -i\kappa$,

$$-\frac{1}{2i}(-2\pi i)\frac{e^{-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}}{-2\kappa^2} = -\frac{\pi}{2\kappa^2}e^{-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}.$$

In the first term the integration path is shifted all the way up, so that the integral turns out to be equal to the sum of the residues in $k = i\kappa$ and in $k = 0$,

$$-\frac{\pi}{2\kappa^2}e^{-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}+\frac{\pi}{\kappa^2}.$$

Collecting results one finds

$$\langle\delta\varphi(\boldsymbol{r})\,\delta\varphi(\boldsymbol{r}')\rangle = \frac{4\pi n_0 e^2}{\kappa^2}\frac{1-e^{-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}.$$

The coefficient in front may also be written kT. Note that this correlation does not decrease as fast as the potential of a Debye sphere, but contains a long range term of Coulomb type. The corresponding correlation for the electric field is

$$\langle\delta E_i(\boldsymbol{r})\,\delta E_j(\boldsymbol{r}')\rangle = -kT\partial_i\partial_j\frac{1-e^{-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|},$$

and has also a long range. It can be shown, however, that when the transverse field is added the correlation of the total field is

$$\langle\delta E_i(\boldsymbol{r})\,\delta E_j(\boldsymbol{r}')\rangle = kT\left[4\pi\delta_{ij}\,\delta(\boldsymbol{r}-\boldsymbol{r}')+\partial_i\partial_j\frac{e^{-\kappa|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\right],$$

which has no long range part.*

* B. U. Felderhof, Physica 31, 295 (1965).

6. Alternative Calculation of the Free Energy of the Electron Gas

The preceding treatment of the statistical mechanics of the electron gas made use of the subdivision of the total volume in small cells Δ. This made it possible to treat the gas more like a continuum than like a collection of particles, in line with Debye's original derivation of the Debye shielding. It is also possible, however, to obtain the same results from the general theory of imperfect gases, if appropriate approximations are made. This method will now be demonstrated on the derivation of the expression (27) for the free energy.

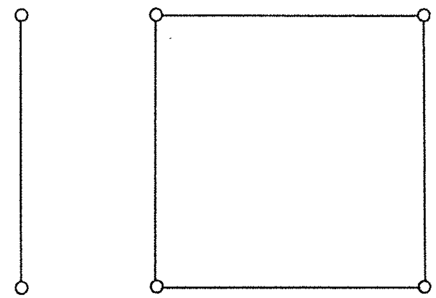

Fig. 19. A disconnected graph ($l = 5$).

We start from Mayer's expansion for the free energy of an imperfect gas*,

$$F = F^{\text{id}} - NkT\sum_{l=1}^{\infty}\frac{\beta_l}{l+1}\left(\frac{N}{V}\right)^l. \tag{30}$$

The coefficients β_l are the "irreducible cluster integrals", defined as follows. Let $v_{12} = v(|\boldsymbol{r}_1-\boldsymbol{r}_2|)$ be the interaction potential between particles 1 and 2, and put

$$f_{12} = e^{-\beta v_{12}}-1.$$

Take the $l+1$ particles 1, 2, ..., $l+1$, and form all products

$$f_{12}f_{13}f_{23}f_{34}\cdots$$

that can be formed using only subscripts referring to these particles. Each product can be represented graphically by a figure consisting of $l+1$ numbered dots, and a connecting line between any two dots i and j for which the factor f_{ij} occurs in the product. Next erase all "disconnected products", i.e., those products whose graphs consist of two or more parts that are not

* See e.g. J. E. Mayer and M. Goeppert Mayer, *Statistical Mechanics* (Wiley, New York 1940) p. 291, eq. (13.45).

mutually connected by any lines. Subsequently erase all "reducible products", i.e., all products whose graphs consist of two parts connected only through a single dot. For each of the remaining "irreducible products" form the integral

$$\int f_{12}f_{13}f_{23}\ldots\, dr_1\, dr_2 \ldots dr_{l+1}\,. \tag{31}$$

Note that the integrand tends to zero as soon as some or all particles are far apart. Consequently, if r_1 is kept fixed, all other integrations extend over a finite region with a diameter of the order of the range of the interaction force $v(r)$. The final integration over r_1 gives a factor V, so that (31) is proportional to the volume. The irreducible cluster integral β_l is now defined by

$$\beta_l = \frac{1}{l!\,V}\sum_{[l+1]}\int f_{12}f_{13}f_{23}\ldots dr_1\, dr_2 \ldots dr_{l+1}\,,$$

where the summation extends over all irreducible products involving the particles 1, 2, .. , $l+1$.

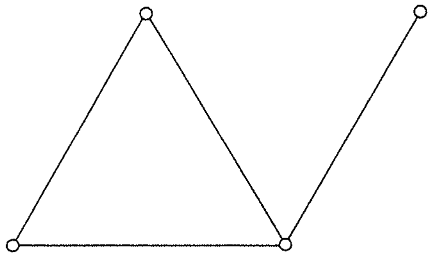

Fig. 20. A reducible graph ($l = 3$).

Equation (30) is exact, but it is very hard to compute the β_l beyond the first two or three. Moreover in the case of Coulomb interaction the integrals do not converge since $v_{12} = e^2\,|\,r_1 - r_2\,|^{-1}$ does not decrease sufficiently rapidly. These difficulties are overcome by making the following approximation. In each β_l all products are neglected except those corresponding to "ring graphs", i.e., the products of the type

$$f_{12}f_{23}f_{34}\ldots f_{l,\,l+1}f_{l+1,\,1}\,.$$

There are $\frac{1}{2}l!$ such products in β_l and they have obviously the same value, so that

$$\beta_l \approx \frac{1}{2V}\int f_{12}f_{23}f_{34}\ldots f_{l+1,\,1}\, d^3\boldsymbol{r}_1 \ldots d^3\boldsymbol{r}_{l+1}\,. \tag{32}$$

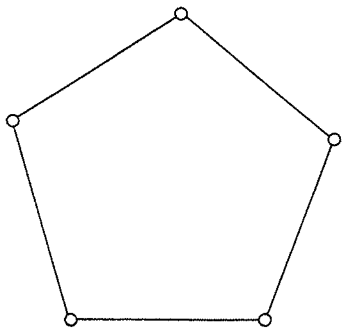

Fig. 21. A ring graph ($l = 4$).

In order to evaluate these products we note that they have the form of convolution integrals. Hence we introduce the Fourier transform of f:

$$f_{12} \equiv f(|\,\boldsymbol{r}_1 - \boldsymbol{r}_2\,|) = (2\pi)^{-3}\int g(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)}\, d^3\boldsymbol{k}\,.$$

On substituting this in (32) one easily finds

$$\beta_l \approx \tfrac{1}{2}(2\pi)^{-3}\int \{g(\boldsymbol{k})\}^{l+1}\, d^3\boldsymbol{k}\,.$$

Substitution in (30) yields

$$F = F^{\mathrm{id}} - \frac{NkT}{16\pi^3}\int\sum_{l=1}^{\infty}\frac{n_0^l g(\boldsymbol{k})^{l+1}}{l+1}\, d^3\boldsymbol{k} \tag{33}$$

$$= F^{\mathrm{id}} + \frac{VkT}{16\pi^3}\int\Big[\log\{1 - n_0 g(\boldsymbol{k})\} + n_0 g(\boldsymbol{k})\Big]\, d^3\boldsymbol{k}\,.$$

It remains to compute

$$g(\boldsymbol{k}) = \int (e^{-\beta e^2/r} - 1)\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3\boldsymbol{r}\,.$$

A second approximation is made by replacing the first exponential by two terms of its expansion,

$$g(\boldsymbol{k}) \approx -\int \frac{\beta e^2}{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3\boldsymbol{r} = -\frac{4\pi\beta e^2}{k^2}\,.$$

The result is

$$F = F^{\mathrm{id}} + \frac{VkT}{16\pi^3} \int \left[\log\left(1 + \frac{\kappa^2}{k^2}\right) - \frac{\kappa^2}{k^2} \right] d^3\boldsymbol{k}.$$

Evaluation of the integral yields

$$F = F^{\mathrm{id}} - \tfrac{2}{3}\pi^{\frac{1}{2}} e^3 n_0^{\frac{3}{2}} V (kT)^{-\frac{1}{2}}.$$

This is identical with (27) but for the self-energy term, which, however, does not affect the thermodynamic behavior. (The reason why this term is not found here is that we now started from (14), from which the self-energy of the electron is excluded by the condition $s \neq s'$, whereas the previous calculation was based on (17), where this self-energy is not excluded.)

Problem. As the integral of each separate term in (33) diverges for $k \to 0$, the integration can only be performed after interchange of summation and integration. Show that this situation can be remedied by replacing the Coulomb interaction with $v(\boldsymbol{r}) = e^2(e^{-\alpha r}/r)$ and taking $\alpha = 0$ in the final result.

General references

A. A. Vedenov in: *Reviews of Plasma Physics*, Vol. 1, M. A. Leontovich ed. (Consultants Bureau, New York 1965).
R. Balescu, *Statistical Mechanics of Charged Particles* (Interscience, New York 1963) ch. XII.
E. Meeron in: *Plasma Physics*, J. E. Drummond ed. (McGraw-Hill, New York 1961).
R. Brout and P. Carruthers, *Lectures on the Many-Electron Problem* (Interscience, New York 1963).

For the quantummechanical theory see Brout and Carruthers, and also:
S. Raimes, *The Wave Mechanics of Electrons in Metals* (North-Holland, Amsterdam 1961) ch. X.
D. Pines, *The Many-Body Problem* (W. A. Benjamin, New York 1961).

CHAPTER IX

TWO-COMPONENT THEORY

In the magnetohydrodynamic approximation the plasma was regarded as a conducting fluid which can be completely described by the quantities ρ, $\boldsymbol{V}$, $\boldsymbol{H}$. All other quantities could be expressed in terms of these; in particular the electric field was expressed in $\boldsymbol{V}$, $\boldsymbol{H}$ through Ohm's law. This picture, however, is not adequate for the description of rapid variations in a low density, high temperature plasma. In these circumstances collisions between electrons and ions are no longer sufficiently frequent to maintain Ohm's law. Rather the electron gas is a separate fluid, having its own degrees of freedom and obeying its own equations of motion. This creates the possibility for the electrons to oscillate with respect to the ions: plasma waves. Moreover for these rapid phenomena the Maxwell term in the equation for curl $\boldsymbol{H}$ can no longer be neglected.

In order to describe such phenomena the two-component theory is used. In this theory the plasma is regarded as two interpenetrating fluids, the electron gas and the ion gas. Each obeys its own MHD equations, but of course there is a single electromagnetic field. Hence the two components interact through this field. In addition there may be a mutual friction due to collisions. Admittedly the physical reality of this picture is questionable, as will be discussed in the next chapter. The reason why two-component theory is used nevertheless is that it arrives at qualitatively correct results in a less cumbersome way than the Vlasov theory. Accordingly, the work in the present chapter should be regarded as a first orientation in the large variety of wave phenomena in a plasma.

1. The equations of two-component theory

Let m_e be the mass of an electron, $-e$ its charge; let n_e be the number of electrons per unit volume, $\boldsymbol{V}_e$ the local velocity of the electron gas, p_e its pressure. Then the hydrodynamic equations for this fluid are (comp. (II, 21a))

$$\frac{\partial n_e}{\partial t} + \nabla \cdot n_e \boldsymbol{V}_e = 0 \tag{1}$$

$$n_e m_e \left\{\frac{\partial \boldsymbol{V}_e}{\partial t} + (\boldsymbol{V}_e \cdot \nabla)\boldsymbol{V}_e\right\} = -\nabla p_e - n_e e \boldsymbol{E} - \frac{n_e e}{c} \boldsymbol{V}_e \wedge \boldsymbol{H} + \boldsymbol{P}. \tag{2}$$

We have omitted the gravitational force, but on the other hand we have added a term $\boldsymbol{P}$, which represents the momentum transfer per unit time from the ion gas to the electron gas through collisions. Similarly the ion gas obeys the equations

$$\frac{\partial n_i}{\partial t} + \nabla \cdot n_i \boldsymbol{V}_i = 0 \tag{3}$$

$$n_i m_i \left\{\frac{\partial \boldsymbol{V}_i}{\partial t} + (\boldsymbol{V}_i \cdot \nabla)\boldsymbol{V}_i\right\} = -\nabla p_i + n_i Z e \boldsymbol{E} + \frac{n_i Z e}{c} \boldsymbol{V}_i \wedge \boldsymbol{H} - \boldsymbol{P}. \tag{4}$$

The charge of each ion is taken to be Ze.

These equations have to be supplemented with Maxwell's equations and with two equations of state connecting the pressures with the densities,

$$p_e = f_e(n_e), \qquad p_i = f_i(n_i). \tag{5}$$

Finally one has to find $\boldsymbol{P}$; we shall assume

$$\boldsymbol{P} = \zeta n_e n_i (\boldsymbol{V}_i - \boldsymbol{V}_e). \tag{6}$$

The constant ζ is positive but unknown. The justification for (6) is that it is the simplest expression in n_e, n_i, $\boldsymbol{V}_e$, $\boldsymbol{V}_i$ which obeys the obvious requirements that $\boldsymbol{P}$ must be a vector in the direction of the relative velocity, and that it must vanish at zero relative velocity and when either density is zero.

The total set of equations obtained in this way permits in principle to describe the behavior of the two-component plasma. However, the equations (1), (2), (3), (4) are usually replaced by certain combinations of them, which are equivalent in content, but resemble more closely the MHD equations. Multiply (1) with m_e and (3) with m_i and add:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{V} = 0. \tag{7}$$

Here ρ is the total mass density

$$\rho = n_i m_i + n_e m_e \tag{8}$$

and $\boldsymbol{V}$ the average velocity

$$\boldsymbol{V} = \frac{n_i m_i \boldsymbol{V}_i + n_e m_e \boldsymbol{V}_e}{n_i m_i + n_e m_e}. \tag{9}$$

Similarly one finds

$$\frac{\partial \tau}{\partial t} + \nabla \cdot \boldsymbol{j} = 0, \tag{10}$$

where τ and $\boldsymbol{j}$ are the net charge and current densities

$$\tau = n_i Z e - n_e e, \qquad \boldsymbol{j} = n_i Z e \boldsymbol{V}_i - n_e e \boldsymbol{V}_e. \tag{11}$$

The combination of (2) and (4) is less straightforward owing to the non-linear terms in $\boldsymbol{V}_i$ and $\boldsymbol{V}_e$. However, anticipating the fact that these equations are only employed to describe small deviations from the static equilibrium, we shall use the following approximations.

(i) Quadratic terms in the velocities are neglected. It is then easy to derive

$$\rho \frac{\partial \boldsymbol{V}}{\partial t} = -\nabla(p_e + p_i) + \tau \boldsymbol{E} + \frac{1}{c} \boldsymbol{j} \wedge \boldsymbol{H} \tag{12}$$

$$\frac{\partial \boldsymbol{j}}{\partial t} = \nabla\left(\frac{e}{m_e} p_e - \frac{Ze}{m_i} p_i\right) + \left(\frac{n_e e^2}{m_e} + \frac{n_i Z^2 e^2}{m_i}\right)\boldsymbol{E} + \frac{1}{c}\left(\frac{n_e e^2}{m_e} \boldsymbol{V}_e + \frac{n_i Z^2 e^2}{m_i} \boldsymbol{V}_i\right) \wedge \boldsymbol{H} - \left(\frac{e}{m_e} + \frac{Ze}{m_i}\right)\boldsymbol{P}. \tag{13}$$

Thus we have found equations for the rate of change of ρ, τ, $\boldsymbol{V}$, $\boldsymbol{j}$, but the right-hand side of (13) still contains the old variables n_i, n_e, $\boldsymbol{V}_i$, $\boldsymbol{V}_e$. It is of course possible to express these in terms of ρ, τ, $\boldsymbol{V}$, $\boldsymbol{j}$ by solving the equations (8), (9), (11). However, one usually introduces two additional approximations, which greatly facilitate the algebra.

(ii) m_e is neglected with respect to m_i, so that one has

$$n_i = \frac{\rho}{m_i}, \qquad n_e = \frac{Z\rho}{m_i} - \frac{\tau}{e}, \qquad \boldsymbol{V}_i = \boldsymbol{V} + \frac{m_e}{\rho e}\boldsymbol{j}, \qquad \boldsymbol{V}_e = \frac{n_i Z}{n_e}\boldsymbol{V} - \frac{\boldsymbol{j}}{n_e e}. \tag{14}$$

With this approximation (13) reduces to

$$\frac{\partial \boldsymbol{j}}{\partial t} = \frac{e}{m_e}\nabla p_e + \frac{n_e e^2}{m_e}\boldsymbol{E} + \left(\frac{\rho Z e^2}{c m_e m_i}\boldsymbol{V} - \frac{e}{c m_e}\boldsymbol{j}\right) \wedge \boldsymbol{H} - \frac{e}{m_e}\boldsymbol{P}. \tag{15}$$

(iii) Quasi-neutrality: τ is considered small and is omitted when it occurs multiplied with another quantity that vanishes in equilibrium. Thus in (12) the term $\tau \boldsymbol{E}$ is omitted, and in (15) n_e in the coefficient of $\boldsymbol{E}$ is replaced with

$$n_e \approx Z\rho/m_i = Zn_i. \tag{16}$$

Moreover, the term with $\boldsymbol{P}$ reduces to

$$-\frac{e}{m_e}\boldsymbol{P} = -\frac{e}{m_e}\zeta n_e n_i \frac{\boldsymbol{j}}{n_e e} = -\frac{\zeta\rho}{m_e m_i}\boldsymbol{j}.$$

Equation (15) then becomes the "*generalized Ohm's law*"

$$\frac{\partial \boldsymbol{j}}{\partial t} = \frac{e}{m_e}\nabla p_e + \frac{\rho Z e^2}{m_e m_i}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{V}\wedge\boldsymbol{H} - \frac{1}{\sigma}\boldsymbol{j}\right) - \frac{e}{m_e c}\boldsymbol{j}\wedge\boldsymbol{H}. \tag{17}$$

We have introduced a new constant,

$$\sigma = Ze^2/\zeta, \tag{18}$$

which is clearly the conductivity of the old Ohm's law (II, 21d).

The most drastic innovation of the generalized Ohm's law is the term with $\partial \boldsymbol{j}/\partial t$. It has the consequence that the electric current is no longer uniquely tied to the instantaneous values of the other quantities describing the plasma. This is of course due to the fact that the electron gas is now treated as a separate fluid, having its own degrees of freedom. It is also through this term that the mass m_e enters into the equations.

Summary. In the two-component theory the plasma is described by ρ, $p_e, p_i, \boldsymbol{V}, \boldsymbol{j}, \tau, \boldsymbol{E}, \boldsymbol{H}$ as functions of $\boldsymbol{r}$ and t governed by the equations

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\boldsymbol{V} = 0 \tag{19a}$$

$$\rho\frac{\partial \boldsymbol{V}}{\partial t} = -\nabla(p_e+p_i) + \frac{1}{c}\boldsymbol{j}\wedge\boldsymbol{H} \tag{19b}$$

$$\frac{m_e m_i}{\rho Z e^2}\frac{\partial \boldsymbol{j}}{\partial t} + \frac{1}{\sigma}\boldsymbol{j} = \boldsymbol{E} + \frac{1}{c}\boldsymbol{V}\wedge\boldsymbol{H} + \frac{m_i}{\rho Z e}\left(\nabla p_e - \frac{1}{c}\boldsymbol{j}\wedge\boldsymbol{H}\right) \tag{19c}$$

$$-\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} = \nabla\wedge\boldsymbol{E}, \qquad \nabla\cdot\boldsymbol{H} = 0 \tag{19d}$$

$$-\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{j} - \nabla\wedge\boldsymbol{H}, \qquad \nabla\cdot\boldsymbol{E} = 4\pi\tau \tag{19e}$$

$$p_i = f_i\left(\frac{\rho}{m_i}\right), \qquad p_e = f_e\left(\frac{Z\rho}{m_i} - \frac{\tau}{e}\right). \tag{19f}$$

The charge conservation equation (10) need not be included since it is a consequence of (19e). The equations (19f) are the equations of state (5) in which n_i and n_e are expressed in the proper variables ρ, τ. It should be remembered that the equations (19) have only been derived for small deviations from a static equilibrium. Accordingly in (19c) one may treat ρ as a constant.

Problem. Derive the balance equation for the total energy

$$\int \left\{\tfrac{1}{2}\rho V^2 + \rho\Psi_i(\rho) + n_e\Psi_e(n_e) + \frac{m_i m_e}{2\rho Z e^2}j^2 + \frac{1}{8\pi}(E^2+H^2)\right\} d^3\boldsymbol{r}. \tag{20}$$

Here we have set

$$\Psi_i(\rho) = \int \frac{f_i(\rho/m_i)}{\rho^2}\,d\rho, \qquad \Psi_e(n_e) = \int \frac{f_e(n_e)}{n_e^2}\,dn_e,$$

and n_e is to be regarded as a function of ρ and τ determined by (14).

2. Plasma waves in a cold plasma

At low temperatures the pressure terms are unimportant, as can be seen from the ideal gas law. Hence the term "cold plasma" is used for the case that one puts $p_e = p_i = 0$. We shall first study the waves in such a cold plasma; it will afterwards turn out that the addition of the pressure terms does not seriously influence the plasma waves. We shall not systematically investigate all possible wave modes, but only the most important ones. It will be supposed that there is a homogeneous external field $\boldsymbol{H}^0$ independent of $\boldsymbol{r}$ and t.*

It is easily verified that a static solution of the equations of two-component theory obtains by setting

$$\boldsymbol{V} = \boldsymbol{j} = \boldsymbol{E} = 0, \quad \rho = \rho^0 = \text{constant}, \quad \boldsymbol{H} = \boldsymbol{H}^0. \tag{21}$$

We now linearize the equations by putting

$$\rho = \rho^0 + \rho^1, \qquad \boldsymbol{H} = \boldsymbol{H}^0 + \boldsymbol{H}^1, \tag{22}$$

and neglecting all terms of second order in ρ^1, $\boldsymbol{H}^1$, $\boldsymbol{V}, \boldsymbol{j}, \boldsymbol{E}$. At the same time we specialize to infinite conductivity. The resulting set of equations is

$$\frac{\partial\rho^1}{\partial t} + \rho^0\nabla\cdot\boldsymbol{V} = 0 \tag{23a}$$

* In most applications $\boldsymbol{H}^0$ is not really homogeneous, e.g. in the ionosphere or the corona. However, as long as $\boldsymbol{H}^0$ is practically constant over many wave lengths our results are applicable.

$$\rho^0 \frac{\partial V}{\partial t} = \frac{1}{c} j \wedge H^0 \tag{23b}$$

$$\frac{m_e m_i}{\rho^0 Z e^2} \frac{\partial j}{\partial t} = E + \frac{1}{c} V \wedge H^0 - \frac{m_i}{\rho^0 Z e c} j \wedge H^0 \tag{23c}$$

$$-\frac{1}{c}\frac{\partial H^1}{\partial t} = \nabla \wedge E \tag{23d}$$

$$-\frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c} j - \nabla \wedge H^1 . \tag{23e}$$

Since these are linear equations with constant coefficients, solutions are obtained by putting

$$\rho^1 = \rho_k e^{i(k \cdot r - \omega t)}, \quad \text{etc.}$$

This leads to the set of algebraic equations

$$\omega \rho_k = \rho^0 k \cdot V_k \tag{24a}$$

$$\omega V_k = \frac{i}{\rho^0 c} j_k \wedge H^0 \tag{24b}$$

$$-i\omega \frac{m_e m_i}{\rho^0 Z e^2} j_k = E_k + \frac{1}{c} V_k \wedge H^0 - \frac{m_i}{\rho^0 Z e c} j_k \wedge H^0 \tag{24c}$$

$$\frac{\omega}{c} H_k = k \wedge E_k \tag{24d}$$

$$\frac{\omega}{c} E_k = -\frac{4\pi i}{c} j_k - k \wedge H_k . \tag{24e}$$

It is convenient to eliminate H_k from (24d) and (24e) to obtain

$$\left(\frac{\omega^2}{c^2} - k^2\right) E_k + k(k \cdot E_k) = -\frac{4\pi i \omega}{c^2} j_k . \tag{24f}$$

These equations are exact consequences of (23) and hence contain all wave modes possible in the two-component theory. They constitute a set of linear homogeneous equations for the constants ρ_k, V_k, j_k, E_k, H_k with the eigenvalue parameter ω. All one has to do is determine the eigenvalues and eigensolutions. However, we select only the most important ones for further study. Moreover, the algebra may be simplified by a judicious manipulation of the equations.

I. Cold plasma with $H^0 = 0$.
It follows from (24b) and (24a) that $V_k = \rho_k = 0$, so that there is no net motion of the plasma. Furthermore, according to (24c)

$$j = \sigma' E, \quad \text{where} \quad \sigma' = \frac{i}{\omega} \frac{n_e^0 e^2}{m_e} . \tag{25}$$

This relation differs from Ohm's law inasmuch as σ' is imaginary and a function of ω.

Combination of (25) with (24f) leads to an equation for E_k alone:

$$\left[\frac{\omega^2}{c^2} + \frac{4\pi i \omega}{c^2} \sigma'\right] E_k = k^2 E_k - k(k \cdot E_k) . \tag{26}$$

One is now easily led to the following possible modes.

Ia. Longitudinal plasma waves: $E_k // k$. The right-hand side of (26) is zero, so that the factor [ ] must also be zero. This yields the dispersion law

$$\omega^2 = \frac{4\pi n_e^0 e^2}{m_e} . \tag{27}$$

Note that there are two eigenfrequencies, which do not depend on k, but only on the electron density. The constant

$$\omega_p = \sqrt{\frac{4\pi n_e^0 e^2}{m_e}} = 5.6 \times 10^4 \sqrt{n_e^0} \ \mathrm{sec}^{-1} \tag{28}$$

is called the "*plasma frequency*".

Longitudinal plasma waves were first found experimentally by Penning.* They can be visualized as follows. The ions are at rest, because in our approximation $V_i = V$, which is zero. At each moment the electron density is a sinusoidal function in space. In the regions of high density the electrons repel each other and hence move out to the regions of low density, so that after half a period the latter have become regions of high density. Thus the longitudinal plasma wave resembles somewhat the sound wave in an ordinary gas, but the elastic resilience counteracting compression is provided by electric repulsion instead of gas pressure. The electrons move back and forth in the direction of propagation, while the ions are virtually fixed.

* F. M. Penning, Nature **118**, 301 (1926); Physica **6**, 241 (1926); see also L. Tonks and I. Langmuir, Phys. Rev. **33**, 195, 990 (1929), and H. J. Merrill and H. W. Webb, Phys. Rev. **55**, 1191 (1939). An early theoretical description was given by Tonks and Langmuir, loc. cit., and by J. J. Thomson; see J. J. Thomson and G. P. Thomson, *Conduction of Electricity through Gases*, 3rd ed. (Cambridge University Press, Cambridge 1933) Vol. **2**, p. 353.

Problem. The fact that the frequency is independent of the wave number k implies that the group velocity is zero. This suggests that there is no propagation (although the phase velocity is not zero). Indeed, show that any arbitrary initial charge distribution oscillates with the plasma frequency ω_P without displacement or change of shape. (For this reason some authors prefer to call these waves "plasma oscillations".)

Problem. Show that if one does not make the approximation $m_e \ll m_i$ the only difference is that ω_P is replaced with

$$\left(\frac{4\pi n_e e^2}{m_e} + \frac{4\pi n_i Z^2 e^2}{m_i}\right)^{\frac{1}{2}}.$$

Problem. Show that the effect of finite conductivity can be taken into account by replacing ω_P with ω_P', determined by

$$\frac{1}{\omega_P'^2} = \frac{1}{\omega_P^2} + \frac{i}{4\pi\omega\sigma}.$$

Ib. Transverse waves: $E_k \perp k$. Equation (26) yields immediately the dispersion law for transverse plasma waves,

$$\omega^2 = \omega_P^2 + c^2k^2. \tag{29}$$

These transverse waves may be regarded as electromagnetic waves modified by the presence of charged particles. The elastic resilience responsible for counteracting the shear is provided by the electromagnetic field, just like in vacuum. The influence of the plasma is often described in terms of a refractive index

$$n(\omega) \equiv \frac{ck}{\omega} = \sqrt{1 - \frac{\omega_P^2}{\omega^2}}. \tag{30}$$

Problem. Show that these longitudinal and transverse plasma waves exhaust all possible oscillatory motions for a cold two-component plasma without external magnetic field.

Problem. Calculate the values of j_k, E_k, H_k in a transverse wave and show that there are two different polarizations possible.

Problem. What happens to the solution when $\omega < \omega_P$?

II. Cold plasma with $H^0 \neq 0$ and $k \,//\, H^0$.

It follows from (24b) that $V_k \perp k$ and hence from (24a) that $\rho_k = 0$: no density variations of the total plasma, or, what amounts to the same, of the ions.

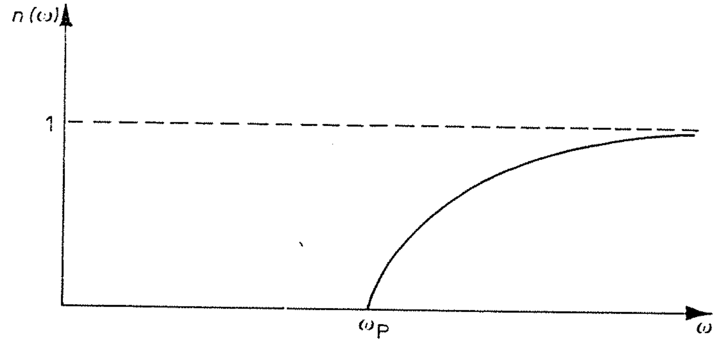

Fig. 22. The refractive index for transverse waves in a cold plasma without external field, equation (30).

IIa. Longitudinal plasma waves. These are identical with the waves **Ia**, because in those waves the electrons move in the direction of $\boldsymbol{k}$, hence of $\boldsymbol{H}^0$, so that $\boldsymbol{H}^0$ exerts no force on them.

Problem. Show explicitly that the solutions **Ia** obey (24) when $\boldsymbol{H}^0 // \boldsymbol{k}$.

IIb. Transverse waves with $k // H^0$. Since in this case $\boldsymbol{E}_k \perp \boldsymbol{k}$, equation (24f) reduces to

$$(\omega^2 - c^2k^2)\boldsymbol{E}_k = -4\pi i\omega \boldsymbol{j}_k.$$

It follows that $\boldsymbol{j}_k$ is parallel to $\boldsymbol{E}_k$ and therefore perpendicular to $\boldsymbol{H}^0$. Hence (24b) yields

$$\omega \boldsymbol{V}_k \wedge \boldsymbol{H}^0 = -\frac{i}{\rho^0 c}(H^0)^2 \boldsymbol{j}_k.$$

Using these two equations to eliminate $\boldsymbol{E}_k$ and $\boldsymbol{V}_k$ from (24c) one arrives at an equation for the vector $\boldsymbol{j}_k$ alone, involving ω:

$$\left\{-i\omega \frac{m_e}{n_e^0 e^2} + \frac{4\pi i\omega}{\omega^2 - c^2k^2} + \frac{i}{\omega}\frac{(H^0)^2}{\rho^0 c^2}\right\} \boldsymbol{j}_k = -\frac{1}{n_e^0 ec}\boldsymbol{j}_k \wedge \boldsymbol{H}^0.$$

The left-hand member is parallel to $\boldsymbol{j}_k$ and the right-hand member is perpendicular to $\boldsymbol{j}_k$; hence at first sight there is no other solution than $\boldsymbol{j}_k = 0$. However, $\boldsymbol{j}_k$ may be complex (since it is the amplitude of a complex wave) and can therefore be perpendicular to itself without being zero. In order to find $\boldsymbol{j}_k$ one has to solve this set of three linear homogeneous equations* for the three components of $\boldsymbol{j}_k$; the corresponding eigenvalue equation is

* Actually the set reduces to two linear equations, as the component of $\boldsymbol{j}_k$ along $\boldsymbol{H}^0$ vanishes.

$$\left\{-\frac{m_e\omega}{n_e^0 e^2}+\frac{4\pi\omega}{\omega^2-c^2k^2}+\frac{(H^0)^2}{\omega n_e^0 m_i c^2/Z}\right\}^2=\frac{(H^0)^2}{(n_e^0 ec)^2}.$$

It is customary to use the following abbreviations:

$$\frac{eH^0}{m_e c}=\omega_{ce}\quad \text{("electron cyclotron frequency"} = 1.8\times 10^7\, H^0\, \text{sec}^{-1}\text{)}$$

$$\frac{ZeH^0}{m_i c}=\omega_{ci}\quad \text{("ion cyclotron frequency"} \ll \omega_{ce}\text{)}.$$

The equation for ω can then be written in the form

$$\omega^2-c^2k^2=\frac{\omega_P^2}{1-\dfrac{\omega_{ce}}{\omega}\left(\dfrac{\omega_{ci}}{\omega}\mp 1\right)},\tag{31}$$

which is therefore *the dispersion law for transverse waves propagating along* H^0. This result may also be expressed in terms of a refractive index

$$n^2(\omega)=1-\frac{\omega_P^2}{\omega^2\pm\omega_{ce}\omega-\omega_{ce}\omega_{ci}}.\tag{32}$$

It will be shown in section 4 that the same dispersion law applies to hot plasmas.

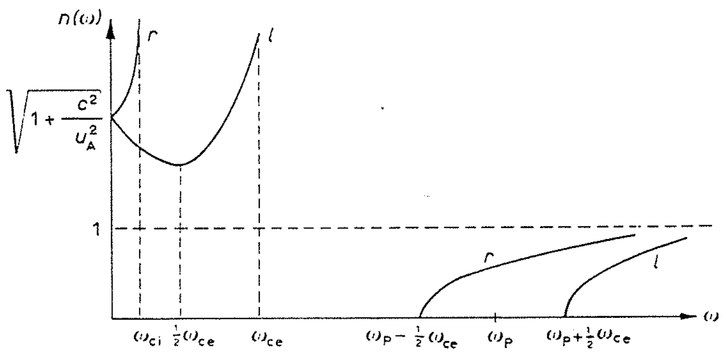

Fig. 23. The refractive index for transverse waves in a cold plasma with external field, equation (32).

Problem. Show that $n^2(0) = 1+c^2/u_A^2$.
Problem. Find the resonances in (32), i.e., the zeros of the denominator.
Problem. Obtain the same result (31) in the following alternative way.

Take the z-axis along $\boldsymbol{H}^0$ and write the six equations for $j_{kx}, j_{ky}, E_{kx}, E_{ky}, V_{kx}, V_{ky}$. The corresponding eigenvalue equation is (31).
Problem. Proceed as in the previous problem but reduce the matrix by using the variables $j_{kx}\pm ij_{ky}$, etc.
Problem. Show that *the waves found above are circularly polarized.* For waves propagating in the direction of $\boldsymbol{H}^0$ the $-$ and $+$ in (31) refer to right and left circular polarization.*
Problem. Show that the resonances in (32) occur when the field rotates with the same speed as the cyclotron motion of the particles (within the present approximation).

III. Cold plasma with $\boldsymbol{H}^0 \neq 0$ and $\boldsymbol{k} \perp \boldsymbol{H}^0$.
IIIa. Longitudinal waves: $\boldsymbol{E}_k//\boldsymbol{k}$. By solving (24) in a similar way as above one finds that no such waves are possible.

Problem. Show that for $\boldsymbol{k}\perp\boldsymbol{H}^0$ a particular wave is possible with $\boldsymbol{j}_k//\boldsymbol{k}$ whose frequency and wave length are given by

$$\omega^2=c^2k^2=\omega_P^2\left(1+\frac{(H^0)^2}{4\pi\rho^0c^2}\right).$$

Note that in this wave $\boldsymbol{E}_k$ is not parallel to $\boldsymbol{j}_k$, but makes an angle ϑ given by $\tan\vartheta = i\omega H^0/4\pi n_e^0 ec$.

IIIb. Transverse waves: $\boldsymbol{E}_k\perp\boldsymbol{k}$. Under **Ib** we found transverse waves with two different polarizations. Now if the field $\boldsymbol{H}^0$ is added, the polarization in which $\boldsymbol{j}_k$ is parallel to $\boldsymbol{H}^0$ remains a possible solution. The polarization in which $\boldsymbol{j}_k$ is perpendicular to $\boldsymbol{H}^0$ is no longer a solution.

Problem. Prove that there is no plasma wave in which $\boldsymbol{H}^0$, $\boldsymbol{k}$ and $\boldsymbol{j}_k$ are perpendicular to each other.

3. Discussion of the transverse waves in a cold plasma

The propagation of transverse waves in a cold plasma without external magnetic field is described by the refractive index (30), see Fig. 22. For $\omega<\omega_P$ the wave number k is purely imaginary, so that no waves can propagate. If an electromagnetic wave in vacuum with frequency ω impinges on a plasma whose ω_P is greater than ω, only an exponentially decreasing

* In a right circularly polarized wave the electric vector rotates clockwise when looking in the direction *from which the light is coming*; see M. Born and E. Wolf, *Principles of Optics* (Pergamon, London 1959) p. 27. This is the traditional convention, but in plasma physics the opposite convention is also used, for instance by Spitzer.

field penetrates into the plasma, while all energy is reflected into the vacuum. This is the reason why long radio waves are reflected by the ionosphere, while short waves (television, 21 cm line) go through. By measuring the heights at which different wave lengths are reflected it is possible to determine the electron density n_e^0 of the various layers. When a space capsule re-enters the atmosphere it heats up the surrounding air to a temperature at which ionization is appreciable. Thus it is surrounded by a plasma cloud, which disrupts the radio contact with the ground stations.

Problem. An electromagnetic wave impinges at right angles on a half space filled with a cold plasma with constant electron density. Compute the reflected and the transmitted intensities as functions of the ω of the incident wave and the ω_P of the plasma.

If a constant, homogeneous external magnetic field $\boldsymbol{H}^0$ is present, right and left circularly polarized transverse waves may propagate along the field lines. Their refractive indexes are given by (32), see Fig. 23. The resonance of the right polarized wave $\omega \approx \omega_{ci}$ is due to the fact that at this frequency the electric field vector rotates with the same velocity as the ions in their cyclotron rotation about $\boldsymbol{H}^0$. Similarly the resonance of the left polarized wave at $\omega \approx \omega_{ce}$ is due to the coincidence of the rotation of the electric field with the cyclotron motion of the electrons.

Comparison of Fig. 23 with Fig. 22 shows that presence of $\boldsymbol{H}^0$ has the effect of increasing the frequency range in which transmission of transverse waves is possible. Not only is the minimum frequency lowered from ω_P to $\omega_P - \frac{1}{2}\omega_{ce}$, but in addition a low frequency "window" is created for frequencies below ω_{ce}. These low frequency waves are observed in the phenomenon of "whistlers" and in "helicons".

Whistlers are produced by electrical discharges in the atmosphere (thunderstorms), which excite electromagnetic waves in the audiofrequency range. These waves propagate mainly along the field lines of the earth's magnetic field, so that they re-enter the atmosphere at a point on the opposite hemisphere. Since these frequencies obey $\omega_{ci} \ll \omega \ll \omega_{ce}$ they propagate as left circular transverse waves with a refractive index approximately given by

$$n(\omega) = \frac{\omega_P}{\sqrt{(\omega_{ce}\omega)}}. \tag{33}$$

Accordingly the group velocity is given by

$$\frac{d\omega}{dk} = \frac{2c}{\omega_P}\sqrt{(\omega_{ce}\omega)},$$

and therefore increases with the frequency. Consequently the highest frequencies arrive first, the lower frequencies later. This is the reason why whistlers are received as short whistles, whose pitch is high at the beginning and subsequently drops. The same whistler may be received several times, because it is reflected back and forth along the field lines.*

Helicons are waves of the same type in the electron gas of a metal.** Clearly in this case one should take $\omega_{ci} = 0$, so that for sufficiently low ω the refractive index is again given by (33). Owing to the high density $n(\omega)$ can be very large; in fact in a field of 10000 gauss waves have been observed*** with frequency 30 sec^{-1}, wave length 1 cm, so that the refractive index is 10^9.

Below ω_{ci} right polarized waves are also possible and for $\omega \ll \omega_{ci}$ the refractive indexes of both waves tend to the same value $\sqrt{(1+c^2/u_A^2)}$. Since for usual field strengths and densities $u_A \ll c$, one may put $n \approx c/u_A$, so that the phase velocity equals the Alfvén velocity u_A. In fact in this limit the transverse waves reduce to the Alfvén waves of magnetohydrodynamics.

Problem. Compare the orders of magnitude of the separate terms in (24c) and show that for $\omega \ll \omega_{ci}$ the term with $\boldsymbol{V}_k$ is much larger than the term originating from $\partial \boldsymbol{j}/\partial t$, so that the MHD equations apply.

Problem. Compute the Alfvén waves from the MHD equations without neglecting the term with $\partial \boldsymbol{E}/\partial t$ in the Maxwell equations. Show that this leads to the corrected value of the refractive index $\sqrt{(1+c^2/u_A^2)}$, rather than c/u_A.

Problem. Find the dispersion law for damped transverse plasma waves by including the term $\boldsymbol{j}/\sigma$.

4. Plasma Waves in a Hot Plasma

The basic equations for waves in a hot two-component plasma differ from the equations (23) for a cold plasma only by the addition of the pressure terms, as given by (19). The terms ∇p_i and ∇p_e should be expressed in ρ and τ,

$$\begin{aligned} \nabla p_i &= f_i'(n_i^0)\nabla n_i^1 = f_i'(n_i^0)\,\frac{1}{m_i}\,\nabla\rho^1, \\ \nabla p_e &= f_e'(n_e^0)\nabla n_e^1 = f_e'(n_e^0)\left(\frac{Z}{m_i}\,\nabla\rho^1 - \frac{1}{e}\,\nabla\tau\right). \end{aligned} \tag{34}$$

* See e.g. R. A. Helliwell in: *Propagation and Instabilities in Plasmas*, W. I. Futterman ed. (Stanford University Press, Stanford 1963).
** S. J. Buchsbaum and R. Bowers in: *Plasma Effects in Solids* (Dunod, Paris 1965).
*** R. Bowers, C. Legendy and F. Rose, Phys. Rev. Letters **7**, 339 (1961).

We shall use the abbreviations Θ_e and Θ_i for the constants $f_e'(n_e^0)$ and $f_i'(n_i^0)$. Since the oscillations we are interested in are very rapid it is reasonable to suppose that the variations obey the adiabatic equation of state (II, 19), so that

$$\Theta_e \equiv f_e'(n_e^0) = \gamma_e k T_e, \qquad \Theta_i \equiv f_i'(n_i^0) = \gamma_i k T_i .$$

γ_e and γ_i are the values of c_p/c_V for the electron and the ion gas.

Remark. The possbility is left open that the electron gas and the ion gas have different temperatures. This is actually the case in experimental plasmas, such as gas discharges. The reason is that the energy transfer between both gases by collisions is very slow, owing to the large mass difference. Of course, if the unperturbed state is really in equilibrium both temperatures are the same; but most experimental plasmas do not reach this equilibrium, either because they are short-lived or because they are constantly perturbed by the external agency that serves to maintain them (like in gas discharges).

Problem. Estimate the average energy transfer per collision between two hard sphere gases with different masses m_e, m_i and different temperatures T_e, T_i.

The addition of these pressure terms does not affect (23a) but changes (23b) into

$$\rho_0 \frac{\partial V}{\partial t} = -\frac{Z\Theta_e + \Theta_i}{m_i} \nabla \rho^1 + \frac{\Theta_e}{4\pi e} \nabla(\nabla \cdot E) + \frac{1}{c} j \wedge H^0 . \tag{35}$$

Also (23c) is changed into

$$\frac{m_e m_i}{\rho^0 Z e^2} \frac{\partial j}{\partial t} = E + \frac{1}{c} V \wedge H^0 - \frac{m_i}{\rho^0 Z e c} j \wedge H^0 + \frac{\Theta_e}{\rho^0 e} \nabla \rho^1 - \frac{m_i \Theta_e}{4\pi \rho^0 Z e^2} \nabla(\nabla \cdot E), \tag{36}$$

while (23d) and (23e) remain unaffected. We shall briefly discuss the various wave solutions of these equations.

I′. Hot plasma with $H^0 = 0$.
The equations for the waves are

$$\omega \rho_k = \rho^0 \boldsymbol{k} \cdot V_k \tag{37a}$$

$$\omega V_k = \boldsymbol{k} \frac{Z\Theta_e + \Theta_i}{\rho^0 m_i} \rho_k - i\boldsymbol{k} \frac{\Theta_e}{4\pi e \rho^0} (\boldsymbol{k} \cdot E_k) \tag{37b}$$

$$-\frac{4\pi i \omega}{\omega_P^2} j_k = E_k + i\boldsymbol{k} \frac{\Theta_e}{\rho^0 e} \rho_k + \boldsymbol{k} \frac{\Theta_e}{m_e \omega_P^2} (\boldsymbol{k} \cdot E_k) \tag{37c}$$

$$\left(\frac{\omega^2}{c^2} - k^2\right) E_k + \boldsymbol{k}(\boldsymbol{k} \cdot E_k) = -\frac{4\pi i \omega}{c^2} j_k . \tag{37d}$$

I′a. Longitudinal waves: $E_k // \boldsymbol{k}$. On solving the above equations for this case one obtains the dispersion law

$$\left(\omega^2 - \omega_P^2 - \frac{\Theta_e}{m_e} k^2\right)\left(\omega^2 - \frac{Z\Theta_e + \Theta_i}{m_i} k^2\right) = \frac{Z\Theta_e^2}{m_e m_i} k^4 . \tag{38}$$

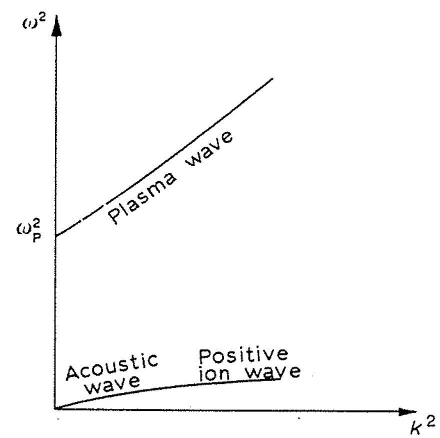

Fig. 24. The longitudinal waves in a hot two-component plasma.

The curve of ω^2 as a function of k^2 is a hyperbola, consisting of two branches (Fig. 24). The upper branch obeys for long wave lengths the approximate dispersion law

$$\omega^2 = \omega_P^2 + \frac{\Theta_e}{m_e} k^2 = \omega_P^2 + \frac{\gamma_e k T_e}{m_e} k^2 . \tag{39}$$

Clearly this is the longitudinal plasma wave **Ia**, modified by the presence of an additional resilience due to the gas pressure of the electron gas. The pressure term $(\gamma_e k T_e/m_e)k^2$ becomes of the same order as the plasma term ω_P^2 for wave lengths of the order of the Debye length. However, for such short wave lengths (39) is no longer valid, as will be shown in chapter XII.

Problem. Show that the phase velocity corresponding to (39) is larger than the velocity of sound in the electron gas, and that the group velocity is smaller.
Problem. Derive (38).

The lower branch in Fig. 24 obeys for long wave lengths the approximate dispersion law

$$\omega^2 = \frac{Z\Theta_e + \Theta_i}{m_i} k^2 = \frac{Z\gamma_e kT_e + \gamma_i kT_i}{m_i} k^2 .$$

This is the formula for a sound wave in a mixture of two gases at different temperatures. There is no influence of the charges, since electrons and ions slowly oscillate together. Accordingly this is called the *acoustic wave* mode. For shorter wave lengths this sound wave is modified by the electrostatic forces and is called *positive-ion wave*.

Problem. Find from (38) the approximate dispersion laws at short wave lengths:

$$\frac{\omega}{k} = \sqrt{\frac{\Theta_e}{m_e}} \quad \text{and} \quad \frac{\omega}{k} = \sqrt{\frac{\Theta_i}{m_i}}$$

for both branches respectively.

Problem. Derive (39) without calculation by the following argument. For $k = 0$ one has $\omega^2 = \omega_P^2$ and for small k one must have a series expansion of the form $\omega^2 = \omega_P^2 + ak^2 + \ldots$. Find the value of a (apart from a numerical constant) by dimensional analysis.

I′b. Transverse waves: $\boldsymbol{E}_k \perp \boldsymbol{k}$. One now finds from (37) the same dispersion law (29), (30) as for the cold plasma. In fact, in this wave mode $\rho^1 = 0$, so that the pressure terms are immaterial.

II′. Hot plasma with $\boldsymbol{H}^0 \neq 0$ and $\boldsymbol{k} \,//\, \boldsymbol{H}^0$.

Like in the cold plasma the longitudinal waves are not affected by the presence of a magnetic field in the direction of propagation, so that the results of **I′a** remain valid.

For transverse waves propagating in the direction of the field lines the pressure terms are irrelevant, so that the results of **IIb** remain valid.

III′. Hot plasma with $\boldsymbol{H}^0 \neq 0$ and $\boldsymbol{k} \perp \boldsymbol{H}^0$.

Again one finds that longitudinal waves do not exist.

Of the transverse waves mentioned under **Ib** the one with $\boldsymbol{j}_k // \boldsymbol{H}^0$ is unaffected by either the field or the pressure terms. The other polarization direction does not survive when the magnetic field is applied.

Of course there is a large variety of waves when $\boldsymbol{k}$ has an arbitrary angle with $\boldsymbol{H}^0$, but we shall not treat these cases. In general, they cannot be uniquely separated in longitudinal and transverse waves, since $\boldsymbol{E}_k$ may turn out to make any angle with $\boldsymbol{k}$.

Problem. Write the linearized equations for a hot plasma with external field $\boldsymbol{H}^0$ and verify the result mentioned above.

5. Alternative derivation of waves in a cold plasma

So far the electromagnetic field has been treated as a vacuum field, interacting with the charges in the plasma. An alternative approach consists in regarding the plasma as a dielectric medium, and using Maxwell's equations for a medium. The movement of the electrons in the plasma is responsible for the polarization of the medium. We shall briefly outline this approach.

The ions are supposed fixed, and in the equilibrium state the electrons are supposed at rest, so that the plasma is neutral and unpolarized. In the presence of an oscillating electric field

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_\omega(\boldsymbol{r}) e^{-i\omega t}$$

an electron at $\boldsymbol{r}$ is displaced by a small amount $\boldsymbol{s}$, given by

$$m\ddot{\boldsymbol{s}} = -e\boldsymbol{E}_\omega(\boldsymbol{r}) e^{-i\omega t} .$$

Disregarding initial conditions one finds

$$\boldsymbol{s} = \boldsymbol{s}_\omega e^{-i\omega t}, \qquad \boldsymbol{s}_\omega = \frac{e}{m\omega^2} \boldsymbol{E}_\omega(\boldsymbol{r}) .$$

This causes a polarization of the medium per unit volume

$$\boldsymbol{P}_\omega(\boldsymbol{r}) = -\frac{n_e^0 e^2}{m\omega^2} \boldsymbol{E}_\omega(\boldsymbol{r}) .$$

Accordingly one has

$$\boldsymbol{D}_\omega = \boldsymbol{E}_\omega + 4\pi \boldsymbol{P}_\omega = \left(1 - \frac{4\pi n_e^0 e^2}{m\omega^2}\right) \boldsymbol{E}_\omega = \varepsilon(\omega) \boldsymbol{E}_\omega .$$

The electromagnetic field may now be described by Maxwell's equations with a dielectric constant $\varepsilon(\omega)$:

$$\operatorname{curl} \boldsymbol{E}_\omega = \frac{i\omega}{c} \boldsymbol{H}_\omega \qquad \operatorname{div} \boldsymbol{H}_\omega = 0$$

$$\operatorname{curl} \boldsymbol{H}_\omega = -\frac{i\omega}{c} \varepsilon(\omega) \boldsymbol{E}_\omega \qquad \operatorname{div} \varepsilon(\omega) \boldsymbol{E}_\omega = 0 .$$

There are no terms $4\pi\rho$ or $4\pi\boldsymbol{j}/c$ since all charges have been taken into account as a polarization of the medium.*

It is immediately seen that either div $\boldsymbol{E} = 0$ or $\varepsilon(\omega) = 0$. In the former case one obtains the wave equation

$$\nabla^2 \boldsymbol{E}_\omega + \frac{\omega^2}{c^2}\,\varepsilon(\omega)\boldsymbol{E}_\omega = 0,$$

which leads to the dispersion law (29) for transverse plasma waves. In the latter case the equation $\varepsilon(\omega) = 0$ is identical with the dispersion law (27) for longitudinal waves.

Problem. The condition $\varepsilon(\omega) = 0$ is necessary for the existence of a solution with div $\boldsymbol{E} \neq 0$; show that it is also sufficient.

Problem. Derive the equation

$$\text{grad div } \boldsymbol{E}_\omega - \nabla^2 \boldsymbol{E}_\omega = \frac{\omega^2}{c^2}\,\varepsilon(\omega)\boldsymbol{E}_\omega$$

and deduce from it the same conclusions.

Problem. Obtain the same results by taking the moving electrons in the plasma into account as a current density $\boldsymbol{j} = -n_e^0 e\dot{\boldsymbol{s}}$ in the Maxwell equations.

Problem. Show that a necessary condition for the validity of this treatment is $eE/m\omega \ll c$; that is, the velocity of the electron must be small compared to c.

General references

L. Spitzer, *Physics of Fully Ionized Gases*, 2nd ed. (Interscience, New York 1962) chs. II, III.
J. F. Denisse and J. L. Delcroix, *Théorie des Ondes dans les Plasmas* (Dunod, Paris 1961).
W. P. Allis, S. J. Buchsbaum and A. Bers, *Waves in Anisotropic Plasmas* (M.I.T. Press, Cambridge, Mass. 1963).
I. B. Bernstein and S. K. Trehan, Nuclear Fusion **1**, 1 (1960).
S. J. Buchsbaum in: *Propagation and Instabilities in Plasmas*, W. J. Futterman ed. (Stanford University Press, Stanford 1963).
R. Lüst, Fortschr. Physik **7**, 503 (1959).
L. Oster, Rev. Mod. Phys. **32**, 141 (1960).

For the application to wave propagation in the ionosphere see:
K. G. Budden, *Radiowaves in the Ionosphere* (Cambridge University Press, Cambridge 1961).
V. L. Ginzburg, *The Propagation of Electromagnetic Waves in Plasma* (Pergamon Press, London 1964).
J. A. Ratcliffe, *The Magneto-Ionic Theory and its Applications to the Ionosphere* (Cambridge University Press, Cambridge 1959).

* We deliberately avoid the confusing terms "true charge" and "free charge".

CHAPTER X

KINETIC FOUNDATIONS OF THE MANY-COMPONENT THEORY

Both in magnetohydrodynamics and in the two-component theory the equations of motion were obtained by making use of the macroscopic equations for fluids. In this chapter we start from the kinetic equations for an assembly of electrons and various kinds of ions and possibly neutral atoms. From these equations the macroscopic equations of motion for the many-component system are derived at the expense of a number of admittedly rather doubtful assumptions. On specializing to two components (electrons and one kind of ions) it is possible to rewrite the equations in the form used in the previous chapter (IX, 19). Finally the assumptions made will be discussed.

1. Kinetic equations for the many-component plasma

As is customary in the kinetic theory of gases we introduce a *distribution function* $f_\nu(\boldsymbol{r}, \boldsymbol{v}, t)$ for each kind of particles,

$f_\nu(\boldsymbol{r}, \boldsymbol{v}, t)d^3\boldsymbol{r}\, d^3\boldsymbol{v}$ = number of particles of kind ν in volume element $d^3\boldsymbol{r}$ at $\boldsymbol{r}$ and with velocity in $d^3\boldsymbol{v}$ at $\boldsymbol{v}$, at time t.

Each distribution function obeys a "Boltzmann equation"

$$\frac{\partial f_\nu}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f_\nu}{\partial \boldsymbol{r}} + \boldsymbol{a}_\nu(\boldsymbol{r}, \boldsymbol{v}, t)\cdot\frac{\partial f_\nu}{\partial \boldsymbol{v}} = \left[\frac{\partial f_\nu}{\partial t}\right]_{\text{collisions}}, \tag{1}$$

where the acceleration $\boldsymbol{a}_\nu$ is given by

$$\boldsymbol{a}_\nu(\boldsymbol{r}, \boldsymbol{v}, t) = \frac{e_\nu}{m_\nu}\left\{\boldsymbol{E}(\boldsymbol{r}, t) + \frac{1}{c}\,\boldsymbol{v}\wedge\boldsymbol{H}(\boldsymbol{r}, t)\right\} - \nabla\Phi_\nu(\boldsymbol{r}, t). \tag{2}$$

The fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are determined by Maxwell's equations, see (II, 6, 7, 8, 9). The source functions τ and $\boldsymbol{j}$ are given as functionals of the f_ν,

$$\tau(r,t) = \sum_\nu e_\nu \int f_\nu\, d^3 \,, \qquad j(r,t) = \sum_\nu e_\nu \int v f_\nu\, d^3 v\,.$$

The collision term on the right of (1) contains in principle all interactions that have not yet been taken into account by the smeared-out fields E and H. However, we shall make the additional assumption that these interactions have a short range, so that the collisions are practically instantaneous and localized at a point in space. That is, we treat the plasma as a dilute gas. This is certainly not realistic for a liquid metal, and even in a more dilute plasma the local deviations of the microscopic field from the smeared-out field gives rise to an interaction whose range is of the order of the inter-particle distance. However, at high temperatures one may expect that this is unimportant, since the kinetic effects will predominate, so that the plasma is almost an ideal gas (comp. the fundamental condition (VIII, 7) and the resulting equation of state (VIII, 28)). We shall not go into these complications, however, because it is not needed for our present purpose, namely to bring out the formal connection between the fluid treatment in the previous chapters and the kinetic theory.

Suppose then that the right-hand side of (1) only involves instantaneous collisions. If it would be possible to express this term in terms of the functions f_ν, the equations (1) together with the Maxwell equations would constitute a closed set of equations, i.e., they would uniquely determine the evolution of the f_ν and the field for any given initial values. In fact, Boltzmann gave a specific expression for $[\partial f_\nu/\partial t]_c$ in terms of the f_ν.* However, he had to introduce the assumption that there is no statistical correlation between the positions and momenta of two molecules that are about to collide with each other ("Stosszahlansatz"). A more general assumption was made by Bogolyubov, see chapter XV. Rigorously speaking, however, the collision term is not fully determined by the one-particle distribution functions f_ν alone, because these do not contain sufficient information to determine the probability for finding two particles with given positions and velocities. Here we shall not specify the collision term, but we employ (1) to derive some conservation laws for macroscopic quantities.

* His specific expression for the collision term is the essential part of the celebrated "Boltzmann equation". Yet plasma physicists often use this name for the much more trivial equation (1).

2. Conservation equations for macroscopic quantities

We are concerned with the following macroscopic quantities

particle density of component ν	$n_\nu(r,t) = \int f_\nu\, d^3 v$	(3a)
average velocity of component ν	$V_\nu(r,t) = \int v f_\nu\, d^3 v / n_\nu$	(3b)
total mass density	$\rho(r,t) = \sum_\nu n_\nu m_\nu$	(3c)
charge density	$\tau(r,t) = \sum_\nu n_\nu e_\nu$	(3d)
electrical current density	$j(r,t) = \sum_\nu n_\nu e_\nu V_\nu$	(3e)
overall average velocity	$V(r,t) = \dfrac{\sum_\nu n_\nu m_\nu V_\nu}{\sum_\nu n_\nu m_\nu}\,.$	(3f)

We shall now derive equations for the rate of change of these quantities.

I. Conservation of particles. Integrating (1) over v one obtains

$$\frac{\partial n_\nu(r,t)}{\partial t} + \frac{\partial}{\partial r}\cdot \int v f_\nu\, d^3 v - \int f_\nu \left(\frac{\partial}{\partial v}\cdot a_\nu\right) d^3 v = \left[\frac{\partial n_\nu}{\partial t}\right]_c. \tag{4}$$

The third term vanishes by virtue of (2). If no "reactions" occur (i.e. if in collisions the outgoing particles are the same as the incoming particles), the right-hand side also vanishes. Then one finds for each component separately the continuity equation

$$\frac{\partial n_\nu}{\partial t} + \nabla\cdot(n_\nu V_\nu) = 0\,. \tag{5}$$

Problem. Derive

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho V) = 0\,. \tag{6}$$

Show that this equation remains true even if reactions occur, such as ionization and recombination.

II. Conservation of charge. The total charge is not altered by a collision. Therefore one obtains on multiplying (4) by e_ν and adding

$$\frac{\partial \tau}{\partial t} + \nabla \cdot \boldsymbol{j} = 0. \tag{7}$$

Of course, this equation also follows as a consequense of the Maxwell equations. What has been shown here is that it is not inconsistent to use the τ and $\boldsymbol{j}$ defined in (3d) and (3e) as source functions for the electromagnetic field.

III. Momentum balance. Multiply (1) by $m_\nu \boldsymbol{v}$ and integrate over $\boldsymbol{v}$. Define the "stress tensor of the ν-th component" as*

$$\Pi^{\nu}_{ij}(\boldsymbol{r},t) = -\, m_\nu \int (v_i - V_{\nu i})(v_j - V_{\nu j}) f_\nu(\boldsymbol{r}, \boldsymbol{v}, t)\, d^3v$$

$$= n_\nu m_\nu V_{\nu i} V_{\nu j} - m_\nu \int v_i v_j f_\nu d^3 \boldsymbol{v}. \tag{8}$$

The second term in (1) then yields

$$m_\nu \int \boldsymbol{v} \left(\boldsymbol{v} \cdot \frac{\partial f_\nu}{\partial \boldsymbol{r}} \right) d^3\boldsymbol{v} = -\nabla \cdot \Pi^\nu + \boldsymbol{V}_\nu (\nabla \cdot \rho_\nu \boldsymbol{V}_\nu) + m_\nu n_\nu (\boldsymbol{V}_\nu \cdot \nabla) \boldsymbol{V}_\nu.$$

With the aid of (5) and some partial integration one finds

$$m_\nu n_\nu \frac{D_\nu \boldsymbol{V}_\nu}{Dt} = \nabla \cdot \Pi^\nu + n_\nu e_\nu \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{V}_\nu \wedge \boldsymbol{H} \right) - m_\nu n_\nu \nabla \Phi + \boldsymbol{P}_\nu. \tag{9}$$

Here D_ν / Dt is the substantial derivative for component ν,

$$\frac{D_\nu}{Dt} = \frac{\partial}{\partial t} +_\nu \boldsymbol{V}_\nu \cdot \nabla,$$

and $\boldsymbol{P}_\nu$ represents the momentum transferred by collisions from the other components to component ν.

Problem. Show that the left-hand side of (9) is also equal to

$$\frac{\partial}{\partial t} (m_\nu n_\nu \boldsymbol{V}_\nu) + \nabla \cdot m_\nu n_\nu \boldsymbol{V}_\nu ; \boldsymbol{V}_\nu, \tag{10}$$

where $\boldsymbol{V}; \boldsymbol{V}$ represents the tensor with components $V_i V_j$. Hence derive

* Actually this is only the kinetic part of the stress tensor, which discribes the momentum transfer due to the motion of the particles. In dense gases there is also momentum transfer through the interaction, but we may put this part equal to zero owing to our assumption that the plasma may be treated as a dilute gas with short range interaction (after subtraction of the smeared-out field).

$$\rho \frac{D\boldsymbol{V}}{Dt} = \nabla \cdot \Pi + \tau \boldsymbol{E} + \frac{1}{c} \boldsymbol{j} \wedge \boldsymbol{H} - \rho \nabla \Phi, \tag{11}$$

where $D/Dt = \partial/\partial t + (\boldsymbol{V} \cdot \nabla)$ and

$$\Pi = \sum_\nu \Pi^\nu - \sum_\nu n_\nu m_\nu (\boldsymbol{V}_\nu - \boldsymbol{V}); (\boldsymbol{V}_\nu - \boldsymbol{V}) = \rho \boldsymbol{V}; \boldsymbol{V} - \sum_\nu m_\nu \int \boldsymbol{v}; \boldsymbol{v} f_\nu \, d^3\boldsymbol{v}. \tag{12}$$

Problem. Derive an equation expressing local energy conservation.

IV. Equation for the electrical current. In order to find the fourth equation, which determines the rate of change of $\boldsymbol{j}$, we multiply (9) with e_ν/m_ν and sum over ν; after some manipulations similar to those leading to (10) one finds

$$\frac{\partial \boldsymbol{j}}{\partial t} + \nabla \cdot \sum_\nu e_\nu n_\nu \boldsymbol{V}_\nu ; \boldsymbol{V}_\nu = \nabla \cdot \sum_\nu \frac{e_\nu}{m_\nu} \Pi^\nu$$

$$+ \sum_\nu \frac{n_\nu e_\nu^2}{m_\nu} \boldsymbol{E} + \frac{1}{c} \sum_\nu \frac{n_\nu e_\nu^2}{m_\nu} \boldsymbol{V}_\nu \wedge \boldsymbol{H} - \tau \nabla \Phi + \sum_\nu \frac{e_\nu}{m_\nu} \boldsymbol{P}_\nu . \tag{13}$$

The occurrence of the term $\partial \boldsymbol{j}/\partial t$ is due to the fact that the electrons are treated as a separate gas, and constitutes the main difference with MHD.

Problem. Show that the left-hand side of (13) can also be written in the form

$$\frac{D}{Dt}(\boldsymbol{j} - \tau \boldsymbol{V}) + \nabla \cdot \sum_\nu n_\nu e_\nu (\boldsymbol{V}_\nu - \boldsymbol{V}); (\boldsymbol{V}_\nu - \boldsymbol{V})$$

$$+ (\boldsymbol{j} - \tau \boldsymbol{V})(\nabla \cdot \boldsymbol{V}) + \tau \frac{D\boldsymbol{V}}{Dt} + \{(\boldsymbol{j} - \tau \boldsymbol{V}) \cdot \nabla\} \boldsymbol{V}. \tag{14}$$

Problem. Where has the assumption been used that the collisions only involve short range interactions?

3. THE EQUATIONS OF MANY-COMPONENT THEORY

The equations (5) and (9) for n_ν and $\boldsymbol{V}_\nu$ are rigorous consequences of (1), but they do not constitute a closed set of equations from which n_ν, $\boldsymbol{V}_\nu$ can be computed as functions of time. There are two reasons. First, the effect of collisions is still not specified; this shows up in the occurrence of the unknown quantities $\boldsymbol{P}_\nu$ and Π^ν in (9) and (13). In fact, the macroscopic quantities n_ν, $\boldsymbol{V}_\nu$ provide a much less detailed description of the state of the plasma than the distribution functions f_ν. Hence one cannot expect them to fully determine the future evolution of the system. Likewise the equations (6), (7), (11), (13) do not constitute a closed set of equations for ρ, τ, V, j. They provide an even less detailed description.

Yet the classical theory of ordinary gases and liquids demonstrates the remarkable fact that, *to a very good approximation, the behavior can be described in terms of macroscopic quantities alone.* This fact was also the foundation of magnetohydrodynamics. Similarly we shall now introduce a number of approximations and assumptions such as to obtain a closed set of equations for the macroscopic quantities n_ν, V_ν.

The equation (6) contains no unknown quantities and may therefore be taken over without alterations. In (9) the unknown stress tensors Π^ν are assumed to be of the form

$$\Pi^\nu_{ij} = -p_\nu \delta_{ij}, \tag{15}$$

where p_ν is related to n_ν by the ideal gas law. This corresponds to the assumption (II, 4), with the additional restriction to an ideal gas, and neglect of viscosity. Finally we write

$$\boldsymbol{P}_\nu = \sum_\mu \boldsymbol{P}_{\nu,\mu},$$

where $\boldsymbol{P}_{\nu,\mu}$ is the momentum transfer from the μ-th component to the ν-th component, and assume

$$\boldsymbol{P}_{\nu,\mu} = \zeta_{\nu\mu} n_\nu n_\mu (V_\mu - V_\nu). \tag{16}$$

The coefficients $\zeta_{\nu\mu}$ are positive and must obey $\zeta_{\nu\mu} = \zeta_{\mu\nu}$, but are otherwise unknown. The justification for (16) is that it is the simplest expression in n_ν, V_ν which obeys the obvious requirements that $\boldsymbol{P}_{\nu,\mu}$ must be a vector in the direction of the relative velocity, and that it must vanish at zero relative velocity and when either density is zero.

By these assumptions we have obtained the basic equations of many-component theory

$$\frac{\partial n_\nu}{\partial t} + \nabla \cdot n_\nu V_\nu = 0 \tag{17a}$$

$$m_\nu n_\nu \frac{DV_\nu}{Dt} = -\nabla p_\nu + n_\nu e_\nu \left(E + \frac{1}{c} V_\nu \wedge H \right) - m_\nu n_\nu \nabla \Phi + \sum_\mu \zeta_{\nu\mu} n_\nu n_\mu (V_\mu - V_\nu) \tag{17b}$$

$$-\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} = \nabla \wedge \boldsymbol{E} \qquad \nabla \cdot \boldsymbol{H} = 0 \tag{17c}$$

$$-\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c} \sum_\nu e_\nu n_\nu V_\nu - \nabla \wedge \boldsymbol{H} \qquad \nabla \cdot \boldsymbol{E} = 4\pi \sum_\nu e_\nu n_\nu \tag{17d}$$

$$p_\nu = f_\nu(n_\nu). \tag{17e}$$

Owing to our simplifying assumption that collisions are instantaneous this derivation only applies to dilute gases, so that (17e) must have the ideal gas form; that is, the isothermal equation $p_\nu = kT_\nu n_\nu$ for static solutions, and the adiabatic equation $p_\nu^1 = \gamma_\nu k T_\nu n_\nu^1$ for rapid oscillations.

Problem. Derive the equation

$$\rho \frac{DV}{Dt} = -\nabla p + \tau E + \frac{1}{c} j \wedge H - \rho \nabla \Phi,$$

where p is the sum of the partial pressures, and where quadratic terms in the relative velocities $V_\nu - V$ are neglected.

4. Discussion of the two-component theory

Suppose the plasma consists of only two components, electrons (ν = e) and a single kind of ions (ν = i), so that

$$e_e = -e, \qquad e_i = Ze.$$

This case is special, inasmuch as it permits to express the four quantities n_e, n_i, V_e, V_i in the four quantities ρ, τ, V, j. Hence it is possible to rewrite the equations (17) as equations for ρ, τ, V, j, which are the same quantities that occurred in the magnetohydrodynamic description. The exact equations, however, are very unwieldy, and are therefore simplified by a number of assumptions. The resulting simplified set of equations are the same equations that were derived in the previous chapter from the model of two interpenetrating fluids; see (IX, 19). We shall now discuss the various assumptions made.

(*i*) *There are only two components.* This is realized only when there is just one kind of nuclei present, which are completely stripped of their electrons. That is by no means true for gas discharges, the ionosphere or the corona. However, it is reasonable to expect that those features that are common to the various kinds of plasma, can be found by studying a particularly simple one. Of course, in the general case the values of m_i and Z have to be replaced with some average ion mass and some average degree of ionization.

(*ii*) *Quadratic terms in the velocity V_ν are omitted.* This greatly simplifies the equations and does not introduce an additional error when the resulting equations are only used in linear approximation anyway.

(*iii*) *Quasi-neutrality.* In magnetohydrodynamics this has been justified by the fact that the plasma is a good conductor, comp. (II, 17). In the present case, however, it is simply part of the linear approximation.

(*iv*) $m_e \ll m_i$. This only introduces a numerical error, which at present is well within the experimental uncertainty.

(*v*) *The momentum transfer is given by (16)*, which gives rise to the term j/σ in (IX, 19). Actually one usually assumes $\sigma = \infty$, so that we only have to verify that the momentum transfer due to collisions between ions and electrons is negligible. The momentum transfer per collision is of the order $m_e | V_i - V_e |$; the number of collisions per sec per cm^3 is $\nu_c n_e$, where ν_c is the "collision frequency", i.e. the number of collisions per sec of a given electron.* Hence

$$\boldsymbol{P}_{e,i} \sim \nu_c n_e m_e \,|\, \boldsymbol{V}_i - \boldsymbol{V}_e \,|,$$

so that according to (16) and (IX, 18)

$$\zeta_{ei} \sim \frac{\nu_c m_e}{n_i}, \qquad \sigma \sim \frac{n_e e^2}{m_e \nu_c}.$$

The ratio of the two terms on the left of (IX, 19c) is

$$\frac{m_e}{n_e e^2} \frac{\partial j}{\partial t} \bigg/ \frac{j}{\sigma} \sim \frac{\omega}{\nu_c}, \tag{18}$$

where ω is a representative frequency of the phenomenon one is studying. Thus we have found that for times short compared to the mean time between two successive collisions of an electron the plasma may be treated as an ideal conductor – which is immediately clear.

In order to obtain a quantitative estimate of ν_c consider an electron moving with velocity $v_T = \sqrt{(kT_e/m_e)}$ through a gas of ions at rest. It will undergo an appreciable deviation from its path only if it approaches an ion within a sufficiently short distance a. The order of magnitude of a can be estimated by putting $Ze^2/a = \frac{1}{2} m_e v_T^2$. The number of such close collisions per second is $\pi a^2 v_T n_i$, so that

* This is a rather qualitative concept, since actually the electron interacts continually with all surrounding particles, even after subtraction of the smeared-out electromagnetic field $\boldsymbol{E}, \boldsymbol{H}$. A more quantitative meaning of ν_c will be given in the next paragraph.

$$\nu_c \sim \pi a^2 v_T n_i \sim \frac{e^4}{m_e^2} \left(\frac{m_e}{kT_e}\right)^{\frac{3}{2}} n_e \sim \frac{\omega_P}{n_e \kappa^{-3}}. \tag{19}$$

The plasma frequency ω_P is representative of the frequencies occurring in plasma waves. The number $n_e \kappa^{-3}$ equals the number of electrons in a Debye sphere and must be large according to the condition (VIII, 7). Hence, for plasma waves in a hot, dilute plasma (18) is actually a small number, as required.

The most crucial assumption, however, is

(*vi*) *Each separate component obeys the hydrodynamic equations*; more specifically, for each component the stress tensor is given by (15), where p_ν is connected with the density by an equation of state (17e). In ordinary gas theory one justifies this assumption by arguing that collisions are so frequent that to first approximation a local Maxwell distribution is maintained

$$f_\nu(\boldsymbol{r}, \boldsymbol{v}, t) = n_\nu(\boldsymbol{r}, t) \left(\frac{m_\nu}{2\pi kT_\nu(\boldsymbol{r}, t)}\right)^{\frac{3}{2}} \exp\left[-\frac{m_\nu\{\boldsymbol{v} - \boldsymbol{V}_\nu(\boldsymbol{r}, t)\}^2}{2kT_\nu(\boldsymbol{r}, t)}\right]. \tag{20}$$

This formula determines the entire velocity distribution of the ν-th component in terms of the macroscopic quantities n_ν, $\boldsymbol{V}_\nu$, T_ν. It is therefore also possible to compute Π^ν from it, and the result is (15), together with the ideal gas law $p_\nu = n_\nu kT_\nu$.

However, it is hard to see how the assumption of frequent collisions between particles of the same component can be reconciled with the assumption (*v*), that collisions between different components are negligible. In both cases the interaction is a Coulomb force, and in fact the same estimate (19) applies to collisions between electrons. It has been argued that energy transfer between electrons and ions is a slow process owing to the large mass difference, but that does not apply to momentum transfer.* Hence in those conditions in which (20) is plausible one should also expect that all $\boldsymbol{V}_\nu$ are equal, so that there is no electrical current. On the other hand, if collisions are so rare that the different components can freely move through each other, there is no reason to believe (20) or (15).**

Now suppose there is a strong external magnetic field $\boldsymbol{H}^0$ in the direction of the z-axis. In this case the cyclotron motion of the plasma particles will cause the velocity distribution to have cylindrical symmetry about the

* In the static case there is no reason to doubt the existence of a local Maxwell distribution, so that the calculation of the Debye sphere remains valid.

** Mathematically speaking, (20) is a sufficient but not a necessary condition for (15); however, it is the only physically reasonable justification for believing (15).

z-axis, as is shown explicitly in ch. XIV. This is sometimes taken into account by assuming a diagonal, but anisotropic stress tensor*

$$\Pi^{\nu} = \begin{pmatrix} -p_{\nu}^{\perp} & 0 & 0 \\ 0 & -p_{\nu}^{\perp} & 0 \\ 0 & 0 & -p_{\nu}^{//} \end{pmatrix}$$

However, this still does not take into account that the plasma waves one is interested in will alter the velocity distribution, such that (20) is not valid and that non-diagonal terms appear in the stress tensor.

Clearly the only way out of these difficulties is to solve the Boltzmann equations for the velocity distributions f_{ν} themselves. There is no good reason to hope that only a small number of macroscopic quantities suffices to describe the plasma; that is only true in ordinary gas theory, as in that case the collision term of the Boltzmann equation is predominant.** Thus one is led to the Vlasov approach in plasma theory, which is treated in the following chapters.

Problem. Verify that the estimate (19) also applies to collisions between electrons.

Problem. Compute the stress tensor from (20).

Remark. In Chapter XI we shall find that the Vlasov theory leads to the same dispersion formula (IX, 39) for longitudinal plasma waves as the two-component theory, but with the number 3 instead of the constant $\gamma_e = \frac{5}{3}$. In the framework of two-component theory this 3 is sometimes explained as follows. "For these rapid plasma waves the heat produced by the adiabatic compression has no time to become evenly distributed among the three velocity components, but remains confined to the longitudinal component. Hence the electron gas should be treated as a one-dimensional ideal gas, and for such a gas $\gamma \equiv c_p/c_V = 3$." However, in order to use hydrodynamics and thermodynamics at all one must assume that collisions are frequent; and each collision transfers energy from one velocity component to the others (unless it happens to be a head-on collision). The actual reason for the difference between both results is that two-component theory and Vlasov theory are based on entirely different models for the plasma.

Problem. Prove that for an m-dimensional gas $c_p/c_V = (m+2)/m$.

* G. F. Chew, M. L. Goldberger and F. E. Low, Proc. Roy. Soc. (London) **A236**, 112 (1956).

** S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge 1939).

General references

D. J. Rose and M. Clark, *Plasmas and Controlled Fusion* (Wiley, New York 1961) ch. VI.

J. L. Delcroix, *Introduction à la Théorie des Gaz Ionisés* (Dunod, Paris 1959).

R. Jancel and T. Kahan, *Electrodynamique des Plasmas* (Dunod, Paris 1963).

E. H. Holt and R. E. Haskell, *Foundations of Plasma Dynamics* (MacMillan, New York 1965).

I. P. Shkarofsky, T. W. Johnston and M. P. Bachynsky, *The Particle Kinetics of Plasmas* (Addison Wesley, Reading 1966).

CHAPTER XI

THE VLASOV THEORY

If a plasma is sufficiently dilute and hot, collisions are relatively unimportant (compare (x, 19)). When studying rapid phenomena it is therefore a good approximation to neglect them altogether, that is, to set the right-hand side of (x, 1) equal to zero. The model obtained in doing so is called the collisionless or Vlasov plasma. It is diametrically opposed to the MHD model, which presupposes that collisions have a dominant effect. As discussed in chapter x, the two-component model is a hybrid combination of both.

1. The Vlasov equation

Consider a gas of electrons (mass m, charge $-e$, average number density n_0) with a homogeneous background of charge density $n_0 e$. When collisions are neglected the distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ obeys the equation

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{r}} - \frac{e}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\wedge\boldsymbol{H}\right)\cdot\frac{\partial f}{\partial \boldsymbol{v}} = 0. \tag{1}$$

This is called the "collisionless Boltzmann equation" or "Vlasov equation".* It has to be supplemented by Maxwell's equations

$$\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} = -\operatorname{curl}\boldsymbol{E} \quad (2a) \qquad \operatorname{div}\boldsymbol{E} = -4\pi e(n-n_0) \quad (2b)$$

$$\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \operatorname{curl}\boldsymbol{H} - \frac{4\pi}{c}\boldsymbol{j} \quad (2c) \qquad \operatorname{div}\boldsymbol{H} = 0 \quad (2d)$$

where the number density n and the current density $\boldsymbol{j}$ are defined in terms of f by

$$n(\boldsymbol{r}, t) = \int f(\boldsymbol{r}, \boldsymbol{v}, t)\, d^3v, \qquad \boldsymbol{j}(\boldsymbol{r}, t) = -e\int \boldsymbol{v} f(\boldsymbol{r}, \boldsymbol{v}, t)\, d^3v. \tag{3}$$

There may be other charges and currents, for example the current in the

* A. Vlasov, J. Phys. USSR 9, 25 (1945).

coil that surrounds a gas discharge. They have to be added to the right-hand member of (2b) and (2c). An easier way to take these into account, however, is by decomposing the total field into an "internal" and an "external" field,

$$\boldsymbol{E} = \boldsymbol{E}^{\text{in}} + \boldsymbol{E}^{\text{ex}} \qquad \boldsymbol{H} = \boldsymbol{H}^{\text{in}} + \boldsymbol{H}^{\text{ex}}$$

where $\boldsymbol{E}^{\text{in}}$, $\boldsymbol{H}^{\text{in}}$ obey the equations (2) and (3), while $\boldsymbol{E}^{\text{ex}}$, $\boldsymbol{H}^{\text{ex}}$ obey the Maxwell equations with the external charges and currents as sources.

These external sources are not necessarily outside the plasma; for example a beam of charged particles shot through the plasma may be regarded as an external charge and current. Thus any source that is not included in (2) is called external. Usually the external field is treated as a given quantity, that is, the reaction of the plasma on the external source is neglected. Examples are the charge e_0 in the calculation of the Debye shielding (ch. VIII, sec. 1) and the field $\boldsymbol{H}^0$ in the derivation of MHD waves (ch. VI).

Problem. Write the analogs of equations (1), (2), (3) for a mixture of many components.

2. Properties of the Vlasov equation

We shall now discuss some of the properties of the set of equations (1), (2), (3) without external sources.

a. Each electron can be represented by a dot in six-dimensional $(\boldsymbol{r}, \boldsymbol{v})$-space (which is the phase space for one electron); $f(\boldsymbol{r}, \boldsymbol{v}, t)$ is the density of these dots. If $\boldsymbol{E}$ and $\boldsymbol{H}$ would be prescribed functions of $\boldsymbol{r}$, t, each dot would move along a prescribed trajectory, independent of the others. The resulting change of density is described by the "Liouville equation" (1). The solution of this equation would then be equivalent to solving the equations of motion of a single electron in the ***EH***-field.

However, the field is not prescribed, but depends on all particles through (2) and (3). In this respect (1) differs from the familiar Liouville equation in phase space of kinetic theory. In particular, it is not linear: the last term of (1) is quadratic in f. This term contains the interaction between the electrons through the smeared-out field.

Yet one conclusion of kinetic theory remains true, namely *Liouville's theorem*: the density f remains constant for an observer who moves along with the dots in phase space. From this follows the property: if $f(\boldsymbol{r}, \boldsymbol{v}, 0) \geqslant 0$ for all $\boldsymbol{r}$, $\boldsymbol{v}$, then $f(\boldsymbol{r}, \boldsymbol{v}, t) \geqslant 0$ for all $\boldsymbol{r}$, $\boldsymbol{v}$, t. This property is clearly necessary when we expect our equations to describe the evolution of the particle density.

Problem. Deduce from (II, 1) that ρ cannot become negative when it is non-negative at $t = 0$.

b. *Conservation of energy*. The total energy of the plasma in a fixed volume Ω is

$$W = \tfrac{1}{2}m \int_\Omega \int v^2 f \, d^3v \, d^3r + \frac{1}{8\pi} \int_\Omega (E^2 + H^2) \, d^3r . \tag{4}$$

Using (1) and (2) one obtains

$$\frac{dW}{dt} = - \oint_S dS \cdot \int \tfrac{1}{2} m v^2 v f d^3 v - \frac{c}{4\pi} \oint_S (E \wedge H) \cdot dS .$$

S is the surface enclosing Ω. These two boundary terms are clearly the kinetic energy flow and the electromagnetic energy radiation through S.

Problem. Derive similar equations for the conservation of mass, momentum, and angular momentum.

c. Equations (1), (2a) and (2c) are "equations of motion": they determine the quantities f, $\boldsymbol{E}$, $\boldsymbol{H}$ at all times as soon as they are given at $t = 0$. Equations (2b) and (2d) are "condition equations": they restrict the possible values at every instant. (3) will be regarded as defining the abbreviations n and $\boldsymbol{j}$ used in (2).

The instantaneous state of our Vlasov plasma is determined by three functions $f(\boldsymbol{r}, \boldsymbol{v})$, $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{H}(\boldsymbol{r})$ obeying the condition equations (2b) and (2d). Such a triplet of functions we shall call a "state" and denote by ψ. Equations (1), (2a), (2c) may be written together as

$$\dot{\psi} = O[\psi] , \tag{5}$$

where O is a nonlinear operator. On solving this equation with given state $\psi(0)$ at $t = 0$ one obtains a sequence of states, which is called a *trajectory* in state space. The general solution of (5) may be written symbolically

$$\psi(t) = U_t[\psi(0)] . \tag{6}$$

The evolution operator U_t has the obvious properties

$$U_0 = 1 , \qquad U_{t_1} U_{t_2} = U_{t_1 + t_2} . \tag{7}$$

Problem. Verify that the condition equations (2b) and (2d) are compatible with the equations of motion.

Problem. Prove that the states ψ form a linear space.

d. *Invariance for time reversal.* In the space of all states ψ we define a "time reversal operator" T as follows:

$$\text{if } \psi = \{f(\boldsymbol{r}, \boldsymbol{v}), \boldsymbol{E}(\boldsymbol{r}), \boldsymbol{H}(\boldsymbol{r})\}, \text{ then } T\psi = \{f(\boldsymbol{r}, -\boldsymbol{v}), \boldsymbol{E}(\boldsymbol{r}), -\boldsymbol{H}(\boldsymbol{r})\} .$$

Clearly T is linear and $T^2 = 1$. One readily verifies

$$O[T\psi] = -TO[\psi] .$$

It follows that if $\psi(t)$ is a solution of (4), so is $T\psi(-t)$, or

$$U_t[T\psi(0)] = T\psi(-t) = TU_{-t}[\psi(0)] .$$

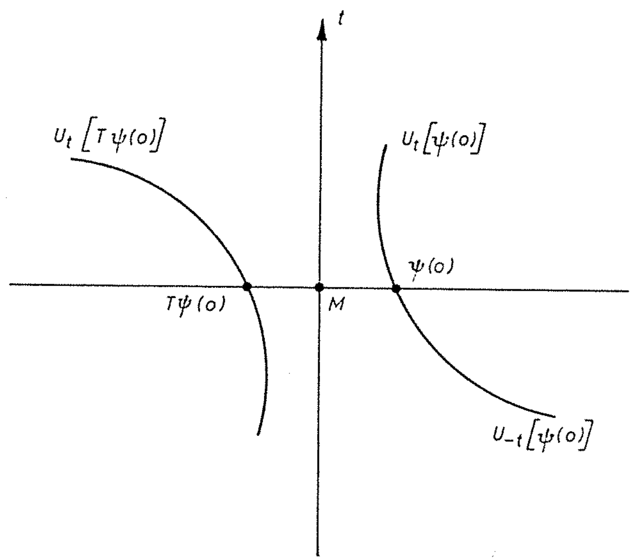

Fig. 25. Schematic representation of time reversal invariance. The operation T is here represented by a rotation of 180° about the point M. Clearly $U_t[T\psi(0)] = TU_{-t}[\psi(0)]$.

Thus we have found an automorphism of state space, which maps the successive states of a trajectory onto the states of another trajectory, in reversed time order. The existence of such an automorphism is expressed by saying that the equations (1), (2) are "invariant for time reversal" or "reversible".

A consequence of this invariance is that *it cannot be true that all solutions* *tend to a certain equilibrium state* (in contrast with the ordinary Boltzmann equation with collisions). For, if $\psi(t)$ would be a solution that approaches equilibrium, $T\psi(-t)$ is another solution, which moves away from it.

In order to make this argument precise we first formulate what is meant by "approach to equilibrium":

(i) for given values of the macroscopic constants of the motion (e.g. energy) there is one and only one time-independent solution ψ^{eq} of (5), that is $O[\psi^{eq}] = 0$;

(ii) there is a functional $H[\psi]$ such that $H[\psi] > H[\psi^{eq}]$ for all $\psi \neq \psi^{eq}$;

(iii) $H[\psi(t)]$ decreases monotonically towards $H[\psi^{eq}]$ for all trajectories $\psi(t)$.

We claim that such a functional does not exist. This can easily be proved for equations with time-invariance, provided one adds the requirement

(iv) H is invariant for time reversal, i.e.,

$$H[T\psi] = H[\psi].$$

It then follows immediately that if $H[\psi(t)]$ is monotone decreasing, $H[T\psi(-t)]$ is monotone increasing, so that (iii) cannot be true.

Actually we shall see that already (i) is not true, because there are infinitely many macroscopically indistinguishable solutions of $O[\psi] = 0$.

Problem. Prove $TU_tT = U_t^{-1}$.

Problem. Show that Boltzmann's H-theorem obeys (i), (ii), (iii), (iv).

Problem. Formulate an "approach to equilibrium" for the case that there are many solutions of $O[\psi] = 0$.

Problem. Prove that $T\psi^{eq} = \psi^{eq}$ when all macroscopic constants of the motion needed to make ψ^{eq} unique are invariant for time reversal. Generalize this theorem for the case that some of these constants (e.g. momentum or angular momentum) change sign when time is reversed.

Problem. Show that the Vlasov equation is invariant for space reflection: $\boldsymbol{r} \to -\boldsymbol{r}$.

3. Vlasov equation for longitudinal waves

One easily verifies that the following time-independent state is a solution of (1), (2), (3).

$$f = n_0 f^0(v), \quad \boldsymbol{E} = 0, \quad \boldsymbol{H} = 0,$$

where $f^0(v)$ may be any function of $v = |\boldsymbol{v}|$ obeying

$$\int f^0(v)\, d^3\boldsymbol{v} = 1.$$

When a plasma (without external fields) is left alone for a very long time the effect of collisions (not included in the Vlasov equation) will be that this state is reached, with f^0 equal to the Maxwell distribution.

Consider a small deviation from this unperturbed state

$$f(\boldsymbol{r}, \boldsymbol{v}, t) = n_0 f^0(v) + f^1(\boldsymbol{r}, \boldsymbol{v}, t), \quad \boldsymbol{E} = \boldsymbol{E}^1(\boldsymbol{r}, t), \quad \boldsymbol{H} = \boldsymbol{H}^1(\boldsymbol{r}, t).$$

Omitting all higher orders of f^1 one obtains the *linearized Vlasov equation*

$$\frac{\partial f^1}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f^1}{\partial \boldsymbol{r}} - \frac{n_0 e}{m} \boldsymbol{E}^1 \cdot \frac{\partial f^0}{\partial \boldsymbol{v}} = 0. \tag{8}$$

Note that the term with $\boldsymbol{H}$ disappears owing to the isotropy of f^0. That does not mean that the magnetic field drops out entirely, because it still occurs in Maxwell's equation (2a). In this and the next chapter, however, we study longitudinal waves, for which $\boldsymbol{H}^1 = 0$, as can readily be shown.

Problem. Show this by proving that, if $\boldsymbol{E}^1$ and $\boldsymbol{j}$ oscillate in the z-direction and depend on z alone, one has $4\pi\boldsymbol{j} + \partial\boldsymbol{E}^1/\partial t = 0$, and hence curl $\boldsymbol{H}^1 = 0$. From this relation together with div $\boldsymbol{H}^1 = 0$ follows $\boldsymbol{H}^1 = 0$, since it has been assumed that there are no external fields.

Thus $\boldsymbol{E}^1$ is determined by

$$\text{curl}\, \boldsymbol{E}^1 = 0, \qquad \text{div}\, \boldsymbol{E}^1 = -4\pi e n^1, \tag{9}$$

where

$$n^1(\boldsymbol{r}, t) = \int f^1(\boldsymbol{r}, \boldsymbol{v}, t)\, d^3\boldsymbol{v}. \tag{10}$$

The solution of (9) is the Coulomb field

$$\boldsymbol{E}^1(\boldsymbol{r}, t) = -e \int \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} n^1(\boldsymbol{r}', t)\, d^3\boldsymbol{r}'. \tag{11}$$

If this is substituted in (8), a linear integro-differential equation for f^1 is obtained. This equation was solved by Landau* by means of Laplace transforms. We shall use the method of stationary solutions.** Accordingly we make the "Ansatz"

* L. D. Landau, J. Phys. USSR **10**, 25 (1946).
** N. G. van Kampen, Physica **21**, 949 (1955).

$$f^1(\boldsymbol{r}, \boldsymbol{v}, t) = g(\boldsymbol{v})\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)},$$

where $\boldsymbol{k}$, ω, and $g(\boldsymbol{v})$ are to be determined. One finds

$$n^1(\boldsymbol{r}, t) = e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \int g(\boldsymbol{v})\, d^3\boldsymbol{v}$$

$$\boldsymbol{E}^1(\boldsymbol{r}, t) = \frac{4\pi i e}{k^2}\, \boldsymbol{k}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \int g(\boldsymbol{v})\, d^3\boldsymbol{v}$$

so that (8) becomes

$$(\omega - \boldsymbol{k}\cdot\boldsymbol{v})g(\boldsymbol{v}) = -\omega_P^2 \frac{\boldsymbol{k}}{k^2}\cdot\frac{\partial f^0}{\partial \boldsymbol{v}} \int g(\boldsymbol{v}')\, d^3\boldsymbol{v}'. \tag{12}$$

Problem. Verify this result and prove the mathematical identity used, viz.,

$$\int \frac{\boldsymbol{r}}{r^3} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3\boldsymbol{r} = \frac{4\pi i}{k^2}\boldsymbol{k},$$

which is the Fourier transformation of the Coulomb force.

Remark. The linearized Vlasov equation (8), with the Coulomb expression (11) for the field, has a different structure than the nonlinear Vlasov equation (1) with the full Maxwell equations (2), inasmuch as the state of the plasma at a given time t is now fully determined by the distribution function $f^1(\boldsymbol{r}, \boldsymbol{v}, t)$ alone. To put it differently, we now work in that subspace of all states ψ in which $\boldsymbol{H}$ and the transverse part of $\boldsymbol{E}$ are zero.

Problem. Prove that the equation (8) with (11) is invariant for time reversal and for space reflection.

Problem. In one dimension the linearized Vlasov equation (8) may be written

$$\frac{\partial f^1(x, v, t)}{\partial t} = -v\frac{\partial f^1(x, v, t)}{\partial x} + \frac{n_0 e}{m} E(x, t)\frac{d f^0(v)}{dv}. \tag{13}$$

From $\partial E/\partial x = -4\pi e n^1(x, t)$ follows

$$E(x, t) = -4\pi e \int \varepsilon(x - x')\, dx' \int f^1(x', v')\, dv', \tag{14}$$

where $\varepsilon(x)$ is defined as $\varepsilon(x) = \frac{1}{2}\operatorname{sgn} x = \frac{1}{2}x/|x|$, so that $\varepsilon'(x) = \delta(x)$, $\varepsilon(-x) = -\varepsilon(x)$. In the space of functions $f(x, v)$ we define the scalar product

$$(g, f) = \iint \frac{g^*(x, v) f(x, v)}{f^0(v)}\, dx\, dv.$$

When (13) is written as $\partial f^1/\partial t = -iLf^1$ the linear operator L is hermitian in terms of this scalar product. Prove this, and conclude from it that for any solution f^1 the quantities $(f^1, L^n f^1)$ are conserved for $n = 0, 1, 2, \ldots$.*

Problem. Write the analogs of equations (8) and (9) for a mixture of many components. Show that the function $\Sigma_\nu e_\nu f_\nu^1(\boldsymbol{r}, \boldsymbol{v}, t)$ obeys the equations (8) and (9) (with different f^0). Verify that once this function has been determined the separate f_ν^1 can readily be found.

We shall now solve (12) in a slightly naive fashion, which is not quite correct. However, the approximate dispersion law we shall find, (18), will be justified by the correct treatment in the next chapter.

Dividing (12) by $\omega - \boldsymbol{k}\cdot\boldsymbol{v}$ one has

$$g(\boldsymbol{v}) = -\omega_P^2 \frac{1}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} \frac{\boldsymbol{k}}{k^2}\cdot\frac{\partial f^0}{\partial \boldsymbol{v}} \int g(\boldsymbol{v}')\, d^3\boldsymbol{v}'. \tag{15}$$

This almost completely determines g as a function of $\boldsymbol{v}$, inasmuch as the only unknown quantities on the right are two numbers: ω and the integral.** The latter, however, is nothing but a constant normalization factor, which can be chosen freely since the equation (12) is homogeneous in $g(\boldsymbol{v})$. However, its value must be consistent with the value one finds by integrating (15) with respect to $\boldsymbol{v}$,

$$\int g(\boldsymbol{v})\, d^3\boldsymbol{v} = -\omega_P^2 \int \frac{1}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} \frac{\boldsymbol{k}}{k^2}\cdot\frac{\partial f^0}{\partial \boldsymbol{v}}\, d^3\boldsymbol{v} \int g(\boldsymbol{v}')\, d^3\boldsymbol{v}'.$$

This consistency condition yields an equation for ω,

$$1 = -\omega_P^2 \frac{\boldsymbol{k}}{k^2}\cdot \int \frac{\partial f^0}{\partial \boldsymbol{v}} \frac{d^3\boldsymbol{v}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}, \tag{16}$$

which is the dispersion law for ω as a function of $\boldsymbol{k}$. This result is due to Vlasov.***

By partial integration it may be written in the more elegant form

* K. M. Case, Phys. Fluids **8**, 96 (1965).
** The reason why this integral equation can be solved so easily is that it is of a type called "degenerate"; see e.g. R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York 1953) Vol. I, p. 114.
*** *Loc. cit.* The same result was derived in an entirely different way by Bohm and Gross.

$$\int \frac{f^0(v)}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^2}\, d^3v = \frac{1}{\omega_P^2}. \tag{17}$$

From this it is easy to deduce an approximate dispersion law for small k by expanding the denominator

$$\omega^2 = \omega_P^2 + k^2\langle v^2\rangle + O(k^4). \tag{18}$$

where $\langle v^2\rangle$ is the mean square velocity of the unperturbed distribution f^0.

Problem. Derive (18) from (17) and show that the next term is

$$(k^4/\omega_P^2)\{\langle v^4\rangle - \langle v^2\rangle^2\}.$$

Problem. Show that, if f^0 is a Maxwell distribution, (18) takes the form (IX, 39) with $\gamma = 3$.

Problem. Verify that (17) remains valid even if f^0 is not an isotropic function of $\boldsymbol{v}$, provided that one neglects the magnetic field, if any.

The flaw in this treatment is apparent from the fact that the denominator in (17) vanishes at $\boldsymbol{k}\cdot\boldsymbol{v} = \omega$, so that the integral is meaningless. Vlasov conjectured that the Cauchy principal value should be taken, and we shall find in the next chapter that this is correct. However, we shall also see that the waves are slightly damped, as was first found by Landau. This Landau damping is not due to collisions and is not found in two component theory or in the above naive treatment of the Vlasov equation.

General references

D. Bohm and E. P. Gross, *Phys. Rev.* **75**, 1851 (1949).

The quantummechanical theory was first given in:
D. Bohm and D. Pines, *Phys. Rev.* **92**, 609 (1953).

CHAPTER XII

CORRECT TREATMENT OF LONGITUDINAL PLASMA WAVES

1. Mathematical preliminaries

To solve (XI, 12) correctly a few mathematical results and notations are needed, which will be summarized in this section.

In order to assign a meaning to integrals with a zero in the denominator one defines* the "principal value", denoted by P, as

$$\mathrm{P}\int_{-a}^{b} \frac{F(x)}{x}\,dx = \lim_{\varepsilon\to 0}\left[\int_{-a}^{-\varepsilon} \frac{F(x)}{x}\,dx + \int_{\varepsilon}^{b} \frac{F(x)}{x}\,dx\right].$$

a, b, ε are all supposed positive. Thus one cuts out an interval $(-\varepsilon, \varepsilon)$ *symmetric around the singular point*, which subsequently tends to zero. Suppose one would cut out an asymmetric interval $(-\varepsilon, \varepsilon')$ and let ε and ε' tend to zero with a fixed ratio; it is easily seen that in this way one obtains the principal value plus the additional term $F(0)\log(\varepsilon/\varepsilon')$. All possible values obtained in this way are comprised in the formula

$$\int_{-a}^{b} \frac{F(x)}{x}\,dx = \mathrm{P}\int_{-a}^{b} \frac{F(x)}{x}\,dx + \rho F(0), \tag{1}$$

where ρ is an arbitrary real number. This may be written symbolically

$$\frac{1}{x} = \mathrm{P}\frac{1}{x} + \rho\delta(x). \tag{2}$$

Problem. Prove that the principal value exists if $F(x)$ is continuous and at $x = 0$ satisfies a Hölder condition:

$$|F(x) - F(0)| \leqslant M|x|^\alpha \quad \text{for some positive } M, \alpha.$$

* See e.g. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge 1927) p. 75.

Problem. Prove

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{e^{ipx'}}{x-x'}\,dx' = -i\pi e^{ipx} \quad \text{for} \quad p > 0$$
$$= \quad i\pi e^{ipx} \quad \text{for} \quad p < 0. \tag{3}$$

Problem. The integral on the left in (1) could also be taken along a path in the complex plane slightly below or slightly above the pole, provided that $F(x)$ is analytic in some neighborhood. Show that this amounts to taking ρ equal to πi or $-\pi i$ respectively, so that the principal value is half their sum.

Problem. Using the result of the previous problem define the principal value for higher order poles and prove

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{F(x)}{x^2}\,dx = \mathrm{P}\int_{-\infty}^{\infty} \frac{F'(x)}{x}\,dx. \tag{4}$$

Suppose $F(x)$ can be written as a Fourier transform,

$$F(x) = \int_{-\infty}^{\infty} \Phi(p)\, e^{ipx}\, dp. \qquad (-\infty < x < \infty)$$

Then $F(x)$ may be decomposed in a "*positive-frequency part*" $F_+(x)$ and a "*negative-frequency part*" $F_-(x)$ defined by

$$F_+(x) = \int_0^{\infty} \Phi(p)\, e^{ipx}\, dp, \qquad F_-(x) = \int_{-\infty}^{0} \Phi(p)\, e^{ipx}\, dp. \tag{5}$$

Clearly $F_+(x)+F_-(x) = F(x)$. Moreover we define

$$F_+(x)-F_-(x) = iF_*(x).$$

Thus one has

$$F_\pm = \tfrac{1}{2}(F \pm iF_*), \qquad (F_*)_* = -F.$$

F and F_* are called *Hilbert transforms* of each other. The reason why they are of interest to us is the identity

$$F_*(x) = \frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{F(x')}{x-x'}\,dx'. \tag{6}$$

It is also convenient to note the equation

$$F_\pm(x) = \tfrac{1}{2}F(x) \mp \frac{1}{2\pi i}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{F(x')}{x-x'}\,dx'. \tag{7}$$

Problem. Prove (6) and (7) with the aid of (3).

Problem. Verify the symbolic formulae

$$\delta_\pm(x) = \tfrac{1}{2}\delta(x) \mp \frac{1}{2\pi i}\,\mathrm{P}\frac{1}{x}$$

and prove for square-integrable F

$$F_\pm(x) = \int_{-\infty}^{\infty} \delta_\pm(x-x')\, F(x')\, dx'. \tag{8}$$

Problem. Derive the inverse formula of (6),

$$F(x) = -\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{F_*(x')}{x-x'}\,dx',$$

and conclude from this the "Poincaré-Bertrand lemma":

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{dx'}{(x-x')(x'-x'')} = -\pi^2\delta(x-x''). \tag{9}$$

In the integral (5) defining $F_+(x)$ we replace x by $z = x+iy$ with $y > 0$. It is easily seen that the resulting integral is uniformly convergent, so that

$$F_+(z) = \int_0^{\infty} \Phi(p)\, e^{ipz}\, dp$$

is an analytic function in the upper half of the complex z-plane (henceforth denoted by I_+). It is holomorphic in I_+, that is, analytic without singularities; it tends to zero for $y \to +\infty$; and it tends to $F_+(x)$ for $y \to 0$.* In the same way $F_-(x)$ can be continued analytically into the lower half of the z-plane, I_-. In this way one arrives at the

* For almost all x, and also in mean, which is good enough for our purpose. Note that F_+ need not be analytic on the real axis. Yet we shall refer to $F_+(z)$ as an "analytic continuation of $F_+(x)$".

Theorem of Titchmarsh.* Any square-integrable function $F(x)$ can be written as a sum of two functions $F_+(x)$ and $F_-(x)$, which have holomorphic analytic continuations in I_+ and I_- respectively that tend to zero for $y \to \pm\infty$. This decomposition is unique.

Problem. Prove the uniqueness.
Problem. Prove that if z lies in I_+,

$$F_+(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(x')}{x'-z}\,dx' . \qquad (z \in I_+)$$

Problem. If $F(x)$ has an analytic continuation in a certain strip below the real axis $(-a < y < 0)$, then $F_+(x)$ has an analytic continuation in the same strip given by

$$F_+(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(x')}{x'-z}\,dx' + F(z) . \qquad (-a < y < 0)$$

2. Exact stationary solutions

We shall now solve again (XI, 12) in a more rigorous fashion. First we reduce it to an integral equation for a function of one variable. Choose the z-axis in the direction of $\boldsymbol{k}$ and integrate (XI, 12) with respect to the transverse components v_x, v_y:

$$(\omega - kv_z)\iint g(\boldsymbol{v})\,dv_x\,dv_y = -\frac{\omega_P^2}{k}\iint \frac{\partial f^0(v)}{\partial v_z}\,dv_x\,dv_y \int g(\boldsymbol{v}')\,d^3\boldsymbol{v}' . \qquad (10)$$

We introduce the following abbreviations

$$g_z(v_z) = \iint g(v_x, v_y, v_z)\,dv_x\,dv_y ,$$

$$F(v_z) = -\iint \frac{\partial f^0(v)}{\partial v_z}\,dv_x\,dv_y ,$$

$$u = \omega/k , \qquad u_P = \omega_P/k .$$

Equation (10) then becomes

$$(u - v_z)g_z(v_z) = u_P^2 F(v_z)\int g_z(v_z')\,dv_z' . \qquad (11)$$

* The rigorous statement and proof are given in: E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Clarendon Press, Oxford 1937) ch. v.

$F(v_z)$ is a given function, u_P a given constant, and u, the phase velocity of the wave, replaces ω as eigenvalue parameter.

Problem. Prove that since f^0 is isotropic one has for $v_z > 0$

$$F(v_z) = 2\pi v_z f^0(v_z) . \qquad (12)$$

This equation remains true for $v_z < 0$ if one defines $f^0(-v) = f^0(v)$, which we shall do henceforth.
Problem. Derive (11) from (XI, 8) by assuming only that f^1 does not depend on x, y.

Again one can find $g_z(v_z)$ from (11) simply by dividing by $u - v_z$. However, it is possible to add a term with $\delta(u - v_z)$, owing to the fact that $(u - v_z)\delta(u - v_z) = 0$. Hence *we now write as the result of dividing (10) by* $u - v_z$:

$$g_z(v_z) = u_P^2\,\mathrm{P}\,\frac{F(v_z)}{u - v_z} + \lambda\delta(u - v_z) . \qquad (13)$$

For convenience the arbitrary normalization factor is supposed to be chosen such that

$$\int g_z(v_z)\,dv_z = 1 . \qquad (14)$$

λ is an arbitrary constant. In order that the solution be consistent, the integral of (13) must satisfy (14),

$$1 = u_P^2\,\mathrm{P}\int \frac{F(v_z)}{u - v_z}\,dv_z + \lambda . \qquad (15)$$

One is free to choose the principal value for the integral, because any other choice would simply amount to a different definition of λ.

Remark. Alternatively one may arrive at (15) without adding the $\lambda\delta$-term in (13), but noting that the integral is only defined up to a constant, comp. (1). The actual justification of this somewhat formal procedure is given in the theory of singular integral equations.* One also arrives at the same result by first restricting the possible values of v_z to a discrete set, and letting this set become very dense.**

The consistency equation (15) no longer determines the eigenvalue para-

* N. I. Muskhelishvili, *Singular Integral Equations* (Noordhoff, Groningen 1953).
** N. G. van Kampen, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **26**, no 15 (1951).

meter u, but is rather an equation for λ. Thus it is no longer true that ω is connected with $\boldsymbol{k}$, but they can both be chosen freely: *no dispersion law exists.*

The situation is fundamentally different from the familiar state of affairs in classical continuum mechanics, magnetohydrodynamics and two-component theory. The reason is that, instead of a few degrees of freedom at each point in space, we now have infinitely many of them, embodied in the velocity distribution $f(\boldsymbol{r}, \boldsymbol{v}, t)$. To clarify this, let us consider for a moment a gas of non-interacting particles. Such a gas may be looked at as a superposition of different "beams" of particles, each beam being the collection of all particles with the same velocity. The distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ can easily be found from the initial state according to

$$f(\boldsymbol{r}, \boldsymbol{v}, t) = f(\boldsymbol{r} - \boldsymbol{v}t, \boldsymbol{v}, 0).$$

Now suppose at $t = 0$ there is a given periodic variation of the density, proportional to $\exp(i\boldsymbol{k}_0 \cdot \boldsymbol{r})$. This does not at all determine the initial state, but only asserts that

$$f(\boldsymbol{r}, \boldsymbol{v}, 0) = n_0 f^0(v) + g(\boldsymbol{v}) e^{i\boldsymbol{k}_0 \cdot \boldsymbol{r}}.$$

The function $g(\boldsymbol{v})$, which describes how the given density variation is shared by the various beams, is still unknown. As a consequence the future is *not* uniquely determined by giving the density at $t = 0$. For example, we may choose $g(\boldsymbol{v}) = A\delta(\boldsymbol{v} - \boldsymbol{v}_0)$, which means that the entire density variation is carried by the single beam with velocity $\boldsymbol{v}_0$. Clearly in this case

$$f(\boldsymbol{r}, \boldsymbol{v}, t) = n_0 f^0(v) + A\delta(\boldsymbol{v} - \boldsymbol{v}_0)\, e^{i\boldsymbol{k}_0 \cdot (\boldsymbol{r} - \boldsymbol{v}_0 t)}, \tag{16}$$

so that $\omega = \boldsymbol{k}_0 \cdot \boldsymbol{v}_0$. By taking a different $\boldsymbol{v}_0$ one obtains a different ω with the same $\boldsymbol{k}_0$. Hence there is no dispersion law.

Note that the existence of the frequency ω is only caused by the fact that the periodic density is carried along by the beam with velocity $\boldsymbol{v}_0$. It is *not* due to any internal resilience of the fluid counteracting the deviation from equilibrium. We shall see that in the case of the Vlasov plasma the picture is essentially the same, but modified by the Coulomb interaction.

Problem. Find (16) by solving the Boltzmann equation for non-interacting particles by means of the above $\lambda\delta$-technique.

Problem. Find the particle density $n(\boldsymbol{r}, t)$ from (16). Consider also the case that $g(\boldsymbol{v})$ consists of two delta peaks and show that this leads to a different $n(\boldsymbol{r}, t)$.

Problem. Show that the solution $g_z(v_z)$ for given k and u may be written in the form

$$g_z^{k,u}(v_z) = \delta(v_z - u) + u_P^2\, \mathrm{P} \int \frac{F(v_z')\delta(v_z - u) + F(v_z)\delta(v_z' - u)}{v_z' - v_z}\, dv_z'. \tag{17}$$

Problem. Derive the identities

$$\int_{-\infty}^{\infty} v_z g_z^{k,u}(v_z)\, dv_z = u,$$

$$\int_{-\infty}^{\infty} v_z^2 g_z^{k,u}(v_z)\, dv_z = u^2 - u_P^2.$$

Also find a formula for the higher moments of v_z with respect to $g_z^{k,u}(v_z)$.

3. The initial value problem

We have found a stationary solution of the Vlasov equation, which varies periodically with z but does not depend on x and y. We shall now construct the general form of all solutions that do not depend on x and y. For any $f^1(z, \boldsymbol{v}, t)$ one may define the *reduced distribution function*

$$f_z(z, v_z, t) = \iint f^1(z, \boldsymbol{v}, t)\, dv_x\, dv_y.$$

By integrating (XI, 8) one finds that it obeys the reduced Vlasov equation

$$\frac{\partial f_z}{\partial t} + v_z \frac{\partial f_z}{\partial z} + \frac{n_0 e}{m} E_z F(v_z) = 0.$$

The work in the preceding section provides the stationary solutions of this equation,

$$f_z(z, v_z, t) = e^{ikz - ikut} g_z^{k,u}(v_z).$$

A more general solution obtains by taking a superposition

$$f_z(z, v_z, t) = \iint_{-\infty}^{\infty} C(k, u)\, g_z^{k,u}(v_z)\, e^{ik(z - ut)}\, dk\, du, \tag{18}$$

with arbitrary $C(k, u)$. We shall show that this solution is *complete*, in the sense that any initial distribution $f_z(z, v_z, 0)$ can be reproduced by a suitable choice of $C(k, u)$. (That does not yet mean that we have found the complete solution of the three-dimensional linear Vlasov equation (XI, 8). This more general completeness will be proved in chapter XIII, when the transverse waves have also been found.)

The problem is to show that the integral equation

$$\int_{-\infty}^{\infty}\int C(k, u)\, g_z^{k,u}(v_z)\, e^{ikz}\, dk\, du = f_z(z, v_z, 0) \tag{19}$$

has a solution $C(k, u)$ for every given $f_z(z, v_z, 0)$. With respect to the z-dependence (19) is simply a Fourier transformation. Accordingly we may put

$$f_z(z, v_z, 0) = \int_{-\infty}^{\infty} \varphi(k, v_z)\, e^{ikz}\, dk\,,$$

so that (19) reduces to

$$\int_{-\infty}^{\infty} C(k, u)\, g_z^{k,u}(v_z)\, du = \varphi(k, v_z)\,.$$

k enters only as a parameter and need no longer be written as a variable. The problem is to solve for each fixed value of k and every given function $\varphi(v_z)$ the integral equation

$$\int_{-\infty}^{\infty} C(u)\, g_z^{u}(v_z)\, du = \varphi(v_z)\,. \tag{20}$$

The kernel $g_z^u(v_z)$ is determined by (13) with (15), or alternatively by (17).

Substitute (13) in (20) and use (15):

$$C(v_z)\,\{1-\pi u_{\rm P}^2 F_*(v_z)\} - \pi u_{\rm P}^2 F(v_z) C_*(v_z) = \varphi(v_z)\,.$$

This may also be written

$$(1+2\pi i u_{\rm P}^2 F_+)C_+ + (1-2\pi i u_{\rm P}^2 F_-)C_- = \varphi_+ + \varphi_-\,.$$

The two terms on the left are holomorphic in I_+ and I_- respectively, and tend to zero at infinity in their respective half planes. Since the decomposition of φ in two such functions is unique one may conclude

$$(1+2\pi i u_{\rm P}^2 F_+)C_+ = \varphi_+ \quad \text{and} \quad (1-2\pi i u_{\rm P}^2 F_-)C_- = \varphi_-\,.$$

It follows that *if* there is a solution it must be equal to $C = C_+ + C_-$ where C_+ and C_- are defined by

$$C_+ = \frac{\varphi_+}{1+2\pi i u_{\rm P}^2 F_+}\,, \qquad C_- = \frac{\varphi_-}{1-2\pi i u_{\rm P}^2 F_-}\,. \tag{21}$$

However, the solution constructed in this way is only admissible if C_+ and C_- are actually holomorphic in their respective half planes, that is if $1+2\pi i u_{\rm P}^2 F_+$ has no zeros in I_+ and $1-2\pi i u_{\rm P}^2 F_-$ has no zeros in I_-. We shall prove that this is so when the undisturbed distribution $f^0(v_z)$ does not vanish for any value of v_z (for example the Maxwell distribution).

Problem. Prove that the behavior of C_+ and C_- at infinity is automatically correct, as the denominators in (21) tend to 1.

Problem. Prove that it suffices to study C_+, as C_- can be obtained from it by complex conjugation.

Problem. Verify that $C_+ + C_-$ is actually a solution of (20), once it has been established that C_+ is holomorphic in I_+.

Proof that the denominators (21) have no zeros. The function $Z(u)$ defined in I_+ by

$$Z(u) = 1+2\pi i u_{\rm P}^2 F_+(u) \qquad (u \in I_+) \tag{22}$$

is holomorphic in I_+, tends to 1 at infinity, and when u approaches the real axis $Z(u)$ tends to a limiting value given by (comp. (12))

$$\operatorname{Re} Z(u) = 1 - u_{\rm P}^2\, \mathrm{P}\int \frac{F(u')}{u-u'}\, du' \tag{23}$$

$$\operatorname{Im} Z(u) = \pi u_{\rm P}^2 F(u) = 2\pi^2 u_{\rm P}^2 u f^0(u)\,. \tag{24}$$

In order to show that it has no zeros in I_+ we consider a closed contour consisting of the interval $(-L, L)$ on the real axis and the semicircle of radius L in I_+ (Fig. 26a). We shall show for large L that, when u moves once around this contour, arg $Z(u)$ returns to its initial value. According to a wellknown theorem* it then follows that there are no zeros inside the contour.

Consider the image of the contour in the complex Z-plane. One easily verifies that it has the following features (Fig. 26b).

(i) The semicircle is mapped in a small neighborhood of $Z = 1$, provided L is large enough.

(ii) The interval $0 < u \leqslant L$ is mapped on a curve above the real axis, because (24) is positive (remember that we assumed $f^0(v_z) > 0$).

* See e.g. E. C. Titchmarsh, *The Theory of Functions*, 2nd ed. (Oxford University Press, Oxford 1939) p. 116.

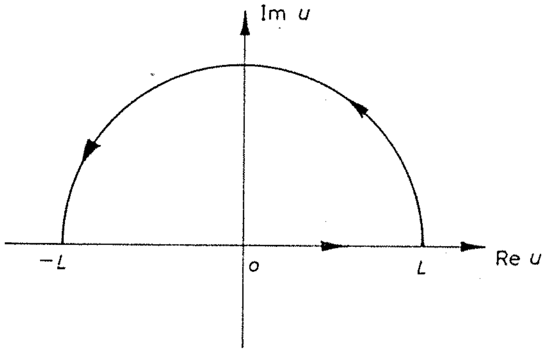

Fig. 26a. The contour described by u in the complex u-plane.

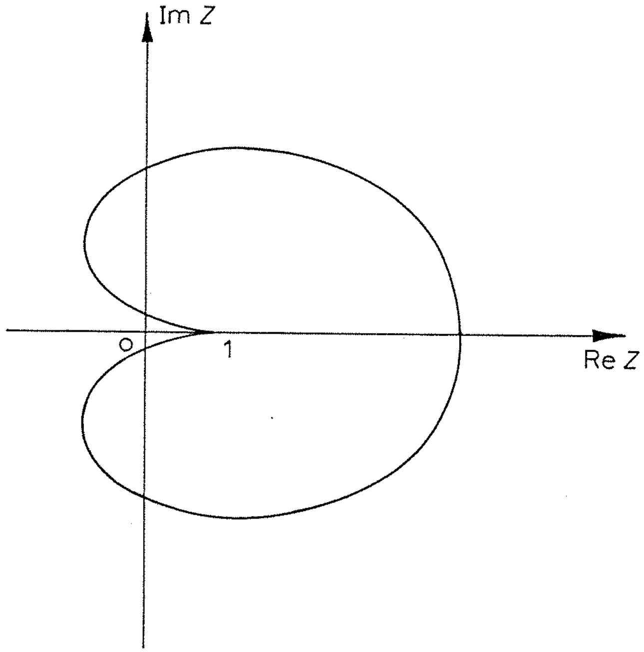

Fig. 26b. The image contour in the complex Z-plane ("Nyquist diagram").

(iii) The interval $-L \leqslant u < 0$ is mapped below the real axis.

(iv) The point $u = 0$ is mapped in a point of the positive real axis, because (23) is positive for $u = 0$.

It now follows immediately that the image contour does not enclose the

origin, so that arg $Z(u)$ is single-valued on this contour. This completes the proof that (21) actually provides a solution of (20).

Problem. Prove that the image contour in the Z-plane is symmetric with respect to the real axis.

Problem. For the case that f^0 is the Maxwell distribution express $Z(u)$ as defined by (22) in terms of the function Z tabellated in B.D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic Press, New York, 1961).

Problem. Show that, if f^0 is analytic in a strip below the real axis, $Z(u)$ has an analytic continuation in the same strip given by

$$Z(u) = 1 + 4\pi^2 i u_P^2 u f^0(u) + 2\pi u_P^2 \int \frac{u' f^0(u')}{u' - u} \, du' . \qquad (u \in I_-) \tag{25}$$

Problem. Calculate $Z(u)$ for the case $f^0(v) = (a/\pi^2)(a^2+v^2)^{-2}$, and verify the properties of $Z(u)$ mentioned in the text. Sketch the image contour in the Z-plane. Also verify the assertion of the previous problem.

Finally, our solution (18) may be written in such a way that $f_z(z, v_z, t)$ is explicitly expressed in terms of its initial value $f_z(z, v_z, 0)$. Inserting (21) into (18) one has

$$f_z(z, v_z, t) = \frac{1}{2\pi} \iint e^{ik(z-z')} \int g_z^{k,u}(v_z)\, e^{-ikut} \times$$

$$\times \left[\frac{f_{z+}(z', u, 0)}{Z(k, u)} + \frac{f_{z-}(z', u, 0)}{Z^*(k, u)} \right] dk\, dz'\, du . \tag{26}$$

$Z(k, u)$ is defined by (22) and $Z^*(k, u)$ is its complex conjugate. f_{z+} and f_{z-} are the positive and negative frequency parts of the initial distribution $f_z(z', u, 0)$ with respect to the variable u.

Problem. Write (26) in the form

$$f_z(z, v_z, t) = \iint U_t(z, v_z \mid z', v_z') f_z(z', v_z', 0)\, dz'\, dv_z'$$

and verify that the evolution operator (or "Green's function") U_t has the properties (XI, 7).

Problem. Show that the perturbation of the density is given by

$$n^1(z, t) = \frac{1}{2\pi} \iiint e^{ik(z-z')-ikut} \left[\frac{f_{z+}(z', u, 0)}{Z(k, u)} + \frac{f_{z-}(z', u, 0)}{Z^*(k, u)} \right] dk\, dz'\, du\,. \quad (27)$$

4. The evolution of the density

For convenience we suppose that the initial disturbance is periodic in space with wave number $k_0 > 0$,

$$f_z(z, v_z, 0) = \varphi(v_z)\, e^{ik_0 z}\,. \quad (28)$$

Then $f_z(z, v_z, t)$ has the same periodicity and is given by

$$f_z(z, v_z, t) = e^{ik_0 z} \int_{-\infty}^{\infty} e^{-ik_0 ut}\, g_z^{k_0, u}(v_z) \left[\frac{\varphi_+(u)}{Z(k_0, u)} + \frac{\varphi_-(u)}{Z^*(k_0, u)} \right] du\,. \quad (29)$$

Integration with respect to v_z yields with the aid of (14)†

$$n^1(z, t) = e^{ik_0 z} \int_{-\infty}^{\infty} e^{-ik_0 ut} \left[\frac{\varphi_+(u)}{Z(u)} + \frac{\varphi_-(u)}{Z^*(u)} \right] du\,.$$

The variable k_0 in Z has again been dropped. The second term in [ ] is a negative-frequency function of u, because $\varphi_-(u)$ is such a function and Z^* is holomorphic in I_- and tends to 1 at infinity. Hence the integral of $\exp(-ik_0 ut)$ times this term vanishes when $t > 0$. Thus

$$n^1(z, t) = \int_{-\infty}^{\infty} e^{ik_0(z - v_z t)} \frac{\varphi_+(v_z)}{Z(v_z)}\, dv_z\,. \qquad (t > 0) \quad (30)$$

This result can be interpreted as follows. The initial disturbance (28) has one wave number k_0, but contains many different velocities v_z. The amount to which each v_z is present in the initial disturbance is $\varphi(v_z)$. After a time t the beam of all electrons with one particular velocity v_z has moved over a distance $v_z t$, so that the phase at each fixed point in space is changed by a factor $\exp(-ik_0 v_z t)$. This much would also be true for non-interacting particles. The interaction is embodied in $Z(u)$, which modifies the amount to which each v_z contributes to the density. Thus one has a kind of response function

$$\frac{1}{Z(v_z)} = \frac{k_0^2}{k_0^2 + 2\pi i \omega_P^2 F_+(v_z)}\,.$$

† Of course this can also be obtained by inserting (28) into (27).

For large $|v_z|$ this factor tends to 1.

Problem. Derive (30) for non-interacting particles (so that $Z = 1$).

Problem. Find the analogous formula to (30) for the momentum density,

$$p_z = m \int_{-\infty}^{\infty} v_z f_z(z, v_z, t)\, dv_z\,.$$

Verify that the continuity equation is satisfied.

Problem. Similarly, compute the first order perturbation of the kinetic energy density of the z-component of $\boldsymbol{v}$

$$\varepsilon_z = \tfrac{1}{2} m \int_{-\infty}^{\infty} v_z^2 f_z(z, v_z, t)\, dv_z\,,$$

and verify the equation

$$\frac{\partial p_z}{\partial t} = -\frac{\partial \varepsilon_z}{\partial z} - u_P^2 \frac{\partial n^1}{\partial z}\,.$$

Now suppose $Z(u)$ can be continued analytically in some strip *below* the real axis $0 \geqslant \operatorname{Im} u \geqslant -a$. Moreover let there be a zero $u_0 = u_B - i\gamma$ near the real axis (u_B real, $0 < \gamma < a$, γ small in a sense to be specified presently). Then the integration path of (30) may be shifted downwards to a straight line parallel to the real axis below u_0 ($\operatorname{Im} u = -b$, $\gamma < b < a$). Since u_0 is a pole of the integrand of (30) it contributes a residue:

$$-2\pi i\, e^{ik_0(z - u_0 t)} \frac{\varphi_+(u_0)}{Z'(u_0)}\,. \quad (31)$$

This contains a factor $\exp(-k_0\gamma t)$ and is therefore damped. The integration along the parallel $\operatorname{Im} u = -b$ contains the factor $\exp(-k_0 bt)$, and is therefore more strongly damped. Hence, after an initial time of the order of $(k_0 b)^{-1}$ the integral (30) reduces to (31), or, approximately for small γ,

$$n^1(z, t) = -2\pi i \frac{\varphi_+(u_B)}{Z'(u_B)}\, e^{ik_0(z - u_B t)}\, e^{-k_0 \gamma t}\,. \quad (32)$$

Thus we have found again a traveling wave with a definite phase velocity associated with the wave number k_B, like in the previous chapter. However, it is now only an approximation, valid after an initial transient time $(k_0 b)^{-1}$.

Moreover it is damped with a decay time $(k_0\gamma)^{-1}$; this is called *Landau damping*.

It remains to determine u_B and γ. Using (23) and (24) we find

$$0 = Z(u_B - i\gamma) = Z(u_B) - i\gamma Z'(u_B) + O(\gamma^2)$$

$$= 1 - u_P^2 \mathrm{P} \int_{-\infty}^{\infty} \frac{F(u')}{u_B - u'} du' + i\pi u_P^2 F(u_B)$$

$$- i\gamma u_P^2 \mathrm{P} \int_{-\infty}^{\infty} \frac{F(u')}{(u_B - u')^2} du' + \gamma\pi u_P^2 F'(u_B) + O(\gamma^2). \tag{33}$$

The real part of this equation yields, to zeroth order in γ,

$$2\pi u_P^2 \mathrm{P} \int_{-\infty}^{\infty} \frac{u f^0(u)}{u_B - u} du = 1. \tag{34}$$

This equation determines the phase velocity u_B for given k_0, and is therefore a dispersion law.

Problem. Verify that this is identical with (XI, 16), but for the P that indicates the principal value. This prescription for dealing with the pole, conjectured by Vlasov, has now been deduced mathematically.

Problem. Show that the expansion (XI, 18) for small k_0 can be derived from (34) without cheating with the pole (like when deriving (XI, 18) in the previous chapter).

Problem. Verify that the approximations made in the calculation amount to regarding γ as small, but not γt; on the other hand γt should be much less than bt.

The imaginary part of equation (33) yields

$$\pi u_B f^0(u_B) = \gamma \mathrm{P} \int \frac{u' f^0(u')}{(u_B - u')^2} du'. \tag{35}$$

This determines the magnitude of the Landau damping. For small k_0 the u_B determined by (34) is large, and the same expansion as in (XI, 18) can be used. In this way one finds for γ in lowest approximation

$$\gamma = \pi^2 u_B^4 f^0(u_B). \tag{36}$$

In the same approximation $u_B \approx u_P = \omega_P/k_0$. The corresponding decay time for Landau damping is therefore*

$$\tau_L = (k_0\gamma)^{-1} = \frac{k_0^3}{\pi^2 \omega_P^4 f^0(\omega_P/k_0)}. \tag{37}$$

Problem. Compute numerically u_B and τ_L for a plasma of 10^8 electrons per cm^3 at temperature 10^6 °K for 20 cm waves.

Problem. Write a dispersion law that incorporates both (XI, 18) and (37).

Remark. It is possible to avoid the use of an analytical continuation of $Z(u)$ into the lower half plane. One first defines u_B by (34), that is: Re $Z(u_B) = 0$. On the real axis in the neighborhood of u_B one has

$$\frac{\varphi_+(u)}{Z(u)} = \frac{\varphi_+(u_B)}{i \,\mathrm{Im}\, Z(u_B) + (u - u_B)\,\mathrm{Re}\, Z'(u_B)}. \tag{38}$$

This is an ordinary resonance denominator of the type $(u - u_B + i\gamma)^{-1}$ with

$$\gamma = \mathrm{Im}\, Z(u_B)/\mathrm{Re}\, Z'(u_B),$$

which is the same as (35). That such a resonance denominator gives rise to the damped wave (32) can be shown by a standard calculation.

Of course this does not mean that one has materially weakened the conditions for the validity of the result. The use of the linear approximation in (38) is only justifiable when the higher terms in the Taylor expansion are small. More precisely

$$\frac{\gamma^n \,|\, Z^{(n)}(u_B) \,|}{n!} \ll |\, Z(u_B) \,|.$$

However, this is roughly the condition that Z has an analytic continuation beyond the point $u_B - i\gamma$, which was assumed in the previous treatment. The same applies to the numerator $\varphi_+(u)$ in (38).

5. Comments on the irreversible behavior of the density

a. Our process of solving the linear Vlasov equation for longitudinal waves, (XI, 8) with (XI, 11), first led to the stationary solutions $g_z^{k,u}(v_z)$, given by (13) with (15), or by (17). The general solution is an arbitrary superposition

* For experiments see A. Y. Wong, R. W. Motley and N. d'Angelo, Phys. Rev. **133A**, 436 (1964); H. Derfler and T. C. Simonen, Phys. Rev. Letters **17**, 172 (1966) and J. H. Malmberg and C. B. Wharton, Phys. Rev. Letters **17**, 175 (1966).

(18), involving all values of k and u. For the present discussion it suffices to take a single value k_0 of k, so that $f_z(z, v_z, t)$ is given by the superposition (29) of stationary solutions. These stationary solutions have the same k_0 but different phase velocities u; that is, there is no dispersion law.

The density was obtained by integrating over v_z with the result (30), when k_0 and t are taken positive. Owing to the fact that the stationary solutions in this superposition all travel with different phase velocities the initial density disturbance is eventually smeared out.* Even without going into the more detailed calculation of the previous section, this follows directly from (30). For, if t is very large, the factor $\exp(-ik_0v_zt)$ is a very rapidly oscillating function of v_z, so that the integral will be small.

Remark. Mathematically this fact is expressed by the Riemann–Lebesgue theorem, which asserts that the integral tends to zero as $t \to \infty$, provided that the function $\varphi_+(v_z)/Z(v_z)$ is Riemann integrable and the integral converges absolutely.** However, from the fact that $\varphi(v_z)$ and $f^0(v_z)$ are density distributions, we can only conclude that they are integrable in the sense of distributions. For example $\varphi(v_z)$ may contain a delta function, in which case (30) does not tend to zero. Thus the proviso of Riemann or Lebesgue integrability imposes a smoothness condition on the initial distribution, which, however weak, is indispensable. Moreover it requires that Z shall have no zero *on* the real axis. The same things are said in more physical terms below.

Problem. Find the asymptotic behavior of $n^1(z, t)$ as $t \to \infty$ for the case $\varphi(v_z) = \delta(v_z - v_0)$.

b. The rate at which the integral (30) tends to zero depends on the smoothness of $\varphi_+(v_z)/Z(v_z)$. If this function varies little over a range U, the cancellation effect of the rapidly varying factor $\exp(-ik_0v_zt)$ becomes effective for $t \sim 2\pi/k_0U$. This "*variation length*" U is determined by three effects:

(i) The variation length U_φ, being the range over which $\varphi(v_z)$ and hence $\varphi_+(v_z)$ remain practically constant;

(ii) The variation length U_Z of $Z(v_z)$, which is roughly equal to the variation length of f^0;

* This is often called "phase mixing".

** E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge 1927) p. 172. Instead of "Riemann integrable and absolutely convergent" one may also read "Lebesgue integrable"; see E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, Oxford 1939) p. 404.

(iii) Even if $Z(v_z)$ is very smooth, $1/Z(v_z)$ will have a sharp peak when $Z(v_z)$ has a zero near the real axis. The width of this peak is the distance γ of the zero from the real axis, so that $U \leqslant \gamma$.

The work in section 4 was based on the tacit assumption that U_φ and U_Z were much greater than γ. This made it possible to treat the effect of the zero separately, with the following result. There is first a transient time $t < (k_0b)^{-1}$, where $b = \text{Min}(U_\varphi, U_z)$; during which the variation of $n^1(z,t)$ depends on the detailed behavior of $\varphi_+(v_z)/Z(v_z)$. After this, $t > (k_0b)^{-1}$, the behavior is mainly determined by the peak, and is found to be the damped wave (32). The phase velocity u_B of this wave is connected with the wave length by the dispersion law (34), while the damping is given by Landau's formula (36) or (37).

If, however, the initial disturbance is *not* so smoothly distributed over the various velocities, so that U_φ is *not* much greater than γ, there is no way of separating the transient from the damped wave, so that (32) is no longer true. Yet it remains true that as $t \to \infty$ the density disturbance n^1 tends to zero, as long as φ is integrable. The only exception occurs when φ contains a delta function, as remarked above; it is natural to characterize this case by the variation length $U_\varphi = 0$.

c. The result that the density tends to its equilibrium value is satisfactory from a physical point of view, but seems to contradict the invariance for time reversal. Yet there is no contradiction, because reversibility only led to the conclusion that f^1 cannot approach an equilibrium, but tells nothing about n^1, which is the integral of f^1 over the velocity. This is analogous to the state of affairs in the kinetic theory of gases: Although the microscopic equations of motion of the molecules of the gas are reversible, there exist nevertheless macroscopic quantities, made up by many molecules, which do approach an equilibrium value.

In fact, the linearized Vlasov equation is an interesting illustration of the general question: How do irreversible phenomena arise from reversible equations?* A given $f_z(z, v_z, 0)$ uniquely determines $f_z(z, v_z, t)$ at later times; and, vice versa, if $f_z(z, v_z, t)$ is known it is possible to calculate backwards to find $f_z(z, v_z, 0)$. In other words, $f_z(z, v_z, t)$ still contains all the information

* The literature on this question is enormous. The most celebrated discussion is: P. and T. Ehrenfest, Enzyklopädie Mathem. Wissens. Band IV, Nr 32 (Teubner, Leipzig 1911). English translation: Paul and Tatiana Ehrenfest, *The Conceptual Foundations of the Statistical Approach in Mechanics* (Cornell University Press, Ithaca, N.Y. 1959). For a modern review see: I. E. Farquhar, *Ergodic Theory in Statistical Mechanics* (Interscience, New York 1964).

that is contained in $f_z(z, v_z, 0)$. It follows that $f_z(z, v_z, t)$ cannot approach an equilibrium value, since in equilibrium all information concerning previous states would be lost, as argued more explicitly in ch. XI, sec. 2.

However, if one integrates $f_z(z, v_z, t)$ over v_z to obtain $n^1(z, t)$ a great deal of information concerning the state of the plasma is discarded. There are

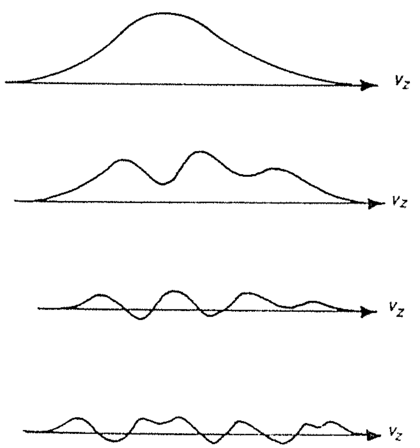

Fig. 27. An artist's impression of the velocity distribution $f_z(z, v_z, t)$ for successive times (taken in a fixed point $z = 0$, say). The function becomes progressively more corrugated so that its *integral* n^1 tends to zero. Yet the information needed to compute the initial f_z from its later form is not lost.

many different functions $f_z(z, v_z, t)$ that give rise to the same $n^1(z, t)$. Thus the density $n^1(z, t)$ only provides a rough, or "*coarse-grained*" description of the state of the plasma, which is insufficient to reconstruct the initial state. To put it differently, when in each point of space the total number of electrons is observed at a time t, one cannot reconstruct the initial state; but when also the breakdown of these total numbers according to velocities is observed, one can reconstruct the initial state (Fig. 27). Thus the reason mentioned in the previous paragraph, why f_z cannot approach equilibrium, does not apply to n^1.

Problem. Show that if $f_z(z, v_z, 0)$ is a smooth function of v_z, the density $n^1(z, t)$ tends to zero exponentially as $t \to \infty$.

Problem. Study in a similar way the case of non-interacting particles and show again that in general the density tends to its equilibrium value.

Remark. The situation in Vlasov theory is slightly different from the one familiar in statistical mechanics, because in that case there usually are

differential equations for the macroscopic quantities. Examples are the diffusion equation, Ohm's law, and the hydrodynamical equations, which were used in previous chapters. Thus, in addition to the deterministic description of the microscopic equations there exists another deterministic description on the macroscopic or coarse-grained level. This macroscopic determinism is due to the fact that the collisions maintain a local equilibrium – and is therefore absent in the Vlasov case. The formal analogy between classical statistical mechanics and Vlasov theory is exhibited in Table III.

TABLE III

	classical gas	Vlasov plasma
reversible equations	microscopic equations of motion (Hamilton eqns.)	Vlasov equation
macroscopic or coarse grained quantities	local density, local velocity, local temperature	local density $n^1(z, t)$, momentum density, etc.
irreversible behavior	macroscopic equations (hydrodynamic eqns., Ohm's law etc.)	no differential equation for $n^1(z, t)$

d. Yet another objection may be raised. Suppose it is true that for the solution $f_z(z, v_z, t)$ the density $n^1(z, t)$ decreases exponentially. Then $f_z(z, -v_z, -t)$ is a solution of the Vlasov equation for which $n^1(z, t)$ clearly *increases* exponentially.*

We formulate this objection more carefully. Let $f_z(z, v_z, t)$ be a solution, whose initial state $f_z(z, v_z, 0)$ is a smooth function of v_z, so that $n^1(z, t)$ after some transient time decreases exponentially according to (32). Take the state $f_z(z, v_z, t_1)$ at some large time t_1. The time reversed state of this is

$$Tf_z(z, v_z, t_1) = f_z(z, -v_z, t_1).$$

Let $\bar{f}_z(z, v_z, t)$ be that solution of the Vlasov equation that has this time reversed state as initial value

$$\bar{f}_z(z, v_z, 0) = f_z(z, -v_z, t_1). \tag{39}$$

Owing to the invariance for time reversal one has

$$\bar{f}_z(z, v_z, t) = f_z(z, -v_z, t_1 - t).$$

Hence

$$\bar{n}^1(z, t) = n^1(z, t_1 - t),$$

* This is the analog of Loschmidt's "Umkehreinwand"; see e.g. Ehrenfest, loc. cit.

which shows that $\bar{f}_z$ is a solution for which the density *increases* exponentially.

The answer to this objection is that the initial state (39) is not smooth, as was tacitly assumed when deriving (32). The solution $f_z(z, v_z, t)$ is a smooth function of v_z at $t = 0$, but as t increases it becomes more and more corrugated, owing to the factor $\exp(-ik_0ut)$ in (29). That is the reason why $n^1(z, t)$ being the integral over v_z, tends to zero. It is also the reason why the derivation of (32) does not apply to $\bar{f}_z(z, v_z, t)$, From the way this solution has been constructed it is clear that $\bar{f}_z(z, v_z, 0)$ is very corrugated, but becomes smoother as t increases from 0 to t_1. However, when t increases beyond t_1, it gradually becomes more corrugated again, so that $n^1(z, t)$ decreases and ultimately tends to zero, in agreement with the Riemann–Lebesgue theorem. Thus for the reversed solution $\bar{f}_z$ the density $\bar{n}^1$ first *increases* exponentially; for t near t_1 it has a somewhat messy behavior depending on the specific form of $f_z(z, v_z, 0)$; for $t \gg t_1$ it *decreases* again exponentially towards zero.

Problem. Show that this ultimate decrease of $n^1(z, t)$ is again exponential in agreement with (32).

General references

N. G. van Kampen, Physica **23**, 641 (1957).
I. B. Bernstein, S. K. Trehan and M. P. Weenink, Nuclear Fusion **4**, 61 (1964).
R. Balescu, *Statistical Mechanics of Charged Particles* (Interscience, New York 1963) chs. III, IV, V.
D. C. Montgomery and D. A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill, New York 1964) chs. V, XII.
H. Motz, *Reports on Progress in Physics* **29** (Institute of Physics and The Physical Society, London 1966) Part II, p. 623.

CHAPTER XIII

TRANSVERSE WAVES IN THE VLASOV THEORY

1. The equations for tranverse waves

In chapter XI the Vlasov equation (XI, 1) was formulated together with the field equations (XI, 2). Subsequently the equation was linearized with the result (XI, 8), the unperturbed distribution function $f^0(v)$ being supposed isotropic. Moreover, for longitudinal waves the field equations could be replaced by the Coulomb expression (XI, 11). In the present chapter we study the transverse waves.* Accordingly we start again from the linearized Vlasov equation (XI, 8), but use the full set of Maxwell's equations:

$$\frac{\partial f^1}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f^1}{\partial \boldsymbol{r}} - \frac{n_0 e}{m} \boldsymbol{E}^1 \cdot \frac{\partial f^0}{\partial \boldsymbol{v}} = 0 \tag{1}$$

$$\frac{1}{c} \frac{\partial \boldsymbol{H}^1}{\partial t} = -\operatorname{curl} \boldsymbol{E}^1 \tag{2a}$$

$$\operatorname{div} \boldsymbol{E}^1 = -4\pi e n^1 = -4\pi e \int f^1(\boldsymbol{r}, \boldsymbol{v}, t)\, d^3 v \tag{2b}$$

$$\frac{1}{c} \frac{\partial \boldsymbol{E}^1}{\partial t} = \operatorname{curl} \boldsymbol{H}^1 - \frac{4\pi}{c} j^1 = \operatorname{curl} \boldsymbol{H}^1 + \frac{4\pi e}{c} \int \boldsymbol{v} f^1(\boldsymbol{r}, \boldsymbol{v}, t)\, d^3 v \tag{2c}$$

$$\operatorname{div} \boldsymbol{H}^1 = 0\,. \tag{2d}$$

In the present chapter external fields are still excluded.

We search for plane wave solutions and put accordingly

$$f^1(\boldsymbol{r}, \boldsymbol{v}, t) = f_k(\boldsymbol{v}, t)\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$
$$\boldsymbol{E}^1(\boldsymbol{r}, t) = \boldsymbol{E}_k(t)\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$
$$\boldsymbol{H}^1(\boldsymbol{r}, t) = \boldsymbol{H}_k(t)\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,.$$

* B. U. Felderhof, Physica **29**, 293 (1963).

For each fixed $\boldsymbol{k}$ we choose the z-axis in the direction of $\boldsymbol{k}$ and subsequently define the *reduced distribution functions*

$$f_x(v_z, t) = \iint v_x f_{\boldsymbol{k}}(\boldsymbol{v}, t)\, dv_x\, dv_y$$

$$f_y(v_z, t) = \iint v_y f_{\boldsymbol{k}}(\boldsymbol{v}, t)\, dv_x\, dv_y \tag{3}$$

$$f_z(v_z, t) = \iint f_{\boldsymbol{k}}(\boldsymbol{v}, t)\, dv_x\, dv_y .$$

Note that f_z contains no factor v_z, and that it is defined in analogy with the function $g_z(v_z)$ in the previous chapter. If one now multiplies (1) by v_x, v_y, 1 successively and integrates over v_x and v_y three separate sets of equations are obtained.

First three equations involving f_x, E_{kx}, H_{ky}:

$$\frac{\partial f_x}{\partial t} + ikv_z f_x + \frac{n_0 e}{m} E_{kx}\bar{F}(v_z) = 0 \tag{4a}$$

$$\frac{\partial E_{kx}}{\partial t} = -ick\, H_{ky} + 4\pi e \int f_x(v_z, t)\, dv_z \tag{4b}$$

$$\frac{\partial H_{ky}}{\partial t} = -ick\, E_{kx} \tag{4c}$$

where

$$\bar{F}(v_z) = \iint_{-\infty}^{\infty} f^0(v_x, v_y, v_z)\, dv_x\, dv_y .$$

Second three equations involving f_y, E_{ky}, H_{kx}:

$$\frac{\partial f_y}{\partial t} + ikv_z f_y + \frac{n_0 e}{m} E_{ky}\bar{F}(v_z) = 0 \tag{5a}$$

$$\frac{\partial E_{ky}}{\partial t} = ick\, H_{kx} + 4\pi e \int f_y(v_z, t)\, dv_z \tag{5b}$$

$$\frac{\partial H_{kx}}{\partial t} = ick\, E_{ky} . \tag{5c}$$

Third two equations involving f_z, E_{kz}:

$$\frac{\partial f_z}{\partial t} + ikv_z f_z + \frac{n_0 e}{m} E_{kz} F(v_z) = 0 \tag{6a}$$

$$ikE_{kz} = -4\pi e \int f_z(v_z, t)\, dv_z , \tag{6b}$$

where $F(v_z)$ is the same as in (XII, 11).

This decomposition in three separate sets is the separation of three polarizations. In (6) one recognizes the equations (XI, 8) and (XI, 9) for the longitudinal plasma waves. Incidentally, we have proved again that it was correct to put $\boldsymbol{H} = 0$ and to take for $\boldsymbol{E}$ the Coulomb field.

The two sets (4) and (5) describe transverse modes since they only involve the field components perpendicular to $\boldsymbol{k}$. Clearly they transform into each other by a rotation about the z-axis. Hence it is sufficient to solve (4).

Problem. Verify that $\bar{F}(v_z)$ is an even, non-negative function of v_z, normalized to 1 and non-increasing for $v_z > 0$, and that $F(v_z) = -\bar{F}'(v_z)$.

Problem. For anisotropic $f^0(\boldsymbol{v})$ the separation in these three polarizations is no longer generally possible. Find in which special case it can still be done.

2. Stationary solutions of equation (4)

In order to find stationary solutions we put

$$f_x(v_z, t) = g_x(v_z)\, e^{-i\omega t}$$

$$E_{kx}(t) = E_x\, e^{-i\omega t}$$

$$H_{ky}(t) = H_y\, e^{-i\omega t} .$$

E_x and H_y are constants and g_x is a function of v_z yet to be determined. Substitution in (4) yields (with $u = \omega/k$)

$$(u - v_z)\, g_x(v_z) = -i \frac{n_0 e}{mk} E_x \bar{F}(v_z) \tag{7a}$$

$$uE_x = cH_y + \frac{4\pi i e}{k} \int g_x(v_z)\, dv_z \tag{7b}$$

$$uH_y = cE_x . \tag{7c}$$

Eliminating H_y from (7b) and (7c),

$$(u^2-c^2)\,E_x = \frac{4\pi ie}{k}\,u \int g_x(v_z)\,dv_z\,. \tag{8}$$

The value of the integral may be arbitrarily fixed since we have a normalization constant at our disposal. It is clearly not expedient to take this integral equal to 1 (like in the previous chapter), because at the special values $u = \pm c$ it must vanish on account of (8). Hence we choose the normalization

$$\int g_x(v_z)\,dv_z = u^2-c^2\,. \tag{9}$$

Then (8) reduces to $E_x = 4\pi ieu/k$, which substituted in (7a) yields an equation for $g_x(v_z)$ alone,

$$(u-v_z)\,g_x(v_z) = u_{\mathrm{P}}^2 u\bar{F}(v_z)\,.$$

From this we find

$$g_x(v_z) = u_{\mathrm{P}}^2 u\ \mathrm{P}\,\frac{\bar{F}(v_z)}{u-v_z} + \lambda\delta(u-v_z) \tag{10a}$$

$$E_x = \frac{4\pi ie}{k}\,u \tag{10b}$$

$$H_y = \frac{4\pi ie}{k}\,c\,. \tag{10c}$$

The requirement that (10a) is consistent with (9) yields an equation for λ,

$$\lambda(u) = u^2-c^2+u_{\mathrm{P}}^2 u\ \mathrm{P}\int \frac{\bar{F}(v_z)}{v_z-u}\,dv_z\,. \tag{11}$$

Problem. Show that the stationary solution for given u is

$$g_x^u(v_z) = (u^2-c^2)\delta(v_z-u) + $$
$$+\, u_{\mathrm{P}}^2\ \mathrm{P}\int \frac{v_z\bar{F}(v_z')\delta(v_z-u)+v_z'\bar{F}(v_z)\delta(v_z'-u)}{v_z'-v_z}\,dv_z'\,. \tag{12}$$

Problem. Solve (7) directly for the case $u = c$.

3. The initial value problem for equation (4)

The most general solution of (4) that we can now construct is an arbitrary superposition of the stationary solutions

$$f_x(v_z, t) = \int C(u)g_x^u(v_z)\,e^{-ikut}\,du$$

$$E_{kx}(t) = (4\pi ie/k)\int C(u)u\,e^{-ikut}\,du$$

$$H_{ky}(t) = (4\pi ie/k)\int C(u)c\,e^{-ikut}\,du\,.$$

Our next problem is to show that this solution is complete in the sense that any initial state can be reproduced by a suitable choice of $C(u)$. In other words, it must be shown that the equations

$$f_x(v_z, 0) = \int_{-\infty}^{\infty} C(u)g_x^u(v_z)\,du \tag{13a}$$

$$E_{kx}(0) = (4\pi ie/k)\int_{-\infty}^{\infty} uC(u)\,du \tag{13b}$$

$$H_{ky}(0) = (4\pi iec/k)\int_{-\infty}^{\infty} C(u)\,du \tag{13c}$$

have a solution $C(u)$ for given $f_x(v_z, 0)$, $E_{kx}(0)$, $H_{ky}(0)$.

First note that it follows from (13b) that $C(u)$ must tend to zero as $u \to \pm\infty$ so rapidly that $uC(u)$ can be integrated. Put $C = C_+ + C_-$ and take u in I_+; then*

$$C_+(u) = \frac{1}{2\pi i}\int \frac{C(u')}{u'-u}\,du' = \frac{1}{2\pi i}\int C(u')\left\{-\frac{1}{u}-\frac{u'}{u^2}+\frac{u'^2}{u^2(u'-u)}\right\}du'$$
$$= \frac{b}{u}+\frac{a}{u^2}+o\left(\frac{1}{u^2}\right), \tag{14}$$

where

* The symbol $o(1)$ denotes terms that tend to zero as u goes to infinity in I_+.

$$b = -\frac{1}{2\pi i}\int C(u')\,du' = \frac{k}{8\pi^2 ec} H_{ky}(0) \tag{15}$$

$$a = -\frac{1}{2\pi i}\int u'C(u')\,du' = \frac{k}{8\pi^2 e} E_{kx}(0). \tag{16}$$

Problem. Prove that the remainder in (14) is actually $o(u^{-2})$.
Problem. Show

$$C_-(u) = -b/u - a/u^2 + o(u^{-2}). \tag{17}$$

Next we define a function $\bar{Z}(u)$ in analogy with the function $Z(u)$ in chapter XII

$$\bar{Z}(u) = u^2 - c^2 + 2\pi i u_p^2 u \bar{F}_+(u).$$

Clearly this function is holomorphic in I_+ and it can easily be shown that for $|u| \to \infty$

$$\bar{Z}(u) = u^2 - c^2 - u_p^2 + o(u^{-1}). \tag{18}$$

For real u its complex conjugate is

$$\bar{Z}^*(u) = u^2 - c^2 - 2\pi i u_p^2 u \bar{F}_-(u).$$

After these preliminary remarks, substitute (10a) with (11) (or, alternatively, (12)) in the equation (13a); by means of some manipulations one finds first

$$f_x(v_z, 0) = C(v_z)\{v_z^2 - c^2 - \pi u_p^2 v_z \bar{F}_*(v_z)\} - u_p^2 \bar{F}(v_z)\{2\pi i b + \pi v_z C_*(v_z)\},$$

which subsequently leads to

$$f_x(u, 0) + 2\pi i u_p^2 b \bar{F}(u) = \bar{Z}(u)C_+(u) + \bar{Z}^*(u)C_-(u).$$

The two terms on the right are holomorphic in I_+ and I_- respectively. Nevertheless this is not yet a decomposition in positive and negative frequency parts, because each term does not vanish at infinity. Rather one has according to (14), (17) and (18)

$$\bar{Z}(u)C_+(u) = bu + a + o(1),$$

$$\bar{Z}^*(u)C_-(u) = -bu - a + o(1).$$

Hence by subtracting $(bu+a)$ from the first term and adding the same amount to the second term one obtains a decomposition in positive and negative frequency parts. Since this decomposition is unique we obtain two equations

$$f_{x+}(u, 0) + 2\pi i u_p^2 b \bar{F}_+(u) = \bar{Z}(u)C_+(u) - a - bu$$

$$f_{x-}(u, 0) + 2\pi i u_p^2 b \bar{F}_-(u) = \bar{Z}^*(u)C_-(u) + a + bu.$$

These equations can be solved for C_+ and C_-, with the result

$$C(u) = \frac{f_{x+}(u, 0) + a + b\{u + 2\pi i u_p^2 \bar{F}_+(u)\}}{\bar{Z}(u)} + \frac{f_{x-}(u, 0) - a - b\{u - 2\pi i u_p^2 \bar{F}_-(u)\}}{\bar{Z}^*(u)}. \tag{19}$$

Thus $C(u)$ has been expressed in the initial reduced velocity distribution $f_x(u, 0)$ and the constants a and b, which through (15) and (16) are connected with the initial field components.

Problem. Verify the consistency of the solution (19) by showing that $\bar{Z}(u)$ has no zeros in I_+, and that $\bar{Z}^*(u)$ has no zeros in I_-.
Problem. Show that the explicit expression for $E_{kx}(t)$ at $t > 0$ in terms of the initial data is

$$E_{kx}(t) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{uE_{kx}(0) + cH_{ky}(0) + (8\pi^2 e/k)uf_{x+}(u, 0)}{\bar{Z}(u)} e^{-ikut}\,du \tag{20}$$

and find similar expressions for $H_{ky}(t)$ and $f_x(v_z, t)$.
Problem. Prove the following mathematical theorem, which was used implicitly in the preceding work. Let the square integrable function $F(u)$ be decomposed, $F = F_1 + F_2$, in such a way that F_1 is holomorphic in I_+ and F_2 in I_-. Let F_1 be of the form $F_1 = P_1 + f_1$ where P_1 is a polynomial and f_1 tends to zero at infinity in I_+. Similarly $F_2 = P_2 + f_2$. Then $P_1 = -P_2$ and $f_1 = F_+$ and $f_2 = F_-$.

4. The initial value problem for the linearized Vlasov equation

By collecting the results obtained, we are now in a position to solve the initial value problem for the full Vlasov equation, i.e., equations (1) and (2). Suppose $f^1(\boldsymbol{r}, \boldsymbol{v}, 0)$, $\boldsymbol{E}^1(\boldsymbol{r}, 0)$, $\boldsymbol{H}^1(\boldsymbol{r}, 0)$ are given arbitrarily, subject to the conditions (2b) and (2d). Our problem is to find $f^1(\boldsymbol{r}, \boldsymbol{v}, t)$, $\boldsymbol{E}^1(\boldsymbol{r}, t)$, $\boldsymbol{H}^1(\boldsymbol{r}, t)$

at all t. The unperturbed distribution $f^0(v)$ is fixed and will be supposed nowhere zero. We shall solve this problem in four steps.

1. First expand $f^1(r, v, 0)$, $E^1(r, 0)$, $H^1(r, 0)$ in Fourier integrals in space. Each value of k may be studied separately, so that henceforth k is a fixed vector. Take the z-axis in the direction of k.

2. Next form the reduced distribution functions $f_x(v_z, 0)$, $f_y(v_z, 0)$, $f_z(v_z, 0)$ according to (3). The evolution of these reduced distribution functions together with the field components is determined by (4), (5), (6). Note that $H_{kz} = 0$, because of (2d), while E_{kz} is uniquely coupled to f_z by (2b).

3. The work in the previous chapter now permits to find $f_z(v_z, t)$ and $E_{kz}(t)$ at all t. The work in the present chapter permits to find $f_x(v_z, t)$, $E_{kx}(t)$, $H_{ky}(t)$ and $f_y(v_z, t)$, $E_{ky}(t)$, $H_{kx}(t)$ at all t. Hence, by adding up the Fourier components the fields $E^1(r, t)$ and $H^1(r, t)$ are obtained for all t.

4. Finally the total distribution function $f^1(r, v, t)$ can be found by substituting the result for $E^1(r, t)$ in the original equation (1). The last member then becomes a known function, $K(r, v, t)$ say, and one has to solve the inhomogeneous equation

$$\left(\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r}\right) f^1(r, v, t) = K(r, v, t).$$

The solution is readily checked to be

$$f^1(r, v, t) = f^1(r - vt, v, 0) + \int_0^t K(r - vt + vt', v, t')\, dt'. \qquad (21)$$

This completes the solution of the initial value problem of the equations (1), (2).

Problem. Find the solution (21) by the method of characteristics.
Problem. Verify that the reduced distribution functions corresponding to (21) do obey (4), (5), (6).
Problem. Visualize (21) in physical terms.

5. CUT-OFF AT THE VELOCITY OF LIGHT

So far we have assumed $f^0(v) > 0$ for all v, which is unrealistic, since velocities greater than c cannot occur. The same objection could be raised against the treatment of longitudinal waves in the previous chapter, but in that case the particles with velocities greater than c have little effect. In the present

case of transverse waves, however, c occurs in the equations themselves and in fact, the zeros of $\bar{Z}(u)$ just below the real axis lie near $\pm c$. This is not surprising, because these transverse waves are ordinary electromagnetic waves modified by the presence of charged particles. In this situation it would be unrealistic to ignore the fact that $\bar{F}(v_z) = 0$ for $|v_z| > c$.

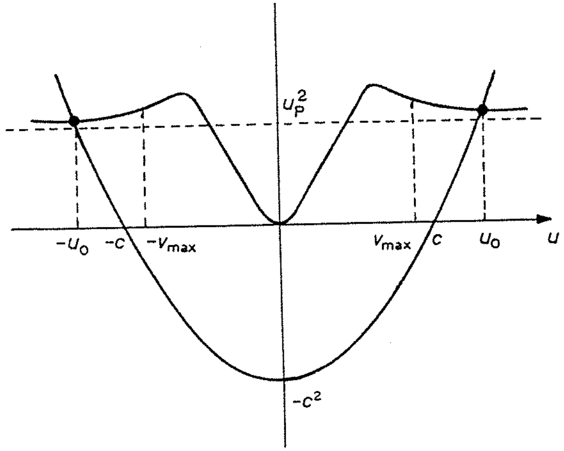

Fig. 28. Graphs of the functions $u^2 - c^2$ and

$$-u_P^2 u \mathrm{P} \int_{-c}^{c} \frac{\bar{F}(u')}{u' - u}\, du',$$

with their intersections at $\pm u_0$.

The condition $\bar{F}(v_z) > 0$ was used in the proof that $\bar{Z}(u)$ has no zeros in I_+ and on the real axis, owing to

$$\mathrm{Im}\, \bar{Z}(u) = \pi u_P^2 u \bar{F}(u) \qquad \text{for real } u.$$

We now have $\mathrm{Im}\, \bar{Z}(u) = 0$ for $|u| > c$, and hence $\bar{Z}(u)$ will have a zero on the real axis for each $u > c$, determined by

$$\mathrm{Re}\, \bar{Z} \equiv u^2 - c^2 + u_P^2 u \int_{-c}^{c} \frac{\bar{F}(u')}{u' - u}\, du' = 0. \qquad (22)$$

The graph in Fig. 28 makes clear that there are two zeros $\pm u_0$, where $u_0 > \sqrt{(c^2 + u_P^2)}$. It follows from (10) and (11) that there are two stationary undamped transverse wave solutions of the Vlasov equation, with phase velocities $\pm u_0$,

$$g_x^{\pm}(v_z) = u_p^2 u_0 \frac{\bar{F}(v_z)}{u_0 \mp v_z}$$

$$E_x^{\pm} = \pm 4\pi i e u_0/k \tag{23}$$

$$H_y^{\pm} = 4\pi i e c/k .$$

Problem. Prove that $\bar{Z}(u)$ has no other zeros on the real axis.

Problem. Show that there are transverse modes with any phase velocity $(-\infty < u < \infty)$, but that the only ones in which no electrons with velocity greater than c occur are those with $|u| < c$, and the two modes (23).

Problem. Study the longitudinal plasma waves for the case of a cut off velocity distribution $f^0(v) = 0$ for $v > v_{\max}$. Show that, if $v_{\max} < u_p$, the two Landau-damped waves are replaced with undamped waves, in agreement with (XII, 36). Show that the only longitudinal modes in which no electrons with velocity greater than $v_{\max}$ occur are those with $|u| < v_{\max}$, and the two above-mentioned undamped waves.

In section 3 the completeness was proved of the set of stationary solutions given by (10), where u took all real values. When we do not admit particle velocities greater than c the only available stationary solutions are the solutions (10) for $|u| < c$, plus the two solutions (23). It cannot be expected that this smaller set has the same completeness property. Nor would that be necessary: it is sufficient if this smaller set is complete for all initial distributions $f_x(v_z, 0)$ in the domain $-c < v_z < c$. In other words, it must be shown that the equations

$$f_x(v_z, 0) = \int_{-c}^{c} C(u) g_x^u(v_z)\, du + C^+ g_x^+(v_z) + C^- g_x^-(v_z), \qquad (|v_z| < c), \tag{24a}$$

$$E_{kx}(0) = \frac{4\pi i e}{k} \left[\int_{-c}^{c} u C(u)\, du + u_0 (C^+ - C^-) \right], \tag{24b}$$

$$H_{ky}(0) = \frac{4\pi i e c}{k} \left[\int_{-c}^{c} C(u)\, du + C^+ + C^- \right] \tag{24c}$$

have a solution $C(u)$, C^+, C^- for given $f_x(v_z, 0)$, $E_{kx}(0)$, $H_{ky}(0)$. This new

completeness property can easily be proved as a limiting case of the proof in section 3.

First we introduce a modified $\tilde{F}(v_z)$, which differs from $\bar{F}(v_z)$ only by the addition of two positive tails in the intervals $v_z > c$ and $v_z < -c$, so that $\tilde{F}(v_z) > 0$ for $-\infty < v_z < \infty$. The corresponding $\tilde{Z}(u)$ has two zeros $\pm u_0 - i\gamma$, where γ is very small when the tails are very low.

With this $\tilde{F}$ and $\tilde{Z}$ the problem (13) can be solved with the result (19). In particular this can be applied to an $f_x(v_z, 0)$ which is zero for $|v_z| > c$. In this case $f_{x+}(v_z, 0) + f_{x-}(v_z, 0) = 0$ for $|v_z| > c$, so that (19) gives

$$\tilde{C}(u) = \{f_{x+}(u, 0) + a + bu - \pi u_p^2 b \tilde{F}_*(u)\} \left(\frac{1}{\tilde{Z}(u)} - \frac{1}{\tilde{Z}^*(u)} \right)$$

$$+ i\pi u_p^2 b \tilde{F}(u) \left(\frac{1}{\tilde{Z}(u)} + \frac{1}{\tilde{Z}^*(u)} \right). \qquad (|u| > c) \tag{25}$$

Finally let the height of the two tails in F tend to zero. Clearly the second line in (25) vanishes in the limit. Moreover, for each u outside the interval $(-c, c)$ the limit of $\tilde{Z}(u)$ is $\bar{Z}(u)$, and hence real. In the vicinity of u_0 one has

$$\tilde{Z}(u) \approx (u - u_0 + i\gamma) \tilde{Z}'(u_0 - i\gamma) \approx (u - u_0 + i\gamma) \bar{Z}'(u_0)$$

$$\tilde{Z}^*(u) \approx (u - u_0 - i\gamma) \tilde{Z}'(u_0 + i\gamma) \approx (u - u_0 - i\gamma) \bar{Z}'(u_0)$$

so that

$$\frac{1}{\tilde{Z}(u)} - \frac{1}{\tilde{Z}^*(u)} \to \frac{1}{\bar{Z}'(u_0)} \left(\frac{1}{u - u_0 + i\gamma} - \frac{1}{u - u_0 - i\gamma} \right)$$

$$\to [-2\pi i / \bar{Z}'(u_0)] \delta(u - u_0) .$$

Hence in this vicinity (25) tends to

$$\tilde{C}(u) \to C(u) = \text{cst.}\ \delta(u - u_0) + \text{cst.}\ \delta(u + u_0) . \qquad (|u| > c)$$

It follows that the expansion (13) takes the form (24) in the limit when the artificial tail tends to zero.

Problem. Prove the symbolic formula

$$\lim_{\gamma \to 0} \left(\frac{1}{u - u_0 - i\gamma} - \frac{1}{u - u_0 + i\gamma} \right) = 2\pi i \delta(u - u_0) .$$

Problem. Verify that the equations of section 3 remain applicable in the case that the velocities are cut off at c, provided the two real zeros of

$\bar{Z}(u)$ are treated as if they lie an infinitesimal distance below the real axis.

6. The relativistic Vlasov equation

The simple device of cutting off the velocity distribution at $|\boldsymbol{v}| = c$ of course does not do full justice to relativistic effects. In this section we shall therefore start from the relativistic equations for the particles and fields, and show that with a minor change in notation the equations (4), (5), (6) are still valid.

The relativistic equations of motion of a particle with charge $-e$ and rest mass m in a given electromagnetic field may be written in the form

$$\frac{d\boldsymbol{r}}{dt} \equiv \boldsymbol{v} = \frac{\boldsymbol{p}}{\sqrt{(m^2+p^2/c^2)}}, \qquad \frac{d\boldsymbol{p}}{dt} = -e\boldsymbol{E} - \frac{e}{c}\boldsymbol{v}\wedge\boldsymbol{H}. \tag{26}$$

The density $f(\boldsymbol{r}, \boldsymbol{p}, t)$ of particles in $(\boldsymbol{r}, \boldsymbol{p})$-space obeys the Liouville equation

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{r}} - e\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\wedge\boldsymbol{H}\right)\cdot\frac{\partial f}{\partial \boldsymbol{p}} = 0. \tag{27}$$

Now express the same quantity f as a function of $\boldsymbol{r}, \boldsymbol{v}, t$, using the transformation formula

$$\frac{\partial f}{\partial \boldsymbol{p}} = \frac{1}{m}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}\left\{\frac{\partial f}{\partial \boldsymbol{v}} - \frac{1}{c^2}\boldsymbol{v}\left(\boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{v}}\right)\right\}.$$

Note that according to (26) $\boldsymbol{v}$ only takes values inside the sphere $|\boldsymbol{v}| = c$. This leads to the nonlinear relativistic Vlasov equation for the density in phase space, written as a function of $\boldsymbol{r}, \boldsymbol{v}, t$

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{r}} - \frac{e}{m}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\wedge\boldsymbol{H}\right)\cdot\left\{\frac{\partial f}{\partial \boldsymbol{v}} - \frac{\boldsymbol{v}}{c}\left(\frac{\boldsymbol{v}}{c}\cdot\frac{\partial f}{\partial \boldsymbol{v}}\right)\right\} = 0. \tag{28}$$

The charge density is

$$\tau(\boldsymbol{r}, t) = -e\int f\,d^3\boldsymbol{p} + en_0 = -e\int (1-v^2/c^2)^{-\frac{5}{2}} f\,d^3\boldsymbol{v} + en_0$$

and the electrical current density

$$\boldsymbol{j}(\boldsymbol{r}, t) = -e\int \boldsymbol{v}f\,d^3\boldsymbol{p} = -e\int (1-v^2/c^2)^{-\frac{5}{2}}\boldsymbol{v}f\,d^3\boldsymbol{v}.$$

The fields $\boldsymbol{E}$ and $\boldsymbol{H}$ obey Maxwell's equations with these sources (we suppose again that there are no external fields).

Problem. Derive the equations (26) from the Hamilton function

$$\mathscr{H} = \sqrt{\{m^2c^4 + (pc+eA)^2\}} - e\varphi,$$

where φ and A represent the scalar and vector potential of the field.
Problem. From the fact that (26) can be written in Hamiltonian form follows that Liouville's theorem (ch. XI, sec. 2) holds. Hence derive (27).

An obvious solution of (28) is again

$$f = n_0 f^0(v), \quad \boldsymbol{E} = 0, \quad \boldsymbol{H} = 0,$$

where $f^0(v)$ may be any function of $v = |\boldsymbol{v}|$ obeying

$$\int f^0(v)\,d^3\boldsymbol{p} \equiv \int (1-v^2/c^2)^{-\frac{5}{2}} f^0(v)\,d^3\boldsymbol{v} = 1.$$

The linearized equation for small deviations from this obvious solution is

$$\frac{\partial f^1}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f^1}{\partial \boldsymbol{r}} - \frac{n_0 e}{m}\boldsymbol{E}^1\cdot\frac{\partial f^0}{\partial \boldsymbol{v}}\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}} = 0.$$

We substitute a plane wave, choose the z-axis along $\boldsymbol{k}$, and define the reduced distribution functions

$$f_x(v_z, t) = \iint v_x f_k(\boldsymbol{v}, t)\,(1-v^2/c^2)^{-\frac{5}{2}}\,dv_x\,dv_y$$

$$f_y(v_z, t) = \iint v_y f_k(\boldsymbol{v}, t)\,(1-v^2/c^2)^{-\frac{5}{2}}\,dv_x\,dv_y$$

$$f_z(v_z, t) = \iint f_k(\boldsymbol{v}, t)\,(1-v^2/c^2)^{-\frac{5}{2}}\,dv_x\,dv_y.$$

These functions obey the equations

$$\frac{\partial f_j}{\partial t} + ikv_z f_j + \frac{n_0 e}{m}E_{kj}F_j(v_z) = 0, \qquad (j = x, y, z) \tag{29}$$

where

$$F_x(v_z) \equiv F_y(v_z) = \left(1 - \frac{v_z^2}{c^2}\right) \iint f^0(v) \left(1 - \frac{v^2}{c^2}\right)^{-2} dv_x\, dv_y$$

$$F_z(v_z) = - \iint \frac{\partial f^0(v)}{\partial v_z} \left(1 - \frac{v^2}{c^2}\right)^{-1} dv_x\, dv_y .$$

The charge and current densities are given by

$$\tau_k = -e \int f_z(v_z)\, dv_z , \qquad j_{k(x,y)} = -e \int f_{(x,y)}(v_z) dv_z .$$

The three equations (29), together with the Maxwell equations for the field, have the same form as (4), (5), (6). The only difference is that the functions $\bar{F}$ and F have been replaced with the slightly different functions $F_x \equiv F_y$ and F_z. Yet the new F_x is still an even, non-negative function of v_z, monotone decreasing when $v_z > 0$, and zero for $| v_z | > c$. Hence the theory in section 5 applies to the two equations obtained from (29) by taking $j = x$ and $j = y$.

F_z is again an odd function of v_z, non-negative and monotone decreasing when $v_z > 0$, and zero for $| v_z | > c$. Hence equation (29) with $j = z$ is covered by the theory of longitudinal waves with cut off $f^0(v)$.

Problem. Derive (29) in detail.
Problem. Show that for $0 \leqslant v_z \leqslant c$

$$F_x(v_z) \equiv F_y(v_z) = 2\pi \left(1 - \frac{v_z^2}{c^2}\right) \int_{v_z}^{c} f^0(v) \left(1 - \frac{v^2}{c^2}\right)^{-2} v\, dv$$

$$F_z(v_z) = -2\pi v_z \int_{v_z}^{c} f^{0\prime}(v) \left(1 - \frac{v^2}{c^2}\right)^{-1} dv .$$

Problem. Verify that the initial value problem for the linearized relativistic Vlasov equation can now be fully solved.

7. Dispersion law and index of refraction

The general solution for transverse waves is an arbitrary superposition of the stationary solutions found above. In particular, the electric field is, for fixed $\boldsymbol{k}$,

$$E_{kx}(t) = \frac{4\pi i e}{k} \left[\int_{-c}^{c} u C(u)\, e^{-ikut} du + u_0 C^+\, e^{-iku_0 t} - u_0 C^-\, e^{iku_0 t} \right] .$$

If $C(u)$ is a reasonably smooth function* the integral tends to zero for large t. After this transient has died out one is left with an undamped oscillation with frequency $\omega = ku_0$, determined by (22), or

$$\omega^2 - c^2k^2 - \omega_P^2 \omega \int_{|v| < c} \frac{f_0(v)}{\omega - \boldsymbol{k} \cdot \boldsymbol{v}}\, d^3\boldsymbol{v} = 0 . \tag{30}$$

Thus, like in the longitudinal case, *although there is no dispersion law for the individual stationary solutions, the asymptotically surviving oscillation* (which in this case is one of the stationary modes itself) *does obey a dispersion law*, viz. (30).

Problem. Show that the dispersion law (22) for the nonrelativistic case with cut off f^0 is equivalent to (30).
Problem. Express (30) in terms of the refractive index and show that it agrees with (IX, 30) when the wave length is long, or when the plasma is cold.
Problem. It is possible to satisfy (30) with an imaginary value for $\boldsymbol{k}$ and a real value for ω (which turns out to be less than ω_P). Show that this actually corresponds to a solution of (4). In this solution the refractive index is also imaginary and agrees with (IX, 30).
Problem. Show that for the *relativistic* case (30) changes into

$$\omega^2 - c^2k^2 - \omega_P^2 \omega \int \frac{1 - (\boldsymbol{k} \cdot \boldsymbol{v})^2/k^2c^2}{(1 - v^2/c^2)^2} \frac{f^0(v)}{\omega - \boldsymbol{k} \cdot \boldsymbol{v}}\, d^3\boldsymbol{v} = 0 . \tag{31}$$

Note that this reduces to (30) when the average velocity in f^0 is not too high; in particular, for Maxwellian f^0, if $kT \ll mc^2$.
Problem. By expanding the dispersion formula in the previous problem find (for real $\boldsymbol{k}$)

$$\omega^2 = \omega_P^2 \left(1 - \frac{5}{6} \frac{\langle v^2 \rangle}{c^2}\right) + c^2k^2 + \tfrac{1}{3}\langle v^2 \rangle \frac{\omega_P^2 k^2}{\omega_P^2 + c^2k^2} + \mathrm{O}\left(\frac{\langle v^2 \rangle}{c^2}\right)^2 .$$

Note that the limiting frequency for $k \to 0$ is lowered by a relativistic correction.
Problem. Find the exact relativistic value of ω for $k = 0$ and show that it coincides with the frequency of longitudinal plasma waves in the limit $k = 0$.

* Compare the discussion in ch. XII, sec. 5.

In the ordinary theory of propagation of light through a medium the modification of the electromagnetic waves by the presence of the medium is described by introducing the refractive index $n(\omega) = ck/\omega$. This quantity not only determines the angle of refraction, but also the intensities of the refracted and the reflected light (Fresnel formulas). The same formulas can be used to compute the refraction and reflection of electromagnetic radiation incident on a plasma, the refractive index being given by (IX, 30) or (30). In particular one finds that for $|\omega| < \omega_P$ the wave length in the medium is imaginary, so that no energy can penetrate and total reflection occurs. Also the bending of rays in an inhomogeneous plasma with slowly varying refractive index (ionosphere, corona) can be treated in the same way as light in an inhomogeneous medium.

However, the description of a plasma in terms of a refractive index is not quite correct. There are two reasons why Fresnel's formulas for reflection and refraction do not readily apply. *The first reason* is that these formulas presuppose a sharp boundary, i.e., the refractive index must drop from its value in the medium to its vacuum value in a distance short compared to the wave length. This is not true for most plasmas, but it is true for electrons in a metal or semiconductor.

Remark. Instead of confining the electron gas by an electrostatic force one may also envisage the possibility of magnetic confinement. Let there be a constant magnetic field $\boldsymbol{H}^0$ outside the plasma. In order that the magnetic pressure balances the gas pressure one must have

$$(\boldsymbol{H}^0)^2/8\pi = n_0 kT.$$

The electrons arriving at the surface are curved back into the plasma with a radius of curvature of the order of the gyromagnetic radius in the field $\boldsymbol{H}^0$:

$$\frac{m_e c}{eH^0}\left(\frac{kT}{m_e}\right)^{\frac{1}{2}} = \left(\frac{m_e c^2}{8\pi n_0 e^2}\right)^{\frac{1}{2}} \approx \frac{c}{\omega_P}. \tag{32}$$

This is roughly the thickness of the boundary layer; it is small compared to the wave length $c/\sqrt{(\omega^2-\omega_P^2)}$ when ω is close to ω_P.

However, the problem of the plasma boundary is more complicated for several reasons. First, the presence of ions cannot be neglected. Since they have a larger gyromagnetic radius than the electrons, they will penetrate deeper in the magnetic field; hence a charge separation occurs, which gives rise to an electrostatic field perpendicular to the surface.

Secondly, the actual problem is not merely to compute the motion of the particles in the electric and magnetic fields, but also to choose these fields in such a way that they are compatible with the charge and current densities due to the particles. Thirdly, in addition to the particles arriving at the boundary from the interior of the plasma, there may be "trapped particles", i.e., particles which never move far away from the boundary. No satisfactory solution of the whole problem has yet been given.*

Problem. Suppose for the sake of argument that the magnetic field has a sharp boundary, $H_z(x) = 0$ for $x < 0$, $H_z(x) = H^0 =$ const. for $x > 0$. Show that the surface current due to the particles that are curved back into the plasma is just the one needed for the drop in the magnetic field strength. Why is this nonetheless not a consistent solution of the boundary layer problem?

Problem. Give a mathematical formulation for the plasma boundary layer problem by writing down two (non-linear, time-independent) Vlasov equations for the electrons and for the ions, and assuming all quantities independent of y and z.

If there is a sharp boundary one may assume that the electrons undergo specular reflection at that boundary, that is, the normal component of the velocity changes sign and the tangential components remain unaltered. This amounts to a boundary condition for $f(\boldsymbol{r}, \boldsymbol{v}, t)$, which has the form

$$f(x, y, 0, v_x, v_y, v_z, t) = f(x, y, 0, v_x, v_y, -v_z, t),$$

when the surface is taken as xy-plane. If the boundary is not sharp, it cannot be described simply in terms of a boundary condition for $f(\boldsymbol{r}, \boldsymbol{v}, t)$.

The second reason why Fresnel's formulas are, strictly speaking, not applicable even to plasmas with sharp boundaries, is that the description of the transverse plasma waves by means of a refractive index is incomplete. For each direction of propagation and each frequency ω there is one stationary solution of the Vlasov equation that obeys the dispersion law. In addition, however, there exists a continuum of other stationary modes with phase velocities ranging from 0 to c. It has been shown in this and the previous chapter that these cannot be omitted when the state of the plasma is calculated from a given initial state. Similarly they cannot be omitted when the situation inside the plasma is calculated with given boundary conditions.

Owing to these additional modes, the state of the plasma is not uniquely

* C. L. Longmire, *Elementary Plasma Physics* (Interscience, New York 1963) ch. v.

determined by the incident wave, but further conditions have to be imposed to specify the state. These conditions express the requirement that the plasma is at rest before the incident radiation has hit the plasma. That is, in the unperturbed plasma one must have $f^1 = 0$, $\boldsymbol{E}^1 = 0$, $\boldsymbol{H}^1 = 0$. For, if that were not the case, there would in general be some radiation emerging from the plasma, even if there is no incident wave. Consequently the reflection problem would not be well defined. The condition that no radiation should emerge before the incident waves have reached the plasma is called "*causality condition*".

Clearly the application of this causality condition requires a *time-dependent* description of the reflection and refraction process. On the other hand, the most convenient description of such a process is in terms of a *stationary* state, consisting of a superposition of stationary modes of one and the same frequency ω but different wave lengths. Fortunately there is an equivalent condition which may be imposed instead of the causality condition, namely the condition that in the stationary superposition only waves may participate that propagate in a direction from the surface into the plasma. Moreover, for $|\omega| < \omega_P$ those solutions that increase exponentially into the interior must be excluded.

It has been shown* that this condition, together with the boundary condition for $f(\boldsymbol{r}, \boldsymbol{v}, t)$ at the surface determine uniquely the situation in the plasma when the incident wave is given. (The direction of the incident wave is taken normal to the surface**.) If one subsequently constructs a time-dependent solution by adding such stationary solutions, each multiplied with its own time factor $e^{-i\omega t}$, this time-dependent solution obeys the causality condition.

It is actually possible to find the stationary states with a prescribed incident electromagnetic wave and obeying the above mentioned requirement. It turns out that the single mode that obeys the dispersion law predominates, but there is an admixture of the other modes with the same ω. As a consequence the reflection and transmission coefficients deviate slightly from the values given by Fresnel's formulas. The physical reason is that the electrons in a plasma move freely under influence of the field, while the electrons in a dielectric medium are elastically bound to their equilibrium positions. This deviation from classical optics is called the *anomalous skin effect*.

* B. U. Felderhof, Physica **29**, 662 (1963); a slightly different approach was used by F. C. Shure, Thesis, University of Michigan (1962).
** The case of oblique incidence has been studied by V. P. Silin and E. P. Fetisov, Soviet Phys.-JETP **14**, 115 (1962), and by V. H. Weston, Phys. Fluids **10**, 632 (1967).

The normal skin effect in metals (or other electrical conductors) is an electromagnetic effect and can be fully described by Maxwell's equations and Ohm's law. For the anomalous skin effect, however, it is necessary to take the electron gas into account as a separate entity, described by the distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$. This function obeys the Vlasov equation, supplemented with a term which roughly describes collisions with impurities. The theory of the anomalous skin effect in metals has been worked out by Reuter and Sondheimer* in a different way. They also allowed for the possibility of diffuse reflection of the electrons at the boundary, which agrees better with experiments than pure specular reflection.

Problem. A plasma fills the half space $z > 0$, and we assume specular reflection at the boundary $z = 0$. Show that a solution of the Vlasov equation with the Maxwell equations can be obtained by taking a solution $\tilde{f}$, $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{H}}$ in the entire space, and putting

$$f(x, y, z, v_x, v_y, v_z, t) = \tilde{f}(x, y, z, v_x, v_y, v_z, t) + \tilde{f}(x, y, -z, v_x, v_y, -v_z, t),$$

with similar equations for $\boldsymbol{E}$ and $\boldsymbol{H}$. Find the corresponding field in the empty half space $z < 0$. Show that this solution does not obey the causality condition.

General references

T. Pradhan, Phys. Rev. **107**, 1222 (1957).
R. S. Zelazny, Ann. Phys. (N.Y.) **20**, 261 (1962).
B. U. Felderhof, Physica **29**, 293, 662 (1963).

* G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) **195A**, 336 (1949).

CHAPTER XIV

ANISOTROPY AND EXTERNAL MAGNETIC FIELD

Anisotropy arises when either $f^0(\boldsymbol{v})$ is not an isotropic function of $\boldsymbol{v}$, or when there is an external magnetic field* $\boldsymbol{H}^0$, or both. We shall assume throughout that $\boldsymbol{H}^0$ is constant, independent of $\boldsymbol{r}$ and t, and parallel to the z-axis. In that case one has according to the Vlasov equation (XI, 1)

$$(\boldsymbol{v}\wedge\boldsymbol{H}^0)\cdot\frac{\partial f^0}{\partial \boldsymbol{v}} = 0 .$$

When $\boldsymbol{H}^0 \neq 0$ the general solution is

$$f^0(\boldsymbol{v}) = f^0(v_x^2+v_y^2,\, v_z) . \qquad (1)$$

Thus, *when there is a constant external field, f^0 is necessarily isotropic about the direction of the field.* The reason is that each electron gyrates about this direction, in addition to its velocity along the field lines.

In true thermal equilibrium, of course, f^0 is isotropic in all three directions, but in many practical cases time is too short for the collisions to establish local thermal equilibrium. In such cases f^0 will in general depend on v_z in a different way than on v_x and v_y. For example, consider the pinch discharge described in ch. IV, sec. 3. The plasma column is rapidly compressed by the azimuthal field outside the plasma. As a result the velocity component perpendicular to the plasma boundary is increased by each collision with the boundary. If there is also an external field $\boldsymbol{H}^0$ inside the plasma parallel to the z-axis, the distribution in v_x and v_y will therefore change but must remain isotropic, while v_z will not increase until a sufficient number of collisions have occurred.

Thus the presence of an external field $\boldsymbol{H}^0$ and the anisotropy of f^0 often occur simultaneously; but we shall study them separately in sections 2 and 3 and discuss their combined effect in section 4. An important feature of anisotropy of f^0 is that it gives rise to various instabilities. A few of them will be mentioned, without explicit calculation.

* "External" means due to given external sources, see ch. XI, sec. 1.

In the presence of anisotropy the linearized Vlasov equation cannot be reduced to three separate sets of equations for the three polarizations. It is still true that for an arbitrary propagation vector $\boldsymbol{k}$ there are three modes, but in general, none of them is purely longitudinal or purely transverse. As a consequence the general solution is very involved.* However, we shall only study two special cases in which there still is isotropy about the direction of $\boldsymbol{k}$:

(i) $\boldsymbol{H}^0 \neq 0$, so that f^0 has the form (1); and $\boldsymbol{k}$ also in the z-direction,
(ii) $\boldsymbol{H}^0 = 0$, $\boldsymbol{k}$ in the z-direction and *assuming* an f^0 of the form (1).

In these cases the separation in one longitudinal and two transverse modes is still possible.

Problem. Suppose that a plasma is so dilute that all interaction between the particles may be neglected.** Give the equation for $f(\boldsymbol{r}, \boldsymbol{v}, t)$ in the presence of a constant, homogeneous external field $\boldsymbol{H}^0$. Find its complete solution.

1. The linearized Vlasov equation with external field

The total magnetic field is

$$\boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{H}^0+\boldsymbol{H}^1(\boldsymbol{r}, t),$$

where $\boldsymbol{H}^0$ is a constant vector in the z-direction. The unperturbed static solution $f^0(\boldsymbol{v})$ must be of the form (1) if $\boldsymbol{H}^0 \neq 0$, and is assumed to be of that form if $\boldsymbol{H}^0 = 0$. The linearized equation for $f^1(\boldsymbol{r}, \boldsymbol{v}, t)$ is

$$\frac{\partial f^1}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f^1}{\partial \boldsymbol{r}} - \frac{e}{mc}(\boldsymbol{v}\wedge\boldsymbol{H}^0)\cdot\frac{\partial f^1}{\partial \boldsymbol{v}} - \frac{n_0 e}{m}\left(\boldsymbol{E}^1 + \frac{1}{c}\,\boldsymbol{v}\wedge\boldsymbol{H}^1\right)\cdot\frac{\partial f^0}{\partial \boldsymbol{v}} = 0 . \qquad (2)$$

In addition one has Maxwell's equations (XIII, 2) which determine $\boldsymbol{E}^1$ and $\boldsymbol{H}^1$ in terms of f^1. It is again possible to find wave solutions by setting f^1, $\boldsymbol{E}^1$, $\boldsymbol{H}^1$ proportional to $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$. We only consider the case $\boldsymbol{k}//\boldsymbol{H}^0$.

We use again the reduced distribution functions f_x, f_y, f_z defined in (XIII, 3) and put

$$f_r = f_x+if_y, \qquad f_l = f_x-if_y$$
$$E_r = E_{kx}+iE_{ky}, \qquad E_l = E_{kx}-iE_{ky}$$

* I. B. Bernstein, Phys. Rev. **109**, 10 (1958) treats the general case for isotropic Maxwellian distribution f^0 in a field $\boldsymbol{H}^0$.
** According to the definition in chapter I this should not be called a plasma, but rather an ensemble of particles.

$$H_r = H_{kx} + iH_{ky}, \qquad H_l = H_{kx} - iH_{ky}.$$

On multiplying (2) with $v_x + iv_y$ and $v_x - iv_y$ and integrating over v_x and v_y one finds in place of (XIII, 4) and (XIII, 5) two new sets of equations for f_r, E_r, H_r and f_l, E_l, H_l respectively. Their solutions will be right and left circularly polarized waves. The first set is

$$\frac{\partial f_r}{\partial t} + ikv_z f_r - i\omega_{ce} f_r = -\frac{n_0 e}{m}\left\{E_r \bar{F}(v_z) + \frac{1}{c} H_r G(v_z)\right\}, \tag{3a}$$

$$\frac{\partial E_r}{\partial t} = -ckH_r + 4\pi e \int f_r(v_z, t)\, dv_z, \tag{3b}$$

$$\frac{\partial H_r}{\partial t} = ckE_r. \tag{3c}$$

Here ω_{ce} is the electron cyclotron frequency, $\bar{F}(v_z)$ is the function occurring in (XIII, 4a), and

$$G(v_z) \equiv v_z \bar{F}(v_z) + \frac{1}{2}\frac{d}{dv_z}\iint (v_x^2 + v_y^2) f^0\, dv_x\, dv_y. \tag{4}$$

The second set of three equations has the same form as (3), but the subscript r is replaced with l and i is replaced with $-i$. The third set obtains on integrating (2) over v_x and v_y; it is clear that the term with $\boldsymbol{H}^0$ disappears, and one is left with the same equation (XIII, 6) as in the isotropic case. The solutions of this set are again longitudinal waves, which are clearly not affected by the external magnetic field parallel to their direction of propagation; the only difference with the work in chapter XII is that $f^0(\boldsymbol{v})$ is not isotropic.

Problem. Show that $G(v_z) = 0$ for isotropic f^0.

Problem. Find the analog of (3) for the isotropic case with $\boldsymbol{H}^0 = 0$, and show that its solutions are right polarized waves (compare the definition on page 111).

2. Isotropic equilibrium distribution in external field

Since in this case $G(v_z) \equiv 0$, the equations (3) can be solved in the same way as (XIII, 4). The result is again that there is a stationary wave solution for all real values of ω and k. If $k > 0$ and $\omega > 0$ (i.e., if the wave propagates in the direction of $\boldsymbol{H}^0$) the expression for the electric field, analogous with (XIII, 20), is

$$E_r(t>0) = -\frac{1}{2\pi i}\int \frac{e^{-ikut}}{\bar{Z}_r(u)}\,[uE_r(0) - icH_r(0) + (8\pi^2 e/k)uf_{r+}(u+u_{ce}, 0)]\, du. \tag{5}$$

We have used the notations $u_{ce} = \omega_{ce}/k$ and

$$\bar{Z}_r(u) \equiv u^2 - c^2 + 2\pi i u_P^2 u \bar{F}_+(u + u_{ce}). \tag{6}$$

This function has no zeros in the upper half plane. It may have zeros in the lower half plane; if they lie near to the real axis they represent damped waves that survive after an initial transient effect has died out. The dispersion law for these waves is therefore

$$\bar{Z}_r(k, u) = 0. \tag{7}$$

If $f^0(\boldsymbol{v})$ is cut off (i.e., $f^0(\boldsymbol{v}) = 0$ for $v > v_{max}$) it may happen that a zero of (7) lies on the real axis; in that case the surviving wave is undamped. The integration path in (5) must then be taken slightly above this pole.

The solutions described by (5), and the damped waves obeying (7) are right circularly polarized. The second set of equations lead to left circularly polarized waves obeying the dispersion law

$$\bar{Z}_l(k, u) \equiv u^2 - c^2 + 2\pi i u_P^2 u \bar{F}_+(u - u_{ce}) = 0.$$

For a mixture of different kinds of charged particles, labeled by ν, the dispersion law for the right circularly polarized waves takes the form

$$u^2 - c^2 + 2\pi i u \sum_\nu u_{P\nu}^2 \bar{F}_{\nu+}(u \mp u_{c\nu}) = 0, \tag{8}$$

with $-$ for particles of positive charge and $+$ for particles with negative charge. For left circularly polarized waves the same equation applies with these signs interchanged.

Problem. Consider a plasma of electrons and one kind of positive ions, and suppose it is cold, i.e., $\bar{F}_e(v_z)$ and $\bar{F}_i(v_z)$ are delta-functions. In this case (8) reduces to

$$u^2 - c^2 - \frac{u_{Pe}^2 u}{u + u_{ce}} - \frac{u_{Pi}^2 u}{u - u_{ci}} = 0. \tag{9}$$

Show that for $m_e \ll m_i$ this is equivalent with (IX, 31).

For most frequencies the dispersion law (8) only adds a small correction to the refractive index given by the two-component theory, (IX, 32). However, whereas (IX, 32) has two poles on the real axis near ω_{ce} and ω_{ci}, the corresponding poles of (8) lie slightly below the real axis. As a consequence the refractive index no longer becomes infinite at the electron and ion resonances, but takes a large complex value. This corresponds to a damping of the same kind as the Landau damping; these effects are called the *electron cyclotron damping* and the *ion cyclotron damping*.

The effect of this damping is to dissipate energy, that is, to transfer energy from the electromagnetic field of the wave to the disordered heat motion of the particles. This effect is utilized to raise the temperature of the plasma (which is of course important for thermonuclear reactions). Since the energy transfer from the electron gas to the ion gas is slow (cf. the Remark in ch. IX, sec. 4), in most cases the ion cyclotron damping is utilized in order that the ions profit directly from the dissipated energy.

An experimental way for achieving this is the following. $\boldsymbol{H}^0$ is made not entirely homogeneous, but slowly decreasing in strength in the z-direction. As a consequence the plasma density slowly increases, owing to (IV, 10). One now generates electromagnetic waves in the low density region, whose frequency corresponds to the ion cyclotron frequency of the high density region. These waves travel along the z-direction into the high density region, where resonance occurs and their energy is dissipated. This device is called "*magnetic beach*".

3. Anisotropic equilibrium distribution, no external field

Suppose that $\boldsymbol{H}^0 = 0$, but that nevertheless f^0 has the form (1). This situation occurs for instance when two electron streams move through each other. The equations (3) apply, with $\omega_{ce} = 0$ but $G(v_z) \neq 0$. Both right and left circularly polarized waves obey the same dispersion law. For the electron gas this law has the form $\bar{Z}(u) = 0$, where

$$\bar{Z}(u) \equiv u^2 - c^2 + 2\pi i u_P^2 \{u\bar{F}_+(u) - G_+(u)\}. \tag{10}$$

For a mixture of several components the dispersion law for transverse waves is

$$u^2 - c^2 + 2\pi i \sum_\nu u_{P\nu}^2 \{u\bar{F}_{\nu+}(u) - G_{\nu+}(u)\} = 0. \tag{11}$$

Similarly the dispersion law for longitudinal waves in a mixture is (comp. (XII, 22))

$$1 + 2\pi i \sum_\nu u_{P\nu}^2 F_{\nu+}(u) = 0.$$

For anisotropic f^0 it is no longer possible to prove that (10) has no zeros in I_+, and in fact such zeros may occur when the anisotropy is strong enough. That means that there may be solutions of the type $e^{ik(z-ut)}$, where u is a complex number in I_+; they grow exponentially in time and are therefore unstable. These solutions can not be discarded, because without them the remaining stationary solutions do not form a complete set.† An arbitrary initial distribution is a superposition of *all* modes, including these exponentially growing ones. Hence, as soon as there exists such an exponentially increasing solution of the linearized Vlasov equation, the plasma is unstable. This kind of instability, due to the anisotropy of f^0, is clearly outside the scope of magnetohydrodynamics, and is therefore a micro-instability.

Problem. In chapter XII it was proved for longitudinal waves and isotropic $f^0(v)$ that $Z(u)$ has no zeros in I_+. At which point does this proof break down for anisotropic $f^0(\boldsymbol{v})$?

Problem. In addition to the zeros of $\bar{Z}(u)$ in I_+ one must also take into account the zeros of $\bar{Z}^*(u)$ in I_-, in order to complete the set of eigensolutions. Show that the latter are the complex conjugates of the former, and that all these zeros together can be found from the equation

$$u^2 - c^2 + u_P^2 \int \frac{u\bar{F}(v_z) - G(v_z)}{v_z - u} dv_z = 0$$

(for the case of transverse waves in the electron gas).

Problem. Prove that (3) is invariant for time reversal and hence that to any eigensolution with frequency ω there corresponds an eigensolution with frequency ω^*.

As a simple example we consider the longitudinal waves in an electron gas consisting of two interpenetrating streams of electrons with velocities V and $-V$ in the z-direction, each having a density $\frac{1}{2}n_0$ and zero temperature:

$$f^0(\boldsymbol{v}) = \tfrac{1}{2}\delta(v_x)\delta(v_y)\{\delta(v_z - V) + \delta(v_z + V)\}. \tag{12}$$

On substituting this f^0 in (XI, 17) one obtains for longitudinal waves in the z-direction the dispersion law

$$\frac{1}{(\omega - kV)^2} + \frac{1}{(\omega + kV)^2} = \frac{2}{\omega_P^2}.$$

† K. M. Case, Ann. Phys. (N.Y.) 7, 349 (1959).

This equation has four real roots when $kV > \omega_P$, but for $kV < \omega_P$ there are two conjugate complex roots. Hence this plasma is unstable for long wave longitudinal oscillations. This is called the "*two-stream instability*".

Next suppose the velocity distributions of both streams are no longer delta functions, but have a certain width, i.e., both streams have positive temperature. In that case it is found that the plasma is stable if the mutual velocity V of the two streams is small, and unstable if V is larger than a certain critical velocity V_{cr}. This V_{cr} is of the order of the thermal velocity in the streams.

A plasma of electrons and ions is unstable when the electrical current density is too high – which is another form of two-stream instability. Again the critical velocity is zero when both the electrons and ions have zero temperature. When they both have the same positive temperature, V_{cr} is of the order of the thermal velocity of the electrons. When the ions have zero temperature, V_{cr} is of the order of the thermal velocity of the electrons times $\sqrt{(m_e/m_i)}$.*

Problem. Prove the "Penrose criterion"**: Longitudinal waves with growing amplitude exist if and only if $\bar{F}(v_z)$ has at least one local maximum $v_z = v_z^m$ for which

$$\int_{-\infty}^{\infty} \frac{\bar{F}(v_z)-\bar{F}(v_z^m)}{(v_z-v_z^m)^2}\, dv_z > 0 .$$

(*Hint*: This condition states that the curve $Z(u)$ in the Nyquist diagram intersects the real axis somewhere to the left of $Z = 1$, and hence it encircles the origin for some value of k.)

4. Anisotropic equilibrium distribution with external field

Constant homogeneous $\boldsymbol{H}^0$ in the z-direction and f^0 of the form (1). The equations (3) apply, both ω_{ce} and $G(v_z)$ being different from zero. The dispersion law for right circularly polarized transverse waves in an electron gas is

$$\bar{Z}_r(u) \equiv u^2-c^2+2\pi i u_P^2\{u\bar{F}_+(u+u_{ce})-G_+(u+u_{ce})\} = 0 . \qquad (13)$$

For left circularly polarized waves u_{ce} in this equation must be replaced

* E. A. Jackson, Phys. Fluids 3, 786 (1960).
** O. Penrose, Phys. Fluids 3, 258 (1960).

with $-u_{ce}$. For right circularly polarized waves in a mixture of several components,

$$u^2-c^2+2\pi i \sum_\nu u_{P\nu}^2\{u\bar{F}_{\nu+}(u \mp u_{c\nu})-G_{\nu+}(u \mp u_{c\nu})\} = 0; \qquad (14)$$

the $-$ sign is to be taken when component ν carries a positive charge, and the $+$ for components with negative charge. For left circularly polarization these signs have to be interchanged.

Again there may occur complex roots of the dispersion law, corresponding to unstable modes. Suppose that the electrons and the ions drift along the field lines in opposite directions, so that there is an electrical current in the direction of $\boldsymbol{H}^0$. (The magnetic field due to this current is neglected.) The same two-stream instability that was mentioned in the previous section exists in this case when the relative velocity of electrons and ions is larger than some critical V_{cr}. However, it has been shown* that additional unstable modes occur when wave vectors $\boldsymbol{k}$ are considered that are not parallel to $\boldsymbol{H}^0$. Moreover these have a smaller critical velocity than those with $\boldsymbol{k}//\boldsymbol{H}^0$, so that instability sets in already at a much lower value of the electrical current density than in the absence of $\boldsymbol{H}^0$. As mentioned above, the dispersion law for these modes is very involved, since a separation of longitudinal and transverse modes is no longer possible.

Problem. Show with the aid of (13) that in the presence of $\boldsymbol{H}^0$ the distribution (12) is also unstable for transverse waves.

In order to have an explicit form of an anisotropic f^0 one sometimes writes

$$f^0(v) = \left(\frac{m}{2\pi kT_\perp}\right)\left(\frac{m}{2\pi kT_\parallel}\right)^{\frac{1}{2}} \exp\left[-\frac{m(v_x^2+v_y^2)}{2kT_\perp}-\frac{mv_z^2}{2kT_\parallel}\right]. \qquad (15)$$

The parameters $T_\perp$ and $T_\parallel$ are called the transverse and longitudinal temperatures, and correspondingly one has transverse and longitudinal pressures $p_\perp = n_0kT_\perp$, $p_\parallel = n_0kT_\parallel$. The form (15) for f^0 is at best an approximation, because it is hard to see why the collisions should be more effective in establishing a Maxwell distribution in two directions than in all three.

On substituting (15) in (14) one finds that in a plasma of electrons and a single kind of ions the Alfvén waves are unstable if

$$p_\parallel > p_\perp+(H^0)^2/4\pi .$$

* W. E. Drummond and M. N. Rosenbluth, Phys. Fluids 5, 1507 (1962).

The reason is that a small bulge in the magnetic field lines has a tendency to grow, owing to the centrifugal force exerted by the particles that move along the field lines. This instability is therefore appropriately called the *fire hose* or *garden hose instability*. The above criterion can be roughly understood by noting that $p_{\parallel}$ is a measure for the particle velocity along the field line, $H^2/4\pi$ represents the magnetic force tending to straighten the field line, and $p_{\perp}$ is an additional stiffness due to the particles gyrating around the field line.

A related kind of instability is the *mirror instability*. In MHD we have found longitudinal waves with $\boldsymbol{k} \perp \boldsymbol{H}^0$ (magnetosonic waves, ch. VI, sec. 5). In the Vlasov theory with *isotropic* f^0, analogous waves are found, which obey, however, a more complicated dispersion law. This law also includes damping ("transit time damping"), which is due to the transfer of particles in the direction of the wave, similar to Landau damping. In Vlasov theory with *anisotropic* f^0 of the form (15) the dispersion law has imaginary solutions for ω if

$$p_{\parallel} < \frac{p_{\perp}^2}{p_{\perp} + (H^0)^2/8\pi}.$$

This is therefore the criterion that the plasma suffers from mirror instability.

Problem. Since in the magnetosonic wave $\boldsymbol{k} \perp \boldsymbol{H}^0$, these waves are not covered by the work in this chapter. Derive from (2) an equation for waves propagating perpendicular to $\boldsymbol{H}^0$.

The instabilities mentioned in this chapter cannot be derived from the MHD equations and are therefore called micro-instabilities. On the other hand, they can only be derived from the Vlasov equation at the expense of a number of simplifying approximations. A complete list of these micro-instabilities for different f^0 and different angles between $\boldsymbol{k}$ and $\boldsymbol{H}^0$ has not yet been given. A new class of micro-instabilities is found when the Vlasov theory is applied to a spatially inhomogeneous plasma, i.e., when f^0 also depends on $\boldsymbol{r}$; these are the so-called "universal instabilities".*

* N. A. Krall and M. N. Rosenbluth, Phys. Fluids **8**, 1488 (1965).

General references

T. Pradhan, Phys. Rev. **107**, 1222 (1957).
I. B. Bernstein, Phys. Rev. **109**, 10 (1958).
I. B. Bernstein, S. K. Trehan and M. P. H. Weenink, Nucl. Fusion **4**, 61 (1964).
B. U. Felderhof, Physica **29**, 317 (1963).
W. P. Allis, S. J. Buchsbaum and A. Bers, *Waves in Anisotropic Plasmas* (M.I.T. Press, Cambridge, Mass. 1963).
D. C. Montgomery and D. A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill, New York 1964) chs. IX, X.
E. G. Harris, J. Nucl. Energy **C2**, 138 (1961).
A. A. Vedenov, E. P. Velikhov and R. Z. Sagdeev, Soviet Phys.-Usp. **4**, 332 (1961).
T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill, New York 1962) ch. IX.

CHAPTER XV

COLLISIONS

All effects of the interaction between particles that have not yet been taken into account by the smeared-out field in the Vlasov equation are comprised in the term "collisions". They give rise to the right-hand side of the equation (x, 1), which for the one-component electron gas takes the form

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{e}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \wedge \boldsymbol{H}\right) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \left[\frac{\partial f}{\partial t}\right]_c . \tag{1}$$

When the collision term predominates, the fluid description of magnetohydrodynamics is applicable. When the collisions are negligible the Vlasov theory of chapters XI through XIV applies.

When the collisions are not negligible but relatively rare, they can be treated as corrections to be added to the Vlasov theory. This is the most realistic description of hot dilute plasmas. In the present chapter we give Landau's derivation of the collision term and outline the method of Bogolyubov; futher information can be found in the monographs of Balescu and of Montgomery and Tidman.

Remark. As a crude approximation one sometimes writes

$$\left[\frac{\partial f}{\partial t}\right]_c = -\frac{f(\boldsymbol{r}, \boldsymbol{v}, t) - f^0(v)}{\tau}, \tag{2}$$

where f^0 is the equilibrium distribution and τ is a constant of the order of the average time between collisions ("time of free flight"). It then seems surprisingly easy to find damped plasma waves without the elaborate work of chapters XII and XIII. However, this approximation is not only crude but also inconsistent, because (2) violates the conservation laws for particles, momentum, and energy.

The term (2) is also sometimes added to the Vlasov equation as a mathematical device to derive plasma waves in a less elaborate way; one then puts $\tau = \infty$ in the final result. However, the solutions obtained in this

simple way are not a complete set: they permit to reproduce at $t = 0$ an arbitrary field, but not an arbitrary particle distribution $f^1(\boldsymbol{r}, \boldsymbol{v}, 0)$.

An improved approximate expression for the collision term, which does not violate the conservation laws, can be constructed by replacing $f^0(v)$ in (2) by a Maxwell distribution in which the particle density, average velocity, and temperature are chosen equal to the corresponding five moments of $f(\boldsymbol{r}, \boldsymbol{v}, t)$.* This model has been much studied in rarefied gas theory. It may be used to obtain a qualitative insight in the effect of collisions on plasma waves, but the analysis is of course more involved than with the use of (2).

Problem. Show that the addition of (2) gives rise to an extra damping term $e^{-t/\tau}$ in the solution, but does not otherwise affect it.

Problem. Write the Vlasov equation down with the above-mentioned "improved approximate expression" for the collision term and verify that the conservation laws are obeyed.

1. THE LANDAU EQUATION

Detailed kinetic considerations (including a statistical assumption in the form of the "Stosszahlansatz"**) led Boltzmann to his celebrated result (for a dilute gas of particles with spherically symmetric short-range pair interaction)

$$\left[\frac{\partial f(\boldsymbol{v}_1)}{\partial t}\right]_c = -\int f(\boldsymbol{v}_1) f(\boldsymbol{v}_2)\, |\, \boldsymbol{v}_1 - \boldsymbol{v}_2 \,|\, b\, db\, d\varepsilon\, d^3\boldsymbol{v}_2 + \int f(\boldsymbol{v}_1^0) f(\boldsymbol{v}_2^0)\, |\, \boldsymbol{v}_1 - \boldsymbol{v}_2 \,|\, b\, db\, d\varepsilon\, d^3\boldsymbol{v}_2 . \tag{3}$$

Actually, f depends not only on the velocity, but also on $\boldsymbol{r}$ and t; these variables have not been written explicitly, however, because they have the same value throughout the formula, owing to the assumption that the interaction has a short range. The first term on the right is the rate of decrease of $f(\boldsymbol{v}_1)$ owing to collisions with particles with another velocity $\boldsymbol{v}_0$. b is the impact parameter and ε the azimuth, which is the only remaining parameter needed to specify an arbitrary collision.*** The probability of such a collision

* P. L. Bhatnagar, E. P. Gross and M. Krook, Phys. Rev. **94**, 511 (1954).

** See ch. x, sec. 1.

*** More precisely, ε is the angle between the plane of the collision (i.e. a plane through the vectors of the relative velocities before and after the collision) and the plane through the relative velocity $\boldsymbol{v}_1 - \boldsymbol{v}_2$ before the collision and a fixed direction, e.g. the vertical. – For a derivation of (3) see any textbook on kinetic gas theory, e.g. D. ter Haar, *Elements of Statistical Mechanics* (Holt, Rinehart and Winston, New York 1961), or Chapman and Cowling.

per unit time is proportional to: (i) the relative velocity $|\boldsymbol{v}_1-\boldsymbol{v}_2|$; (ii) the target area $b\,db\,d\varepsilon$ corresponding to specified values of the parameters b and ε; (iii) $f(\boldsymbol{v}_1)$ and $f(\boldsymbol{v}_2)$ which are the numbers of particles per unit volume having the velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

The second term on the right of (3) is the rate of increase of $f(\boldsymbol{v}_1)$ due to those collisions between two particles with initial velocities $\boldsymbol{v}_1^0$ and $\boldsymbol{v}_2^0$ and parameters b and ε, for which after the collision one of the particles has a velocity $\boldsymbol{v}_1$. The vector quantities $\boldsymbol{v}_1^0$ and $\boldsymbol{v}_2^0$ are complicated but uniquely defined functions of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, b, ε. Note that $|\boldsymbol{v}_1^0-\boldsymbol{v}_2^0| = |\boldsymbol{v}_1-\boldsymbol{v}_2|$. On substituting (3) in (1) one arrives at a complicated nonlinear differential-integral equation for $f(\boldsymbol{r}, \boldsymbol{v}, t)$.*

Problem. Two particles of mass m, charge e, velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, collide with the result that, owing to their Coulomb interaction, they are deflected by an angle Θ in their center-of-mass system. Derive

$$\operatorname{cotg}\tfrac{1}{2}\Theta = \frac{m(\boldsymbol{v}_1-\boldsymbol{v}_2)^2}{2e^2}\,b\,. \tag{4}$$

Problem. The differential cross-section $\sigma(\Theta)$ is defined by the statement that out of a beam of I_0 particles per cm^2 per sec, $I_0\sigma(\Theta)d\Omega$ will be scattered in the solid angle $d\Omega$ in a direction at an angle Θ (measured in the center-of-mass-system) with the incoming beam. Derive from (4) Rutherford's formula

$$\sigma(\Theta) = \left(\frac{e^2}{m(\boldsymbol{v}_1-\boldsymbol{v}_2)^2}\right)^2 \frac{1}{\sin^4\frac{1}{2}\Theta}\,. \tag{5}$$

Problem. How is Θ related to the angle of deflection ϑ in the laboratory system?

Landau has applied Boltzmann's expression (3) to the case of a plasma.** He used the long range of the Coulomb force to his advantage by the following remark. Owing to the long range the number of collisions resulting in a small deflection Θ ("soft collisions") are much more numerous than the "hard collisions" with large Θ. Hence the change of velocity $\boldsymbol{v}_1-\boldsymbol{v}_1^0$ will be

* A second form of the Boltzmann collision term is obtained by replacing v_1^0, v_2^0 with the *final* velocities v'_1, v'_2 after the collision of two particles with initial data v_1, v_2, b, ε. This second form can be derived from (3), but we shall not need it.

** L. D. Landau, Physik. Z. Sowjetunion **10**, 154 (1936); *Collected Papers*, D. ter Haar ed. (Pergamon, Oxford 1965) p. 163.

small for most collisions, so that it is reasonable to expand (3) in powers of $\boldsymbol{v}_1-\boldsymbol{v}_1^0$ and keep only the first and second powers. This will now be carried out.

The work is facilitated by the following artifice. Multiply both members of (3) by a test function $\varphi(\boldsymbol{v}_1)$ and integrate over $\boldsymbol{v}_1$; in the final result φ can again be eliminated owing to the fact that it is an entirely arbitrary function. Thus one finds

$$\int \varphi(\boldsymbol{v}_1)\left[\frac{\partial f(\boldsymbol{v}_1)}{\partial t}\right]_c d^3\boldsymbol{v}_1 = -\int \varphi(\boldsymbol{v}_1)f(\boldsymbol{v}_1)f(\boldsymbol{v}_2)\,|\boldsymbol{v}_1-\boldsymbol{v}_2|\,b\,db\,d\varepsilon\,d^3\boldsymbol{v}_1\,d^3\boldsymbol{v}_2$$
$$+\int \varphi(\boldsymbol{v}_1)f(\boldsymbol{v}_1^0)f(\boldsymbol{v}_2^0)\,|\boldsymbol{v}_1-\boldsymbol{v}_2|\,b\,db\,d\varepsilon\,d^3\boldsymbol{v}_1\,d^3\boldsymbol{v}_2\,. \tag{6}$$

In the second term on the right we transform to $\boldsymbol{v}_1^0$, $\boldsymbol{v}_2^0$ as integration variables, using the fact that for fixed b, ε the Jacobian determinant equals unity:

$$\frac{d(\boldsymbol{v}_1, \boldsymbol{v}_2)}{d(\boldsymbol{v}_1^0, \boldsymbol{v}_2^0)} = 1\,,$$

as is proved in kinetic theory. Hence this term may be written

$$\int \varphi(\boldsymbol{v}_1)f(\boldsymbol{v}_1^0)f(\boldsymbol{v}_2^0)\,|\boldsymbol{v}_1^0-\boldsymbol{v}_2^0|\,b\,db\,d\varepsilon\,d^3\boldsymbol{v}_1^0\,d^3\boldsymbol{v}_2^0\,.$$

If in this integral we *rename* the integration variables by writing $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ instead of $\boldsymbol{v}_1^0$, $\boldsymbol{v}_2^0$, and $\boldsymbol{v}_1' = \boldsymbol{v}_1+\boldsymbol{u}$ instead of $\boldsymbol{v}_1$, the right-hand side of (6) becomes

$$\int [\varphi(\boldsymbol{v}_1+\boldsymbol{u})-\varphi(\boldsymbol{v}_1)]f(\boldsymbol{v}_1)f(\boldsymbol{v}_2)\,|\boldsymbol{v}_1-\boldsymbol{v}_2|\,b\,db\,d\varepsilon\,d^3\boldsymbol{v}_1\,d^3\boldsymbol{v}_2\,.$$

We are now in a position to expand $\varphi(\boldsymbol{v}_1+\boldsymbol{u})$ to second order in $\boldsymbol{u}$. The result is

$$\int\left[\{u_i\}\frac{\partial\varphi(\boldsymbol{v}_1)}{\partial v_{1j}} + \tfrac{1}{2}\{u_iu_j\}\frac{\partial^2\varphi(\boldsymbol{v}_1)}{\partial v_{1i}\partial v_{1j}}\right] f(\ _1)f(\ _2)\,|\boldsymbol{v}_1-\boldsymbol{v}_2|\,d^3\boldsymbol{v}_1\,d^3\boldsymbol{v}_2\,. \tag{7}$$

The subscripts i, j refer to the cartesian coordinates and summation over repeated subscripts is implied. Furthermore the following abbreviations have been used

$$\{u_i\} = \int u_i b\,db\,d\varepsilon\,,\qquad \{u_iu_j\} = \int u_iu_j b\,db\,d\varepsilon\,.$$

By partial integration the result may be cast in the form

$$\int \varphi(\boldsymbol{v}_1)\left[-\frac{\partial}{\partial v_{1i}}\{u_i\}f(\boldsymbol{v}_1)f(\boldsymbol{v}_2)\mid \boldsymbol{v}_1-\boldsymbol{v}_2\mid + \frac{1}{2}\frac{\partial^2}{\partial v_{1i}\partial v_{1j}}\{u_iu_j\}f(\boldsymbol{v}_1)f(\boldsymbol{v}_2)\mid \boldsymbol{v}_1-\boldsymbol{v}_2\mid\right] d^3\boldsymbol{v}_1\, d^3\boldsymbol{v}_2 .$$

This is equal to the left-hand side of (6) for arbitrary choice of the test function φ, so that one may conclude

$$\left[\frac{\partial f(\boldsymbol{v}_1)}{\partial t}\right]_c = -\frac{\partial}{\partial v_{1i}}f(\boldsymbol{v}_1)\int\{u_i\}f(\boldsymbol{v}_2)\mid \boldsymbol{v}_1-\boldsymbol{v}_2\mid d^3\boldsymbol{v}_2 + \frac{1}{2}\frac{\partial^2}{\partial v_{1i}\partial v_{1j}}f(\boldsymbol{v}_1)\int\{u_iu_j\}f(\boldsymbol{v}_2)\mid \boldsymbol{v}_1-\boldsymbol{v}_2\mid d^3\boldsymbol{v}_2 . \quad (8)$$

It remains to evaluate $\{u_i\}$ and $\{u_iu_j\}$ where $\boldsymbol{u}$ is the change of velocity of a particle with initial velocity $\boldsymbol{v}_1$, due to a collision (characterized by parameters b, ε) with another particle with velocity $\boldsymbol{v}_2$. The relative velocity before the collision is $\boldsymbol{g} = \boldsymbol{v}_1-\boldsymbol{v}_2$. The relative velocity after the collision is $\boldsymbol{g}' = \boldsymbol{v}_1'-\boldsymbol{v}_2' = (\boldsymbol{v}_1+\boldsymbol{u})-(\boldsymbol{v}_2-\boldsymbol{u}) = \boldsymbol{g}+2\boldsymbol{u}$. The angle between $\boldsymbol{g}$ and $\boldsymbol{g}'$ is Θ. It is easily seen from Fig. 29 that the components of $\boldsymbol{u}$ parallel and perpendicular to $\boldsymbol{g}$ are

$$u_{\parallel} = -g\sin^2\tfrac{1}{2}\Theta\,, \qquad u_{\perp} = g\sin\tfrac{1}{2}\Theta\cos\tfrac{1}{2}\Theta\,.$$

The evaluation of $\{u_i\}$ implies an integration over ε, which obviously reduces the perpendicular component to zero. Hence one finds

$$\{u_i\} = -g_i\int\sin^2\tfrac{1}{2}\Theta\; b\, db\, d\varepsilon\,. \quad (9)$$

So far no use has been made of the specific form of the interaction. If one now inserts (4) the result is

$$\{u_i\} = -2\pi g_i\int\frac{b\,db}{1+(mg^2/2e^2)^2b^2} = -\frac{8\pi e^4}{m^2}\frac{g_i}{g^4}J\,, \quad (10)$$

where J stands for the integral

$$J = \int\frac{\lambda\,d\lambda}{1+\lambda^2}\,, \qquad \lambda = \frac{mg^2}{2e^2}b\,. \quad (11)$$

As b, and hence also λ, should run from 0 to ∞, this integral diverges; this divergence will presently be discussed.

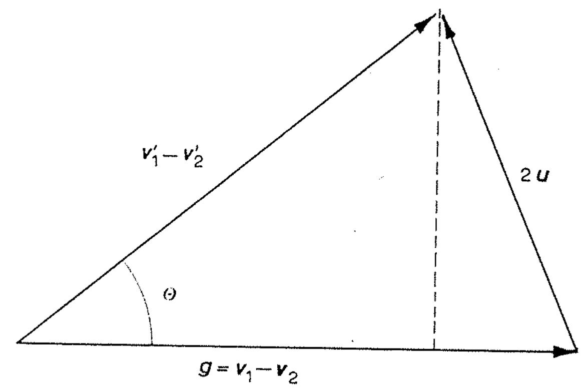

Fig. 29

In order to evaluate the tensor $\{u_iu_j\}$ we take temporarily the z-axis along $\boldsymbol{g}$, and the x-axis in the vertical plane through $\boldsymbol{g}$. The components of $\boldsymbol{u}$ in this coordinate system are

$$u_1 = g\sin\tfrac{1}{2}\Theta\cos\tfrac{1}{2}\Theta\cos\varepsilon\,,$$
$$u_2 = g\sin\tfrac{1}{2}\Theta\cos\tfrac{1}{2}\Theta\sin\varepsilon\,, \quad (12)$$
$$u_3 = -g\sin^2\tfrac{1}{2}\Theta\,.$$

Owing to the integration over ε one has $\{u_iu_j\} = 0$ for $i \neq j$. Furthermore one finds using (4),

$$\{u_1^2\} = \{u_2^2\} = \frac{4\pi e^4}{m^2g^2}J'\,, \qquad \{u_3^2\} = \frac{8\pi e^4}{m^2g^2}(J-J')\,, \quad (13)$$

where J' stands for another divergent integral:

$$J' = \int\frac{\lambda^3\,d\lambda}{(1+\lambda^2)^2}\,.$$

Equation (13) determines the elements of the tensor $\{u_iu_j\}$ in a special coordinate system whoze z-axis lies along $\boldsymbol{g}$. It follows that in an arbitrary coordinate system

$$\{u_iu_j\} = \frac{4\pi e^4}{m^2g^2}\left[J'\left(\delta_{ij}-\frac{g_ig_j}{g^2}\right)+2(J-J')\frac{g_ig_j}{g^2}\right].$$

For, both members of this equation transform like tensors, and in one coordinate system they are equal, according to (13); hence they are equal in every coordinate system. It will be shown presently that with sufficient approximation $J = J'$, so that we finally find

$$\{u_i u_j\} = \frac{4\pi e^4}{m^2 g^4} J(g^2\delta_{ij} - g_i g_j). \tag{14}$$

Substituting the results (10) and (14) in (8) one obtains for the collision term

$$\left[\frac{\partial f(\boldsymbol{v}_1)}{\partial t}\right]_c = \frac{8\pi e^4}{m^2} J \left[\frac{\partial}{\partial v_{1i}} f(\boldsymbol{v}_1) \int \frac{g_i}{g^3} f(\boldsymbol{v}_2)\, d^3\boldsymbol{v}_2 + \right.$$
$$\left. + \frac{1}{4}\frac{\partial^2}{\partial v_{1i}\partial v_{1j}} (\ \ _1) \int \frac{g^2\delta_{ij} - g_i g_j}{g^3} f(\boldsymbol{v}_2)\, d^3\boldsymbol{v}_2\right].$$

With the aid of the identity

$$\frac{g_i}{g^3} = \frac{1}{2}\frac{\partial}{\partial v_{2j}}\frac{g^2\delta_{ij} - g_i g_j}{g^3} \tag{15}$$

the above expression can be cast in the form

$$\left[\frac{\partial f(\boldsymbol{v}_1)}{\partial t}\right]_c = \frac{2\pi e^4}{m^2} J \frac{\partial}{\partial v_{1i}} \int \frac{g_i g_j - g^2\delta_{ij}}{g^3} \left\{f(\boldsymbol{v}_1)\frac{\partial f(\boldsymbol{v}_2)}{\partial v_{2j}} - \frac{\partial f(\boldsymbol{v}_1)}{\partial v_{1j}} f(\boldsymbol{v}_2)\right\} d^3\boldsymbol{v}_2. \tag{16}$$

Together with (1) this constitutes the *Landau equation.* It has been used to compute transport coefficients like thermal and electrical conductivity, etc.* However, we still have to find the numerical constant J.

Problem. Verify (15).
Problem. Prove that the right-hand side of (16) is zero when the Maxwell distribution is inserted for $f(\boldsymbol{v})$.
Problem. Show that (16) obeys the H-theorem by proving that

$$\frac{d}{dt}\int f(\boldsymbol{v}) \log f(\boldsymbol{v})\, d^3\boldsymbol{v} < 0,$$

unless $f(\boldsymbol{v})$ is the Maxwell distribution.

The fact that J diverges for $\lambda \to \infty$, i.e., for large values of the impact parameter b, is a consequence of the long range of the Coulomb potential. Owing to the Debye shielding, however, the actual interaction does not extend far

* B. B. Robinson and I. B. Bernstein, Ann. Phys. (N.Y.) **18**, 110 (1962); I. P. Shkarovsky, I. B. Bernstein and B. B. Robinson, Phys. Fluids **6**, 40 (1963).

beyond a Debye length $1/\kappa$. Hence two particles approaching one another with an impact parameter $b \gg 1/\kappa$ interact only very little; in other words they do not collide. It is therefore reasonable to cut off the integration interval at $b_{\max} = \kappa^{-1}$, corresponding to

$$\lambda_{\max} = \frac{mg^2}{2e^2}\left(\frac{kT}{4\pi n_0 e^2}\right)^{\frac{1}{2}}.$$

By replacing g^2 with its average value $6kT/m$, we obtain a cut-off Λ, which is independent of the particular collision,

$$\Lambda = 12\pi n_0 \left(\frac{kT}{4\pi n_0 e^2}\right)^{\frac{3}{2}}. \tag{17}$$

Note that the justification of this cut-off is based on the physical argument that the Coulomb potential is screened off by the presence of the Debye sphere, and may therefore be treated as if the interaction potential has the form (VIII, 8). However, the screening is itself a statistical effect and ought to come out of the same kinetic equations that are used to describe the collisions. This more ambitious approach is outlined in section 3.

2. Discussion of the Landau Equation

Apart from a numerical factor of order 10, (17) is the number of particles in a Debye sphere, and hence much larger than 1. With this cut-off,

$$J = \tfrac{1}{2}\log(1+\Lambda^2) \approx \log \Lambda. \tag{18}$$

Some typical values of Λ are listed in Table I, p. 6. We shall show that the approximations made during the derivation of Landau's equation from the Boltzmann equation all amount to omitting terms that do not contain J as a factor, and are therefore of relative order J^{-1}. Of course this does not guarantee that they are actually numerically small for any given plasma, but only that they are unimportant in the limit of high T and/or small n_0.

First, it follows from the inequality

$$J - J' = \int_0^{\Lambda} \frac{\lambda\, d\lambda}{(1+\lambda^2)^2} = \frac{\Lambda^2}{2(1+\Lambda^2)} < \tfrac{1}{2}$$

that the error made by identifying J' with J is of relative order J^{-1}.

Secondly, higher terms in the expansion (7) of φ involve the averages

$\{u_i u_j u_k\}$, etc. On inserting (12) one readily sees that these contain at least four factors $\sin \frac{1}{2}\Theta$; hence the integration no longer diverges at $\Theta = 0$, i.e., for $b \to \infty$. Thus these terms are no longer proportional to $\log \Lambda$ and are therefore of relative order J^{-1}.

The omission of these higher terms amounts to neglecting the effect of hard collisions. The justification is that they are rare compared to the soft collisions. However, it might be true that their effect is not negligible since one hard collision is much more effective in altering the velocity distribution than a soft collision. (This amounts to saying that although each higher term is of relative order J^{-1}, all higher terms together might be of comparable size.) Yet it has been shown that for a sufficiently hot and dilute plasma their effect is negligible.*

Remark. A consistent omission of hard collisions would require that the integration in (11) is also cut off at the lower end, corresponding to a value $b_{\min}$ of the impact parameter, where the collisions become hard. By extending the integration down to zero we have erroneously treated the hard collisions as if they were soft. However, this error is again of relative order J^{-1}.

Problem. Find a reasonable value for $b_{\min}$ and show that the hard collisions ($b < b_{\min}$) are rare.

Thirdly, our expression for the cut-off Λ inevitably involves an arbitrariness in the numerical factor. However, a numerical factor a would only change $\log \Lambda$ into $\log \Lambda + \log a$, which changes J by an amount of order unity. For the same reason it was permissible to replace g^2 by its average value $6kT/m$.

A serious objection against the preceding derivation is that it started from Boltzmann's equation, which has only been derived for the case of short-range interactions. In fact, Boltzmann's equation neglects triple collisions altogether, whereas in a plasma each charged particle is always involved in many simultaneous collisions (about as many as there are particles in a Debye sphere). This objection can be overcome by the following alternative derivation.

First note that the integrals appearing in (8) have a simple physical meaning. The first integral is the average change of velocity per unit time of a

* R. S. Cohen, L. Spitzer and P. M. Routly, Phys. Rev. **80**, 230 (1950).

particle having a velocity $\boldsymbol{v}_1$, and may therefore be written in a more pregnant notation as*

$$\int \{u_i\} f(\boldsymbol{v}_2)\, |\, \boldsymbol{v}_1 - \boldsymbol{v}_2 \,|\, d^3 \boldsymbol{v}_2 = \langle \Delta v_{1i} \rangle / \Delta t . \tag{19}$$

Similarly, the second integral in (8) represents the tensor of the second moments of the change in velocity per unit time. Hence (8) can be rewritten in the form

$$\left[\frac{\partial f(\boldsymbol{v})}{\partial t}\right]_c = -\frac{\partial}{\partial v_i}\frac{\langle \Delta v_i \rangle}{\Delta t} f(\boldsymbol{v}) + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\frac{\langle \Delta v_i \Delta v_j \rangle}{\Delta t} f(\boldsymbol{v}) . \tag{20}$$

This has the form of a multivariate Fokker–Planck equation for the function f of the three variables v_x, v_y, v_z, and of t.

A Fokker–Planck equation is the appropriate tool for describing particles that undergo a large number of very small velocity changes, like Brownian particles. Hence, in order to arrive at (8) it is not necessary to start from the doubtful picture of the plasma as a dilute Boltzmann gas. Instead, one may argue that since each plasma particle undergoes a large number of very soft collisions, the velocity distribution must obey a Fokker–Planck equation of the general form (20). Thus one may postulate (20) straightaway, and it only remains to compute the three "coefficients of dynamical friction" $\langle \Delta v_i \rangle / \Delta t$, and the six "coefficients of dynamical dispersion" $\langle \Delta v_i \Delta v_j \rangle / \Delta t$. This computation consists of first evaluating $\{u_i\}$ and $\{u_i u_j\}$ as in the previous section, and then making up the integrals that occur in (8). In this way one is again led to (16), and even in a much shorter way, although the underlying physical picture is quite different.**

Problem. Derive in this way the Landau equation for a mixture of many components.

Are the conditions for the validity of the Fokker-Planck equation satisfied? To investigate this question let us consider an electron moving with a certain velocity v_0 of the order of the thermal velocity $v_T = \sqrt{(kT/m)}$. The *first condition* is that each individual collision should have relatively little effect on the velocity of the electron considered. A typical collision consists of the passing of another electron at a distance $d = n_0^{-\frac{1}{3}}$ with the velocity v_T. Such an event causes a force e^2/d^2 to act during a time of order d/v_T. Hence it causes a change of velocity $\Delta v = e^2/d v_T m$. This is, indeed, much less than

* On the right-hand side the limit $\Delta t \to 0$ is implied.
** M. Rosenbluth, W. MacDonald and D. Judd, Phys. Rev. **107**, 1 (1957).

the velocity v_T of the electron considered, because $mv_T^2 \gg e^2/d$ is the fundamental condition (VIII, 7), which must be satisfied.

The *second condition* for the validity of the Fokker–Planck equation is that the duration of each individual collision should be short compared to the time τ in which the velocity of the electron considered is materially changed. An estimate of τ is given by $(\langle \Delta v \rangle/\Delta t)\tau = v_T$, or on account of (19) and (10)

$$\frac{1}{\tau} = \frac{1}{v_T}\frac{\langle \Delta v \rangle}{\Delta t} \sim \frac{1}{v_T}\{u_1\}n_0 v_T \sim n_0 \frac{e^4}{m^2}\frac{1}{v_T^3} J.$$

Since the duration of a collision is d/v_T our condition requires

$$\frac{v_T}{d} \gg n_0 \frac{e^4}{m^2}\frac{1}{v_T^3} J = n_0 \frac{e^4}{m^2}\frac{1}{v_T^3} \log \Lambda$$

or

$$\frac{mv_T^2}{e^2/d} \gg \log \frac{mv_T^2}{e^2/d},$$

which is again (VIII, 7).

Finally it should be remarked that the standard derivation of the Fokker–Planck equation* applies to the case where $\langle \Delta v_i \rangle$ and $\langle \Delta v_i \Delta v_j \rangle$ are quantities that may depend on $\boldsymbol{v}$, but not on f. In the present case, however, these averages themselves involve f.** It is therefore not clear that one may still apply the Fokker–Planck equation. At any rate, one neglects statistical correlations that might exist between two particles that are about to collide. To put it differently, a "Stosszahlansatz" is implied, just like in the Boltzmann equation.

Alternative ways of evaluating the coefficients $\langle \Delta v_i \rangle/\Delta t$ and $\langle \Delta v_i \Delta v_j \rangle/\Delta t$ have been given.*** When an electron moves with a certain velocity $\boldsymbol{v}$, its Debye sphere will somewhat lag behind and hence exert a drag on the electron. This decreases the velocity of the electron by an amount $\langle \Delta v_i \rangle$ per unit time. In order to find $\langle \Delta v_i \Delta v_j \rangle$ one has to calculate the fluctuations

* See e.g. S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943) or M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945). Both are reprinted in: *Selected Papers on Noise and Stochastic Processes*, N. Wax ed. (Dover, New York 1954).

** This is also the reason why (16) is nonlinear, whereas the standard Fokker–Planck equation is linear.

*** A. N. Kaufman in: *La théorie des Gaz Neutres et Ionisés* (École d'Été Les Houches 1959; Hermann, Paris 1960); J. Hubbard, Proc. Roy. Soc. (London) **A260**, 114 (1961).

of the local electric field.* The results are the same as found above, apart from terms of relative order J^{-1}.

Remark. Landau did not make use of the exact formulas for the Rutherford scattering, but used the *impact approximation* to describe the two-body collisions. This approximation is analogous to the Born approximation in quantum mechanics and proceeds as follows. One replaces the hyperbolic paths of the particles by rectilinear motion with constant velocity, and computes the force which they exert on each other during this rectilinear motion. The time integral of this force is then taken as an approximation for the total momentum transfer. This momentum transfer is added to the initial momentum vectors to obtain the momentum vectors after the collision.

Clearly this is a good approximation for soft collisions but not for hard collisions. Of course that suffices for the present purpose, since the effect of hard collisions is not correctly described by Landau's equation (16) anyway. However, when this approximation is used the integrals for $\{u_i\}$ and $\{u_i u_j\}$ diverge at small values of the impact parameter too. For this reason Landau used a second cut-off $b_{\min} = e^2/3kT$, which according to (4) corresponds to

$$\operatorname{cotg} \tfrac{1}{2}\Theta = \frac{mg^2}{2e^2}\frac{e^2}{3kT} = 1, \qquad \Theta = \tfrac{1}{2}\pi \tag{21}$$

where g^2 is again replaced with its average value $6kT/m$. The precise choice of this cut-off is again immaterial, since it affects only terms of relative order J^{-1}.

Problem. Show that the impact approximation yields, instead of (4) and (5),

$$\operatorname{cotg} \Theta = \frac{m(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2}{4e^2} b, \qquad \sigma(\Theta) = \left(\frac{4e^2}{m(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2}\right)^2 \frac{\cos \Theta}{\sin^4 \Theta},$$

which, however, in this approximation cannot be distinguished from

$$\Theta = \frac{4e^2}{m(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2}\frac{1}{b}, \qquad \sigma(\Theta) = \left(\frac{4e^2}{m(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2}\right)^2 \frac{1}{\Theta^4}.$$

Problem. Show that using these equations and the cut-off (21) one arrives at exactly the same Landau equation (which is somewhat accidental in view of the arbitrariness in the cut-off).

* These fluctuations also add to $\langle \Delta v_i \rangle$.

3. Kinetic theory

The most fundamental approach to plasma physics starts from the fact that the plasma consists of a large number N of particles with electromagnetic interaction, which obey the equations of motion of mechanics. We shall here simplify the problem by using classical nonrelativistic mechanics, by considering only one kind of particles (electrons), and by omitting magnetic interaction and retardation of the field (so that the interaction is fully described by a Coulomb potential). Then the equations of motion are determined by a Hamilton function

$$H_N(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2, \ldots, \boldsymbol{r}_N, \boldsymbol{p}_N) = \\ = \sum_{\sigma=1}^{N} \frac{p_\sigma^2}{2m} + \sum_{\sigma<\sigma'} \varphi(\boldsymbol{r}_\sigma - \boldsymbol{r}_{\sigma'}) - n_0 \sum_{\sigma=1}^{N} \int \varphi(\boldsymbol{r}_\sigma - \boldsymbol{r}_0)\, d^3\boldsymbol{r}_0 . \tag{22}$$

Here $\varphi(r) = e^2/r$, and the last term is the potential energy due to the fixed background. If the corresponding equations of motion

$$\frac{d\boldsymbol{r}_\sigma}{dt} = \frac{\partial H_N}{\partial \boldsymbol{p}_\sigma} = \frac{\boldsymbol{p}_\sigma}{m}, \qquad \frac{d\boldsymbol{p}_\sigma}{dt} = -\frac{\partial H_N}{\partial \boldsymbol{r}_\sigma} \tag{23}$$

could be solved, there would be no further problems.

Since they cannot be solved one has to resort to statistical methods. Rather than studying a single system of N particles one introduces an ensemble of such systems, defined by a probability distribution $D_N(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2, \ldots, \boldsymbol{r}_N, \boldsymbol{p}_N, t)$ in phase space. D_N is normalized to unity, that is, its integral over the whole phase space equals 1. As there is no need to distinguish between the individual particles, D_N is taken to be a symmetric function of all particles. Since each sample system in this ensemble moves in phase space according to (23), the distribution changes with time according to the *Liouville equation**

$$\frac{\partial D_N}{\partial t} = -\sum_{\sigma, i} \left(\frac{\partial D_N}{\partial r_{\sigma i}} \frac{\partial H_N}{\partial p_{\sigma i}} - \frac{\partial D_N}{\partial p_{\sigma i}} \frac{\partial H_N}{\partial r_{\sigma i}} \right) = -\{D_N, H_N\} . \tag{24}$$

Here $\{D_N, H_N\}$ is a "Poisson bracket", which is nothing but a short-hand way of writing the preceding expression.

Problem. We have tacitly assumed that the following theorems hold. When D_N is normalized at one time, it remains normalized for all t.

* Closely related to, but not identical with "Liouville's theorem", which merely states that the density is constant in a point that moves along with the motion in phase space.

When D_N is chosen symmetrical in all particles at one time, it must remain so for all t. Prove this from (24) and (22).

Next one introduces the *reduced distribution function* for a subset of s particles,

$$f_s(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2, \ldots, \boldsymbol{r}_s, \boldsymbol{p}_s, t) = V^s \int D_N d^3\boldsymbol{r}_{s+1} d^3\boldsymbol{p}_{s+1} \ldots d^3\boldsymbol{r}_N d^3\boldsymbol{p}_N , \tag{25}$$

where V is the volume to which the plasma is confined by the fixed background*. Clearly $f_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t) d^3\boldsymbol{r}_1 d^3\boldsymbol{p}_1/V$ is the probability that at t particle 1 is in $d^3\boldsymbol{r}_1$ at $\boldsymbol{r}_1$, with momentum vector in $d^3\boldsymbol{p}_1$ at $\boldsymbol{p}_1$. Similarly the two-body distribution function $f_2(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2, t)/V^2$ is the joint probability that at t particle 1 is at $\boldsymbol{r}_1$ with momentum $\boldsymbol{p}_1$, and that simultaneously particle 2 is at $\boldsymbol{r}_2$ with momentum $\boldsymbol{p}_2$.

Next the rate of change of the f_s is obtained by integrating (24) over all coordinates and momenta of the particles $s+1, s+2, \ldots, N$. The result is

$$\frac{\partial f_s}{\partial t} = -\{f_s, H_s\} - \frac{N-s}{V} \int \left\{ f_{s+1}, \sum_{\sigma=1}^{s} \varphi(\boldsymbol{r}_\sigma - \boldsymbol{r}_{s+1}) \right\} d^3\boldsymbol{r}_{s+1}\, d^3\boldsymbol{p}_{s+1} . \tag{26}$$

The first Poisson bracket on the right represents the change in f_s due to the mutual interaction of the particles $1, \ldots, s$, and, of course, their interaction with the background. The Poisson bracket under the integral represents the interaction of the s particles with one of the remaining $N-s$ particles.

Problem. Express the local density and the total energy of the system of particles in f_1 and f_2. Show explicitly that the total density and the total energy are constant in time.

Problem. Express the Boltzmann distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ used in chapter x in the present f_1.

The most important feature of this set of equations is that the expression for $\partial f_s/\partial t$ involves f_{s+1}. Consequently, in order to calculate f_1 one needs f_2, and in order to calculate f_2 one needs f_3, and so on. Only f_N obeys an equation by itself, but that is just the Liouville equation (24) itself. Thus one does not have separate equations for the f_s, but only a hierarchy of N coupled equations, the so-called B.B.G.K.Y. hierarchy.**

The exact solution of this hierarchy is just as hard as that of the Liouville equation (24), or of the equations of motion (23). However, it is hoped that

* Boundary effects may be neglected, because we shall presently let V tend to infinity.

** Roughly anti-chronological abbreviation for Yvon, Kirkwood, Born and Green, Bogolyubov.

an approximate solution can be found by terminating the hierarchy at some low value of s. For this it is necessary to express f_{s+1} in terms of $f_1, \ldots, f_s$ in a physically sensible way.

Problem. Investigate in which way the termination of the hierarchy is achieved in Boltzmann's equation.

We shall cast the hierarchy in a more convenient form. First, since we have decided to study only low values of s, we may put $(N-s)/V = n_0$. More precisely this means that we go to the limit $V \to \infty$, $N \to \infty$ with fixed $N/V = n_0$ and fixed s ("thermodynamic limit"). Next we replace the momenta by the velocities, which is more conventional. Then (26) becomes

$$\frac{\partial f_s}{\partial t} + \sum_{\sigma=1}^{s} \boldsymbol{v}_\sigma \cdot \frac{\partial f_s}{\partial \boldsymbol{r}_\sigma} = -\frac{e^2}{m} \sum_{\sigma=1}^{s} \frac{\partial f_s}{\partial \boldsymbol{v}_\sigma} \cdot \Big(\sum_{\substack{\sigma'=1 \\ \sigma' \neq \sigma}}^{s} \boldsymbol{F}_{\sigma,\sigma'} - n_0 \int \boldsymbol{F}_{\sigma,0}\, d^3\boldsymbol{r}_0 \Big)$$

$$- \frac{n_0 e^2}{m} \sum_{\sigma=1}^{s} \int \boldsymbol{F}_{\sigma,s+1} \cdot \frac{\partial f_{s+1}}{\partial \boldsymbol{v}_\sigma}\, d^3\boldsymbol{r}_{s+1}\, d^3\boldsymbol{v}_{s+1}, \qquad (27)$$

where we have used the abbreviation

$$\boldsymbol{F}_{\sigma\sigma'} = -\frac{\partial}{\partial \boldsymbol{r}_\sigma} \frac{1}{|\boldsymbol{r}_\sigma - \boldsymbol{r}_{\sigma'}|} = \frac{\boldsymbol{r}_\sigma - \boldsymbol{r}_{\sigma'}}{|\boldsymbol{r}_\sigma - \boldsymbol{r}_{\sigma'}|^3}.$$

In (27) only two constants occur, n_0 and e^2/m. They can both be put equal to unity by an appropriate choice of the units of length and time. Thus *equation* (27) *contains no dimensionless quantity which could be used as an expansion parameter*. However, if one is interested in solutions in the neighborhood of a thermal equilibrium, an additional constant kT appears. In that case the dimensionless parameter $kT/e^2 n_0^{\frac{1}{3}}$ is available. This quantity must be large according to (VIII, 7). It is more convenient to use instead the parameter Λ defined in (17). If one takes as unit of length the Debye length κ^{-1}, as unit of time ω_P^{-1}, one has $n_0 e^2/m = 1/4\pi$ and $n_0 = \Lambda/12\pi$. In these units (27) becomes

$$\frac{\partial f_s}{\partial t} + \sum_{\sigma=1}^{s} \boldsymbol{v}_\sigma \cdot \frac{\partial f_s}{\partial \boldsymbol{r}_\sigma} = -\frac{3}{\Lambda} \sum_{\sigma=1}^{s} \sum_{\substack{\sigma'=1 \\ \sigma \neq \sigma'}}^{s} \boldsymbol{F}_{\sigma,\sigma'} \cdot \frac{\partial f_s}{\partial \boldsymbol{v}_\sigma}$$

$$+ \frac{1}{4\pi} \sum_{\sigma=1}^{s} \frac{\partial f_s}{\partial \boldsymbol{v}_\sigma} \cdot \int \boldsymbol{F}_{\sigma,0}\, d^3\boldsymbol{r}_0 - \frac{1}{4\pi} \sum_{\sigma=1}^{s} \int \boldsymbol{F}_{\sigma,s+1} \cdot \frac{\partial f_{s+1}}{\partial \boldsymbol{v}_\sigma}\, d^3\boldsymbol{r}_{s+1}\, d^3\boldsymbol{v}_{s+1}. \qquad (28)$$

We are now in a position to attempt an expansion in powers of Λ^{-1}.

To zeroth order in Λ^{-1} the mutual interaction among the s particles drops out and only their interaction with the remaining particles and with the background remains. This suggests that the hierarchy may be solved by a distribution function that contains no correlations between the particles. Absence of correlations means that the distribution function factorizes:

$$f_s(\boldsymbol{r}_1, \boldsymbol{v}_1, \boldsymbol{r}_2, \boldsymbol{v}_2, \ldots, \boldsymbol{r}_s, \boldsymbol{v}_s, t) = \prod_{\sigma=1}^{s} f_1(\boldsymbol{r}_\sigma, {}_\sigma, t). \qquad (29)$$

On substituting this in (28) with $\Lambda = \infty$ one finds that it is actually a solution, provided that f_1 obeys

$$\frac{\partial f_1(\boldsymbol{r}_1, \boldsymbol{v}_1, t)}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial f_1}{\partial \boldsymbol{r}_1} + \frac{1}{4\pi} \frac{\partial f_1}{\partial \boldsymbol{v}_1} \cdot \int \boldsymbol{F}_{1,2} \Big\{ \int f_1(\boldsymbol{r}_2, \boldsymbol{v}_2, t)\, d^3\boldsymbol{v}_2 - 1 \Big\}\, d^3\boldsymbol{r}_2 = 0. \qquad (30)$$

This is identical with the nonlinear Vlasov equation (XI, 1) in electrostatic approximation.* Thus *we have deduced from kinetic theory that the Vlasov equation is correct in the limiting case of a very hot, dilute plasma.*

To first order in Λ^{-1} a solution of the form (29) is no longer possible. The reason is that the equation (28) now contains the mutual interaction among the s particles, which results in a correlation between their positions and velocities. In order to solve (28) to first order in $1/\Lambda$ we now set (in abbreviated but obvious notation)

$$\begin{aligned} f_2(1,2) &= f_1(1)f_1(2) + \Lambda^{-1} g_2(1,2) \\ f_3(1,2,3) &= f_1(1)f_1(2)f_1(3) \\ &\quad + \Lambda^{-1}\{f_1(1)g_2(2,3) + f_1(2)g_2(3,1) + f_1(3)g_2(1,2)\} \\ &\quad + \Lambda^{-2} g_3(1,2,3), \end{aligned} \qquad (31)$$

etc.

This implies the assumption that the successive terms are actually of the order indicated by the powers of Λ. Substituting this in the hierarchy (28) one obtains first an equation for f_1 involving g_2:

$$\frac{\partial f_1(1)}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial f_1}{\partial \boldsymbol{r}_1} = -\frac{1}{4\pi} \frac{\partial f_1}{\partial \boldsymbol{v}_1} \cdot \int \boldsymbol{F}_{1,2} \Big\{ \int f_1(2)\, d^3\boldsymbol{v}_2 - 1 \Big\}\, d^3\boldsymbol{r}_2$$

$$- \frac{1}{4\pi\Lambda} \int \boldsymbol{F}_{1,2} \cdot \frac{\partial g_2(1,2)}{\partial \boldsymbol{v}_1}\, d^3\boldsymbol{r}_2\, d^3\boldsymbol{v}_2. \qquad (32)$$

* The magnetic field has been included by A. Simon and E. G. Harris, Phys. Fluids **3**, 245 (1960); **4**, 586 (1961).

Next one obtains an equation for g_2 involving f_1,

$$\frac{\partial g_2(1,2)}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial g_2}{\partial \boldsymbol{r}_1} + \boldsymbol{v}_2 \cdot \frac{\partial g_2}{\partial \boldsymbol{r}_2} =$$

$$- \frac{1}{4\pi} \frac{\partial g_2}{\partial \boldsymbol{v}_1} \cdot \int \boldsymbol{F}_{1,3} \left\{ \int f_1(3)\, d^3 v_3 - 1 \right\} d^3 r_3$$

$$- \frac{1}{4\pi} \frac{\partial g_2}{\partial \boldsymbol{v}_2} \cdot \int \boldsymbol{F}_{2,3} \left\{ \int f_1(3)\, d^3 v_3 - 1 \right\} d^3 r_3$$

$$- 3 f_1(1) \boldsymbol{F}_{2,1} \cdot \frac{\partial f_1(2)}{\partial \boldsymbol{v}_2} - 3 f_1(2) \boldsymbol{F}_{1,2} \cdot \frac{\partial f_1(1)}{\partial \boldsymbol{v}_1}$$

$$- \frac{1}{4\pi} \frac{\partial f_1(1)}{\partial \boldsymbol{v}_1} \cdot \int \boldsymbol{F}_{1,3}\, g_2(2,3)\, d^3 r_3\, d^3 v_3 - \frac{1}{4\pi} \frac{\partial f_1(2)}{\partial \boldsymbol{v}_2} \cdot \int \boldsymbol{F}_{2,3}\, g_2(1,3)\, d^3 r_3\, d^3 v_3 . \tag{33}$$

The term with g_3 has disappeared because it is of order Λ^{-2}, according to the assumption implied in (31).

Thus we have succeeded in extracting from the hierarchy a closed set of only two equations for f_1 and g_2. These equations, however, constitute a pair of coupled, nonlinear integro-differential equations, which still cannot be solved exactly. Moreover, even if they could be solved exactly, the solution would depend on the initial values of both f_1 *and* g_2; this shows that these two coupled equations provide an entirely different kind of description than the Boltzmann or Landau equations, which involve only f_1. Bogolyubov has invented a method for extracting from the set (32), (33) an approximate equation for f_1 alone. An equation for f_1 alone is sometimes called a *kinetic equation*.

Problem. Show that the set of equations (32), (33) is invariant for time reversal.

4. The Method of Bogolyubov

The basic idea of Bogolyubov's method is that $g_2(1, 2, t)$ rapidly approaches an asymptotic form $\bar{g}_2(1, 2)$, which is uniquely determined by the instantaneous form of the function f_1 of $\boldsymbol{r}_1$ and $\boldsymbol{v}_1$. Thus one has for each single value of t

$$\bar{g}_2(1,2) = \bar{g}_2(\boldsymbol{r}_1, \boldsymbol{v}_1, \boldsymbol{r}_2, \boldsymbol{v}_2 \,|\, f_1(\boldsymbol{r}, \boldsymbol{v}, t)) . \tag{34}$$

This notation is meant to indicate that $\bar{g}_2$ is a function of $\boldsymbol{r}_1$, $\boldsymbol{v}_1$, $\boldsymbol{r}_2$, $\boldsymbol{v}_2$, and moreover a functional in the function space of all possible one-particle distribution functions $f_1(\boldsymbol{r}, \boldsymbol{v})$. In order to find the actual value of the correlation function $\bar{g}_2$ at a particular time t one must insert the actual distribution function f_1 at time t. Thus $\bar{g}_2$ depends on time only through its functional dependence on f_1. When this $\bar{g}_2$ is substituted for g_2 in (32), there results an equation for f_1 alone, which is a kinetic equation analogous to the Boltzmann equation.

It remains to determine the functional $\bar{g}_2(1, 2 \,|\, f_1)$. According to (34) we must find $\bar{g}_2$ for all $\boldsymbol{r}_1$, $\boldsymbol{v}_1$, $\boldsymbol{r}_2$, $\boldsymbol{v}_2$ and for all possible forms of the function $f_1(\boldsymbol{r}, \boldsymbol{v})$. The prescription is as follows. First construct a function $f_1^{(0)}(\boldsymbol{r}, \boldsymbol{v}, t; t_0)$ for $-\infty < t \leqslant t_0$ by requiring that it obeys the zeroth order equation (30) for $-\infty < t \leqslant t_0$, and that for $t = t_0$ it is identical with $f_1(\boldsymbol{r}, \boldsymbol{v}, t_0)$. To put it differently, $f_1(\boldsymbol{r}, \boldsymbol{v}, t_0)$ is extrapolated backwards in time by means of the collisionless Vlasov equation. Substitute this $f_1^{(0)}(\boldsymbol{r}, \boldsymbol{v}, t; t_0)$ in the equation (33) for g_2 and obtain the solution $g_2(1, 2, t; t_0)$ with initial condition $g_2 = 0$ at $t = -\infty$. This initial condition amounts to the assumption that there is no correlation between the particles before they interact with each other, which is again the "Stosszahlansatz". This assumption is responsible for the fact that the final equation is not invariant for time reversal. The solution $g_2(1, 2, t; t_0)$, taken at $t = t_0$, is the desired functional $\bar{g}_2(\boldsymbol{r}_1, \boldsymbol{v}_1, \boldsymbol{r}_2, \boldsymbol{v}_2 \,|\, f_1(\boldsymbol{r}, \boldsymbol{v}, t_0))$.

This procedure can be carried out explicitly in the special case of a *spatially homogeneous plasma*. In this case f_1 does not depend on $\boldsymbol{r}_1$, and g_2 is of the form $g_2(1,2) \equiv g_2(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, t)$. It follows that the charge density and the electrostatic field vanish at all t. This makes it possible to actually carry out Bogolyubov's procedure and to arrive at a kinetic equation for f_1 alone. If one could also solve this kinetic equation, one would know how an initial velocity distribution relaxes towards the Maxwell distribution owing to the collisions.

We apply Bogolyubov's prescription to the spatially homogeneous plasma. First it is very easy to construct the desired $f_1^{(0)}(\boldsymbol{r}_1, \boldsymbol{v}_1, t; t_0)$: as it must obey the Vlasov equation but cannot depend on $\boldsymbol{r}_1$, it does not depend on t either. Hence

$$f_1^{(0)}(\boldsymbol{r}_1, \boldsymbol{v}_1, t; t_0) = f_1^{(0)}(\boldsymbol{v}_1; t_0) = f_1(\boldsymbol{v}_1, t_0) .$$

This must be substituted in (33) and the equation must be solved to find g_2 at $t = t_0$ with the prescribed initial condition $g_2 = 0$ at $t = -\infty$. Since in this case f_1 does not depend on t, one may alternatively impose the condition $g_2 = 0$ at $t = 0$, and find g_2 at $t = +\infty$, which is somewhat more convenient. Actually one does not determine g_2 itself, but rather $\int g_2 d^3 v_2$, which is all that is needed in (32). Moreover, g_2 itself does not tend to a limit, but this integral does; compare the function f^1 and its integral n^1 in the Vlasov equation (chapter XII).

The problem of solving (33) with time-independent $f_1(v)$ turns out to be similar to the problem of solving the linearized Vlasov equation; in particular the function $Z(k, u)$ defined in (XII, 22) now also occurs, although constructed with $f_1^{(0)}(v)$ in place of $f^0(v)$. It is necessary to impose the condition that $f_1^{(0)}(v)$ is such that $Z(k,u)$ has no zeros in the upper half of the complex u-plane, because otherwise g_2 does not tend to a limit for t going to infinity.* In this way one arrives at

$$\left[\frac{\partial f_1(v_1)}{\partial t}\right]_c = -\frac{\partial}{\partial v_{1i}} \int Q_{ij} \left\{ f_1(v_1) \frac{\partial f_1(v_2)}{\partial v_{2j}} - \frac{\partial f_1(v_1)}{\partial v_{1j}} f_1(v_2) \right\} d^3 v_2 , \quad (35)$$

where the tensor $Q_{ij}(v_1, v_2)$ is defined by the integral

$$Q_{ij}(v_1, v_2) = \frac{2e^4}{m^2} n_0 \int \frac{k_i k_j \delta(k \cdot v_1 - k \cdot v_2)}{|Z(k, k \cdot v_1/k)|^2} \frac{d^3 k}{k^4} . \quad (36)$$

Here we have returned to conventional units. Equation (1), together with the expression (35) for the collision term, is called the *Balescu–Lenard equation*.** It can be used to compute relaxation times connected with the approach toward the equilibrium distribution.***

Equation (35) differs from Landau's expression (16) by the form of the tensor Q_{ij}. The interaction is no longer treated as two-body collisions, but collective effects are taken into account, as exhibited by the presence of the function Z. Accordingly it is no longer necessary to take the Debye screening into account by an *ad hoc* cut-off. We shall now verify this in more detail.

The integration variable $\boldsymbol{k}$ has come in because the correlation function $g_2(1, 2) \equiv g_2(r_1 - r_2; v_1, v_2)$ has been written as a Fourier transform with respect to the particle distance $r_1 - r_2$. Hence low values of k correspond to large distance between both particles, which is the analog of soft collisions in Landau's treatment. However, the integral (36) does not diverge at $\boldsymbol{k} = 0$, owing to the $|Z|^2$ in the denominator. Thus, *by including the collective effects one suppresses the divergence due to the long range of the Coulomb potential.*

* The kinetic equation for unstable plasmas has also been investigated. R. Balescu, J. Math. Phys. **4**, 1009 (1963); P. H. Rutherford and E. A. Frieman, Phys. Fluids **6**, 1139 (1963); E. Frieman and P. Rutherford, Ann. Phys. (N.Y.) **28**, 134 (1964).
** R. Balescu, Phys. Fluids 3, 52 (1960); A. Lenard, Ann. Phys. (N.Y.) **10**, 390 (1960).
*** R. L. Rosenberg and T. Y. Wu, Can. J. Phys. **42**, 548 (1964).

Large values of k correspond to small values of the impact parameter and hence to hard collisions. The fact that Z tends to 1 as k goes to infinity, shows that the influence of the collective behavior becomes negligible for hard collisions, as was to be expected. However, the integral (36) diverges logarithmically at $k = \infty$. The reason is that the present derivation inherently employs the impact approximation (compare the Remark in section 2). The remedy is again that one cuts off the integral in (36) at $k_{max} = 3kT/e^2$. With this cut-off at low values of the impact parameter, the result agrees with that of Landau, apart from terms of relative order J^{-1}.*

Problem. Verify the dimensions of (36).
Problem. Verify that (36) diverges logarithmically at $k = \infty$, and converges at $k = 0$. Show that the influence of the denominator $|Z(k, k \cdot v_1/k)|^2$ is appreciable for $k < \kappa$.
Problem. The integral (36) may be evaluated approximately by putting $|Z|^2 = 1$ for $k > \kappa$ and $|Z|^2 = \infty$ for $k < \kappa$. Show that with this approximation it reduces exactly to Landau's result, if also the cut-off $k_{max} = 3kT/e^2$ is used.
Problem. Show that (35) obeys the H-theorem.
Problem. Apply the Bogolyubov method to the equations

$$\frac{df(t)}{dt} = -\varepsilon g(t), \qquad \frac{dg(t)}{dt} = f(t) - g(t),$$

where f and g are two functions of t alone, and ε is a small parameter. Compare the result with the exact solutions.
Problem. Find the errors in this book and write them to the authors.

General references

L. Spitzer, *Physics of Fully Ionized Gases*, 2nd ed. (Interscience, New York 1962) ch. V
J. L. Delcroix, *Introduction à la Théorie des Gaz Ionisés* (Dunod, Paris 1959).
D. J. Rose and M. Clark, *Plasmas and Controlled Fusion* (Wiley, New York 1961) ch. VIII.
C. L. Longmire, *Elementary Plasma Theory* (Interscience, New York 1963) chs. VIII, IX.
R. Jancel and T. Kahan, *Electrodynamique des Plasmas* (Dunod, Paris 1963).

* It is also possible to avoid the divergence by taking the hard collisions into account separately, see E. A. Frieman and D. L. Book, Phys. Fluids **6**, 1700 (1963).

S. Chandrasekhar, *Principles of Stellar Dynamics* (University of Chicago Press, Chicago 1942) ch. II.
S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge 1939).
W. B. Thompson, *An Introduction to Plasma Physics* (Pergamon, Oxford 1962) ch. VIII.
W. B. Thompson in: *Advanced Plasma Theory*, M. N. Rosenbluth ed. (Academic Press, New York 1964).
B. A. Trubnikov and S. I. Braginskii in: *Reviews of Plasma Physics*, Vol. 1, M. A. Leontovich ed. (Consultants Bureau, New York 1965).
N. N. Bogolyubov, *Problems of a Dynamical Theory in Statistical Physics* (Moscow 1946); translated in: *Studies in Statistical Mechanics*, Vol. 1, J. de Boer and G. E. Uhlenbeck eds. (North-Holland, Amsterdam 1962).
N. Rostoker and M. N. Rosenbluth, *Phys. Fluids* 3, 1 (1960).
R. Balescu, *Statistical Mechanics of Charged Particles* (Interscience, New York 1963).
D. C. Montgomery and D. A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill, New York 1964).

AUTHOR INDEX

Alfvén, H., 38, 65
Allan, D. W., 53
Allen, C. W. 49
Allis, W. P., 118, 187
Artsimovich, L. A., 38
Arzimovich, L. A., 7

Backus, G., 53
Bachynsky, M. P., 129
Balescu, R., 100, 158, 206, 208
Bernstein, I. B., 76, 118, 158, 179, 187, 194
Bers, A., 118, 187
Bhatnagar, P. L., 189
Bogolyubov, N. N., 208
Bohm, D., 137, 138
Book, D. L., 207
Born, M., 111
Bowers, R., 59, 113
Braginskii, S. I., 208
Brout, R., 100
Buchsbaum, S. J., 113, 118, 187
Budden, K. G., 118
Bullard, E. C., 52, 53

Callebaut, D. K., 80
Carruthers, P., 100
Case, K. M., 137, 183
Chandrasekhar, S., 7, 41, 53, 81, 198, 208
Chapman, S., 128, 189, 208
Chew, G. F., 128
Clark, M., 27, 35, 38, 81, 129, 207
Cohen, R. S., 196
Cole, G. H. A., 7, 48
Coppi, B., 81
Courant, R., 137
Cowling, T. G., 7, 52, 53, 128, 189, 208

D'Angelo, N., 153
Debye, P., 83
De Groot, S. R., 11
Dekeyser, R., 85
Delcroix, J. L., 118, 129, 207
Denisse, J. F., 118
Derfler, H., 153
Drummond, W. E., 185
Dungey, J. W., 59

Ehrenfest, P. and T., 155
Elliot, H., 4
Elsasser, W. M., 49, 50

Farquhar, I. E., 155
Felderhof, B. U., 96, 159, 176, 177, 187
Ferraro, V. C. A., 7, 49
Fetisov, E. P., 176
Frieman, E. A., 76, 206, 207
Fünfer, E., 7, 38, 79
Furth, H. P., 81

Ginzburg, V. L., 118
Glasstone, S., 59, 78
Goldberger, M. L., 128
Grad, H., 65
Greene, J. M., 38, 81
Gross, E. P., 137, 138, 189

Hadamard, J., 18
Hain, K., 76
Harris, E. G., 187, 203
Hartmann, J., 45
Haskell, R. E., 129
Helliwell, R. A., 113
Herzenberg, A., 53
Hilbert, D., 137
Holt, E. H., 129
Hubbard, J., 198
Hückel, E., 83

Jackson, E. A., 184

Jackson, J. D., 7, 35
Jancel, R., 129, 207
Jeffrey, A., 16, 76, 80
Johnson, J. L., 38, 81
Johnston, T. W., 129
Judd, D., 197

Kahan, T., 129, 207
Kalikhman, L. E., 16
Kaufman, A. N., 198
Knechtli, R. C., 3
Krall, N. A., 186
Krook, M., 189
Kruskal, M. D., 76
Kulikovskiy, A. G., 16, 53
Kulsrud, R. M., 76, 81

Lamb, H., 11, 69
Landau, L. D., 7, 135, 190
Langmuir, I., 107
Lazarus, F., 45
Legendy, C., 113
Lehner, G., 7, 38, 79
Lehnert, B., 4
Lenard, A., 206
Lifshitz, E. M., 7
Linhart, J. G., 7
Longmire, C. L., 7, 175, 207
Lovberg, R. H., 59, 78
Low, F. E., 128
Lowes, F. J., 53
Lundquist, S., 42, 53
Lüst, R., 41, 65, 76, 118
Lyubimov, G. A., 16, 59

MacDonald, W., 197
Malmberg, J. H., 153
Mayer, J. E., 97
Mayer, M. Goeppert, 97
Mazur, P., 11
Meeron, E., 100
Merrill, J. H., 107
Montgomery, D. C., 158, 187, 208
Motley, R. W., 153
Motz, H., 158
Murgatroyd, W., 45
Muskhelishvili, N. I., 143

Newcomb, W. A., 27, 80
Northrop, T. G., 4

Oster, L., 118

Penning, F. M., 107
Penrose, O., 184
Pines, D., 100, 138
Plumpton, C., 7
Pradhan, T., 177, 187
Prandtl, L., 48

Raimes, S., 100
Ratcliffe, J. A., 118
Reuter, G. E. H., 177
Robinson, B. B., 194
Rose, D. J., 27, 35, 38, 81, 129, 207
Rose, F., 113
Rosenberg, R. L., 206
Rosenbluth, M. N., 35, 185, 186, 197, 208
Rostoker, N., 208
Routly, P. M., 196
Rutherford, P. H., 206
Rynn, N., 3

Sagdeev, R. Z., 81, 187
Schlüter, A., 41, 43, 76
Shercliff, J. A., 16, 38, 53
Shih-I Pai, 16, 27, 47, 53
Shkarofsky, I. P., 129, 194
Shure, F. C., 176
Silin, V. P., 176
Simon, A., 203
Simonen, T. C., 153
Sneddon, I. N., 35
Sommerfeld, A., 69
Sondheimer, E. H., 177
Spitzer, L., 81, 111, 118, 196, 207
Stix, T. H., 187
Szabo, J., 49

Taniuti, T., 76, 80
Ter Haar, D., 93, 189, 190
Thompson, W. B., 16, 81, 208
Thomson, G. P., 107
Thomson, J. J., 107
Tidman, D. A., 158, 187, 208
Titchmarsh, E. C., 142, 147, 154
Tonks, L., 107
Trehan, S. K., 118, 158, 187
Trubnikov, B. A., 208

Uhlenbeck, G. E., 198
Uman, M. A., 7

Van Allen, J. A., 4

Van Kampen, N. G., 135, 143, 158
Vedenov, A. A., 81, 100, 187
Velikhov, E. P., 81, 187
Vlasov, A., 4, 130
Voslamber, D., 80

Wada, J. Y., 3
Wang, M. C., 198
Watson, G. N., 139, 154
Webb, H. W., 107
Weenink, M. P., 158, 187
Wergeland, H., 93
Weston, V. H., 176
Wharton, C. B., 153
White, R. S., 4
Whittaker, E. T., 139, 154
Wilkinson, I., 53
Wolf, E., 111
Woltjer, L., 41, 45
Wong, A. Y., 153
Wu, T. Y., 206

Zelazny, R. S., 177

SUBJECT INDEX

acoustic wave, 116
adiabatic compression, 128
Alfvén, 30
Alfvén velocity, 56
Alfvén waves, 7, 57, 113, 185
anisotropy, 178
anomalous skin effect, 176
authors, errors of, 207

balance equations, 18
–, energy, 19, 105
–, mass, 18
–, momentum, 19, 122
Balescu, 188
Balescu–Lenard equation, 206
barometric formula, 38
B.B.G.K.Y. hierarchy, 201
Bennett's relation, 34
Bogolyubov, 188
–, method of, 204, 207
Boltzmann, 189
Boltzmann equation, 119
–, collisionless 130
boundary layer, 174
Bullard, 52

causality condition, 176
Chandrasekhar, 40
chemical potential, 37
circular polarization, 111, 181
cluster integrals, 97
coarse-grained description, 156
cold plasma, 105, 117
collision frequency, 126
collisions, 188
–, hard, 191, 196, 207
–, soft, 190, 196
compatibility, 17
completeness, 17, 145, 163
compressible fluid, 29, 35

condition equations, 17, 132
conservation equations, 121, 137
Coulomb collision, 190
Cowling, 51
Crab nebula, 45
critical velocity, 185
cut-off, 195, 196, 199
cut-off at the velocity of light, 166
cyclotron damping, 182
cyclotron frequency, 110
cylindrical coordinates, 31

damping, by collisions, 188
–, electron cyclotron, 182
–, ion cyclotron, 182
–, transit time, 186
Debye length, 83, 86
Debye potential, 85
Debye shielding, 82, 194
Debye sphere, 83
dielectric constant, 10, 117
dimensionless parameter, 202
dispersion law, 56, 110, 115, 172, 181
dispersion relation, 56
dissipation, 25, 39, 50, 81
dissipation time, 26
distribution function, 119
–, reduced, 145, 160, 201
drift velocity, 24
dynamical dispersion, coefficients of, 197
dynamical friction, coefficients of, 197
dynamo theory, 50

earth, 49
earth's magnetic field, 26, 49
electron cyclotron damping, 182
electron gas, 82, 87
–, free energy, 93
energy balance, 19, 105
– conservation, 132

energy flow, 20
– principle, 69
enthalpy, 37
equation of state, 11, 13
equilibrium, approach to, 134, 153
equipartition, 59
equipartition solution, 40
Euler description, 69
evolution operator, 132, 149
external fields and sources, 131, 178
external magnetic field, 178

Ferraro's law of isorotation, 49
fire hose instability, 186
fluctuations, 85
– of density, 94
– of field, 95, 198
Fokker–Planck equation, 197
force-free fields, 21, 40, 44, 80
free energy 87, 93
Fresnel formulas, 174
frozen-in field, 23
fundamental condition, 84

garden hose instability, 186
gravity, 9, 16
Green's function, 149
group velocity, 56
gyromagnetic radius, 174

H-theorem, 134, 194, 207
Hamilton function, 200
hard collisions, 190, 196, 207
hard core pinch, 80
Hartmann's number, 47
heat, production of, 11
helicons, 113
Hilbert transform, 140
Hölder condition, 139
hot dilute plasmas, 84, 203
hot plasma, 113
hydromagnetics, 5

ideal conductor, 14, 23, 29
impact approximation, 199, 207
impact parameter, 189, 207
incompressible fluid, 15, 29
initial value problem, 145, 163, 165
instabilities, interchange, 77
–, magnetohydrodynamic, 7, 66
–, microinstabilities, 7, 183, 186
–, universal, 186

instability, 2, 66
–, fire hose, 186
–, garden hose, 186
–, kink, 79
–, mirror, 186
–, sausage, 78
–, two-stream, 184
interchange instabilities, 77
invariance for time reversal, 18, 133, 155
ion cyclotron damping, 182
irreversible behavior, 153
isorotation, Ferraro's law of, 49

Joule heat, 10, 20

kinetic equation, 119, 204
kinetic theory, 200
kink instability, 79
Kramers–Kronig relation, 56

Lagrange description, 69
Lagrange function, 67, 71
laminar flow, 47
Landau, 135, 188, 199
Landau damping, 138, 152
Landau equation, 189, 194
–, discussion of, 195
Langmuir, 1, 107
Larmor, 50
Liouville equation, 131, 170, 200
Liouville's theorem, 131, 171, 200
Loschmidt's Umkehreinwand, 157
Lundquist, 59
Lundquist solution, 42, 43
Lyapunov, 66

magnetic beach, 182
– permeability, 10
– pressure, 20, 31
– Reynolds' number, 26
– stiffness, 47, 59
– stress tensor, 19
magnetoacoustic wave, 64
magneto-fluid dynamics, 5
magneto-gas dynamics, 5
magnetohydrodynamic equations, 14
– instabilities, 7
– waves, 5, 54
magnetohydrodynamics, 5
magnetosonic wave, 64, 186
many-component theory, 119
–, equations of, 124

mass balance, 18
Maxwell distribution, local, 127
Maxwell equations, 10, 159
– term, 13, 19, 101
Mayer expansion, 97
metals, 59
micro-instability, 7, 183, 186
mirror instability, 186
mixture of ionized gases, 86
momentum balance, 19, 122
Montgomery, 188
motion of field lines, 24

Navier–Stokes equation, 10
negative-frequency part, 140, 164
neutral equilibrium, 68
Newcomb, 80
Nyquist diagram, 148, 184

Ohm's law, 10
Ohm's law, generalized, 104
overstability, 69

pair distribution function, 86
Penning, 107
Penrose criterion, 184
phase mixing, 154
– velocity, 56
pinch effect, 32
pinched discharge, 31, 77, 80
planar flow, 45
plasma column, 32, 34, 77
plasma, definition, 1
– frequency, 107
– oscillations, 108
plasma waves, 2, 7
–, longitudinal, 107 ff, 139
–, transverse, 108 ff, 159
Poincaré–Bertrand lemma, 141
Poisson bracket, 200
polarization, 117, 161, 179
polarization, circular, 111, 181
positive-frequency part, 140, 164
positive-ion wave, 116
principal value, 139

quantum mechanics, 85
quasi-neutrality, 12, 104, 126

reflection, 174, 177
refraction, 174
refractive index, 108, 172, 181
resonance, 111, 153
Reuter, 177
Reynolds' number, 12, 48
Riemann–Lebesgue theorem, 154
ring graphs, 98
Rosenbluth, 35
Rutherford's formula, 190

sausage instability, 78
screw-shaped instability, 79
self-adjoint operator, 76
self-energy, 100
self-exciting dynamo, 53
sheared field, 77
skin effect, 177
soft collisions, 190, 196
Sondheimer, 177
sound waves, 62, 65, 116
– velocity, 62, 115
spatially homogeneous plasma, 205
specific heat, 11
spiral arm, 41
stability, 66
state space, 132
stationary solutions, 39, 142, 161
statistical mechanics, 82
Stosszahlansatz, 120, 189, 198, 205
stress tensor, 9, 122
–, anisotropic, 128
sunspots, 30, 60
Suydam's criterion, 80

terrestrial magnetism, 39, 49
thermodynamic limit, 83, 202
Tidman, 188
time reversal, 18, 133, 155
Titchmarsh, theorem of, 142
trajectory, 132
transfer, of energy, 127
–, of momentum, 124, 127
transient, 181
transient time, 151, 155
transit time damping, 186
trapped particles, 175
turbulence, 48
two-component theory, 5, 101
–, equations of, 104
–, discussion, 125
two-stream instability, 184

Umkehreinwand, 157
universal instabilities, 186

Van Allen belts, 4
variation length, 154
viscosity, 9, 12, 45, 48
Vlasov, 4, 137, 152
Vlasov equation, 130, 203
–, linearized, 135, 159, 179
–, relativistic, 170
Vlasov theory, 130

Walén, 60
whistlers, 112