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AN INTRODUCTION TO THE THEORY OF PLASMA TURBULENCE

by

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PERGAMON PRESS
OXFORD · NEW YORK · TORONTO
SYDNEY · BRAUNSCHWEIG

Pergamon Press Ltd., Headington Hill Hall, Oxford
Pergamon Press Inc., Maxwell House, Fairview Park, Elmsford, New York 10523
Pergamon of Canada Ltd., 207 Queen's Quay West, Toronto 1
Pergamon Press (Aust.) Pty. Ltd., 19a Boundary Street, Rushcutters Bay, N.S.W. 2011, Australia
Vieweg & Sohn GmbH, Burgplatz 1, Braunschweig

First edition 1972
Library of Congress Catalog Card No. 77-182262

PRINTED IN HUNGARY

08 016587 7

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Note by Series Editor

THIS book is based upon lectures given by Professor Tsytovich at Culham Laboratory. We are grateful to Culham Laboratory for their assistance in getting these lectures published. A preliminary draft of part of the text was published as Culham Report CLM-L19.

1. Comparison of Plasma and Liquid Turbulence

As an introduction to the whole problem of plasma turbulence, we start with a comparison of plasma turbulence and the turbulence in liquids. The turbulence of liquids has been the subject of an enormous number of investigations during the last 30 years, and therefore the basic concepts are well known. The field of plasma turbulence has been developed only in the last decade both experimentally and theoretically. Nevertheless, it is surprising to see how much research is now devoted to plasma-turbulence problems as compared to liquid-turbulence problems. On the one hand, this is due to a very large activity in the experimental and theoretical study of plasmas in recent years; these activities are partly due to the applications and explanations of the phenomena found in laboratory investigations of plasmas in high electric and magnetic fields and of shock waves, as well as those phenomena found in plasma heating or in geophysical and astrophysical applications. On the other hand, there exists a physical reason which allows us to give a complete theoretical description of the most important turbulent motions connected with so-called plasma oscillations. This is the existence of some kind of elasticity in collective motions of plasmas, which is absent in incompressible liquids. For example, if in a plasma, a sheet of plasma electrons is displaced over a distance d (see Fig. 1) the charge separation provides

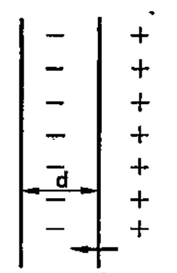

FIG. 1.
Plasma oscillation arising from a charge sheet separation.

a force which tends to prevent the charge separation and oscillations around the neutrality position arise. These are the so-called Langmuir plasma oscillations which have a frequency

$$\omega_{pe} = \sqrt{\left(\frac{4\pi ne^2}{m_e}\right)}, \tag{1.1}$$

where n is the mean density of the electrons and m_e their mass. In the presence of an external magnetic field a frequency of the order of the gyrofrequency of electrons

$$\omega_{He} = \frac{eH}{m_e c}, \tag{1.2}$$

also occurs; other frequencies, involving the ion parameters, are also possible (Stix, 1962; Ginzburg, 1970).

In incompressible liquids the eddies have no special frequency and their frequency is determined by their interaction with the other eddies. Turbulence is usually connected with the non-linear interaction of collective motions. Owing to the elasticity of plasma motions there occurs a small parameter, which is the ratio of a period of oscillation $1/\omega$ to the characteristic time τ of the non-linear interactions, that is, we have

$$\frac{1}{\omega\tau} \ll 1. \tag{1.3}$$

Condition (1.3) is called the condition for weak turbulence. The theory of plasma turbulence was completely developed for the case when (1.3) holds, and this condition seems to be valid in most experimental investigations of plasma turbulence (Kadomtsev, 1964).

In Fig. 2 we show the possible branches of collective plasma motions. Plasmas can also have collective motions similar to those which incompressible liquids have, namely eddies. These motions exist for $\omega \ll \nu$ (ν is the two-particle collision frequency) and for $\omega/k \ll v_s$, where v_s is the sound velocity. This region is the very small shaded region in Fig. 2. Most of the collective motions shown in Fig. 2 are collisionless. Let us mention that ω_{pe}/ν_{ee} is usually a very big number (as a definition of the plasma as a state of matter implies) from 10^4 in laser produced plasmas to 10^8 in most laboratory gas discharge plasmas, and to 10^{10}–10^{12} in astrophysical conditions. We see, therefore, that the concepts of plasma turbulence contain a generalisation of the concepts that were used earlier in the case of liquid turbulence.

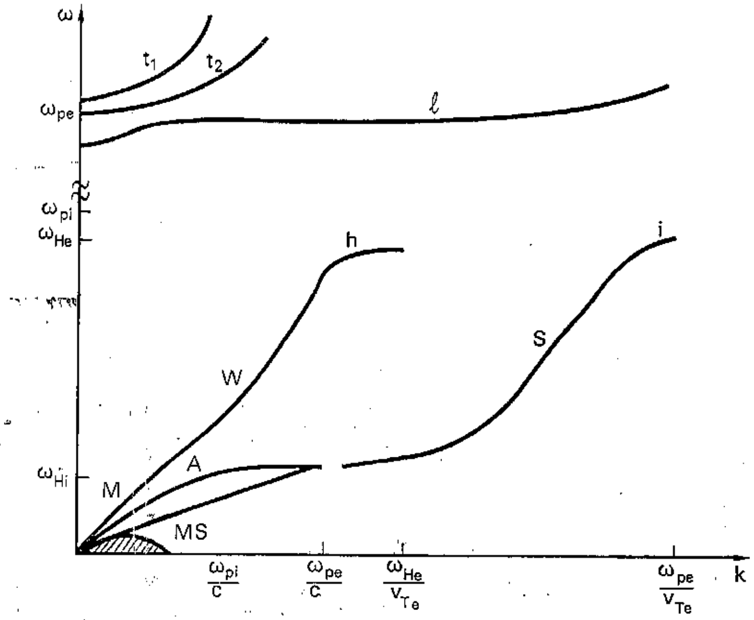

FIG. 2.

Possible branches of collective plasma motions. M and MS indicate the fast ($\omega = kv_A$) and slow ($\omega = kv_s$) magnetohydrodynamic waves, A Alfvén waves ($\omega = k_z v_A$), W whistlers ($\omega = \omega_{He} k_z^2 c^2/\omega_{pe}^2$), S ion-sound waves ($\omega = kv_s$), i plasma ion oscillations, h hybrid plasma oscillations ($\omega = \omega_{He} k_z/k$), l Langmuir plasma oscillations ($\omega \approx \omega_{pe}$), and t_1 and t_2 ordinary and extra-ordinary transverse waves. Here $v_A \gg v_s$, where v_A is the Alfvén velocity ($v_A = H/\sqrt{(4\pi nm_i)}$), v_s the sound velocity ($v_s = \sqrt{(T_e/m_i)}$); $\omega_{pe} \gg \omega_{He}$; and the external magnetic field is assumed to be in the z-direction.

The other essential difference between plasma and liquid turbulence is the presence of electric and magnetic fields in collective motions. Therefore, if in a liquid the stochastic variables are the mean particle velocity of particles $\boldsymbol{v}$ or the density ϱ in plasmas the electric fields also become stochastic properties.

In the general case one can divide the observables into two parts—a regular and a stochastic part

$$\boldsymbol{v} = \boldsymbol{v}^{\text{reg}} + \boldsymbol{v}^{\text{stoch}}, \tag{1.4}$$

$$\boldsymbol{E} = \boldsymbol{E}^{\text{reg}} + \boldsymbol{E}^{\text{stoch}}. \tag{1.5}$$

By definition

$$\langle E^{\text{stoch}}\rangle = 0, \quad \langle v^{\text{stoch}}\rangle = 0, \tag{1.6}$$

where brackets indicate averages (for example, over a statistical ensemble). As in liquids the statistical properties of the stochastic fields are given by correlation functions that are measured in most experiments of plasma turbulence:

$$\frac{1}{8\pi}\langle(\boldsymbol{E}^{\text{stoch}}(\boldsymbol{r}, t)\cdot \boldsymbol{E}^{\text{stoch}}(\boldsymbol{r}', t))\rangle = \int d\omega\, d^3k I_{k,\omega}\, e^{-i\omega(t-t')+i(\boldsymbol{k}\cdot[\boldsymbol{r}-\boldsymbol{r}'])}. \tag{1.7}$$

Equation (1.7) is written for the case of a stationary and homogeneous turbulence, and when $\boldsymbol{r} = \boldsymbol{r}'$ and $t = t'$ the quantity $I_{k,\omega}$ gives the frequency and $\boldsymbol{k}$-dependence of the energy of the electric field of turbulent motions.

Usually as, for example, for all turbulent motions shown in Fig. 2, the whole energy $W_{k,\omega}$ of turbulent motion, which consists of the energy of the turbulent electric fields and the energy of the particle motion, is proportional to $I_{k,\omega}$:

$$W_{k,\omega} = \alpha_{k,\omega} I_{k,\omega}. \tag{1.8}$$

Equation (1.8) is an approximate one. It is approximately valid, if (1.3) is valid. The ω-width of $W_{k,\omega}$, that is, $\Delta\omega$, describes the characteristic time of the correlations:

$$\tau_{\text{corr}} \approx \frac{1}{\Delta\omega}. \tag{1.9}$$

The correlation curve $W_{k,\omega}$ of a collective plasma motion as a function of ω for fixed $\boldsymbol{k}$ has a maximum near the frequency of elastic response (Fig. 3).

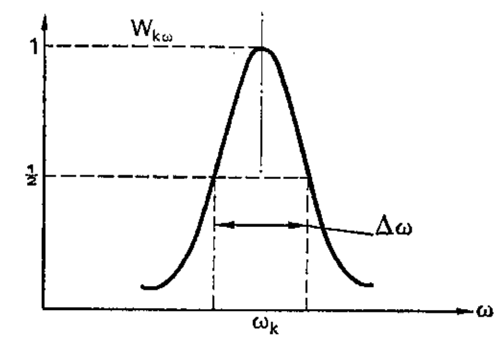

FIG. 3.
Sketch of the correlation curve in a turbulent plasma.

The half-width $\Delta\omega$ of this curve is much less than $\omega(\boldsymbol{k})$, if the condition (1.3) for weak turbulence is fulfilled. As in liquids the integral of $W_{k,\omega}$ over ω is called the turbulent spectrum

$$W_k = \int W_{k,\omega}\, d\omega. \tag{1.10}$$

For the case of isotropic turbulence it is useful to introduce a quantity W_k normalised to the modulus of $\boldsymbol{k}$ ($|\boldsymbol{k}| = k$):

$$W_k = 4\pi k^2 W_{\boldsymbol{k}}, \tag{1.11}$$

$$\int_0^\infty W_k\, dk = W. \tag{1.12}$$

For magnetic types of oscillations it is also useful to introduce a quantity $W_{\omega,\Omega}$, which satisfies the equation

$$\int_0^\infty d\omega \int d^2\Omega W_{\omega\Omega} = W, \tag{1.13}$$

where Ω is the solid angle.

Now let us come to the problem of the spectrum of turbulent liquids. Suppose, as a very simple example, that we have a liquid flow in a pipe of dimension L_0 (Fig. 4).

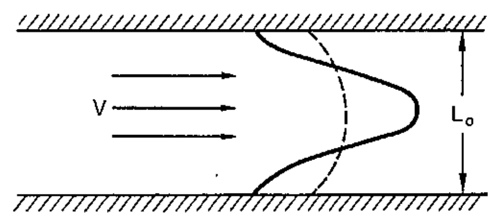

FIG. 4.
Sketch of the velocity distribution in a liquid flowing through a pipe. The solid line corresponds to laminar flow and the dashed line to turbulent flow.

If the mean velocity (or the Reynolds number) of the liquid is low enough, there exists a laminar flow which forms a velocity profile such as shown on Fig. 4. If the velocity reaches some critical value, such a profile becomes unstable and eddies of dimension L_0 are excited. These eddies become also unstable, excite new ones with smaller dimensions, and so on. In the developed turbulent regime, eddies with dimensions less than L_0 are present, so that one can ask for the distribution of the turbulent energy over the dimensions of the eddies, that is, the energy that is carried by eddies with dimensions in the interval between l and $l+dl$. Instead of l one can introduce the wave-number k,

$$k = \frac{2\pi}{l}, \tag{1.14}$$

and ask for the distribution over k. This is the same as that given by equation (1.12). The region $k \sim k_0 = 2\pi/L_0$ is usually called the energy-containing region. The diminishing of the eddy dimensions described above is usually considered as a result of their non-linear interactions and leads to an energy flow in $\boldsymbol{k}$-space from $k = k_0$ to higher k. For very large k viscosity becomes important and the turbulent energy of the eddies is decreased by viscosity (see, for instance, Landau and Lifshitz, 1959).

We show in Fig. 5 the turbulence spectrum of incompressible liquids. This spectrum is stationary. That means that for each k there is an energy

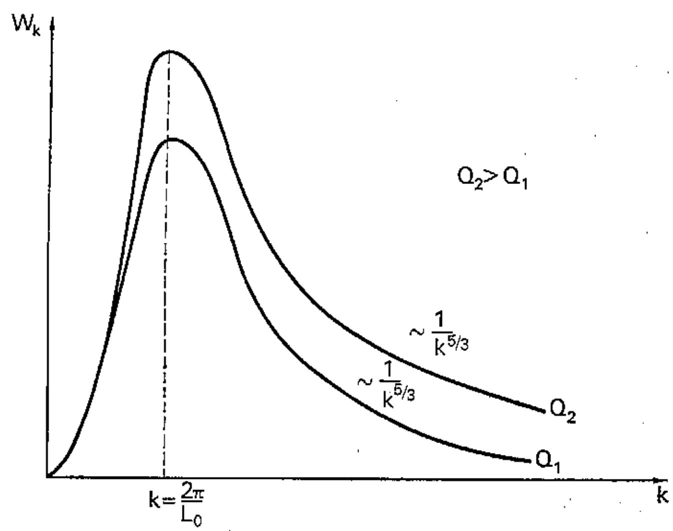

FIG. 5.
The turbulence spectrum of incompressible liquids; Q is the energy input, $W_k \sim k^{-5/3}$ the universal Kolmogorov law.

balance between the eddies which enter and leave the interval dk. In the energy containing region (k of the order of k_0) the balance is maintained by the excitation of eddies and their transformation to higher k by non-linear interactions. In the middle part of the spectrum, called the inertia region, the balance is only caused by the non-linear interactions. That means that the energy which enters the interval dk from lower k-values is compensated by energy outflow from dk to the higher k. In this region the spectrum is universal; it was first found by Kolmogorov (1941) to vary as $\sim k^{-5/3}$ by means of dimensional analysis and is known as the Kolmogorov spectrum. In the region of high k, when the damping of the eddies becomes comparable with the energy input due to the energy inflow from small k-values, there is a balance between energy inflow and damping. If one raises the energy input Q the energy spectrum does not change (see Fig. 5, $Q_2 > Q_1$). All the characteristic elements of the formation of the stationary turbulence shown above for liquids are also found in the case of plasma turbulence (Tsytovich, 1969a). These are

1. The excitation of collective motions in some interval in $\boldsymbol{k}$-space.
2. The non-linear energy transfer from the excitation region to the region where the turbulent oscillations are damped.
3. The damping of the oscillations.

The difference lies, of course, in the actual mechanisms of excitation, energy transfer, and damping.

Since damping is the final result of the history of any turbulent oscillation excited in a plasma, it is useful to classify the possible types of plasma turbulence by mentioning the most important damping mechanisms. There are new mechanisms of damping that have no analogy with the damping of eddies in liquids, and, therefore, in a plasma new types of turbulent motions can exist. First, it should be mentioned that the turbulent oscillations can be damped by binary collisions, and this kind of damping is in some sense analogous to the damping in liquids. (Although in a collisionless plasma $\omega \gg \nu$ and in liquids $\omega \ll \nu$.) This kind of damping can be essential, for example, in a partly ionised plasma, or in the case when the turbulent oscillations are transformed by non-linearity from the regions of intensive collective damping. Such a type of turbulence—turbulence dissipated by binary collisions—can have a large energy stored in the turbulent motions because the damping is usually small.

Secondly, in plasmas there exists a new type of damping—the collective damping. An example of such damping is the well-known Landau damping. The presence of such damping is connected with the electric field that exists in turbulent plasma motions and, therefore, with the possibility of the interaction of such a field with the charged thermal particles of the plasma. It is essential also that one can associate with the turbulent motions definite phase velocities

$$v_{ph} = \frac{\omega(k)}{k}. \tag{1.15}$$

Because of correlation broadening this correspondence is only approximate (see Fig. 3) but the uncertainly is of the order of the small parameter (1.3). If then v_{ph} becomes of the order of the mean thermal particle velocity, all thermal particles are resonant with the waves, that is, it can gain energy from the wave and, therefore, the turbulent oscillations are heavily damped (Landau, 1946).

In a magnetic field there can also exist cyclotron resonance, when the frequency of the field of turbulent oscillations in the reference frame where the particles have zero velocity along the magnetic field is equal to the gyration frequency of the particles or to its harmonics. This is the so-called cyclotron resonance, or cyclotron damping. This kind of damping can exist near the electron–cyclotron or near the ion–cyclotron frequencies.

If the energy of turbulent motions flows in the direction where a region of collective dissipation exists, the heating of plasma is very essential. Such a kind of heating is usually called turbulent heating. From the considerations given above, it is obvious that the rate of such heating depends on the rate of energy transfer and therefore depends on W, or the energy input.

Thirdly, in a plasma there can exist also a new type of damping due to acceleration of charged particles. Such an acceleration is a stochastic one and is also deeply connected with the electromagnetic nature of the plasma oscillations. One can call this also a heating of a small fraction of the particles which are resonant with the oscillations, that is, with the maximum in Fig. 3. The thermal particle can be non-resonant, and then the oscillations are not heavily damped. Because of the long life of the oscillations, fast particles can receive enough energy and as a result the oscillations can be damped. Such particles occur naturally in the tail of a Maxwellian distribution in a plasma. Their injection into an acceleration regime (that is, one in which they are able to resonate with oscillations) can be due to the non-linear interactions of the oscillations or to a magnetic type of turbulent waves.

Fourthly, there can also be a transformation of the turbulent energy into electromagnetic radiation. It is due to the electromagnetic nature of turbulent plasma motions and their non-linear interactions. This kind of turbulence is radiatively dissipative turbulence.

The excitation of turbulence in plasmas can be due to a change of sign of the damping coefficient as a result of the anisotropy of particle distributions. Thus the same collective effects that give damping in the isotropic case can produce instabilities, if the particles are distributed anisotropically. The turbulence can be excited also by fast particles and the best known example of this is the plasma beam instability. Also it is possible to excite turbulent motion by radiation or by a high-frequency electromagnetic field. This has an analogy with the non-linear optics problem of excitation of supersonic waves in solids (Tsytovich, 1967).

The existence of charged particles in a plasma is very essential for the nature of the non-linear energy transfer. In liquids the eddies interact only with one another or, in other words, the whole energy of the turbulent motion is conserved and is only transformed from the biggest eddy to the lowest, or from smaller to larger k. The entropy in this kind of process increases because the phase volume that is proportional to the volume in $\boldsymbol{k}$-space increases.

In the presence of charged particles and due to the electromagnetic nature of plasma oscillations there exists the possibility of interactions and energy exchange between the turbulent motion and the particles. One of the most important of such processes is the induced scattering of oscillations by particles. One such process is shown in Fig. 6. In such a process the particles can gain energy or, in other words, are heated.

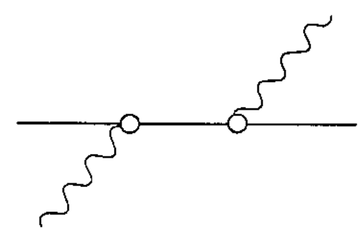

Fig. 6.
Scattering of oscillations by particles.

The particle entropy increases because of this heating, and to compensate this increase the phase volume occupied by the waves can decrease. In other words, the energy of turbulent motions can be transformed from higher to lower k. This is the case, for example, for Langmuir plasma turbulence, the spectrum of which is shown in Fig. 7.

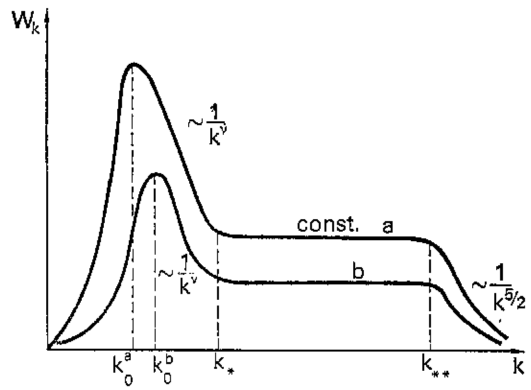

FIG. 7.
The spectrum of Langmuir turbulence ($T_e = T_i$).

The turbulent oscillations created at high k (where the source of the turbulent energy is located) are transformed to lower k. The spectrum $\sim k^{-5/2}$ is due to the energy balance when this transformation is produced by induced scattering by electrons. This spectrum exists only for $k > (m_e/m_i)^{1/5}k_D$ in a narrow region. The spectrum $W_k = \text{const} = k^{-\nu}$, $\nu = 0$, due to the induced scattering by ions is valid up to

$$k > \left(\frac{m_e}{3m_i}\right)^{1/2} k_D = k_*; \qquad k_D = \frac{\omega_{pe}}{v_{Te}}. \tag{1.16}$$

The turbulent energy is transferred to smaller k up to k_* (each step of such a transfer is of order k_*, so that one can only roughly say that the spectrum is flat on the average in a k-interval larger than k_*). In the region $k \ll k_*$ the whole energy has come to small k and the eddies visually speaking must have very rapid collisions with one another because the phase volume is small. This repulsion is described by the interaction of plasmons which is diagramatically shown in Fig. 8.

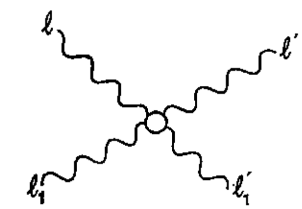

FIG. 8.
Plasmon collisions; l_1 and l describe the plasma state before and l_1' and l' the state after the collision.

As the collisions of particles produce a Maxwellian distribution, the collisions of plasmons produce a maximum in Fig. 7. The essential difference is that there is continuous flow of plasmons from higher k into this region. The damping of plasmons in the region of the maximum in Fig. 7, which can be called an energy-containing region, can be due either to ordinary binary collisions or to radiative losses. In the second case, the Langmuir turbulence can be radiatively dissipative (if the number of fast particles accelerated is small).

The spectrum in the asymptotic region, $k_* \gg k \gg k_0$, is $\sim k^{-\nu}$, where $2{\cdot}8 < \nu < 4$, depending on the energy input Q. If the input of the turbulent energy Q is raised the spectrum remains the same in all above-mentioned regions except the last where ν tends to 4, if Q increases. The position of the maximum of the spectra, $k_0 = 2\pi/L_0$, comes down in k-space if the dissipation is due to ordinary collisions and is unchanged if the dissipation is due to electromagnetic radiation. This picture was found by several authors (Pikelner and Tsytovich, 1968; Liperovskii and Tsytovich, 1969) by an analytical solution of the complicated non-linear equations of weak turbulence and checked by numerical computations by Makhankov *et al.* (1970).

We shall describe the properties of Langmuir turbulence in more detail later on. We mention here that the energy transfer in this case is the opposite of that in liquids, and that this can be possible only if the particles are heated. Such a heating is a stochastic process. One must distinguish this from the turbulent heating where the energy of turbulent oscillations is directly drawn into the dissipative region. For Langmuir turbulence we have no such heating in the absence of magnetic fields because the direction of energy transfer in the case of isotropically distributed particles is such that the turbulent oscillations are drawn away from the region of Landau damping. From the energy-conservation law it is possible to see that the energy of plasmons becomes thermal particle energy. Because the number of plasmons is conserved, the larger the difference between the maximum and the minimum frequencies of the branch to which the energy is transformed the larger the heating. Therefore, the rate of stochastic heating depends on this difference as well as on the time needed for the transfer of waves from maximum to minimum frequencies. As can be seen from Fig. 2, the Langmuir waves undergo a small change of frequency along the branch and the low-frequency waves such as whistler and ion-sound waves a large one. On the other hand, however, the interaction of whistlers is rather small compared with the interaction of ion-sound waves and thus the transfer of the energy across the same frequency difference needs a longer time. The Langmuir turbulence can transfer the energy more quickly than the ion-sound turbulence. Both Langmuir and ion-sound turbulence are appropriate for stochastic heating.

It is well known that for liquids in the turbulent regime the viscosity becomes anomalously large and the profile of a liquid flowing through a pipe is flattened. Similar effects are observed in turbulent plasmas. For example, there occurs an anomalous diffusion in a plasma confined by a magnetic field, an anomalous resistivity when an electric field is applied, and so on. One can, therefore, introduce the effective collisions due to such processes. The physical nature of such collisions in the weak-turbulence case is very obvious, the collision frequency is the inverse of the characteristic time of energy transfer along the turbulent spectra. Thus one can expect that this characteristic collision frequency is detectable if a small-amplitude wave with a frequency smaller than that of the characteristic turbulent collision frequencies is pushed onto the plasma. In this case, the skin effect of the electromagnetic wave can be due to such turbulent collisions. Indeed, a growth of the skin-depth when turbulence is excited was found experimentally.

Generally speaking, one can say that the electromagnetic properties of a turbulent plasma governing the penetration of a small linear perturbation is quite different from those of a non-turbulent plasma if the frequency of the perturbation is much less than the turbulent collision frequency. A new type of wave can appear and also a new stability pattern, that is, the stabilisation of some waves which are unstable in the non-turbulent regime and the existence of new instabilities. As an example, we mention the possibility of the excitation of a magnetic type of perturbations (Tsytovich, 1968a) in pure Langmuir turbulence, that is, in the case when only electrostatic fields are present. This effect has an analogy with the well-known effect of the excitation of a magnetic field by the turbulent motion of conducting liquids—the Batchelor effect. The scale of the characteristic dimensions or wavelengths of the excited magnetic fields in the case of Langmuir turbulence is much smaller than in the case of the hydrodynamic dynamo.

2. General Problems of the Theory of Plasma Turbulence

IN THIS chapter we shall give a summary and general description of some problems of the theory of plasma turbulence.

2.1. Excitation of the Turbulent State

Before discussing turbulence itself, one should define the kinds of motion which can be considered turbulent and how the motions can be excited. Usually the turbulence is connected with a stochastic variation of some measurable quantity, for example the electric potential φ. In experimental investigations the time-dependence of the potential fluctuations can be measured. One can find many examples of such measurements in which φ changes in a very complicated manner with time (see Fig. 9).

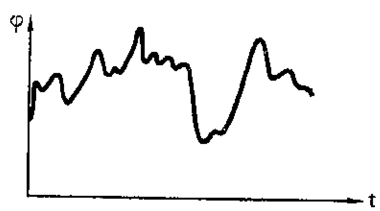

FIG. 9.
A typical experimental variation of the electrical potential in turbulent motion.

Does this mean that the potential is a stochastic variable? Generally not! For φ to be a stochastic variable, the measurement should not be reproducible by repeating the same experiment under the same macroscopic conditions (which must be carefully defined). When φ is stochastic in this sense one can ask: What is the cause of the irreproducibility? The answer is that small changes in the initial conditions change the

behaviour of φ considerably. This is well known in the statistical theory of molecular motion. Plasma-turbulence theory deals with the collective motions of a plasma, such as plasma oscillations. There are no difficulties in describing this kind of motion such as there are in describing the individual particle motions and general statistical concepts can be applied to describe the turbulence of plasmas.

For plasma oscillations, which in zeroth approximation could be considered as propagating waves, the stochastic behaviour lies in the randomness of their phases. Indeed, the initial value to be specified to match the potential of such a wave is its phase. Randomness of phase means

$$\langle\varphi\rangle = 0. \tag{2.1}$$

The average is over the statistical ensemble of the experiments mentioned above. In the general case, there is an applied or regular component of φ as well:

$$\varphi = \varphi^{\text{reg}} + \varphi^{\text{stoch}}, \quad \langle\varphi^{\text{stoch}}\rangle = 0. \tag{2.2}$$

For experimentalists it is necessary to mention that the theory predicts the statistical average value which is to be found by averaging the results from many equivalent experiments.

The question now is, why do small changes in the initial conditions change so appreciably the time behaviour of the measured quantity? The answer is that such initial states are usually unstable. One can find in the Landau description of fluid turbulence the instability necessary for the excitation of turbulence. It is also known, however, that the excitation of turbulence in liquids is one of the most complicated problems. The same is true in plasmas. Many plasma instabilities are known; they can be divided into so-called kinetic and hydrodynamic types. For the first kind the growth-rate γ is much less than the characteristic frequency of collective motions:

$$\frac{\gamma_{\text{kin}}}{\omega(\boldsymbol{k})} \ll 1. \tag{2.3}$$

For the second kind the growth-rate and the frequency are of the same order and for so-called aperiodic instabilities the growth-rate is much larger.

It is not obvious that the presence of instabilities leads to the excitation of turbulence, but it is generally the case. The important factor is the time needed for turbulence to develop. It is known from non-linear optics (when large amplitude waves interact in solids, for example) that the first stage of wave mixing can be described as an instability of the generated waves. The instability is aperiodic, but this only describes the initial energy transfer. The full history is obtained from an exact treatment of three wave interactions.

One can now ask if an aperiodic instability of a plasma leads to only one mode being excited. From the point of view of the uncertainly raised by the growth-rate γ one should say yes, but one should really examine the stochastic behaviour of the exciting field. Let us introduce the time τ needed for a plasma to become stochastic in its behaviour. If only one mode is excited it is obvious that the spread of energy to other modes is due only to non-linear interactions. The turbulent state is reached, by definition, when the energy is distributed over a large number of modes. The time needed to develop turbulence in the case of an aperiodic instability is, therefore, much larger than the growth-time of the instability. It is possible that the parameters which characterise the plasma state are changed sufficiently that the condition (2.3) is fulfilled in the last stages. This is the case, for example, if a low-density beam interacts with a plasma. In the first stage the instability of the beam is a dynamical one and aperiodic. The beam comes to a stochastic regime as the instability becomes kinetic. It is not known quite how this transition occurs: two possibilities are an explosive non-linear instability or the action of trapped particles. One can now not exclude the possibility that in the aperiodic type of instability there is a stage of its development when it cannot be considered a weak one. It is quite probable that the final state of stationary turbulence is weak, but it is not known if the statistical approach can be applied during this transition. If not, this means that although the energy in collective motions during the transition may be very high, this is not a strong turbulent regime. All of these questions could be solved experimentally with the present level of knowledge, but this has not yet been done.

From the general statistical point of view, in an ergodic system the average over a statistical ensemble must be equal to the time average. This can also be checked. From the work of Fermi, Pasta and Ulam (1955), Kolmogorov (1955), and Arnold (1963) one can see that the non-linearity plays a significant role in the stochastisation processes. The smaller the difference between the characteristic frequencies of the system and the higher the amplitude of the field, the sooner the stochastic regime appears. In this sense, one could suppose that the best condition for

stochastisation is the presence of broad branches of frequencies of collective motions. This is, indeed, true for an infinite plasma as can be seen from Fig. 2, and is also true in finite plasmas, if there are modes with wavelengths much smaller than the size of the system. The presence of a broad spectrum means that the energy of the collective motions can be spread by non-linearity, at least in such a broad interval, and stochastic behaviour is quite probable.

2.2. Description of Weak Turbulence by the Concept of "Elementary Excitations"

Suppose that a weakly turbulent state exists in a plasma. It is usually said that weak turbulence has an advantage over strong turbulence in having a small parameter γ/ω (where γ is the linear or non-linear growth-rate) in which everything could be expanded. The presence of a small parameter is a fact, but the conclusion that an expansion procedure is possible is mostly wrong! For example, strictly speaking one cannot expand the non-linear plasma equations in the energy of the turbulence, W, or the strength of the stochastic part of the field connected with such turbulence. We can give many examples of this. Only in an approximate way for some integrated value can such expansions be made, and there is a good physical reason for this statement as will be shown.

If a statistical description is valid and the collective motions of the plasma are of a wave type, as in Fig. 2, one can consider waves as "elementary excitations", generally called plasmons (Tsytovich, 1967, 1968b). The interactions of plasmons can be treated by perturbation theory. For example, one can introduce the number of plasmons, N_k, connected with the energy W_k describing the turbulent spectrum by the relation

$$W_k = \frac{\omega(k)N_k}{(2\pi)^3}, \tag{2.4}$$

where $\omega(k)$ is one of the frequency branches shown in Fig. 2. (This differs from that used in quantum mechanics by a factor $\hbar$ (Dirac's constant). This will not matter as we know the N_k can be described classically and $\hbar$ does not appear.)

One can introduce the probability of, for example, the decay process shown in Fig. 10,

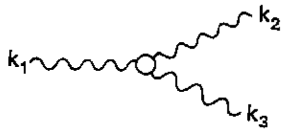

FIG. 10.

Decay of a plasmon (k_1) into two plasmons (k_2 and k_3).

Denoting the probability of this decay by $w(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$, we can write down the balance equation that shows the growth of the N_{k_1} excitations due to the coalescence of N_{k_2}, N_{k_3}, and its decay into N_{k_2}, N_{k_3}:

$$\frac{\partial N_{k_1}}{\partial t} = \int w(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\,[-N_{k_1}(N_{k_2}+1)\,(N_{k_3}+1) + (N_{k_1}+1)N_{k_2}N_{k_3}]\,\frac{d^3\boldsymbol{k}_2\, d^3\boldsymbol{k}_3}{(2\pi)^6}$$

$$= \int w(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\,\{-N_{k_1}N_{k_2} - N_{k_1}N_{k_3} + N_{k_2}N_{k_3}\}\,\frac{d^3\boldsymbol{k}_2\, d^3\boldsymbol{k}_3}{(2\pi)^6}. \quad (2.5)$$

The normalisation of the probability is to the phase volume $d^3\boldsymbol{k}_1\, d^3\boldsymbol{k}_2/(2\pi)^6$. Similar equations can be written for the processes shown diagrammatically in Figs. 11, 12 and 13.

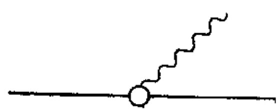

FIG. 11.

Emission of a plasmon by a particle.

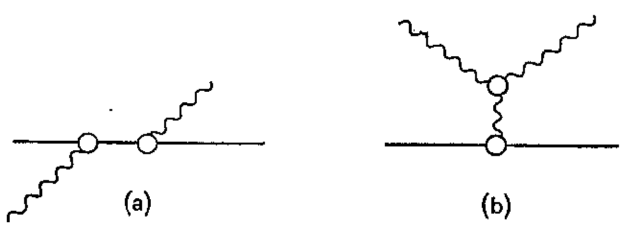

FIG. 12.

Two possible mechanisms of plasmon–particle scattering.

Here the solid line corrresponds to particles (electrons or ions). The process shown in Fig. 11 decribes the quasi-linear effect which can be called induced Cherenkov emission and absorption, first investigated in plasmas by Drummond and Pines (1962) and by Vedenov, Velikhov, and Sagdeev (1962). The process shown in Fig. 12 is an induced scattering which gives a non-linear energy transfer in the spectrum of turbulent waves. Sometimes it is wrongly called a non-linear Landau damping. It is not a damping at all, because it can be shown to conserve the number of quanta. This process was first mentioned by Pauli (as an induced one), but he did not consider the process shown in Fig. 12b, which is very essential in all turbulent plasma-energy transfers. In the case of high-frequency transverse waves the contribution of this effect is small compared to that of Fig. 12a, but it generally exists, if one considers the effects of many particles, even if they are only free electrons. For the ion-sound-wave interaction with the ions the process shown in Fig. 12 was first described by Kadomtsev and Petviashvili (1962), and the non-linear interaction of Langmuir waves was first considered by Sturrock (1957).

The process shown in Fig. 13 is a four-wave decay process—a scattering of plasmons by plasmons.

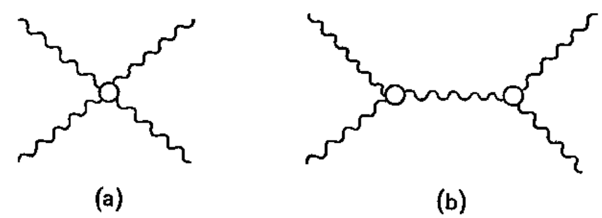

FIG. 13.

Two possible mechanisms of plasmon–plasmon scattering.

After the balance equations are written down the probabilities can be found by the correspondence principle, that is, by finding the power emitted in the N_{k_1} mode, if $N_{k_1} \to 0$. For the case of equation (2.5) this is

$$Q = \int \omega(\boldsymbol{k}_1)\,\frac{\partial N_{k_1}}{\partial t}\,\frac{d^3\boldsymbol{k}_1}{(2\pi)^3}$$

$$= \int w(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\,N_{k_2}N_{k_3}\omega(\boldsymbol{k}_1)\,\frac{d^3\boldsymbol{k}_1\, d^3\boldsymbol{k}_2\, d^3\boldsymbol{k}_3}{(2\pi)^9}. \quad (2.6)$$

One must then compare this with the work done classically, which is

$$Q = -\int (\boldsymbol{j}\cdot\boldsymbol{E})\, d^3\boldsymbol{r}. \tag{2.7}$$

This approach and the interpretation of non-linear interactions as induced scattering processes was developed in detail in papers by the author and collaborators (see Tsytovich, 1968b).

It is possible to find all the required probabilities in the presence of an external magnetic field—for example, the linear and non-linear interactions of drift waves, and also of the modes shown in Fig. 2. This subject will be treated in full in the next section.

Now we must return to the approximations needed to describe the turbulence by means of elementary excitations. First, one can easily see from the non-linear interactions described by Fig. 12 that the plasmons are driven in $\boldsymbol{k}$-space and flow to smaller k (as was mentioned earlier). Such a process has a finite characteristic time, τ_*. It is apparent that the frequency of the plasmons will be uncertain with an uncertainty $\Delta\omega \sim 1/\tau_*$. Therefore, there does not exist a one-to-one correspondence between ω and $\boldsymbol{k}$ (energy and momentum of the plasmons) and for each $\boldsymbol{k}$ one can only have frequencies in some broad area around the dispersion curves shown in Fig. 2. Thus, the physical nature of the plasmons in the turbulent state is quite different from that in statistical equilibrium because the frequency width is determined, not by the decay of the plasmons, but by their non-linear interactions with other plasmons, which is dependent on the level of turbulent energy (which determines the number of other plasmons).

Usually there exist some resonance conditions for the interactions shown in Figs. 11, 12, 13, which are nothing but the laws for conservation of energy and momentum during the interaction. For the process shown in Fig. 11 it is

$$\omega(\boldsymbol{k}) = (\boldsymbol{k}\cdot\boldsymbol{v}), \tag{2.8}$$

where $\boldsymbol{v}$ is the velocity of the particle.

For Figs. 12, 13 respectively:

$$\omega(\boldsymbol{k}) = \omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}-\boldsymbol{k}_1), \tag{2.9}$$

$$\omega(\boldsymbol{k}_1)-(\boldsymbol{k}_1\cdot\boldsymbol{v}) = \omega(\boldsymbol{k}_2)-(\boldsymbol{k}_2\cdot\boldsymbol{v}), \tag{2.10}$$

$$\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_1')+\omega(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_1'). \tag{2.11}$$

Because the frequency $\omega(\boldsymbol{k})$ has an uncertainty these laws only need to be satisfied within the accuracy corresponding to this uncertainty. Thus, resonance interactions such as (2.8) are not proportional to

$$\delta[\omega(\boldsymbol{k})-(\boldsymbol{k}\cdot\boldsymbol{v})], \tag{2.12}$$

but to a broadened function of finite width. This point is essential, for example, for ion-sound interactions with electrons, when the broadening of the resonance (2.8) changes the interaction appreciably. As will be shown later, this effect comes not from the uncertainty in ω, but is due to the effective turbulent collision frequency which broadens the resonance (2.8). The broadening of (2.9) is also essential for ion-sound interactions.

One can see now that it is possible to derive the equations for plasmon interactions as an expansion in the magnitude of the turbulent energy only if one integrates over all ω. This takes into account the whole area around the broadened dispersion curve of $\omega(\boldsymbol{k})$ and neglects approximately the uncertainty due to the turbulent broadening. Indeed, in such an approach it is possible to derive the balance equations given above from the Vlasov kinetic equation. Dividing the distribution function into two parts

$$f = f^{\text{reg}}+f^{\text{stoch}}, \quad \langle f^{\text{stoch}}\rangle = 0, \tag{2.13}$$

and similarly

$$\boldsymbol{E} = \boldsymbol{E}^{\text{reg}}+\boldsymbol{E}^{\text{stoch}}, \quad \langle \boldsymbol{E}^{\text{stoch}}\rangle = 0, \tag{2.14}$$

one then expands in $\boldsymbol{E}^{\text{stoch}}$, constructing an equation for $\langle E^{\text{stoch}}_{\boldsymbol{k},\omega} E^{\text{stoch}}_{\boldsymbol{k}',\omega'}\rangle$ which is then integrated over ω. It can then be seen that the regular part of the distribution function describes the solid lines of Figs. 11 and 12 and the $\int W_{\boldsymbol{k},\omega}\, d\omega = N_{\boldsymbol{k}}\omega(\boldsymbol{k})/(2\pi)^3$ describes the wavy lines. The process shown in Fig. 12b can be interpreted as a scattering of a plasmon by the oppositely charged cloud surrounding the initial charge. Thus, f^{reg} does not describe real particles (they take part in both the free motion indicated by solid lines and the plasma oscillations indicated by wavy lines), but an elementary excitation, that is, it represents charges surrounded by polarisation clouds.

2.3. Problems of Correlation Broadening

Suppose that a stationary state of weak turbulence is achieved; then the principal question is that of the structure of the correlation functions in such a state. To construct a theory of correlation functions it is obvious that an expansion in the energy turbulence, W, is not possible even approximately. Indeed, if one tries to use an expansion similar to that used in deriving the balance equation one finds that, for example, the correlation function for longitudinal waves is proportional to

$$\frac{1}{\varepsilon(k)} \sim \frac{1}{\omega-\omega(k)}. \tag{2.15}$$

Here $\omega(k)$ is the solution of the dispersion relation $\varepsilon(k, \omega(k)) = 0$. This resonance does not matter for the balance equations because, after integration over ω, only the imaginary part of (2.15) contributes. This can be approximated as

$$\pi\delta[\omega-\omega(k)]. \tag{2.16}$$

However, for the correlation function it is necessary to describe the whole correlation curve inside the resonance. As was mentioned above, the width of this resonance depends on the energy of the turbulence. Formally, this divergence comes from using the Maxwell equation,

$$\varepsilon(\varkappa)E_\varkappa = \frac{4\pi i}{\omega}\int S_{\varkappa,\varkappa_1,\varkappa_2}E_{\varkappa_1}E_{\varkappa_2}\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2,$$
$$\varkappa = \{k,\omega\}, \quad d^4\varkappa = d^3k\,d\omega, \tag{2.17}$$

where the $S_{\varkappa,\varkappa_1,\varkappa_2}$ are the components of the non-linear current, to express the average $\langle E_1E_2E_3\rangle$ in terms of the average of four fields $\langle E_1E_2E_3E_4\rangle$. It is obvious now that to divide equation (2.11) by $\varepsilon(\varkappa)$ near the resonance is impossible. It is possible to construct a non-linear integral equation for $\langle E_{\varkappa_1}E_{\varkappa_2}E_{\varkappa_3}\rangle$ by supposing that $[\omega-\omega(k)]/\omega$ and W/nT are of the same order of magnitude. Thus, it is necessary to use a more precise equation then equation (2.17) including higher-order terms in the expansion in $E_\varkappa$. A term proportional to W in $\varepsilon(\varkappa)$ arises and gives the turbulent renormalisation of $\varepsilon(\varkappa)$, which takes into account the turbulent collisions. This kind of approach was derived in a paper by Makhankov and Tsytovich (1970) and in a review paper at the Bucharest Conference (Tsytovich, 1969a).

The complicated non-linear equation for $\langle E_{\varkappa_1}E_{\varkappa_2}E_{\varkappa_3}\rangle$ can be solved approximately if one supposes that in the terms containing the integral over ω, $1/\varepsilon(\varkappa)$ has no resonance behaviour and can be approximated by (2.16). In other words, it is possible to make the same approximation as in the balance equation, but only in the integrated terms. The final result given below in equation (2.18) for the correlation function has a force that can be compared with that found by the expansion procedure and given in equation (2.19).

$$I_\varkappa[\varepsilon(\varkappa)+\varepsilon^{\text{n.l.}}(\varkappa)] = 32\pi^2\int\frac{I_{\varkappa_1}I_{\varkappa_2}|S_{\varkappa,\varkappa_1,\varkappa_2}|^2\,\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2}{\omega^2[\varepsilon(-\varkappa)+\varepsilon^{\text{n.l.}}(-\varkappa)]}, \tag{2.18}$$

$$I_\varkappa[\varepsilon(\varkappa)+\varepsilon^{\text{n.l.}}(\varkappa)] = 32\pi^2\int\frac{I_{\varkappa_1}I_{\varkappa_2}|S_{\varkappa,\varkappa_1,\varkappa_2}|^2\,\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2}{\omega^2\varepsilon(-\varkappa)}, \tag{2.19}$$

where again

$$\varkappa = \{k,\omega\}, \quad d^4\varkappa = d^3k\,d\omega. \tag{2.20}$$

The difference is that the non-linear dielectric constant $\varepsilon^{\text{n.l.}}$ occurs in the resonance denominator. The quantity $\varepsilon^{\text{n.l.}}$ is defined as follows:

$$\varepsilon^{\text{n.l.}}(\varkappa) = \frac{8\pi i}{\omega}\int \Sigma_{\varkappa,\varkappa_1}I_{\varkappa_1}\,d^4\varkappa_1, \tag{2.21}$$

$$\Sigma_{\varkappa,\varkappa_1} = \frac{1}{2}\{\Sigma_{\varkappa,\varkappa_1,\varkappa,-\varkappa_1}+\Sigma_{\varkappa,\varkappa_1,-\varkappa_1,\varkappa}\}-\frac{8\pi i S_{\varkappa,\varkappa_1,\varkappa-\varkappa_1}S_{\varkappa-\varkappa_1,\varkappa-\varkappa_1}}{(\omega-\omega_1)\,\varepsilon(\varkappa-\varkappa_1)}, \tag{2.22}$$

where S and Σ are the components of the non-linear current:

$$j_\varkappa = \sigma_\varkappa E_\varkappa + \int S_{\varkappa,\varkappa_1,\varkappa_2}E_{\varkappa_1}E_{\varkappa_2}\,\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2$$
$$+\int \Sigma_{\varkappa,\varkappa_1,\varkappa_2,\varkappa_3}E_{\varkappa_1}E_{\varkappa_2}E_{\varkappa_3}\,\delta(\varkappa-\varkappa_1-\varkappa_2-\varkappa_3)\,d^4\varkappa_1\,d^4\varkappa_2\,d^4\varkappa_3. \tag{2.23}$$

S and Σ can be found explicitly by the usual expansion of the Vlasov equation. It should be mentioned that the balance equations found from equation (2.18) using approximation (2.16) are practically the same as those found from equation (2.19). Equation (2.18) can be written in a form which shows that $I_\varkappa$ is always positive:

$$I_\varkappa = \frac{32\pi^2}{\omega^2}\int \frac{I_{\varkappa_1} I_{\varkappa_2} |S_{\varkappa,\varkappa_1,\varkappa_2}|^2\, \delta(\varkappa-\varkappa_1-\varkappa_2)\, d^4\varkappa_1\, d^4\varkappa_2}{[\varepsilon(\varkappa)+\varepsilon^{\text{n.l.}}(\varkappa)]^2}. \tag{2.24}$$

This necessary and reasonable condition is not always fulfilled by various approaches to the description of fluid turbulence (that is, strong turbulence).

From equation (2.24) it is easy to see that the correlation curves have a Lorentzian structure near the resonance $\omega = \omega(\boldsymbol{k})$, if one neglects the difference between and ω and $\omega(\boldsymbol{k})$ in all terms except $\varepsilon(\varkappa)$:

$$W_{\boldsymbol{k},\omega} = \frac{W_{\boldsymbol{k}}\gamma_{\boldsymbol{k}}^{\text{n.l.}}}{\pi[(\omega-\omega'(\boldsymbol{k}))^2+(\gamma_{\boldsymbol{k}}^{\text{n.l.}})^2]}, \tag{2.25}$$

where

$$\gamma_{\boldsymbol{k}}^{\text{n.l.}} = -\frac{\operatorname{Im}[\varepsilon(\varkappa)+\varepsilon^{\text{n.l.}}(\varkappa)]}{\partial\varepsilon(\varkappa)/\partial\omega}\bigg|_{\omega=\omega(\boldsymbol{k})}, \tag{2.26}$$

$$\omega'(\boldsymbol{k}) = \omega(\boldsymbol{k}) - \frac{\operatorname{Re}\varepsilon^{\text{n.l.}}(\varkappa)}{\partial\varepsilon(\varkappa)/\partial\omega}\bigg|_{\omega=\omega(\boldsymbol{k})}. \tag{2.27}$$

Equation (2.27) describes the non-linear shift in the frequency.

For weak turbulence $\gamma_{\boldsymbol{k}}^{\text{n.l.}} \ll \omega(\boldsymbol{k})$. If the turbulence is stationary, $\gamma_{\boldsymbol{k}}^{\text{n.l.}}$ may be small because of the non-linear compensation of the linear damping and even smaller than the collision frequency. Therefore, the order of magnitude of $\gamma_{\boldsymbol{k}}^{\text{n.l.}}$ corresponds usually to the slowest decay process allowed by the conservation laws. As can be seen from the balance equations (2.5) the decay balance is not a differential one, but an integral one, the turbulent energy coming from different $\boldsymbol{k}$ values, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$.

2.4. Problems of Stationary Turbulent Spectra

To find the stationary spectrum $W_{\boldsymbol{k}}$ it is possible to avoid the detailed forms of the correlation functions and simply use the balance equations. Such equations were first used by Kadomtsev (1962) to find the spectrum of ion-sound turbulence in a partially ionised plasma. Then in some later papers there were attempts to use such spectra for a fully ionised plasma, which, as we shall see, is sometimes doubtful. The Langmuir turbulence spectrum was calculated by using the balance equations in the papers by Kaplan (?) Pikel'ner and Tsytovich (1968) and by Liperovskii and Tsytovich (1969) and numerically by Makhankov *et al.* (1970). We shall have some special remarks to make later about both Langmuir and ion-sound turbulence spectra.

Here we mention that after finding the spectrum it is possible to calculate the correlation broadening. In particular, it is possible to use equation (2.24) to find the form of the correlation curve outside the resonance, that is, on the tail of the correlation curve. This is shown schematically in Fig. 14.

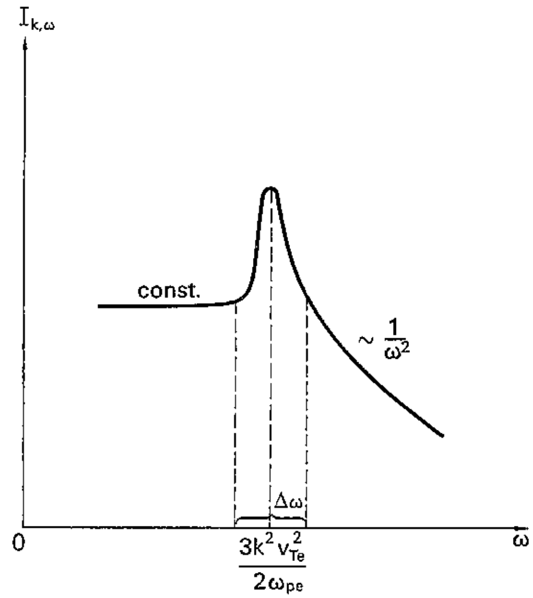

FIG. 14.

The correlation curve of Langmuir turbulence at low frequencies ($\Delta\omega/\omega \sim k_0/k$).

The whole of this curve can be found when the power input Q is known. The maximum in Fig. 14 corresponds to frequencies equal to the difference between the frequencies of the turbulent waves. These correlations have no ω, -dependence. Their action on the particles gives the heating. Because the energy in the tail is proportional to W^2 (see equation (2.24)) the next terms compared to the quasi-linear diffusion, which is proportional to W in the expansion of the diffusion coefficient in $I_\varkappa$ are of the same order of magnitude. Both terms together describe the heating due to induced scattering.

Now, if the stationary turbulent spectrum is known, it is possible to find the general form of (i) the rate of stochastic acceleration of fast particles, (ii) the scattering, amplification and fluctuations of electromagnetic waves propagating in a turbulent plasma, (iii) the anomalous turbulent emissivity, and (iv) macroscopic characteristics such as the anomalous conductivity and anomalous diffusion.

2.5. The Electromagnetic Properties of a Stationary Turbulent Plasma

As was mentioned previously, for low frequencies, $\omega < \nu^{\text{turb}}$ (ν^{turb} is a characteristic turbulent frequency), the electromagnetic properties of a plasma for small signals or small perturbations are changed significantly. This means that the dielectric constant ε as a functional of W_k,

$$\varepsilon = \varepsilon(\omega, \boldsymbol{k}, W_{k_1}), \tag{2.28}$$

cannot be expanded in W_{k_1}. One possible method of deriving (2.28) is to consider a small deviation of this stationary turbulent state due to a regular electric field perturbation, $\boldsymbol{E}^{\text{reg}}$, and expand all variables in the amplitude of $\boldsymbol{E}^{\text{reg}}$. Collision integrals arise which describe the turbulent collisions. It is sometimes possible to expand the kernels of these collision integrals in terms of the turbulent energy, that is, to take into account the highest turbulent collision rate. Nothing is expanded in ν^{turb}/ω, and in this way the essential change in ε is found. Sometimes this expansion of the collision integrals does not work because a resonance denominator arises. Thus, a problem similar to that of the correlation broadening arises. The non-linear integral equation for summing the series in the turbulent energy for such collision integrals can be constructed. It can be solved and the collision integrals found. This give $\varepsilon(\omega, \boldsymbol{k}, W_{k_1})$ and the electromagnetic properties of a turbulent plasma can be investigated.

The first approach to such problems was made by Rudakov and Vedenov (1964), who used a phenomenological approach with a Miller force, and by Gailitis (1966), who used the correspondence principle. The summation in the turbulent energy was done by Tsytovich (1968a, 1969b) and an investigation of drift waves in a turbulent plasma was given by Krivorutskii, Makhankov, and Tsytovich (1969). There is also an approach in which the turbulent collisions are taken into account phenomenologically.

3. The Balance Equation for a Turbulent Plasma

3.1. The Refractive Index for Waves

The balance equations can be found in a simple way using the concept of induced processes. Let us consider a linear mode propagating with small damping, or small excitation, in an inhomogeneous plasma in an external magnetic field, so that

$$\omega = \omega_k^\sigma + i\gamma_k^\sigma, \tag{3.1}$$

where $\gamma_k^\sigma \ll \omega_k^\sigma$. As we mentioned above, γ_k^σ describes a linear or quasi-linear growth-rate or non-linear energy transfer. In the first approximation, one can neglect all these effects if $\gamma \ll \omega$, and say that equation (3.1) describes approximately a linear mode $\omega \approx \omega_k^\sigma$. As we shall see later this ω_k^σ is determined not by the exact distribution function f but by the part f^{reg}, which varies slowly in time. Thus, we consider ω_k^σ as the frequency of an elementary excitation, and drop the superscript "reg", but remember that f describes the distribution of the other elementary excitations—the "dressed" particles. Thus the dispersion law for the elementary wave excitations, the so-called plasmons, can be found from the linear dispersion equation. One can also find the normal unit vector $\boldsymbol{e}_k^\sigma$, which characterises different electric field components. It is useful then to introduce the dielectric constant $\varepsilon_\kappa^\sigma$ which describes such a wave, or the refractive index $n^\sigma = \sqrt{(\text{Re}\, \varepsilon^\sigma)}$, where $n^\sigma = k/\omega_k^\sigma$ and thus we have

$$k^2 = (\omega_k^\sigma)^2 \,\text{Re}\, \varepsilon^\sigma(\omega_k^\sigma, \boldsymbol{k}). \tag{3.2}$$

Thus, ω_k^σ satisfies the usual equation for transverse waves, but the polarisation of the wave is in general arbitrary and is defined by $\boldsymbol{e}_k^\sigma$. The question is how to find $\varepsilon^\sigma(\omega, \boldsymbol{k})$, if $\boldsymbol{e}_k^\sigma$ and the dielectric tensor $\varepsilon_{ij}(\omega, \boldsymbol{k})$ are known. The general linear dispersion relation is ($c = 1$)

$$\sum_j [k^2\delta_{ij} - k_i k_j - \omega^2 \varepsilon_{ij}(\omega, \boldsymbol{k})]\,(E_\kappa)_j = 0. \tag{3.3}$$

If the field has the direction $\boldsymbol{e}_k^\sigma$ one can put $(E_\kappa)_j = (\boldsymbol{e}_k^\sigma)_j E_\kappa^\sigma$ and by multiplying equation (3.3) by e_k^σ, and using

$$(e_k^\sigma \cdot e_k^{\sigma *}) = 1, \tag{3.4}$$

we find

$$[k^2 - \omega^2 \varepsilon^\sigma(\omega, k)]E_k^\sigma = 0, \tag{3.5}$$

where

$$\varepsilon^\sigma(\omega, k) = (e_k^{\sigma *})_i \, \varepsilon_{ij}(\omega, k) \, (e_k^\sigma)_j + \frac{(k \cdot e_k^\sigma)(k \cdot e_k^{\sigma *})}{\omega^2}. \tag{3.6}$$

One may say that to find ε^σ is useless because to find e_k^σ one must solve the dispersion equation (3.3) for $\omega^\sigma(k)$ and put it in (3.3) to find the relations between the electric field components. However, ε^σ describes the energy of plasmons and comes into all probabilities of non-linear interactions. Let us, for example, calculate the average energy density W of the field corresponding to the σ-mode, supposing that the field has a stochastic nature.

From the energy conservation theorem we have

$$\frac{\partial W}{\partial t} = \frac{1}{4\pi}\left(E \cdot \frac{\partial D}{\partial t}\right) + \frac{1}{4\pi}\left(B \cdot \frac{\partial B}{\partial t}\right), \tag{3.7}$$

and putting

$$\left.\begin{aligned} E &= \int e_k^\sigma E_{k,\omega}^\sigma \, e^{i(k \cdot r) - i\omega t} \, d^3k \, d\omega, \\ D_i &= \int \varepsilon_{ij}(\omega, k) \, E_{k,\omega}^\sigma \, (e_k^\sigma)_j \, e^{i(k \cdot r) - i\omega t} \, d^3k \, d\omega, \\ B_{k,\omega} &= \frac{[k \wedge E_{k,\omega}]}{\omega} = \frac{[k \wedge e_k^\sigma]}{\omega} E_{k,\omega}^\sigma, \end{aligned}\right\} \tag{3.8}$$

we find that

$$\begin{aligned} \frac{\partial W}{\partial t} = -\frac{i}{8\pi}\sum_{i,j}\int \Big[&(e_k^\sigma)_i \, \varepsilon_{ij}(\omega, k) \, \omega (e_k^\sigma)_j + (e_k^\sigma)_i \, \omega' \varepsilon_{ij}(\omega', k') \, (e_{k'}^\sigma)_j \\ &+ (\omega + \omega')\left\{\frac{(k \cdot k')}{\omega\omega'}(e_k^\sigma \cdot e_{k'}^\sigma) - \frac{(k \cdot e_{k'}^\sigma)(k' \cdot e_k^\sigma)}{\omega\omega'}\right\}\Big] E_{k,\omega}^\sigma E_{k',\omega'}^\sigma \\ &\times e^{i([k+k'] \cdot r) - i(\omega+\omega')t} \, d^4\varkappa \, d^4\varkappa'. \end{aligned} \tag{3.9}$$

The result is symmetrised in $\varkappa$ and $\varkappa'$. Integrating over t and taking an ensemble average, and using

$$\langle E_{k,\omega}^\sigma E_{k',\omega'}^\sigma \rangle = |E_{k,\omega}^\sigma|^2 \, \delta(\omega + \omega') \, \delta(k + k'), \tag{3.10}$$

one finds that

$$\langle W \rangle = \int \frac{|E_{k,\omega}|^2}{8\pi\omega} \frac{\partial}{\partial \omega}(\varepsilon^\sigma(\omega, k)\omega^2) \, d^3k \, d\omega. \tag{3.11}$$

In the approximation where the correlation broadening is neglected

$$|E_{k,\omega}|^2 = \alpha_k N_k \, \delta(\omega - \omega_k^\sigma) + \alpha_k N_{-k} \, \delta(\omega + \omega_k^\sigma) \tag{3.12}$$

(here $\omega_k^\sigma > 0$), we have

$$\langle W \rangle = \frac{1}{4\pi}\int \alpha_k N_k \frac{1}{\omega} \frac{\partial}{\partial \omega}[\omega^2 \varepsilon^\sigma(\omega, k)]\Big|_{\omega = \omega^\sigma(k)} d^3k. \tag{3.13}$$

On the other hand,

$$\langle W \rangle = \int \frac{\omega_k^\sigma N_k \, d^3k}{(8\pi)^3}, \tag{3.14}$$

so that

$$\alpha_k = \frac{1}{2\pi^2}\left[\frac{1}{(\omega_k^\sigma)^2} \frac{\partial}{\partial \omega}\{\omega^2 \varepsilon^\sigma(\omega, k)\}\right]^{-1}_{\omega = \omega_k^\sigma}. \tag{3.15}$$

3.2. Quasi-linear Equations

The simple balance equation, which describes the induced emission and absorption of plasmons by particles was first derived by Vedenov, Velikhov, and Sagdeev (1962) and by Drummond and Pines (1962). We shall write it in a general form applicable to a plasma in an external magnetic field and including possible inhomogeneities of the plasma perpendicular to the field. It is useful to start from the quantum description of free-particle motion in a magnetic field and then consider the classical limit, or more precisely, the quasi-classical approximation.

The particle motion in a magnetic field $H_0 = H_z$ is described by the energy spectrum of the Landau levels which in the general relativistic case are given by

$$\varepsilon^2 = m^2c^4 + c^2p_z^2 + c^2p_\perp^2, \tag{3.16}$$

where p_z is the particle momentum component along the field and $p_\perp^2$ is

$$p_\perp^2 = 2n\hbar\,|eH_0|, \tag{3.17}$$

where n is an integral quantum number characterising the energy levels and $\hbar = h/2\pi$, where h is Planck's constant;

$$n = 0, 1, 2, \ldots, \infty. \tag{3.18}$$

In the quasi-classical limit the spin does not change and can, therefore, be neglected. The quasi-classical limit corresponds to a very high n such that $n \to \infty$, $\hbar \to 0$, while $n\hbar$ is kept finite and $p_\perp$ is the particle momentum component perpendicular to the magnetic field.

It is also useful to choose the gauge of the magnetic vector potential as in the first Landau paper:

$$A_x = H_0 y, \quad A_z = A_y = 0, \tag{3.19}$$

$$\boldsymbol{H}_0 = [\nabla \wedge A]. \tag{3.20}$$

The momentum component p_x is conserved and the energy spectrum is independent of p_x. In the quasi-classical limit p_x determines the y-coordinate of the centre of the Larmor orbit:

$$y = -\frac{c}{eH_0} p_x. \tag{3.21}$$

The probability of emission of a σ-plasmon by a particle gyrating in a magnetic field (or in the quantum case by a particle in one of the Landau levels) is denoted by

$$w^\sigma(\boldsymbol{k}, p_z, p_\perp, \nu). \tag{3.22}$$

This probability depends, as written, on the momentum of the emitted plasmon $\boldsymbol{k}$, on p_z, the initial energy of the particle (or more precisely n, the integer characterising the Landau level), and on the final n' in which one finds the particle after the emission. Instead of the variables n and n' in equation (3.22), we use $p_\perp$ and ν, where

$$\nu = n' - n, \tag{3.23}$$

and ν varies in the range

$$\nu = -\infty, \ldots, -1, 0, 1, \ldots, +\infty, \tag{3.24}$$

The distribution function f of the particles depends on p_z, p_x, and n:

$$f = f(p_z, p_x, n). \tag{3.25}$$

Expressions (3.22) and (3.25) are supposed to be normalised to the phase volumes $d^3\boldsymbol{k}/(2\pi)^3$ and $d^3\boldsymbol{p}/(2\pi)^3$.

The balance equation for the waves can be written in the form

$$\frac{dN_k^\sigma}{dt} = \sum_{\alpha, n, \nu} \int \frac{dp_z}{2\pi} [(N_k^\sigma + 1) f_\alpha(p_z, p_x, n) - N_k^\sigma f_\alpha(p_z - \hbar k_z, p_x - \hbar k_x, n-\nu)] \times w^\sigma(\boldsymbol{k}, p_z, p_\perp, \nu), \tag{3.26}$$

where $\alpha = $ e, i corresponds to electrons and ions, respectively. The term with the minus sign occurs because of detailed balance arguments and the conservation of momentum in the emission process gives

$$p_z' = p_z - \hbar k_z, \quad p_x' = p_x - \hbar k_x, \tag{3.27}$$

where the prime corresponds to the state after emission. In the quasi-classical limit $\hbar k_z \ll p_z$, $\hbar k_x \ll p_x$, and $\nu \ll n$. Expanding in k_z, k_x and ν it is useful to introduce classical variables for the argument of f: $p_\perp$ instead of n and y instead of p_x. Thus, instead of $k_x\, \partial f/\partial p_x$ we write $-(k_x c/|eH_0|)(\partial f/\partial y)$, and instead of $\nu\, \partial f/\partial n$ we write $\Delta p_\perp^2\, \partial f/\partial p_\perp^2$. From (3.18) one finds that

$$\Delta p_\perp^2 = 2\nu\hbar\,|eH_0| = \hbar\omega_H \nu\varepsilon. \tag{3.28}$$

Thus, equation (3.26) can be written as

$$\frac{dN_k^\sigma}{dt} = \sum_{\nu=-\infty}^{+\infty} \sum_{\alpha, i} \int \frac{d^3p}{(2\pi)^3} \Big[N_k w^\sigma(\boldsymbol{k}, p_z, p_\perp, \nu)\, \Delta\lambda_i \frac{\partial f_\alpha(p_z, p_\perp, y)}{\partial \lambda_i} + w^\sigma(\boldsymbol{k}, p_z, p_\perp, \nu) f_\alpha(p_z, p_\perp, y) \Big]. \tag{3.29}$$

The first term describes the induced process, the second the spontaneous emission, and

$$\lambda_1 = p_z, \quad \lambda_2 = p_\perp, \quad \lambda_3 = y, \tag{3.30}$$

$$\Delta\lambda_1 = k_z, \quad \Delta\lambda_2 = \frac{\omega_{H\alpha}\nu}{v_\perp}, \quad \Delta\lambda_3 = -k_x \frac{c}{|eH_0|}, \tag{3.31}$$

$$v_\perp = \frac{p_\perp}{\varepsilon}. \tag{3.32}$$

Similarly, one finds the balance equations for particles:

$$\frac{df_\alpha(p_z, p_\perp, y)}{dt} = \sum_{i,j} \frac{\partial}{\partial \lambda_i} D^\alpha_{ij} \frac{\partial}{\partial \lambda_j} f_\alpha(p_z, p_\perp, y) + \sum_i \frac{\partial}{\partial \lambda_i} A^\alpha_i f_\alpha(p_z, p_\perp, y), \quad (3.33)$$

where

$$D^\alpha_{ij} = \sum_{\nu=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \Delta\lambda_i \, \Delta\lambda_j w^\sigma_\alpha(k, p_z, p_\perp, \nu) N^\sigma_k, \quad (3.34)$$

$$A^\alpha_i = \sum_{\nu=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \Delta\lambda_i w^\sigma_\alpha(k, p_z, p_\perp, \nu). \quad (3.35)$$

The first term of (3.33) describes the induced process and the second one the spontaneous emission. The d/dt operator on the left-hand sides of equations (3.33) and (3.26) describes the free transitions of particles and plasmons:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (v \cdot \nabla) f, \quad (3.36)$$

$$\frac{dN_k}{dt} = \frac{\partial N_k}{\partial t} + (v_{\text{gr}} \cdot \nabla) N_k, \quad (3.37)$$

where $v_{\text{gr}} = d\omega^\sigma_k/dk$ is the plasmon group velocity. In the case of an inhomogeneous plasma, equation (3.37) is sometimes rather more complicated and contains an additional term

$$-\left(\nabla\omega \cdot \frac{\partial N_k}{\partial k}\right). \quad (3.38)$$

3.3. The Probabilities for Plasmon Emission

Let us consider the power Q emitted by all the particles in the limit as $N^\sigma_k \to 0$, when only the spontaneous emission is essential in equation (3.29):

$$Q = \int \omega^\sigma_k \frac{dN^\sigma_k}{dt} \frac{d^3k}{(2\pi)^3}\bigg|_{N^\sigma_k \to 0} = \int \frac{d^3p}{(2\pi)^3} f(p_z, p_\perp, y) Q_p, \quad (3.39)$$

where

$$Q_p = \sum_{\nu=-\infty}^{+\infty} \int \omega^\sigma_k w^\sigma(k, p_z, p_\perp, \nu) \frac{d^3k}{(2\pi)^3} \quad (3.40)$$

is the power emitted by a single particle. To find Q_p in the classical limit, we shall use a procedure similar to that used by Landau for calculating Cherenkov losses, that is, we will calculate the work done by the field $\boldsymbol{E}$ on the current $\boldsymbol{j}$:

$$Q = -\int (\boldsymbol{j}(\boldsymbol{r}, t) \cdot \boldsymbol{E}(\boldsymbol{r}, t)) \, d^3\boldsymbol{r}. \quad (3.41)$$

The current produced by a particle spiralling in a magnetic field,

$$\boldsymbol{j} = e\boldsymbol{v}(t) \, \delta(\boldsymbol{r} - \boldsymbol{r}(t)), \quad (3.42)$$

where $\boldsymbol{v}(t)$ and $\boldsymbol{r}(t)$ are simple periodic (sine or cosine) functions of time, has Fourier components

$$\boldsymbol{j}_{k,\omega} = \frac{e}{(2\pi)^3} \sum_{\nu=-\infty}^{+\infty} \boldsymbol{\Gamma}^\alpha_k e^{i(\boldsymbol{k}\cdot\boldsymbol{r}_0) + i\varphi_0\nu} \, \delta(\omega - k_z v_z - \nu\omega_{H\alpha}); \quad (3.43)$$

$\boldsymbol{r}_0 = \{x_0, y_0, z_0\}$, x_0 and y_0 are thus the coordinates of the centre of the Larmor orbit, z_0 and φ are the initial z and phase φ of the particles at $t = 0$. The components of the vectors $\boldsymbol{\Gamma}$ in the frame $k_y = 0$ are

$$(\Gamma^\alpha_k)_x = \frac{v_\perp \nu J_\nu(z_\alpha)}{z_\alpha}; \qquad (\Gamma^\alpha_k)_y = -iv_\perp J'_\nu(z_\alpha);$$

$$(\Gamma^\alpha_k)_z = v_z J_\nu(z_\alpha); \qquad z_\alpha = \frac{k_\perp v_\perp}{\omega_{H\alpha}}; \qquad v_\perp = \frac{p_\perp}{\varepsilon}. \quad (3.44)$$

The J_ν are the Bessel functions of order ν. The field produced by the current (3.42) is found from the Maxwell equations (see equation (3.3))

$$[k^2\delta_{ij} - k_i k_j - \omega^2 \varepsilon_{ij}(\omega, \boldsymbol{k})] \, (E_\varkappa)_j = 4\pi i\omega (j_\varkappa)_i. \quad (3.45)$$

Because we are interested in the emission of the σ-wave, we must set

$$(E_\varkappa)_j = (e^\sigma_k)_j E^\sigma_\varkappa \quad (3.46)$$

in equation (3.45). Multiplying (3.45) by $\boldsymbol{e}^{\sigma*}_k$ and using the definition (3.6), we have

$$E^\sigma_\varkappa = 4\pi i\omega \frac{(\boldsymbol{e}^{\sigma*}_k \cdot \boldsymbol{j}_\varkappa)}{(k^2 - \omega^2 \varepsilon^\sigma)}. \quad (3.47)$$

Expanding $\boldsymbol{j}$ and $\boldsymbol{E}$ in equation (3.41) in Fourier series and using equations (3.46) and (3.47) we get

$$Q = -(2\pi)^3 \int \frac{d^3k\, d\omega\, d\omega'}{k^2-\omega^2\varepsilon^\sigma} 4\pi i\omega (j_\kappa^* \cdot e_k^\sigma)(e_k^{\sigma *} \cdot j_\kappa) e^{-i(\omega-\omega')t}$$

$$= -\frac{4\pi i}{(2\pi)^3} \sum_{\nu,\nu'=-\infty}^{+\infty} \int e^{-i(\omega-\omega')+i(\nu-\nu')\varphi_0} (\Gamma_k^* \cdot e_k^\sigma)(\Gamma_k \cdot e_k^{\sigma *})$$

$$\times \frac{\omega}{k^2-\omega^2\varepsilon^\sigma} \delta(\omega - k_z v_z - \nu\omega_{H\alpha})\, \delta(\omega' - k_z v_z - \nu'\omega_{H\alpha})\, d^3k\, d\omega\, d\omega'. \quad (3.48)$$

We substitute here the exact expression (3.43) for the current j_κ. Averaging over the initial phase φ_0,

$$\frac{1}{2\pi}\int e^{i(\nu-\nu')\varphi_0}\, d\varphi_0 = \delta_{\nu,\nu'}, \quad (3.49)$$

and noting that only the imaginary part of $1/(k^2-\omega^2\varepsilon^\sigma)$ contributes to equation (3.48) and that

$$\frac{1}{k^2-\omega^2\varepsilon^\sigma} \approx \pi i\, \delta(k^2-\omega^2\varepsilon^\sigma)$$

$$= \pi i \frac{\delta(\omega-\omega_k^\sigma)+\delta(\omega+\omega_k^\sigma)}{(\partial/\partial\omega)(\omega^2\varepsilon^\sigma)\Big|_{\omega=\omega_k^\sigma}}, \quad (3.50)$$

one finds an expression of the form of equation (3.40) and can immediately find the required probability

$$w_\alpha^\sigma(k, p_z, p_\perp, \nu) = (2\pi)^3 \frac{e^2}{\pi} \frac{|(\Gamma_k^\alpha \cdot e_k^\sigma)|^2\, \delta(\omega_k^\sigma - k_z v_z - \nu\omega_{H\alpha})}{(\partial/\partial\omega)(\omega^2\varepsilon^\sigma)\Big|_{\omega=\omega_k^\sigma}}. \quad (3.51)$$

Using this probability in equations (3.33) and (3.29) gives us general quasilinear equations valid, for example, for a relativistic plasma. The equations describe all processes of interest, including transverse waves, synchrotron emission, synchrotron instability, spontaneous processes, and they take account of plasma inhomogeneities.

In the case of longitudinal oscillations, it is useful to introduce the longitudinal dielectric constant in a way slightly different from equation (3.6):

$$\varepsilon_\kappa^l = \varepsilon_\kappa^\sigma - \frac{k^2}{\omega^2} = \sum_{i,j} \varepsilon_{ij} \frac{k_i k_j}{k^2}. \quad (3.52)$$

The dispersion relation (3.5) then has the form

$$\varepsilon_\kappa^l = 0. \quad (3.53)$$

Using the fact that $e_k = k/k$ and that

$$\frac{\partial}{\partial\omega}\omega^2\varepsilon_\kappa^\sigma\Big|_{\omega=\omega_k^l} = (\omega_k^l)^2 \frac{\partial}{\partial\omega}\varepsilon_\kappa^l\Big|_{\omega=\omega_k^l}, \quad (3.54)$$

we can write down a simple expression for the probability for the emission of longitudinal plasmons:

$$w^l(k, p_z, p_\perp, \nu) = (2\pi)^3 \frac{e^2}{\pi k^2} \frac{J_\nu^2(z_\alpha)\, \delta(\omega_k^l - k_z v_z - \nu\omega_{H\alpha})}{\partial\varepsilon_\kappa^l/\partial\omega\Big|_{\omega=\omega_k^l}}. \quad (3.55)$$

3.4. The Case of Non-magnetic Particles

Even with a magnetic field present, in some cases the Larmor radii of the gyrating particles are much larger than the wavelength, that is, $z_\alpha \gg 1$.† In this case, a large number of cyclotron harmonics are involved in the interaction and the probabilities which we gave above are useless. In this case, one can consider the opposite limit, by supposing that during the emission the particle does not gyrate, but moves rectilinearly. One can use as the particle current the expression

$$j_\kappa = \frac{ev}{(2\pi)^3} e^{i(k\cdot r_0)}\, \delta(\omega - (k\cdot v)). \quad (3.56)$$

The quasi-linear equations have the same form as in the absence of a magnetic field:

$$\frac{df}{dt} = \sum_{ij} \frac{\partial}{\partial p_i} D_{ij} \frac{\partial f}{\partial p_j} + \sum_i \frac{\partial}{\partial p_i} A_i f, \quad (3.57)$$

$$D_{ij} = \int \frac{d^3k}{(2\pi)^3} w(k, p)\, N_k^\sigma k_i k_j, \quad (3.58)$$

$$A_i = \int \frac{d^3k}{(2\pi)^3} k_i w(k, p), \quad (3.59)$$

$$\frac{dN_k^\sigma}{dt} = N_k^\sigma \int \frac{d^3p}{(2\pi)^3} w(k, p) \left(k \cdot \frac{\partial f}{\partial p}\right) + \int \frac{d^3p}{(2\pi)^3} w(k, p) f. \quad (3.60)$$

† We call such particles "non-magnetic particles".

The probability can easily be found from equations (3.40) and (3.56):

$$w(\boldsymbol{k}, \boldsymbol{p}) = (2\pi)^3 \frac{e^2}{\pi} \frac{|(\boldsymbol{v}\cdot\boldsymbol{e}_k^{\sigma*})|^2\, \delta(\omega_k^\sigma - (\boldsymbol{k}\cdot\boldsymbol{v}))}{(\partial/\partial\omega)\,(\omega^2\varepsilon_\varkappa^\sigma)\big|_{\omega=\omega_k^\sigma}}. \tag{3.61}$$

Note that equations (3.57) to (3.60) are also valid for a relativistic plasma or for relativistic particles in a cold plasma.

3.5. Examples of the Quasi-linear Equations

In the case of the Rosenbluth–Post loss-cone instability (Rosenbluth and Post, 1965) the ions are non-magnetic and one can use equations (3.57) and (3.60). Since $k_\perp \gg k_z$, we have

$$\frac{1}{N_k^1}\frac{dN_k^1}{dt} = \frac{e^2}{\pi k^2(\partial\varepsilon_\varkappa^\sigma/\partial\omega)\big|_{\omega=\omega_k^1}} \int \delta(\omega_k^1 - (\boldsymbol{k}\cdot\boldsymbol{v}))\frac{(\boldsymbol{k}_\perp\cdot\boldsymbol{v}_\perp)}{v_\perp}\frac{\partial f}{\partial p_\perp}\, d^2p_\perp\, dp_z. \tag{3.62}$$

Introducing

$$f(\varepsilon_\perp) = \frac{m_i}{4\pi^2 n_0}\int f^{(i)}\, dp_z, \tag{3.63}$$

one immediately finds Galeev's quasi-linear equations for this instability (Galeev, 1967):

$$\frac{1}{N_k^1}\frac{dN_k^1}{dt} = \frac{m_i^{3/2}\omega_{pi}^2\omega_k^1}{2\sqrt{(2)}\,k^2k_\perp(\partial\varepsilon_\varkappa^1/\partial\omega)}\int_{\varepsilon_{\perp,0}}^{\infty}\frac{f(\varepsilon_\perp)\, d\varepsilon_\perp}{\varepsilon_\perp^{3/2}}, \tag{3.64}$$

where

$$\varepsilon_{\perp,0} = \frac{1}{2}m_i\frac{(\omega_k^1)^2}{k^2}, \quad \varepsilon_{\perp,0} \ll \varepsilon_{\perp\,\min}, \tag{3.65}$$

and

$$f(\varepsilon_\perp) = \begin{cases} 0, & \text{if} \quad \varepsilon < \varepsilon_{\perp\,\min},\\ f(\varepsilon_\perp), & \text{if} \quad \varepsilon > \varepsilon_{\perp\,\min},\end{cases} \tag{3.66}$$

$$\frac{\partial f(\varepsilon_\perp)}{\partial\tau} = \frac{\partial}{\partial\varepsilon_\perp}\left(\frac{1}{\sqrt{\varepsilon_\perp}}\frac{\partial f(\varepsilon_\perp)}{\partial\varepsilon_\perp}\right), \tag{3.67}$$

$$\tau = t\int d^3k\,\frac{(\omega_k^1)^2e^2N_k^1\sqrt{m_i}}{2\sqrt{(2)}\,\pi^2k^2k_\perp(\partial\varepsilon_\varkappa^1/\partial\omega)\big|_{\omega=\omega_k^1}}. \tag{3.68}$$

Another example: if the Larmor radii are very small ($z \ll 1$) one can use for the probability for the emission of longitudinal plasmons the following expression, which is easily found from equation (3.55):

$$w(\boldsymbol{k}, p_z, p_\perp, \nu) = \frac{(2\pi)^3e^2\,\delta_{\nu,0}}{\pi k^2(\partial\varepsilon_\varkappa^1/\partial\omega)\big|_{\omega=\omega_k^1}}\,\delta(\omega_k^1 - k_zv_z). \tag{3.69}$$

This immediately gives the quasi-linear equations for drift waves, first found by Galeev and Rudakov (1963):

$$\frac{dN_k}{dt} = \frac{e^2N_k}{\pi m_e^2k^2(\partial\varepsilon_\varkappa^1/\partial\omega)\big|_{\omega=\omega_k^1}}\int dp_z\left\{k_z\frac{\partial f}{\partial v_z} - \frac{k_x}{\omega_{He}}\frac{\partial f}{\partial y}\right\}\delta(\omega_k^1 - k_zv_z), \tag{3.70}$$

$$\frac{df}{dt} = \int d^3k\left(k_z\frac{\partial}{\partial v_z} - \frac{k_x}{\omega_{He}}\frac{\partial}{\partial y}\right)\frac{e^2\,\delta(\omega_k^1 - k_zv_z)N_k}{\pi k^2m_e^2(\partial\varepsilon_\varkappa^1/\partial\omega)\big|_{\omega=\omega_k^1}} \times\left(k_z\frac{\partial}{\partial v_z} - \frac{k_x}{\omega_{He}}\frac{\partial}{\partial y}\right)f. \tag{3.71}$$

It is evident that the method we have described can be used to obtain rapidly results which would normally require a substantial amount of work.

3.6. Non-linear Plasmon-Plasmon Interactions

Let us consider the simplest non-linear wave decay process:

$$\sigma \to \sigma' + \sigma''. \tag{3.72}$$

We shall denote the probability of this process by

$$w_\sigma^{\sigma'\sigma''}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2). \tag{3.73}$$

It is easy to obtain a balance equation similar to that of Section 3.2:

$$\frac{dN_k^\sigma}{dt} = \int w_\sigma^{\sigma'\sigma''}(\boldsymbol{k}, \boldsymbol{k}_1\,\boldsymbol{k}_2)\{N_{k_1}^{\sigma'}N_{k_2}^{\sigma''} - N_k^\sigma N_{k_1}^{\sigma'} - N_k^\sigma N_{k_2}^{\sigma''}\}\frac{d^3k_1\, d^3k_2}{(2\pi)^6}. \tag{3.74}$$

To find the probability let us again use the correspondence principle. The emissivity of σ-waves in the limit as $N_k^\sigma \to 0$ is

$$Q^\sigma = \int \omega_k^\sigma \frac{dN_k^\sigma}{dt} \frac{d^3\boldsymbol{k}}{(2\pi)^3} = \int \omega_k^\sigma w_\sigma^{\sigma'\sigma''}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) N_{k_1}^{\sigma'} N_{k_2}^{\sigma''} \frac{d^3\boldsymbol{k}\, d^3\boldsymbol{k}_1\, d^3\boldsymbol{k}_2}{(2\pi)^9}. \tag{3.75}$$

This emissivity is equal to

$$Q^\sigma = -\langle(\boldsymbol{j}\cdot\boldsymbol{E})\rangle, \tag{3.76}$$

where $\boldsymbol{j}$ is the non-linear current excited in the plasma by the σ'- and σ''-waves, and $\boldsymbol{E}$ is the field produced by the current.

We recall the general expression for the second-order non-linear current:

$$(j_\varkappa)_i = \sum_{j,l} \int S_{ijl}(\varkappa, \varkappa_1, \varkappa_2)\, \delta(\varkappa - \varkappa_1 - \varkappa_2) E_{j,\varkappa_1} E_{l,\varkappa_2}\, d^4\varkappa_1\, d^4\varkappa_2. \tag{3.77}$$

The S_{ijl} can easily be found from the Vlasov equation by means of an expansion in powers of E. In the present case equation (3.77) becomes

$$(j_\varkappa)_i = 2 \int S_i(\varkappa, \varkappa_1, \varkappa_2) E_{\varkappa_1}^{\sigma'} E_{\varkappa_2}^{\sigma''}\, \delta(\varkappa - \varkappa_1 - \varkappa_2)\, d^4\varkappa_1\, d^4\varkappa_2. \tag{3.78}$$

The fields E can be found from equation (3.47). Substituting for $\boldsymbol{j}$ into equation (3.76), we obtain

$$Q = -4\pi i \int \frac{d^4\varkappa\, d^4\varkappa_1}{k^2 - \omega^2 \varepsilon_k^\sigma} \langle\langle(\boldsymbol{e}_k^\sigma \cdot \boldsymbol{j}_{\varkappa_1}^{\text{n.l.}*})\rangle\rangle \langle\langle(\boldsymbol{e}_{k_1}^{\sigma*} \cdot \boldsymbol{j}_\varkappa^{\text{n.l.}})\rangle\rangle \times \exp\,[i(\{\boldsymbol{k} - \boldsymbol{k}_1\}\cdot\boldsymbol{r}) - i(\omega - \omega_1')t]. \tag{3.79}$$

We now use equation (3.50) and (3.78) and average the result by using equations (3.10), (3.12) and (3.15). Comparing these results with equation (3.75) we see that

$$w_\sigma^{\sigma'\sigma''}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = 32\varrho(2\pi)^7\, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\, \delta(\omega_k^\sigma - \omega_{k_1}^{\sigma'} - \omega_{k_2}^{\sigma''}) \times \frac{(\omega_{k_1}^{\sigma'})^2 (\omega_{k_2}^{\sigma''})^2 \left|S_{\sigma\sigma'\sigma''}(\omega_k^\sigma, \boldsymbol{k}, \omega_{k_1}^{\sigma'}, \boldsymbol{k}_1, \omega_{k_1}^{\sigma''}, \boldsymbol{k}_2)\right|^2}{\frac{\partial}{\partial\omega}(\omega^2\varepsilon_k^\sigma)\Big|_{\omega=\omega_k^\sigma} \frac{\partial}{\partial\omega_1}(\omega_1^2\varepsilon_{\varkappa_1}^{\sigma'})\Big|_{\omega_1=\omega_{k_1}^{\sigma'}} \frac{\partial}{\partial\omega_2}(\omega_2^2\varepsilon_{\varkappa_2}^{\sigma''})\Big|_{\omega_2=\omega_{k_2}^{\sigma''}}}, \tag{3.80}$$

where

$$S_{\sigma\sigma'\sigma''} = \sum_{i,j,l} (e_k^\sigma)_i (e_{k_1}^{\sigma'})_j (e_{k_2}^{\sigma''})_l S_{ijl}, \tag{3.81}$$

and

$$\varrho = \begin{cases} 1, & \text{if} \quad \sigma' \neq \sigma'', \\ \frac{1}{2}, & \text{if} \quad \sigma' = \sigma''. \end{cases} \tag{3.82}$$

From this general formula it is easy to find many of the results of other authors.

3.7. Non-linear Plasmon–Particle Interactions

To a first approximation such an interaction is due to the induced scattering of plasmons by particles

$$\sigma + \alpha \rightleftarrows \sigma' + \alpha'. \tag{3.83}$$

Let us denote the probability of this event by

$$w_\sigma^{\sigma'}(\boldsymbol{k}, \boldsymbol{k}', p_z, p_\perp, \nu). \tag{3.84}$$

The balance equation is very similar to that for emission except that the momentum change is now

$$p_z' = p_z' + k_z' - k_z, \quad p_x' = p_x + k_x' - k_x. \tag{3.85}$$

We have

$$\frac{dN_k^\sigma}{dt} = \sum_{\alpha, n, \nu, \sigma'} \int \frac{dp_z}{2\pi} \frac{d^3\boldsymbol{k}'}{(2\pi)^3} w_\sigma^{\sigma'}(\boldsymbol{k}, \boldsymbol{k}', p_z, p_x, \nu) \times [N_{k'}^{\sigma'}(N_k^\sigma + 1) f_\alpha(p_z, p_x, n) - (N_{k'}^{\sigma'} + 1) N_k^\sigma f_\alpha(p_z + k_z' - k_z, p_x + k_x' - k_x, n - \nu)] \tag{3.86}$$

In the quasi-classical limit we obtain

$$\frac{dN_k^\sigma}{dt} = \sum_{\alpha, \sigma', i} \sum_{\nu=-\infty}^{+\infty} \int \frac{d^3\boldsymbol{p}\, d^3\boldsymbol{k}}{(2\pi)^6} N_k^\sigma N_{k'}^{\sigma'} w_\sigma^{\sigma'}(\boldsymbol{k}, \boldsymbol{k}', p_z, p_\perp, \nu)\, \Delta\lambda_i' \frac{\partial}{\partial\lambda_i} f_\alpha(p_z, p_\perp, y) + \sum_{\alpha, \sigma'} \sum_{\nu=-\infty}^{+\infty} \int \frac{d^3\boldsymbol{p}\, d^3\boldsymbol{k}}{(2\pi)^6} N_{k'}^{\sigma'} w_\sigma^{\sigma'}(\boldsymbol{k}, \boldsymbol{k}', p_z, p_\perp, \nu) f_\alpha(p_z, p_\perp, y) - \sum_{\alpha, \sigma'} \sum_{\nu=-\infty}^{+\infty} \int \frac{d^3\boldsymbol{p}'\, d^3\boldsymbol{k}'}{(2\pi)^6} N_k^\sigma w_\sigma^{\sigma'}(\boldsymbol{k}, \boldsymbol{k}', p_z, p_\perp, \nu) f_\alpha(p_z, p_\perp, y), \tag{3.87}$$

$$\Delta\lambda_1' = k_z - k_z', \quad \Delta\lambda_2' = \frac{\omega_{H\alpha} v}{v_\perp}, \quad \Delta\lambda_3' = -\frac{k_x - k_x'}{|eH_0|}. \tag{3.88}$$

In the limit as $N_k \to 0$ equation (3.87) describes spontaneous scattering:

$$\frac{dN_k^\sigma}{dt} = \sum_{\alpha,\sigma'} \sum_{\nu=-\infty}^{+\infty} \frac{d^3p\, d^3k}{(2\pi)^6} N_{k'}^{\sigma'} w_\sigma^{\sigma'}(k, k', p_z, p_\perp, \nu) f_\alpha. \tag{3.89}$$

This can be used to find the emissivity, as follows:

$$Q = \int \frac{d^3p}{(2\pi)^3} f Q_p^{\sigma\sigma'}, \tag{3.90}$$

$$Q_p^{\sigma\sigma'} = \sum_{\nu=-\infty}^{+\infty} \int \frac{d^3k\, d^3k'}{(2\pi)^6} N_{k'}^{\sigma'} w_\sigma^{\sigma'}(k, k', p_z, p, \nu) \omega_k^\sigma. \tag{3.91}$$

In order to find the probability it is necessary to know the current due to emission. This current has two components, corresponding to the two possible mechanisms of scattering, that is, the usual Compton scattering and the non-linear scattering (see Fig. 15).

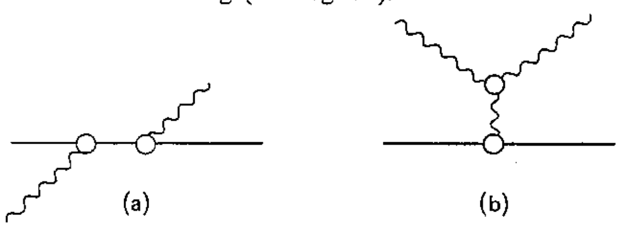

FIG. 15.
Two possible mechanisms of plasmon–particle scattering.

Often these currents cancel one another as, for example, in the case of the scattering of Langmuir waves by electrons, or ion-sound waves by ions, and in many other circumstances.

The current due to Compton scattering is easily found. It is necessary merely to compute the small oscillations of a particle in the E^σ-field ($c = 1$):

$$m \frac{d}{dt} \frac{v}{\sqrt{(1-v^2)}} = e \int \{E_\varkappa^\sigma + [v \wedge H_\varkappa^\sigma]\} e^{i(k\cdot r) - i\omega t}\, d^3k\, d\omega, \tag{3.92}$$

where

$$H_\varkappa^\sigma = \frac{1}{\omega}[k \wedge E_\varkappa^\sigma]. \tag{3.93}$$

In first approximation the particle moves either with constant velocity or spiral in the zero-order constant magnetic field. If this trajectory is substituted on the right-hand side of equation (3.92) the small oscillations of the particle in the field can be determined. The current due to these oscillations is easily found from the relation

$$j = ev(t)\, \delta(r - r(t)). \tag{3.94}$$

Although in the relativistic case the calculations are sometimes tedious, the general result is usually quite simple. We write it in the general form

$$j_\varkappa = \sum_{\alpha,\sigma'} \sum_{\nu=-\infty}^{+\infty} \int \frac{d^4\varkappa'}{(2\pi)^3} \Lambda_{\varkappa,\varkappa',\nu} E_{\varkappa'}^{\sigma'} e^{i(k\cdot r_0) + i\nu\varphi_0}\, \delta(\omega - \omega' - (k_z - k_z')v_z - \nu\omega_{H\alpha}), \tag{3.95}$$

without specifying the coefficients Λ.

The current due to the non-linear scattering is also easy to find. Indeed, such a current stems from the simultaneous presence of the $E_\varkappa^\sigma$-field and the field produced by the particle, neglecting the small oscillations of the particle in the $E_\varkappa^\sigma$-field.

From equation (3.77) we thus obtain:

$$(j_\varkappa)_i = \sum_{\alpha,\sigma} \sum_{j,l} 2 \int S_{ijl}(\varkappa, \varkappa', \varkappa - \varkappa') E_{\varkappa' j}^\sigma E_{\varkappa-\varkappa', l}^\alpha\, d^4\varkappa', \tag{3.96}$$

where

$$\sum_j [k^2\delta_{ij} - k_i k_j - \omega^2 \varepsilon_{ij}(\omega, k)] (E_\varkappa^\alpha)_j = 4\pi i\omega (j_\varkappa^\alpha)_i, \tag{3.97}$$

and $(j_\varkappa^\alpha)_i$ is given by equation (3.43). If one introduces the inverse Maxwellian operator by

$$\sum_i \Pi_{si}(\varkappa) [k^2\delta_{ij} - k_i k_j - \omega^2 \varepsilon_{ij}(\varkappa)] = \delta_{sj}, \tag{3.98}$$

one finds

$$E_{\varkappa-\varkappa', l}^\alpha = 4\pi i(\omega - \omega') \sum_{l,s} \Pi_{ls}(\varkappa - \varkappa') \sum_{\nu=-\infty}^{+\infty} \frac{e}{(2\pi)^3} \Gamma'^{\alpha}_{\varkappa-\varkappa', s}$$
$$\times \exp[i(\{k - k'\} \cdot r_0) + i\nu\varphi_0]\, \delta(\omega - \omega' - (k_z - k_z')v_z - \nu\omega_{H\alpha}); \tag{3.99}$$

Γ' differs from Γ (see equation (3.44)) by a rotation of the xy-axes such that $k_y \neq 0$. Thus we find

$$j_\varkappa^\alpha = \sum_{\nu=-\infty}^{+\infty} \int \Lambda_{\varkappa,\varkappa',\nu}^{\text{n.l.}} E_{\varkappa'}^{\sigma'} e^{i(k\cdot r_0) + i\nu\varphi_0}\, \delta(\omega - \omega' - (k_z - k_z')v_z - \nu\omega_{H\alpha})\, d^4\varkappa', \tag{3.100}$$

where

$$(\Lambda^{\text{n.l.}}_{\varkappa,\varkappa',\nu})_i = \sum_{j,l,s} \frac{8\pi ie(\omega-\omega')}{(2\pi)^3}(e^{\sigma'}_{k'})_j S_{ijl}(\varkappa,\varkappa',\varkappa-\varkappa')\Pi_{ls}(\varkappa-\varkappa')\Gamma'^{\alpha}_{\varkappa-\varkappa',s}. \tag{3.101}$$

The emissivity of such a current can be calculated in a manner analogous to that in the case of plasmon emission. Comparing the result with equation (3.91) we obtain

$$w^{\sigma'}_{\sigma}(k, k', p_z, p_\perp, \nu) = \frac{4(2\pi)^9 \left|\Lambda_{\sigma\sigma'}(\omega^\sigma_k, k, \omega^{\sigma'}_{k'}, k')\right|^2}{\frac{\partial}{\partial\omega}(\omega^2\varepsilon^\sigma_\varkappa)\Big|_{\omega=\omega^\sigma_k}\frac{\partial}{\partial\omega'}(\omega'^2\varepsilon^{\sigma'}_{\varkappa'})\Big|_{\omega'=\omega^{\sigma'}_{k'}}}, \tag{3.102}$$

where

$$\Lambda_{\sigma\sigma'} = \sum_i (e^\sigma_k)_i(\Lambda^1_i + \Lambda^{\text{n.l.}}_i). \tag{3.103}$$

It should be noted that the inverse Maxwellian operator represents the virtual wave in the non-linear scattering diagram. Often such a wave can be considered to be longitudinal, in which case

$$\Pi_{ij} = -\frac{1}{\omega^2\varepsilon^1}\frac{k_ik_j}{k^2}, \tag{3.104}$$

where ε^1 is defined by equation (3.52). If, in addition, the non-linear scattering dominates (as it does for the scattering of high-frequency plasmons by ions) a simple expression for the probability results:

$$w^{1'}_1(k, k', p_z, p_\perp, \nu) = \sum_\alpha \frac{32(2\pi)^6 e^2\left|S^1(\omega^1_k, k, \omega^{1'}_{k'}, k', \omega^1_k-\omega^{1'}_{k'}, k-k')\right|^2}{\pi|k-k'|^2\left|\varepsilon^1(\omega^1_k-\omega^{1'}_{k'}, k-k')\right|^2}$$
$$\times\frac{J^2_\nu(z_\alpha)\,\delta(\omega^1_k-\omega^{1'}_{k'}-(k_z-k'_z)v_z-\nu\omega_{H\alpha})}{(\omega^1_k)^2\frac{\partial}{\partial\omega}\varepsilon^1_\varkappa\Big|_{\omega=\omega^1_k}\frac{\partial}{\partial\omega'}\varepsilon^{1'}_{\varkappa'}\Big|_{\omega'=\omega^{1'}_{k'}}}, \tag{3.105}$$

where

$$z_\alpha = \frac{([k-k']_\perp\cdot v_\perp)}{\omega_{H\alpha}}, \quad S^1 = \sum_{i,l}\frac{k_ik_{1j}k_{2l}}{kk_1k_2}S_{ijl}. \tag{3.106}$$

An equation for the change in the particle distribution function resulting from the scattering can also be written down. It has the same form as equation (3.33) with

$$D_{ij} = \sum_{\sigma,\sigma'}\sum_{\nu=-\infty}^{+\infty}\int\frac{d^3k\,d^3k'}{(2\pi)^6}\Delta\lambda'_i\,\Delta\lambda'_j w^{\sigma'}_\sigma(k, k', p_z, p_\perp, \nu)N^\sigma_k N^{\sigma'}_{k'}, \tag{3.107}$$

$$A_j = \sum_{\sigma,\sigma'}\sum_{\nu=-\infty}^{+\infty}\int\frac{d^3k\,d^3k'}{(2\pi)^6}(N^{\sigma'}_{k'}-N^\sigma_k)\,\Delta\lambda'_i w^{\sigma'}_\sigma(k, k', p_z, p_\perp, \nu). \tag{3.108}$$

If the particles are non-magnetic (or if $H_0 = 0$) then $\lambda_i = p_i$ and $\Delta\lambda_i = k_i$. In the circumstances in which both the interacting and virtual waves are longitudinal, the probability is obtained from (3.105) by making the substitutions:

$$J^2_\nu \to 1, \tag{3.109}$$

$$\delta(\omega^\sigma_k-\omega^\sigma_k-(k_z-k'_z)v_z-\nu\omega_{H\alpha}) \to \delta(\omega^\sigma_k-\omega^{\sigma'}_{k'}-([k-k']\cdot v)). \tag{3.110}$$

For example, in the simplest case, when Langmuir waves are scattered by ions, the probability is

$$w^{1'}_1(p, k, k') = \frac{(2\pi)^3}{m^2_e\omega^2_{pe}}\frac{e^4(k\cdot k')^2\,\delta(\omega^1_k-\omega^{1'}_{k'}-([k-k']\cdot v))}{k^2k'^2(1+T_e/T_i)^2}. \tag{3.111}$$

The method described above enables one to obtain and use the probability for the process of interest.

4. Turbulent Collisions and Resonance Broadening

WE EMPHASISED in the earlier chapters that the balance equations are only an approximation and that turbulent broadening of resonance interactions plays an essential role in the turbulent state. One can say that the broadening of resonances is due to turbulent collisions. The derivation which we have given of the balance equation shows that the particles emit and scatter waves in a linearly stable or unstable plasma as if there were a high level of dissipative processes. The point is that we used

$$\mathrm{Im}\,\frac{1}{k^2-\omega^2\varepsilon^\sigma} = \pi\delta(k^2-\omega^2\varepsilon^\sigma), \tag{4.1}$$

which is only true in the region of transparency where Im (ε^σ) is small and positive. This implies the use of retarded potentials in calculating the waves emitted by particles. In an unstable plasma when the linear part of Im (ε^σ) is negative, one must use equation (4.1), but not the expression with the opposite sign (advanced potentials), because the linear part of Im (ε^σ) has then no meaning for plasmons. It was already emphasised that the plasmons are driven in $\boldsymbol{k}$-space by non-linear interactions which play an effective role as collisions. This means that the denominator in equation (4.1) depends on the turbulent energy which can, therefore, not be used as an expansion parameter. Only in the balance equations where the broadening due to the turbulent collisions is neglected can one use equation (4.1). Also, the transition to the turbulent state involves non-linear interactions and thus a more precise description of the turbulent state is needed to include turbulent collisions.

Maybe one can say that for the balance equations it is an academic question, because it is obvious that in a statistical description one can use a transition probability approach and that the increase in entropy needs a retarded potential. For example, it is obvious also that binary collisions in a plasma could be described by collision probabilities or cross-sections and the exact derivation of the collision integral from the

Bogolyubov scheme can be regarded as an important, but academic problem. However, this is not so because Bogolyubov's scheme is the way to prove the validity of the physical picture used.

Let us emphasise then, that the question of turbulent collisions is not academic, nor is the treatment of the Landau poles in the balance equation, but that they are essential practical questions for ion sound, for example. The study of this problem by Rudakov and Tsytovich (1971) shows that the induced scattering of ion-sound waves by electrons becomes negligible compared to the quasi-linear collisions. Therefore, not all possible diagrams should be taken into account, even in the balance equations, and a full statement of which diagrams are the most important will be given later. Let us only remark that the lifetime of the elementary excitations depends on the turbulence energy and that they are physically quite different from excitations near statistical equilibrium.

4.1. The Balance Equation found by Statistical Averaging

Let us first find the balance equations exactly from the Vlasov and Maxwell equations using an expansion in the stochastic field amplitude. For simplicity let us consider waves in an unmagnetised plasma which are not resonant with the particles, that is $\omega \neq (\boldsymbol{k}\cdot\boldsymbol{v})$, and start from the equations

$$\frac{\partial f_\alpha}{\partial t}+(\boldsymbol{v}\cdot\nabla f_\alpha)+e\left(\boldsymbol{E}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{p}}\right) = 0, \tag{4.2}$$

$$(\nabla\cdot\boldsymbol{E}) = 4\pi\sum_\alpha e_\alpha\int f_\alpha\frac{d^3p}{(2\pi)^3}\,. \tag{4.3}$$

We divide the distribution function and the electric field into regular and stochastic parts as follows:

$$f_\alpha = f_\alpha^{\mathrm{reg}}+f_\alpha^{\mathrm{stoch}}, \quad \langle f_\alpha^{\mathrm{stoch}}\rangle = 0, \tag{4.4}$$

$$\boldsymbol{E} = \boldsymbol{E}^{\mathrm{reg}}+\boldsymbol{E}^{\mathrm{stoch}}, \quad \langle \boldsymbol{E}^{\mathrm{stoch}}\rangle = 0. \tag{4.5}$$

Thus, by averaging equations (4.2) and (4.3) and substracting from the exact equation the averaged one, it is possible to find two sets of coupled equations for the regular and the stochastic variables ($f \equiv f^{\text{reg}}$):

$$\frac{\partial f}{\partial t}+(\boldsymbol{v}\cdot\nabla f)+e\left(\boldsymbol{E}^{\text{reg}}\cdot\frac{\partial f}{\partial \boldsymbol{p}}\right) = -e\left\langle\left(\boldsymbol{E}^{\text{stoch}}\cdot\frac{\partial f}{\partial \boldsymbol{p}}\right)^{\text{stoch}}\right\rangle, \tag{4.6}$$

$$\frac{\partial f^{\text{stoch}}}{\partial t}+(\boldsymbol{v}\cdot\nabla f^{\text{stoch}})+e\left(\boldsymbol{E}^{\text{stoch}}\cdot\frac{\partial f}{\partial \boldsymbol{p}}\right)+e\left(\boldsymbol{E}^{\text{reg}}\cdot\frac{\partial f^{\text{stoch}}}{\partial \boldsymbol{p}}\right)$$
$$+e\left\{\left(\boldsymbol{E}^{\text{stoch}}\cdot\frac{\partial f^{\text{stoch}}}{\partial \boldsymbol{p}}\right)-\left\langle\left(\boldsymbol{E}^{\text{stoch}}\cdot\frac{\partial f^{\text{stoch}}}{\partial \boldsymbol{p}}\right)\right\rangle\right\} = 0, \tag{4.7}$$

$$(\nabla\cdot\boldsymbol{E}^{\text{reg}}) = 4\pi\sum_\alpha e_\alpha\int f_\alpha\frac{d^3p}{(2\pi)^3}, \tag{4.8}$$

$$(\nabla\cdot\boldsymbol{E}^{\text{stoch}}) = 4\pi\sum_\alpha e_\alpha\int f^{\text{stoch}}\frac{d^3p}{(2\pi)^3}. \tag{4.9}$$

Let us consider for simplicity the case $\boldsymbol{E}^{\text{reg}} = 0$, and expand in $\boldsymbol{E}^{\text{stoch}}$. In the Fourier representation equation (4.7) splits into a series of equations:

$$i[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]f_\varkappa^{\text{stoch}(1)}+e\left(\boldsymbol{E}_\varkappa^{\text{stoch}}\cdot\frac{\partial f}{\partial \boldsymbol{p}}\right) = 0, \tag{4.10}$$

$$-i[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]f_\varkappa^{\text{stoch}(2)}+e\int\left\{\left(\boldsymbol{E}_{\varkappa_1}^{\text{stoch}}\cdot\frac{\partial f_{\varkappa_2}^{\text{stoch}(1)}}{\partial \boldsymbol{p}}\right)\right.$$
$$\left.-\left\langle\left(\boldsymbol{E}_{\varkappa_1}^{\text{stoch}}\cdot\frac{\partial f_{\varkappa_2}^{\text{stoch}(1)}}{\partial \boldsymbol{p}}\right)\right\rangle\right\}\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2 = 0, \tag{4.11}$$

$$-i[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]f_\varkappa^{\text{stoch}(3)}+e\int\left\{\left(\boldsymbol{E}_{\varkappa_1}^{\text{stoch}}\cdot\frac{\partial f_{\varkappa_2}^{\text{stoch}(2)}}{\partial \boldsymbol{p}}\right)\right.$$
$$\left.-\left\langle\left(\boldsymbol{E}_{\varkappa_1}^{\text{stoch}}\cdot\frac{\partial f_{\varkappa_2}^{\text{stoch}(2)}}{\partial \boldsymbol{p}}\right)\right\rangle\right\}\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2 = 0, \tag{4.12}$$

....

Here the f^{reg} are for simplicity taken to be time and space independent. The generalisation is simple but this approximation is appropriate when f^{reg} varies slowly in space and time.

The first step is to substitute from equation (4.10) for $f_\varkappa^{\text{stoch}}$, given by

$$f_\varkappa^{\text{stoch}} = \frac{e}{i[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]}\,\frac{E_\varkappa^{\text{stoch}}}{k}\left(\boldsymbol{k}\cdot\frac{\partial f}{\partial \boldsymbol{p}}\right), \tag{4.13}$$

into the "collision integral" of equation (4.6) to get the quasi-linear equation:

$$\frac{\partial f}{\partial t}+\left(\boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{p}}\right)$$
$$= -e\int\left\langle E_{\varkappa_1}^{\text{stoch}}\left(\frac{\boldsymbol{k}_1}{k_1}\cdot\frac{\partial f_{\varkappa_2}^{\text{stoch}}}{\partial \boldsymbol{p}}\right)\right\rangle e^{i((\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{r})-i(\omega_1+\omega_2)t}\,d^4\varkappa_1\,d^4\varkappa_2$$
$$= -\frac{e^2}{i}\sum_{i,j}\frac{\partial}{\partial p_i}\int\frac{k_{1i}k_{2j}}{k_1k_2}\langle E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}\rangle\frac{e^{i((\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{r})-i(\omega_1+\omega_2)t}}{\omega_2-(\boldsymbol{k}_2\cdot\boldsymbol{v})+i\delta}\,\frac{\partial f}{\partial p_j}\,d^4\varkappa_1\,d^4\varkappa_2$$
$$= \sum_{i,j}\frac{\partial}{\partial p_i}\left(D_{ij}\frac{\partial f}{\partial p_j}\right), \tag{4.14}$$

where

$$D_{ij} = \pi e^2\int\frac{k_{1i}k_{1j}}{k_1^2}\,|E_{\varkappa_1}|^2\,\delta(\omega_1-(\boldsymbol{k}_1\cdot\boldsymbol{v})). \tag{4.15}$$

Here we have used the usual expression for the average of the stochastic longitudinal fields, a minus sign arising as $\boldsymbol{k}_1 = -\boldsymbol{k}_2$,

$$\langle E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}\rangle = -|E_{\varkappa_1}|^2\,\delta(\varkappa_1+\varkappa_2). \tag{4.16}$$

The imaginary part of $[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})+i\delta]^{-1}$ has been put equal to $-\pi\delta(\omega-(\boldsymbol{k}\cdot\boldsymbol{v}))$. The essential question is how did we decide on this rule for dealing with the singularities $[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]^{-1}$? One could, of course, introduce a small collision term in the initial equation (4.2), but this is wrong because the turbulent collisions are much larger than the binary collisions. From expression (4.13) one can see there is a divergence near $\omega = (\boldsymbol{k}\cdot\boldsymbol{v})$ and one concludes that near the resonance one cannot use an expansion in $E_\varkappa^{\text{stoch}}$ and the higher-order terms in $E_\varkappa^{\text{stoch}}$ in equation (4.10) are essential. As we shall see this gives a broadening of $\delta(\omega-(\boldsymbol{k}\cdot\boldsymbol{v}))$ in the quasi-linear equation (4.15). Because this function has an appreciable maximum near $\omega = (\boldsymbol{k}\cdot\boldsymbol{v})$, one can use equation (4.15) as an approximate expression for the diffusion coefficient. This can only be done for integrated functions such as the diffusion coefficient, but not for the stochastic part of the distribution function $f_\varkappa^{\text{stoch}}$, so that equation (4.13) is wrong near the resonance.

Let us leave this problem for the moment and consider the case where $\omega \neq (\boldsymbol{k}\cdot\boldsymbol{v})$, when expression (4.15) is approximately zero and equation

(4.13) has a finite source term. An example of this is found in Langmuir turbulence, when the phase velocity of the waves is much larger than the thermal velocity. Substituting expression (4.13) into equations (4.11) and (4.12) one finds the non-linear charge density

$$\varrho^{(2)} = e\int f_\varkappa^{\text{stoch}\,(2)} \frac{d^3p}{(2\pi)^3}$$

$$= -e^3 \int \frac{d^3p}{k_1k_2(2\pi)^3\,[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]}\left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)\frac{1}{\omega_2-(\boldsymbol{k}_2\cdot\boldsymbol{v})}\left(\boldsymbol{k}_2\cdot\frac{\partial f}{\partial \boldsymbol{p}}\right)$$

$$\times[E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}-\langle E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}\rangle]\,\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2$$

$$= \int \varrho^{(2)}_{\varkappa,\varkappa_1,\varkappa_2}[E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}-\langle E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}\rangle]\,\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2\,, \tag{4.17}$$

$$\varrho^{(3)} = e\int f_\varkappa^{\text{stoch}\,(3)}\frac{d^3p}{(2\pi)^3}$$

$$= \int \varrho^{(3)}_{\varkappa,\varkappa_1,\varkappa_2,\varkappa_3}[E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}E_{\varkappa_3}^{\text{stoch}}-E_{\varkappa_1}^{\text{stoch}}\langle E_{\varkappa_2}^{\text{stoch}}E_{\varkappa_3}^{\text{stoch}}\rangle$$

$$-\langle E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}E_{\varkappa_3}^{\text{stoch}}\rangle]\,\delta(\varkappa-\varkappa_1-\varkappa_2-\varkappa_3)\,d^4\varkappa_1\,d^4\varkappa_2\,d^4\varkappa_3\,, \tag{4.18}$$

$$\varrho^{(3)}_{\varkappa,\varkappa_1,\varkappa_2,\varkappa_3} = ie^4\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]}\left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\frac{1}{\omega-\omega_1-([\boldsymbol{k}-\boldsymbol{k}_1]\cdot\boldsymbol{v})}$$

$$\times\left(\boldsymbol{k}_2\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\frac{1}{\omega_3-(\boldsymbol{k}_3\cdot\boldsymbol{v})}\left(\boldsymbol{k}_3\cdot\frac{\partial f}{\partial\boldsymbol{p}}\right)\frac{1}{k_1k_2k_3}. \tag{4.19}$$

One can now write the non-linear Maxwell equation (4.9) in the form

$$ik\varepsilon_\varkappa E_\varkappa^{\text{stoch}} = 4\pi\int \varrho^{(2)}_{\varkappa,\varkappa_1,\varkappa_2}[E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}-\langle E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_1}^{\text{stoch}}\rangle]$$

$$\times\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2$$

$$+4\pi\int\varrho^{(3)}_{\varkappa,\varkappa_1,\varkappa_2,\varkappa_3}[E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}E_{\varkappa_3}^{\text{stoch}}-E_{\varkappa_1}^{\text{stoch}}\langle E_{\varkappa_2}^{\text{stoch}}E_{\varkappa_3}^{\text{stoch}}\rangle$$

$$-\langle E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}E_{\varkappa_3}^{\text{stoch}}\rangle]$$

$$\times\delta(\varkappa-\varkappa_1-\varkappa_2-\varkappa_3)\,d^4\varkappa_1\,d^4\varkappa_2\,d^4\varkappa_3\,, \tag{4.20}$$

where

$$\varepsilon_\varkappa = 1+\frac{4\pi e^2}{k^2}\int\frac{\left(\boldsymbol{k}\cdot\dfrac{\partial f^{(\text{reg})}}{\partial\boldsymbol{p}}\right)}{\omega-(\boldsymbol{k}\cdot\boldsymbol{v})+i\delta}\,\frac{d^3p}{(2\pi)^3}\,. \tag{4.21}$$

Note that $\varepsilon_\varkappa$ contains the average regular distribution f^{reg}, but not the linear stochastic distribution $f_\varkappa^{\text{stoch}}$, so that equation (4.21) does not give a linear dielectric constant. Also it is necessary to mention that expression (4.21) was derived from equation (4.17) which has no exact meaning. But expression (4.21) is an integrated value, and use of Im $[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})+i\delta]^{-1}$ $=\pi\delta(\omega-(\boldsymbol{k}\cdot\boldsymbol{v}))$ that leads in equation (4.21) to the quasilinear growth-rate can be shown to be appropriate. Now $\omega\neq(\boldsymbol{k}\cdot\boldsymbol{v})$ and the imaginary part of $\varepsilon_\varkappa$ is zero. Using equations (4.20) and (4.21), and averaging over an ensemble, one has

$$\varepsilon_\varkappa|E_\varkappa^{\text{stoch}}|^2 = \frac{4\pi i}{k}\int\varrho^{(2)}_{\varkappa,\varkappa_1,\varkappa_2}\langle E_{\varkappa'}^{\text{stoch}}E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}\rangle\,\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa'\,d^4\varkappa_1\,d^4\varkappa_2$$

$$+\frac{4\pi i}{k}\int\varrho^{(3)}_{\varkappa,\varkappa_1,\varkappa_2,\varkappa_3}[\langle E_{\varkappa'}^{\text{stoch}}E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}E_{\varkappa_3}^{\text{stoch}}\rangle$$

$$-\langle E_{\varkappa'}^{\text{stoch}}E_{\varkappa_1}^{\text{stoch}}\rangle\langle E_{\varkappa_2}^{\text{stoch}}E_{\varkappa_3}^{\text{stoch}}\rangle]$$

$$\times\delta(\varkappa-\varkappa_1-\varkappa_2-\varkappa_3)\,d^4\varkappa'\,d^4\varkappa_1\,d^4\varkappa_2\,d^4\varkappa_3\,. \tag{4.22}$$

This equation is written down for the case of stationary turbulence. If the turbulent state depends weakly on time and space it is useful to define $|E_\varkappa^{\text{stoch}}|^2$ by the relation

$$|E_\varkappa^{\text{stoch}}|^2 = \int\langle E_\varkappa^{\text{stoch}}E_{\varkappa'}^{\text{stoch}}\rangle\,e^{i([\boldsymbol{k}+\boldsymbol{k}']\cdot\boldsymbol{r})-i(\omega+\omega')t}\,d^4\varkappa'\,, \tag{4.23}$$

which coincides with that used above, if the turbulence is stationary. Then the equations for $(\varepsilon_\varkappa-\varepsilon_{\varkappa'})\langle E_\varkappa^{\text{stoch}}E_{\varkappa'}^{\text{stoch}}\rangle$ lead to the term

$$\left[\frac{\partial}{\partial t}+(\boldsymbol{v}_{\text{gr}}\cdot\nabla)\right]|E_\varkappa^{\text{stoch}}|^2+\gamma_k|E_\varkappa^{\text{stoch}}|^2\,, \tag{4.24}$$

which occurs in the balance equations. Let us emphasise here that the weak dependence on t and $\boldsymbol{r}$ in the treatment given now is not a result of the smallness of the quasi-linear growth-rate, because the non-linear interactions may well lead to a stationary state.

We shall continue to consider the case of stationary turbulence and use equation (4.22). If the turbulence is weak, the correlations of the fields are rather weak so that one can put in equation (4.22)

$$\langle E^{\text{stoch}}_{\varkappa'} E^{\text{stoch}}_{\varkappa_1} E^{\text{stoch}}_{\varkappa_2} E^{\text{stoch}}_{\varkappa_3} \rangle \approx \langle E^{\text{stoch}}_{\varkappa'} E^{\text{stoch}}_{\varkappa_1} \rangle \langle E^{\text{stoch}}_{\varkappa_2} E^{\text{stoch}}_{\varkappa_3} \rangle + \langle E^{\text{stoch}}_{\varkappa'} E^{\text{stoch}}_{\varkappa_2} \rangle \langle E^{\text{stoch}}_{\varkappa_1} E^{\text{stoch}}_{\varkappa_3} \rangle + \langle E^{\text{stoch}}_{\varkappa'} E^{\text{stoch}}_{\varkappa_3} \rangle \langle E^{\text{stoch}}_{\varkappa_1} E^{\text{stoch}}_{\varkappa_2} \rangle . \tag{4.25}$$

The last term of equation (4.22) is, therefore,

$$\frac{4\pi i}{k} \int [\varrho^{(3)}_{\varkappa, \varkappa_1, \varkappa, -\varkappa_1} + \varrho^{(3)}_{\varkappa, \varkappa_1, -\varkappa_1, \varkappa}] \, |E^{\text{stoch}}_{\varkappa}|^2 \, |E^{\text{stoch}}_{\varkappa_1}|^2 \, d^4\varkappa_1 . \tag{4.26}$$

To find the average of the three fields one needs to express it in terms of four fields. This is possible by using the first term of equation (4.20)

$$E^{\text{stoch}}_{\varkappa} = -\frac{4\pi i}{k\varepsilon_\varkappa} \int \varrho^{(2)}_{\varkappa, \varkappa_1 \varkappa_2} [E^{\text{stoch}}_{\varkappa_1} E^{\text{stoch}}_{\varkappa_2} - \langle E^{\text{stoch}}_{\varkappa_1} E^{\text{stoch}}_{\varkappa_1} \rangle] \times \delta(\varkappa - \varkappa_1 - \varkappa_2) \, d^4\varkappa_1 \, d^4\varkappa_2 . \tag{4.27}$$

Let us mention that this expression has no meaning if $\varepsilon_\varkappa$ is close to zero, as can be seen by analogy with our discussion of equation (4.13). But if it is used, one finds the first term of equation (4.22) in the following form:

$$4\frac{(4\pi)^2}{k} |E^{\text{stoch}}_{\varkappa}|^2 \int \bar{\varrho}^{(2)}_{\varkappa, \varkappa_1, \varkappa, -\varkappa_1} \bar{\varrho}^{(2)}_{\varkappa-\varkappa_1, \varkappa, -\varkappa_1} \frac{|E^{\text{stoch}}_{\varkappa_1}|^2 \, d^4\varkappa_1}{|\boldsymbol{k}-\boldsymbol{k}_1| \, \varepsilon_{\varkappa-\varkappa_1}}$$
$$2\frac{(4\pi)^2}{k^2 \varepsilon_{-\varkappa}} \int \bar{\varrho}^{(2)}_{\varkappa, \varkappa_1, \varkappa_2} \bar{\varrho}^{(2)}_{-\varkappa, -\varkappa_1, -\varkappa_2} |E^{\text{stoch}}_{\varkappa_1}|^2 \, |E^{\text{stoch}}_{\varkappa_2}|^2 \, \delta(\varkappa - \varkappa_1 - \varkappa_2) \, d^4\varkappa_1 \, d^4\varkappa_2 , \tag{4.28}$$

where $\bar{\varrho}^{(2)}$ is the non-linear charge density coefficient symmetrised in $\varkappa_1$ and $\varkappa_2$:

$$\bar{\varrho}^{(2)}_{\varkappa, \varkappa_1, \varkappa_2} = \tfrac{1}{2} (\varrho^{(2)}_{\varkappa, \varkappa_1, \varkappa_2} + \varrho^{(2)}_{\varkappa, \varkappa_2, \varkappa_1}). \tag{4.29}$$

One may assume that the first term of expression (4.28) will cause no trouble, because $\varepsilon^{-1}_{\varkappa-\varkappa_1}$ occurs under the integral sign, but the second term goes to infinity when $\varepsilon_\varkappa \to 0$. This inconsistency comes from equation (4.27). However, let us continue the considerations by neglecting the term (4.28). This is possible, if the turbulent oscillations cannot even roughly satisfy the decay conditions required by the δ-function in the

second term in expression (4.28). This is exactly the case for Langmuir turbulence because $2\omega_{pe} \neq \omega_{pe}$ and the frequency of the waves lies close to ω_{pe}. Now equation (4.22) can be written in the form

$$(\varepsilon_\varkappa + \varepsilon^{\text{n.l.}}_\varkappa) \, |E^{\text{stoch}}_{\varkappa}|^2 = 0, \tag{4.30}$$

where

$$\varepsilon^{\text{n.l.}}_\varkappa = -\int \Sigma_{\varkappa, \varkappa_1} |E^{\text{stoch}}_{\varkappa_1}|^2 \, d^4\varkappa_1 , \tag{4.31}$$

$$\Sigma_{\varkappa, \varkappa_1} = \frac{4\pi i}{k} (\varrho^{(3)}_{\varkappa, \varkappa_1, \varkappa, -\varkappa_1} + \varrho^{(3)}_{\varkappa, \varkappa_1, -\varkappa_1, \varkappa}) + 4\frac{(4\pi)^2}{k\,|\boldsymbol{k}-\boldsymbol{k}_1|} \times \frac{\bar{\varrho}^{(2)}_{\varkappa, \varkappa_1, \varkappa-\varkappa_1} \bar{\varrho}^{(2)}_{\varkappa-\varkappa_1, \varkappa, -\varkappa_1}}{\varepsilon_{\varkappa-\varkappa_1}} = \Sigma^{(3)}_{\varkappa, \varkappa_1} + \Sigma^{(2)}_{\varkappa, \varkappa_1} . \tag{4.32}$$

The quantity $\Sigma^{(3)}$ contains $\varrho^{(3)}$ and the quantity $\Sigma^{(2)}$ similarly $\varrho^{(2)}$. The term $-\text{Im}\, \varepsilon^{\text{n.l.}}_\varkappa / (\partial\varepsilon/\partial\omega)|_{\omega=\omega_k} = \gamma^{\text{n.l.}}_\varkappa$ can be called a non-linear growth-rate and indeed describes the energy transfer of the turbulent waves. Because $\omega \neq (\boldsymbol{k}\cdot\boldsymbol{v})$ the imaginary part of the first term of $\Sigma_{\varkappa, \varkappa_1}$ can easily be found from equation (4.19):

$$\text{Im}\, \Sigma^{(3)}_{\varkappa, \varkappa_1} = \frac{4\pi e^4}{k m_\alpha} \int \frac{d^3p}{(2\pi)^3} \frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)}{[\omega - (\boldsymbol{k}\cdot\boldsymbol{v})]^2} \pi\delta(\omega - \omega_1 - ([\boldsymbol{k}-\boldsymbol{k}_1]\cdot\boldsymbol{v}))$$
$$\times \left[\left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right) \frac{1}{\omega_1 - (\boldsymbol{k}_1\cdot\boldsymbol{v})} \left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial \boldsymbol{p}}\right) - \left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial \boldsymbol{p}}\right) \frac{1}{\omega - (\boldsymbol{k}\cdot\boldsymbol{v})} \left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right) \right] f$$
$$= \sum_\alpha \frac{4\pi^2 e^4}{k m_\alpha^2} \int \frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)^2}{[\omega - (\boldsymbol{k}\cdot\boldsymbol{v})]^4} \left([\boldsymbol{k}-\boldsymbol{k}_1]\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{p}} \right) \times \delta(\omega - \omega_1 - ([\boldsymbol{k}-\boldsymbol{k}_1]\cdot\boldsymbol{v})) \frac{d^3p}{(2\pi)^3} . \tag{4.33}$$

By expressing $|E_{\varkappa_1}|^2$ in terms of N_{k_1} and calculating $\gamma^{\text{n.l.}}_\varkappa$ we easily find that equation (4.33) is exactly the same as the one we find for the usual Compton scattering of particles. If one examines the next term $\Sigma^{(2)}_{\varkappa, \varkappa_1}$, and calculates the total imaginary part that exists in $\varepsilon^{-1}_{\varkappa, \varkappa_1}$, $\varrho^{(2)}_{\varkappa, \varkappa_1, \varkappa-\varkappa_1}$, and $\varrho^{(2)}_{\varkappa-\varkappa_1, \varkappa, -\varkappa_1}$, one finds the non-linear scattering and the interference of the non-linear and the Compton scattering, so that—as must be the case—the scattering cross-section is $|M_1+M_2|^2$ where M_1 and M_2 are, respectively, the matrix elements of the two scattering processes. Thus the result is just the same as found in the scheme described in the previous section.

To finish this kind of consideration, we return to the problem of stochastic heating. Supposing that for the turbulent oscillations $\omega \neq (\boldsymbol{k}\cdot\boldsymbol{v})$ for the thermal particles so that the diffusion (4.15) seems to be absent. This is not quite true because the non-fulfilment of the condition $\omega_k = (\boldsymbol{k}\cdot\boldsymbol{v})$ in the centre of correlation curve does not mean that this condition cannot be fulfilled far away from this centre, and this indeed is possible. It is interesting that it is possible to find the form of this correlation tail by using the secular term (the second term in expression (4.28), which we have neglected in equation (4.30) or near the centre of the correlation curve). Indeed, far from the centre $\varepsilon_{-\varkappa}$ has no singularities and the terms that are proportional to $|E_\varkappa|^2$ become negligible because $|E_\varkappa|^2$ is small. If one takes into account only the second term of expression (4.28) and substitutes it in the diffusion coefficient one finds that $\delta(\omega-(\boldsymbol{k}\cdot\boldsymbol{v}))$ is replaced by $\delta(\omega-\omega_1-([\boldsymbol{k}-\boldsymbol{k}_1]\cdot\boldsymbol{v}))$, and the whole term is proportional to $|E_{\varkappa_1}|^2\,|E_{\varkappa_2}|^2$ and is exactly the same as found from non-linear scattering in the scheme of the previous section.

As the result is proportional to $|E_\varkappa|^4$ one must take all terms of the same order of magnitude into account and these come into the collision integral (right-hand side of equation (4.6)) from the product $f_\varkappa^{\text{stoch}(2)} \times f_\varkappa^{\text{stoch}(3)}$. These give the Compton scattering and the interference of the Compton and the non-linear scattering.

Although we derived the same results by another procedure, it is clear that there exist some questions about the singularities $[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]^{-1}$ and $\varepsilon_\varkappa^{-1}$, which indeed are not formal, but deep and physical. Still we find the stochastic heating and it is very clearly explained; it comes from substitution $\operatorname{Im}\,[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]^{-1} = -\pi\,\delta(\omega-(\boldsymbol{k}\cdot\boldsymbol{v}))$ or a similar substitution for $[\omega-\omega_1-([\boldsymbol{k}-\boldsymbol{k}_1]\cdot\boldsymbol{v})]^{-1}$, and this was done formally. Thus to clarify the physical meaning of the stochastic heating, one needs to treat the singularities correctly.

It is known how this problem of the singularity $[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]^{-1}$ is treated in the linear theory. This is the Landau treatment that gives the Landau damping. It can be found as the limit of $[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})+i\nu]^{-1}$ (where ν is a small collision term) as $\nu \to 0$. However, in the turbulent regime the energy in the fluctuations is much higher than at thermal equilibrium

and is, therefore, dominant. This means that the denominators must have a broadening dependent on the energy of the turbulence. Thus the whole Landau problem must be re-examined. This means that the nature of the Landau damping of plasmons is different from the usual linear damping. This also follows from the well-known fact that the linear Landau damping can be reversible and the plasmon Landau damping is irreversible.

On the other hand, the presence of the energy of the turbulence in denominators that can be expected from this physical picture, means that to construct the corresponding theory it is necessary to sum the whole series in $|E_\varkappa|^2$.

This one can hope to do only because the turbulent collision frequency ν^{turb} is much smaller than the frequency of the plasmons, or at least than kv_T. Thus we must use the small parameter

$$\frac{1}{\omega\tau_*} = \frac{\nu^{\text{turb}}}{\omega}, \tag{4.34}$$

which is appropriate for weak turbulence.

4.2. Turbulent Broadening of the Wave-particle Resonance

To include the turbulent collisions it is necessary to change the perturbation theory, and to start with an initial approximation that already is a sum of the $|E_\varkappa|^2$. This is possible to do as was shown by Rudakov and Tsytovich (1971), by a renormalisation of the equation for the stochastic distribution function, or by a renormalisation of the propagator $[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]^{-1}$. Let us write equation (4.7) for f^{stoch} once more in the Fourier representation:

$$\begin{aligned} -i[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]f_\varkappa^{\text{stoch}} + eE_\varkappa^{\text{stoch}}\left(\frac{\boldsymbol{k}}{k}\cdot\frac{\partial f^{\text{reg}}}{\partial \boldsymbol{p}}\right) &= -e\int\left[E_{\varkappa_1}^{\text{stoch}}\left(\boldsymbol{k}_1\cdot\frac{\partial f_{\varkappa_2}^{\text{stoch}}}{\partial \boldsymbol{p}}\right) - \left\langle E_{\varkappa_1}^{\text{stoch}}\left(\boldsymbol{k}_1\cdot\frac{\partial f_{\varkappa_2}^{\text{stoch}}}{\partial \boldsymbol{p}}\right)\right\rangle\right] \\ &\quad \times \frac{d^4\varkappa_1}{k_1}\,d^4\varkappa_2\,\delta(\varkappa-\varkappa_1-\varkappa_2). \end{aligned} \tag{4.35}$$

Let us extract then from the right-hand side of equation (4.35) the term that is diagonal in $f_\varkappa^{\text{stoch}}$ and denote it by

$$-i[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})+i\hat{\nu}_\varkappa(\boldsymbol{p})]f_\varkappa^{\text{stoch}} = -eE_\varkappa^{\text{stoch}}\left(\frac{\boldsymbol{k}}{k}\cdot\frac{\partial f^{\text{reg}}}{\partial \boldsymbol{p}}\right)+\hat{\nu}_\varkappa(\boldsymbol{p})f_\varkappa^{\text{stoch}}$$
$$-e\int\left[E_{\varkappa_1}^{\text{stoch}}\left(\frac{\boldsymbol{k}_1}{k_1}\cdot\frac{\partial f_{\varkappa_2}^{\text{stoch}}}{\partial \boldsymbol{p}}\right)-\left\langle E_{\varkappa_1}^{\text{stoch}}\left(\frac{\boldsymbol{k}_1}{k_1}\cdot\frac{\partial f_{\varkappa_2}^{\text{stoch}}}{\partial \boldsymbol{p}}\right)\right\rangle\right]$$
$$\times\,\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2. \qquad (4.36)$$

We introduce the operator $\hat{g}_\varkappa(\boldsymbol{p})$ that is the inverse of the left-hand side operator of equation (4.36)

$$\hat{g}_\varkappa(\boldsymbol{p})\,[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})+i\hat{\nu}_\varkappa(\boldsymbol{p})]f_\varkappa^{\text{stoch}} = f_\varkappa^{\text{stoch}}, \qquad (4.37)$$

and write equation (4.36) in the following form

$$f_\varkappa^{\text{stoch}} = \frac{eE_\varkappa^{\text{stoch}}}{i}\,\hat{g}_\varkappa(\boldsymbol{p})\left(\frac{\boldsymbol{k}}{k}\cdot\frac{\partial f^{\text{reg}}}{\partial \boldsymbol{p}}\right)+i\hat{g}_\varkappa(\boldsymbol{p})\,\hat{\nu}(\boldsymbol{p})f_\varkappa^{\text{stoch}}$$
$$+\frac{1}{i}\,\hat{g}_\varkappa(\boldsymbol{p})e\int\left[E_{\varkappa_1}^{\text{stoch}}\left(\frac{\boldsymbol{k}_1}{k_1}\cdot\frac{\partial f_{\varkappa_2}^{\text{stoch}}}{\partial \boldsymbol{p}}\right)-\left\langle E_{\varkappa_1}^{\text{stoch}}\left(\frac{\boldsymbol{k}_1}{k_1}\cdot\frac{\partial f_{\varkappa_2}^{\text{stoch}}}{\partial \boldsymbol{p}}\right)\right\rangle\right]$$
$$\times\,\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2. \qquad (4.38)$$

We insert this relation into the non-linear term on the right-hand side of equation (4.36)

$$-i[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})+i\nu_\varkappa(\boldsymbol{p})]f_\varkappa^{\text{stoch}} = -eE_\varkappa^{\text{stoch}}\left(\frac{\boldsymbol{k}}{k}\cdot\frac{\partial f^{\text{stoch}}}{\partial \boldsymbol{p}}\right)+\hat{\nu}(\boldsymbol{p})f_\varkappa^{\text{stoch}}$$
$$+ie\int\left(E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}-\langle E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}\rangle\right)\left(\frac{\boldsymbol{k}_1}{k_1}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)\hat{g}_\varkappa(\boldsymbol{p})\left(\frac{\boldsymbol{k}_2}{k_2}\cdot\frac{\partial f^{\text{reg}}}{\partial \boldsymbol{p}}\right)$$
$$\times\;\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2$$
$$-ie\int\left[E_{\varkappa_1}^{\text{stoch}}\left(\frac{\boldsymbol{k}_1}{k_1}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)\hat{g}_\varkappa(\boldsymbol{p})\,\hat{\nu}_\varkappa(\boldsymbol{p})f_\varkappa^{\text{stoch}}\right.$$
$$\left.-\left\langle E_{\varkappa_1}^{\text{stoch}}\left(\frac{\boldsymbol{k}_1}{k_1}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)\hat{g}_\varkappa(\boldsymbol{p})\,\hat{\nu}_\varkappa(\boldsymbol{p})f_\varkappa^{\text{stoch}}\right\rangle\right]\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2$$
$$+ie^2\int\left(\frac{\boldsymbol{k}_1}{k_1}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)\hat{g}_{\varkappa-\varkappa_1}(\boldsymbol{p})\left(\frac{\boldsymbol{k}_2}{k_2}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)[E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}f_{\varkappa_3}^{\text{stoch}}(\boldsymbol{p})$$
$$-E_{\varkappa_1}^{\text{stoch}}\langle E_{\varkappa_2}^{\text{stoch}}f_{\varkappa_3}^{\text{stoch}}(\boldsymbol{p})\rangle-\langle E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}f_{\varkappa_3}^{\text{stoch}}(\boldsymbol{p})\rangle]$$
$$\times\,\delta(\varkappa-\varkappa_1-\varkappa_2-\varkappa_3)\,d^4\varkappa_1\,d^4\varkappa_2\,d^4\varkappa_3. \qquad (4.39)$$

In order that the operations that have been made have a meaning, it is necessary that the diagonal term, at least to first order, vanishes on the right-hand side of equation (4.39). This term comes from the last integral on the right-hand side of equation (4.39), that is, from

$$\langle E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}\rangle\,f_{\varkappa_3}^{\text{stoch}} \qquad (4.40)$$

We thus find an integral equation for the turbulent collision operator $\hat{\nu}_\varkappa(\boldsymbol{p})$:

$$\hat{\nu}_\varkappa(\boldsymbol{p})f_\varkappa^{\text{stoch}} = -ie^2\sum_{i,j}\frac{\partial}{\partial p_i}\int d^4\varkappa_1\,\frac{k_{1i}k_{1j}}{k_1^2}\,|E_{\varkappa_1}^{\text{stoch}}|^2\,\hat{g}_{\varkappa-\varkappa_1}(\boldsymbol{p})\,\frac{\partial f_{\varkappa_2}^{\text{stoch}}}{\partial p_j}. \qquad (4.41)$$

Remember that $\hat{g}$ is related to $\hat{\nu}$ by equation (4.37). The new perturbation theory can now be constructed by using as a first approximation

$$-i[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})+i\hat{\nu}_\varkappa(\boldsymbol{p})]f_\varkappa^{\text{stoch}} = -eE_\varkappa^{\text{stoch}}\left(\frac{\boldsymbol{k}}{k}\cdot\frac{\partial f^{\text{reg}}}{\partial \boldsymbol{p}}\right). \qquad (4.42)$$

From this we obtain

$$f_\varkappa^{\text{stoch}} = \frac{e}{i}\,E_\varkappa^{\text{stoch}}\hat{g}_\varkappa(\boldsymbol{p})\left(\frac{\boldsymbol{k}}{k}\cdot\frac{\partial f^{\text{reg}}}{\partial \boldsymbol{p}}\right). \qquad (4.43)$$

This result differs from equation (4.13) in that $[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]^{-1}$ is replaced by a propagator which involves the turbulent collisions. Equation (4.43) leads to a quasi-linear diffusion coefficient:

$$\tilde{D}_{ij} = ie^2\int d^4\varkappa_1\,|E_{\varkappa_1}^{\text{stoch}}|^2\,\frac{k_{1i}k_{1j}}{k_1^2}\,\hat{g}_\varkappa(\boldsymbol{p}) \qquad (4.44)$$

and a dielectric constant

$$\tilde{\varepsilon}_\varkappa = 1+\frac{4\pi e^2}{k^2}\int\frac{d^3p}{(2\pi)^3}\,\hat{g}_\varkappa(\boldsymbol{p})\left(\boldsymbol{k}\cdot\frac{\partial f^{\text{reg}}}{\partial \boldsymbol{p}}\right). \qquad (4.45)$$

By this means we have included the effect of the turbulent collisions in the lowest order of perturbation theory. One can also work out the next approximation, that is, the equation corresponding to equation (4.22) (including equation (4.26) and (4.28)). The result is:

$$\tilde{\varepsilon}_\varkappa\,|E_\varkappa^{\text{stoch}}|^2 = |E_\varkappa^{\text{stoch}}|^2\int\tilde{\Sigma}_{\varkappa,\varkappa_1}\,|E_{\varkappa_1}|^2\,d^4\varkappa_1$$
$$+\frac{2(4\pi)^2}{k^2\tilde{\varepsilon}_{-\varkappa}}\int|\tilde{g}^{(2)}_{\varkappa,\varkappa_1,\varkappa_2}|^2\,|E_{\varkappa_1}|^2\,|E_{\varkappa_2}|^2\,\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2. \qquad (4.46)$$

The quantity $\tilde{\varrho}^{(2)}$ is derived from $\bar{\varrho}^{(2)}_{\sim}$ (see equation (4.28)) by replacing $[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]^{-1}$ by $\hat{g}_{\varkappa}(\boldsymbol{p})$. The quantity $\Sigma_{\varkappa,\varkappa_1}$ is derived from $\Sigma_{\varkappa,\varkappa_1}$ of equation (4.32) by replacing $[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]^{-1}$ by $\hat{g}_k(\boldsymbol{p})$, except that the term corresponding to $(4\pi i/k)\,\varrho^{(3)}_{\varkappa,\varkappa_1,-\varkappa_1,\varkappa}$ must be omitted. It may be shown that terms of this type are now included in the first approximation.

We must now consider in more detail the equation for $\hat{g}$. It is obvious that away from resonance the solution of equation (4.37) is the same as it was for the incorrect expansion procedure, that is, $[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]^{-1}$. So we need only consider the solution near resonance. Introducing $\eta=\omega-(\boldsymbol{k}\cdot\boldsymbol{v})$ and assuming that

$$\eta \ll \mathrm{Max}\,\{\omega(k),\,kv_T\},\tag{4.47}$$

one can consider $\hat{g}$ as a function of η, $\boldsymbol{k}$, and $\boldsymbol{p}$. The equation

$$[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})+i\hat{\nu}_{\varkappa}(\boldsymbol{p})]\,\hat{g}_{\varkappa}(\boldsymbol{p})=1,\tag{4.48}$$

which follows from equation (4.37) is an operator equation and we find that the most important terms are the derivatives $\partial/\partial\eta$ that arise in $\hat{\nu}_{\varkappa}$ when the new variables are introduced. In this case $\hat{g}$ is diagonal and can be replaced by its eigenvalue, so that we obtain

$$\left(\eta+\frac{\partial}{\partial\eta}\,D_\eta\,\frac{\partial}{\partial\eta}\right)g^{(0)}_{\eta,k}(\boldsymbol{p})=1,\tag{4.49}$$

where

$$D_\eta=\frac{e^2}{m^2}\int d^4\varkappa'\,\frac{(\boldsymbol{k}\cdot\boldsymbol{k}')^2}{k'^2}\,|E_{\varkappa'}|^2\,g^{(0)}_{\eta-\eta',\,k-k'}(\boldsymbol{p}).\tag{4.50}$$

Because of condition (4.47) η can be neglected in equation (4.50) so that

$$D_\eta\approx -iD_0,\tag{4.51}$$

$$D_0=\frac{ie^2}{m^2}\int\frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)^2}{k_1^2}\,|E_{\varkappa_1}|^2\,g^{(0)}_{-\eta_1,\,k-k_1}\,d^3k_1\,d\omega_1\,.\tag{4.52}$$

It can be ascertained that the imaginary part of D_0 is small compared with the real part; we shall, therefore, assume D_0 to be real.

If we now perform a Fourier transformation

$$g^{(0)}_{\eta,k}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}g^{(0)}_{\tau,k}\,e^{i\eta\tau}\,d\tau,\tag{4.53}$$

we obtain

$$-\frac{\partial g^{(0)}_{\tau,k}}{\partial\tau}+\tau^2D_0g^{(0)}_{\tau,k}=-2\pi i\,\delta(\tau).\tag{4.54}$$

The solution of equation (4.54) is

$$g^{(0)}_{\tau,k}(\boldsymbol{p})=g_0\exp\left(-\tfrac{1}{3}D_0\tau^3\right),\qquad \tau\neq 0,\tag{4.55}$$

while $g_{0,k}$ is determined by the condition

$$g^{(0)}_{\tau,k}\Big|_{\tau=0+}-g^{(0)}_{\tau,k}\Big|_{\tau=0-}=-2\pi i.\tag{4.56}$$

There are two possibilities:

$$\left.\begin{array}{lllll}(1) & D_0>0, & g^{(0)}_{0-,k}=0, & g_0=-2\pi i; \\ (2) & D_0<0, & g^{(0)}_{0+,k}=0, & g_0=2\pi i.\end{array}\right\}\tag{4.57}$$

In the first case

$$g^{(0)}_{\eta,k}(\boldsymbol{p})=-i\int_0^{\infty}e^{i\eta\tau-\frac{1}{3}D_0\tau^3}\,d\tau.\tag{4.58}$$

In the second case

$$g^{(0)}_{\eta,k}(\boldsymbol{p})=i\int_0^{\infty}e^{i\eta\tau+\frac{1}{3}D_0\tau^3}\,d\tau.\tag{4.59}$$

Introducing expression (4.58) into equation (4.52), one finds

$$D_0(\boldsymbol{k},\boldsymbol{p})=e^2\int d^4\varkappa_1\,\frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)^2}{k_1^2}\int_0^{\infty}d\tau\,\exp\Big[-i\{\omega_1-(\boldsymbol{k}_1\cdot\boldsymbol{v})\}\,\tau-\frac{1}{3}\,D_0(\boldsymbol{k}-\boldsymbol{k}_1,\boldsymbol{p})\tau^3\Big].\tag{4.60}$$

Since the integration in equation (4.60) is over a wide range of $\boldsymbol{k}$ and ω one can approximate D_0 by

$$D_0\approx\frac{\pi e^2}{m^2}\int d^4\varkappa_1\,\frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)^2}{k_1^2}\,|E_{\varkappa_1}|^2\,\delta(\omega-(\boldsymbol{k}\cdot\boldsymbol{v})).\tag{4.61}$$

This confirms that D_0 is real and positive. The second solution (4.57) leads to the same results, if one uses the substitution $t\to -t$, as long as one takes the positive time direction to be always to correspond to increasing entropy.

We can now also estimate the effective collision frequency. From equations (4.58) and (4.61) it is easily seen that

$$\nu_{\text{eff}} \sim \eta \sim D_0^{1/3} \sim \left(\frac{W\omega_{\text{pe}}^2 k^2 v_{Te}^2}{nT_e kv}\right)^{1/3}$$
$$\approx \omega_{\text{pe}}\left(\frac{kv_{Te}}{\omega_{\text{pe}}}\right)^{1/3}\left(\frac{W}{nT_e}\right)^{1/3}. \tag{4.62}$$

This estimate is valid for electron–plasmon interactions. A similar estimate can be made for ion–plasmon interactions.

It must be emphasised that ν_{eff} cannot be expanded in the turbulent energy and that this comes about because of the importance of the turbulent collisions near resonance. Notice also that ν_{eff} does decrease very rapidly with wave-number and that ν_{eff} is associated with the stochastic part of the distribution function.

Let us now return to the problem of stochastic heating and the Landau damping of plasmons. The non-linear $\tilde{\varepsilon}_\varkappa$ now describes such a damping, which in the zeroth approximation is identical with the linear Landau damping. If $\omega \neq (\boldsymbol{k}\cdot\boldsymbol{v})$, we can expand the equation for $\hat{g}$, taking $\hat{\nu}$ as a perturbation. But $\hat{g}$ appears in $\hat{\nu}$ and we now know how to treat it when resonance is possible. This is, in fact, possible for the beat frequencies and we can obtain an approximation by replacing $\hat{g}_{\varkappa-\varkappa_1}(\boldsymbol{p})$ in the integrals by $\delta(\omega-\omega_1-([\boldsymbol{k}-\boldsymbol{k}_1]\cdot\boldsymbol{v}))$. We thus find the result obtained previously and can understand the physical meaning of the stochastic heating.

We have not yet succeeded in avoiding all the singularities. Indeed, $\tilde{\varepsilon}_{-\varkappa}$ in equation (4.46) has the same resonance as before, since it follows from equation (4.62) that the turbulent collisions produce only a small broadening and that the real part of $\tilde{\varepsilon}_{-\varkappa}$ is almost the same as it was previously.

4.3. Broadening of the Wave–Wave Interactions and Correlation Functions in a Turbulent Plasma

We now suppose that the three-wave decay process is possible and that the singularities in equations (4.46) and (4.28) exist. The wave-particle collisions, even if they introduce an imaginary part into the non-linear ε, do not remove the singularity. We must now renormalise the plasmon propagator $(k^2-\omega^2\varepsilon^\sigma)^{-1}$.

This problem was considered by Makhanov and Tsytovich (1970). Here we give only the main ideas. It is obvious that one cannot use equation (4.27) to express the average of the three fields by an average of four fields.

Two steps are essential. The first is the introduction of a non-linear (renormalised) dielectric constant $\varepsilon_\varkappa^{\text{n.l.}} = -\int d^4\varkappa_1 \tilde{\Sigma}_{\varkappa\varkappa_1}|E_{\varkappa_1}|^2$. The second is to split the stochastic field into two parts: the high-frequency part E^{H}, with frequencies of the order of the mean plasma frequency, and the low-frequency part E^{L} with frequencies of the order of the frequency of the plasmon–plasmon interactions (that is, the frequency corresponding to the correlation time of the fluctuations). We rewrite equation (4.20) as follows:

$$(\tilde{\varepsilon}_\varkappa + \tilde{\varepsilon}_\varkappa^{\text{n.l.}})E_\varkappa^{\text{stoch}} = \frac{4\pi}{ik}\int \tilde{\varrho}^{(2)}_{\varkappa,\varkappa_1,\varkappa_2}\left(E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}} - \left\langle E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}\right\rangle\right)$$
$$\times\,\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2 + \frac{4\pi}{ik}\int \tilde{\varrho}^{(3)}_{\varkappa,\varkappa_1,\varkappa_2,\varkappa_3}\Big\{E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}E_{\varkappa_3}^{\text{stoch}}$$
$$-E_{\varkappa_1}^{\text{stoch}}\left\langle E_{\varkappa_2}^{\text{stoch}}E_{\varkappa_3}^{\text{stoch}}\right\rangle - \left\langle E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}E_{\varkappa_3}^{\text{stoch}}\right\rangle\Big\}$$
$$\times\,\delta(\varkappa-\varkappa_1-\varkappa_2-\varkappa_3)\,d^4\varkappa_1\,d^4\varkappa_2\,d^4\varkappa_3 - \int \tilde{\Sigma}_{\varkappa,\varkappa_2}E_{\varkappa_1}^{\text{stoch}}\left\langle E_{\varkappa_2}^{\text{stoch}}E_{\varkappa_3}^{\text{stoch}}\right\rangle$$
$$\times\,\delta(\varkappa-\varkappa_1-\varkappa_2-\varkappa_3)\,d^4\varkappa_1\,d^4\varkappa_2\,d^4\varkappa_3. \tag{4.63}$$

The $\tilde{\varrho}$ include the turbulent collisions considered in the previous section (although this is not crucial for the considerations given below). Thus by averaging equation (4.63) multiplied by $E_{\varkappa'}$ we get a result similar to that of equation (4.22) with $\tilde{\varepsilon}_\varkappa + \tilde{\varepsilon}_\varkappa^{\text{n.l.}}$ on the left-hand side instead of $\tilde{\varepsilon}_\varkappa$ and the term $\varepsilon_\varkappa^{\text{n.l.}}|E_\varkappa|^2$ on the right-hand side. By definition, the last term must vanish to take into account all the terms diagonal in $|E_\varkappa|^2$ on the left-hand side. The average of the three turbulent fields can be written in the form

$$\left\langle E_{\varkappa'}^{\text{stoch}}E_{\varkappa_1}^{\text{stoch}}E_{\varkappa_2}^{\text{stoch}}\right\rangle = \left\langle E_{\varkappa'}^{\text{H}}E_{\varkappa_1}^{\text{H}}E_{\varkappa_2}^{\text{H}}\right\rangle + \left\langle E_{\varkappa'}^{\text{H}}E_{\varkappa_1}^{\text{L}}E_{\varkappa_2}^{\text{H}}\right\rangle + \left\langle E_{\varkappa'}^{\text{H}}E_{\varkappa_1}^{\text{H}}E_{\varkappa_2}^{\text{L}}\right\rangle;$$
$$E_\varkappa^{\text{stoch}} = E_\varkappa^{\text{H}} + E_\varkappa^{\text{L}}. \tag{4.64}$$

The $E_{\varkappa'}$-field in equation (4.22) is always a high-frequency field. Thus one needs to express the low-frequency field in terms of the high-frequency field E^{H}. This can be done if one takes into account that a low frequency can come from the difference of two high frequencies and thus

$$E_{\varkappa}^{\mathrm{L}} = \frac{1}{\tilde{\varepsilon}_{\varkappa}+\tilde{\varepsilon}_{\varkappa}^{\mathrm{n.l.}}} \frac{4\pi}{ik} \int \tilde{\varrho}^{(2)}_{\varkappa,\varkappa_1,\varkappa_2}\left(E^{\mathrm{H}}_{\varkappa_1}E^{\mathrm{H}}_{\varkappa_2}-\langle E^{\mathrm{H}}_{\varkappa_1}E^{\mathrm{H}}_{\varkappa_2}\rangle\right)\delta(\varkappa-\varkappa_1-\varkappa_2)\, d^4\varkappa_1\, d^4\varkappa_2 . \tag{4.65}$$

We can also of course write down the next terms of equation (4.62), but this will not help in the calculation of expression (4.64). Thus, by using equation (4.65) in the last two terms of equation (4.64) and also in equation (4.25) one finds from them one of the terms diagonal in $|E_{\varkappa}|^2$; the other one comes from $\tilde{\varrho}^{(3)}$. The requirement that these diagonal terms do not occur on the right-hand side of the equation which we derived leads to the relation

$$\Sigma_{\varkappa,\varkappa_1} = \frac{4\pi i}{k}\tilde{\varrho}^{(3)}_{\varkappa,\varkappa_1,\varkappa,-\varkappa_1} + \frac{4\pi(4\pi)^2}{k\,|\boldsymbol{k}-\boldsymbol{k}_1|}\frac{\tilde{\varrho}^{(2)}_{\varkappa,\varkappa_1,\varkappa-\varkappa_1}\tilde{\varrho}^{(2)}_{\varkappa-\varkappa_1,\varkappa,-\varkappa_1}}{\tilde{\varepsilon}_{\varkappa-\varkappa_1}+\tilde{\varepsilon}^{\mathrm{n.l.}}_{\varkappa-\varkappa_1}} . \tag{4.66}$$

This equation together with the definition of $\tilde{\varepsilon}^{\mathrm{n.l.}}_{\varkappa}$ gives a non-linear integral equation for $\varepsilon^{\mathrm{n.l.}}$. Because the denominator $\tilde{\varepsilon}_{\varkappa-\varkappa_1}+\tilde{\varepsilon}^{\mathrm{n.l.}}_{\varkappa-\varkappa_1}$ occurs in an integral in the induced decay process, one can sometimes neglect $\tilde{\varepsilon}^{\mathrm{n.l.}}$ and put approximately

$$\mathrm{Im}\,\frac{1}{\tilde{\varepsilon}_{\varkappa}+\tilde{\varepsilon}^{\mathrm{n.l.}}_{\varkappa}} \approx -\pi\,\delta(\varepsilon_{\varkappa}). \tag{4.67}$$

To calculate $\langle E^{\mathrm{H}}_{\varkappa'}E^{\mathrm{H}}_{\varkappa_1}E^{\mathrm{H}}_{\varkappa_2}\rangle$ one needs (i) to write down the equation (4.63) for $E^{\mathrm{H}}_{\varkappa'}$, taking on the right-hand side only the terms linear in $E^{\mathrm{L}}_{\varkappa}$, (ii) multiply the result by $E^{\mathrm{H}}_{\varkappa_1}E^{\mathrm{H}}_{\varkappa_2}$; and (iii) average over a statistical ensemble. Then, by using equation (4.65) and approximating the five-field average terms by a product of the averages of three and two fields, one finds a non-linear equation:

$$\begin{aligned}(\tilde{\varepsilon}_{\varkappa'}+\tilde{\varepsilon}^{\mathrm{n.l.}}_{\varkappa'})\langle E^{\mathrm{H}}_{\varkappa'}E^{\mathrm{H}}_{\varkappa_1}E^{\mathrm{H}}_{\varkappa_2}\rangle &= \frac{8\pi i}{k'}\tilde{\varrho}^{(2)}_{\varkappa',-\varkappa_1,-\varkappa_2}|E_{\varkappa_1}|^2|E_{\varkappa_2}|^2\,\delta(\varkappa'+\varkappa_1+\varkappa_2)\\ &+\frac{2\pi i}{k'}\left\{|E_{\varkappa_1}|^2\int\tilde{\Sigma}_{\varkappa',-\varkappa,\varkappa_2',\varkappa_3'}\langle E^{\mathrm{H}}_{\varkappa_2}E^{\mathrm{H}}_{\varkappa_2'}E^{\mathrm{H}}_{\varkappa_3'}\rangle\,\delta(\varkappa'+\varkappa_1-\varkappa_2'-\varkappa_3')\right.\\ &\left.\times d^4\varkappa_2'\,d^4\varkappa_3'+1\rightleftarrows 2\right\},\end{aligned} \tag{4.68}$$

where

$$\tilde{\Sigma}_{\varkappa,\varkappa_1,\varkappa_2,\varkappa_3} = \tilde{\varrho}^{(3)}_{\varkappa,\varkappa_1,\varkappa_2,\varkappa_3} - \frac{8\pi i\tilde{\varrho}^{(2)}_{\varkappa_2+\varkappa_3,\varkappa_2,\varkappa_3}\tilde{\varrho}^{(2)}_{\varkappa,\varkappa_1,\varkappa_2+\varkappa_3}}{|\boldsymbol{k}_2+\boldsymbol{k}_3|\,(\tilde{\varepsilon}_{\varkappa_2+\varkappa_3}+\tilde{\varepsilon}^{\mathrm{n.l.}}_{\varkappa_2+\varkappa_3})} . \tag{4.69}$$

One can solve this equation by a perturbation method using only the first term of the right-hand side of equation (4.68) in first approximation and one finds that the other terms give small corrections of the order of $W/nT \ll 1$. Thus the equation for the correlation function takes the form

$$(\tilde{\varepsilon}_{\varkappa}+\tilde{\varepsilon}^{\mathrm{n.l.}}_{\varkappa})\,|E_{\varkappa}|^2 = 2\frac{(4\pi)^2}{k^2}\int\frac{|\tilde{\varrho}^{(2)}_{\varkappa,\varkappa_1,\varkappa_2}|^2}{\tilde{\varepsilon}_{-\varkappa}+\tilde{\varepsilon}^{\mathrm{n.l.}}_{-\varkappa}}|E_{\varkappa_1}|^2|E_{\varkappa_2}|^2 \times\delta(\varkappa-\varkappa_1-\varkappa_2)\,d^4\varkappa_1\,d^4\varkappa_2 . \tag{4.70}$$

This has (i) a soluble form as $|E_{\varkappa}|^2 > 0$ and there are no secular terms, (ii) gives the same expression for the balance equation, and (iii) gives also the same tail of the correlation function as without $\tilde{\varepsilon}^{\mathrm{n.l.}}$.

Thus we have seen that the turbulent collisions are essential in the treatment of the properties of a turbulent plasma and the appropriate treatment of them can be found by summing an infinite series of some of the terms which are essential near resonances. The turbulent broadening can be neglected only for such average values as the diffusion coefficients or the expression $\int |E_{\varkappa}|^2\,d\omega$. In the next chapter, we shall show through the example of ion-sound turbulence how essential this turbulent broadening is, which is due both to wave-particle interactions and to wave–wave interactions.

5. The Spectrum and Correlation Functions of Ion-sound Turbulence

5.1. Introduction

Ion-sound turbulence can be excited by different processes. For example, it is well known that such turbulence can be excited in shock waves propagating perpendicular to an external magnetic field or arising from an external electric field along the magnetic field lines. In both cases, excitation arises if the mean drift velocity u of the electrons is larger than the ion-sound velocity v_s and $T_e \gg T_i$. A detailed numerical calculation of the growth-rate of this instability was given by Stringer (1964). The essential behaviour of this kind of instability is that, if $u > v_s$, the growth-rate exceeds the Landau damping for each frequency. Dissipation possibly exists only for the waves that propagate outside the Cherenkov cone or, in the case when $u \gg v_s$, in the direction opposite to the applied electric field. This means that an exactly stationary turbulence can be created, if there is energy transfer in the angles of the turbulent waves. Another example is the excitation by non-linear creation of waves that can be due to the decay of waves with higher frequencies (for instance, Langmuir waves). In this case, the excitation can exist in a narrow interval of frequencies so that the lowering of the frequencies of the ion-sound wave can lead the system to a stationary turbulent state.

An essential question is what kind of non-linear interactions can be involved in such an energy transfer. Usually, ion-sound turbulence is accompanied by an anomalous resistivity of the plasma, due to the scattering of the electrons by the turbulent ion-sound waves. This is usually observed and measured experimentally. The value of the anomalous resistivity depends, of course, on the nature of the non-linear energy transfer.

In principle, one can suppose that there can exist the three following non-linear processes:

$$\mathrm{e}+\mathrm{s} \rightleftharpoons \mathrm{e}'+\mathrm{s}', \tag{5.1}$$

$$\mathrm{i}+\mathrm{s} \rightleftharpoons \mathrm{i}'+\mathrm{s}', \tag{5.2}$$

$$\mathrm{s} \rightleftharpoons \mathrm{s}'+\mathrm{s}''. \tag{5.3}$$

The first two processes are induced scattering processes and the last one is a decay process. The process (5.2) was first considered by Kadomtsev (1964) in a partially ionised plasma. The condition that the ionisation of the plasma is very small seems to be a very essential point in the Kadomtsev theory, that is, the binary collisions must play a significant role. Indeed, as a result of the presence of a high number of collisions, a region where damping exists is created at not very low frequencies. Energy transfer to this region due to a lowering of the frequencies is possible. Still, in the Kadomtsev treatment as given in his book (1964) it is supposed that the waves mostly go into a direction near to that of the applied field because, for example, the plasma is confined in a long tube, and the waves that move at larger angles strike the wall of the tube and are, therefore, damped. The turbulent energy is not very high because of the binary collisions. These two restrictions (the collisions and a long tube) are very essential. Nevertheless, some authors tried later to apply the same equations to a fully ionised plasma, or to the case where the change in the angle of the wave can be rather large. Indeed, from the probabilities derived earlier we find the following non-linear growth-rate "due to the scattering of ions":

$$\gamma_\varkappa^{\mathrm{n.l.}} = -\frac{\mathrm{Im}\,\varepsilon_\varkappa^{\mathrm{n.l.}}}{\left.\dfrac{\partial \varepsilon_\varkappa}{\partial \omega}\right|_{\omega=\omega_k}}$$

$$= k^2 \frac{\partial}{\partial k}\,\frac{T_i}{T_e} \int dx_1\, \frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)^2([\boldsymbol{k}\wedge\boldsymbol{k}_1]\cdot[\boldsymbol{k}\wedge\boldsymbol{k}_1])}{k^4 k_1^4 n T_e}\, W_{k_1} 2\pi k^3 v_s\,,$$

$$x_1 = \frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)}{k k_1}, \quad v_s = \sqrt{(T_e/m_i)} \tag{5.4}$$

One can see from equation (5.4) that if there exist waves propagating in a direction close to that of the applied electric field, waves with frequencies sufficiently lower than the original ones are growing in any direction including the direction opposite to that of the field. Thus, the distribution of turbulent waves must be more or less isotropic. This seems to contradict the known experimental results (Jančarik and Hamberger, 1970; Daughney *et al.*, 1970). On the other hand, the interaction (5.4) seems to be rather weak. That means that to compensate the high Landau damping one needs a high level of turbulent energy. Indeed, if one takes into account that the linear decrement is of the order of

$\sqrt{(\frac{1}{2}\pi m_e/m_i)}$ and requires that it balances expression (5.4), one finds the following rough estimate for the turbulent energy:

$$\frac{W}{nT_e} \approx 4\sqrt{\left(\frac{m_e}{m_i}\right)}\frac{T_e}{T_i}\ln^2\left[\frac{\omega_{max}}{\omega_{min}}\right], \tag{5.5}$$

which shows that W seems to be too high (as we assumed that $W/nT_e \ll 1$). Estimating W from balancing the non-linear interaction with the linear growth-rate of the current driven turbulence—which is of the order of $\sqrt{(\frac{1}{2}\pi)}\cdot(m_e/m_i)(u/v_s)\,\omega$—and putting it in the quasi-linear equation makes it possible to estimate the turbulent collision frequency of the electrons and, therefore, the turbulent conductivity. This was done by Sagdeev (1967), who found that

$$\tau^{turb} = \frac{1}{\nu^{turb}} \approx \frac{u}{v_s}\frac{1}{\omega_{pi}}\alpha, \tag{5.6}$$

where α, when calculated from equation (5.4), is of the order of unity or even less. To obtain agreement with the observed thickness of shocks it is necessary to put in equation (5.6) an experimental value for α of the order of 10^2. This means that the non-linear interaction must be much more effective, if one believes that the observed anomalous resistivity is due to the development of the ion-sound instability.

The problem became much more sophisticated after Drummond's suggestion that the most important non-linear interactions may be due to the scattering of ion-sound waves by electrons, that is, to the process (5.1). Indeed the rough estimates by Sisonenko and Stepanov (1969) and by Krall and Book (1969) show that such an interaction produces a balance for the linear growth-rate, even if the level of the turbulent energy is very small,

$$\frac{W}{nT_e} \approx \frac{m_e}{m_i}\frac{u}{v_s}. \tag{5.7}$$

This also contradicts the observations that show the turbulent energy level to be high (Jančarik and Hamberger, 1970; Daughney *et al.*, 1970).

The interaction (5.3) is usually neglected because the ion-sound waves do not satisfy the conservation laws

$$\omega(\boldsymbol{k}) = \omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2), \tag{5.8}$$

$$\boldsymbol{k} = \boldsymbol{k}_1+\boldsymbol{k}_2. \tag{5.9}$$

If this process is allowed, it dominates the processes (5.1) and (5.2), because the scattering arises as a tail of the decay process.

To finish this introduction, it is necessary to mention the work of Rudakov and Korablev (1966) and of Kovrizhnyk (1967) who considered the quasi-linear development of ion-sound turbulence. The argument that the quasi-linear interaction may be a most important one can be seen as follows. The non-linear interaction (5.4) is rather weak. Therefore, one may assume that the quasi-linear change in the spectrum is unimportant. On the other hand, the quasi-linear development can also change the angular distribution or, in other words, bring the oscillations from the excitation region to the damping region. However, the best known quasi-linear effect is to form a kind of plateau. In this case, it means that the growth-rate comes near to the threshold, that is, $\gamma_k \approx 0$. This means that u is of the order of v_s. On the other hand, as shown by the above authors, the spectrum of the turbulent frequencies must be very narrow. The two statements ($u \sim v_s$ and narrow spectrum) seem to contradict the above-mentioned observations (broad spectrum and essential difference between u and v_s).

5.2. The Influence of the Turbulent Collisions

Now let us consider from the general point of view of the turbulent collisions the interactions with ion-sound waves, and illustrate the essential physical change of the picture of the turbulent state, if these are taken into account. First of all, the interaction of ion-sound with the electrons is taken into account in the quasi-linear interaction. We cannot expect that the next term in the expansion, that is, the non-linear interaction, can be expanded in terms of the turbulent energy, because the turbulent collisions were already taken into account, and the result cannot be expanded in terms of W. In other words, only the first rough approximation corresponds to the quasi-linear one as was shown above, and the exact expression cannot be expanded and, therefore, the corrections to this rough approximation can also not be expanded. This means that the broadening of the resonance must be supposed to depress the non-linear interaction, that is, the difference between the rough and the exact solution as well as higher-order effects. Indeed, in any consistent theory, in which there is a small parameter, the next order (perturbation effect) must be small, so that one can expect the effect of process (5.1) to be negligible. This was indeed shown by Rudakov and Tsytovich (1971) to be the case.

The next question is the possibility of the process (5.3). It is forbidden, but not strictly so. Indeed, if one neglects the curvature of the ion-sound wave branch and supposes that

$$\omega_s = kv_s, \tag{5.10}$$

then for three waves propagating in the same direction the fulfilment of the conservation law (5.9) means the fulfilment of the conservation law (5.8). However, if there is a curvature, this is no longer possible. What is the effect of the turbulent collisions associated with the wave–wave interactions? If the spreading-out of the dispersion curve due to these turbulent collisions is larger than the effect of the curvature, one must say that such a process is allowed. Because the probability of this process is large, one can suppose that the non-linear interactions become much stronger. This is the effect needed to obtain agreement between the theory and the observed thickness of the shock waves.

5.3. The Electron Ion-sound Non-linear Interactions

It is easy to see that the large cross-section for electron ion-sound interactions found by Krall and Book (1969) and by Sisonenko and Stepanov (1969) can exist for a very small level of the turbulent energy, much smaller than that for which this non-linear interaction can balance the linear growth-rate. Indeed, the perturbation theory expansion used in these papers can only be supposed to have any meaning if the particle oscillation velocity in the turbulent field $v_\sim$ is much smaller than the mean particle velocity. In the case of ion-sound, one can write

$$v_\sim \approx \frac{eE^{\text{stoch}}}{m[\omega_k-(\boldsymbol{k}\cdot\boldsymbol{v})]} \sim \frac{eE^{\text{stoch}}}{m\omega_k}, \tag{5.11}$$

because the Cherenkov condition means that $(\boldsymbol{k}\cdot\boldsymbol{v}) > \omega_k$, while the lowest $(\boldsymbol{k}\cdot\boldsymbol{v})$ is of the order of ω_k. Therefore,

$$\frac{v_\sim^2}{v_{Te}^2} \approx \frac{e^2(E^{\text{stoch}})^2}{m_e^2\omega_k^2 v_{Te}^2} \approx \frac{\omega_{pe}^2}{\omega_{pi}^2}\frac{W}{nT_e} \approx \frac{m_i}{m_e}\frac{W}{nT_e}. \tag{5.12}$$

Here ω_k is of the order of ω_{pi}. We find thus that even from these arguments one can see that the limit (5.7) which is necessary to balance the linear growth-rate cannot be reached. However, one can see that the non-linear interactions are lowered by the turbulent collisions even for much smaller W.

Indeed, the theory of turbulent collisions developed in the previous section is a theory that uses the small parameter

$$\lambda = \frac{e^2(E^{\text{stoch}})^2}{m_e^2(\nu^{\text{turb}})^2 v_{Te}^2} \approx \frac{\omega_{pe}^2 W}{nT_e(\nu^{\text{turb}})^2}. \tag{5.13}$$

If one uses the estimate for $k \sim \omega_{pe}/v_{Te}$ (the most important k in the ion-sound spectrum) one finds that

$$\lambda = \left(\frac{W}{nT_e}\right)^{1/3}. \tag{5.14}$$

Therefore, we have for weak turbulence that $\lambda \ll 1$. As we shall see now, the corrections to the first approximations, that is, to the quasi-linear result, are larger than the result (5.14). Thus, the second approximation need not be considered. Therefore, the non-linear interactions of ion-sound waves with electrons come from the difference between the exact expression of the first approximation, taking into account the turbulent collisions and the approximate one when Im g is replaced by $-\pi\delta(\omega_i-(\boldsymbol{k}\cdot\boldsymbol{v}))$. We thus find that

$$\frac{\gamma_e^{\text{n.l.}}}{\gamma_k} = \frac{\int\left(\boldsymbol{k}\cdot\frac{\partial f^{\text{reg}}}{\partial \boldsymbol{p}}\right)[\text{Im } g+\pi\delta(\omega_k-(\boldsymbol{k}\cdot\boldsymbol{v}))]\frac{d^3p}{(2\pi)^3}}{\int\left(\boldsymbol{k}\cdot\frac{\partial f^{\text{reg}}}{\partial \boldsymbol{p}}\right)\pi\,\delta(\omega_k-(\boldsymbol{k}\cdot\boldsymbol{v}))\frac{d^3p}{(2\pi)^3}}, \tag{5.15}$$

where γ_k is an approximate growth-rate. Using the expression for g one finds, for example, for a Maxwellian distribution

$$\frac{\gamma_e^{\text{n.l.}}}{\gamma_k} = \frac{1}{\pi}\int_0^\infty \exp(-y^2)\,dy^2 \int_0^\infty \frac{\sin \tau d\tau}{\tau}$$

$$\times\left\{\left[\cos\left(\frac{\omega\tau}{kv_Ty}\right)-\sin\left(\frac{\omega\tau}{kv_Ty}\right)\left(\frac{\alpha\tau^3}{y^4}\bigg/\frac{\omega\tau}{kv_Ty}\right)\right]\exp\left(-\frac{\alpha\tau^3}{3y^4}\right)-1\right\}, \tag{5.16}$$

where

$$\alpha = \frac{D_0 v}{k^3 v_{Te}^4}. \tag{5.17}$$

In τ there exist two essential cut-offs, namely $\tau = \tau_1 \approx ykv_{Te}/\omega$ and $\tau = \tau_2 \approx y^{4/3}/\alpha^{1/3}$. If $\tau_2 \gg \tau_1$, one can put the exponential equal to unity and one gets

$$\frac{\gamma_e^{\text{n.l.}}}{\gamma_k} \approx \frac{W}{nT_e}\frac{m_i}{m_e}. \tag{5.18}$$

This is approximately the result of the rough estimates of Krall and Book (1969) and of Sisonenko and Stepanov (1969).

If $\tau_2 \ll \tau_1$, one finds

$$\frac{\gamma_e^{\text{n.l.}}}{\gamma_\varkappa} \approx \left(\frac{W}{nT_e}\right)^{1/2}. \tag{5.19}$$

These corrections, indeed, are larger than expression (5.14). The result (5.19) shows that the non-linear corrections due to the electrons never can reach the linear effect, if the turbulence is weak, $W/nT_e \ll 1$. One can now estimate the value of W for which $\tau_1 = \tau_2$, and one finds that it corresponds to

$$\frac{W}{nT_e} \approx \left(\frac{m_e}{m_i}\right)^2. \tag{5.20}$$

This is much smaller than the estimate found from equation (5.12):

$$\frac{W}{nT_e} \ll \frac{m_e}{m_i}. \tag{5.21}$$

One can now illustrate the result by plotting $\gamma_e^{\text{n.l.}}/\gamma_k$ against W/nT_e (Fig.

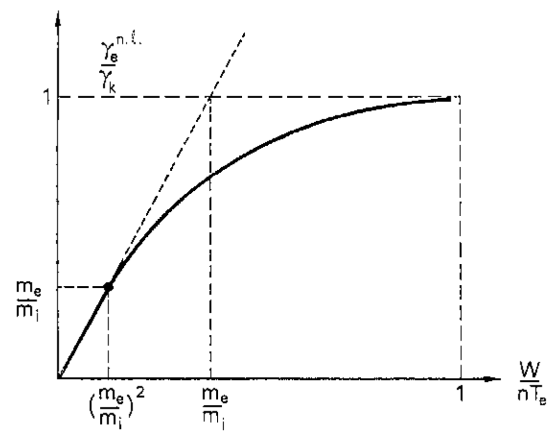

FIG. 16.

The ratio of the non-linear correction $\gamma_e^{\text{n.l.}}$ to the quasi-linear growth-rate γ_k as function of the ratio of the turbulent energy W to the thermal particle energy nT_e. The solid curve takes into account the turbulent collision broadening.

16). If one linearly extrapolates the results found for very low W, that is,

$$\frac{W}{nT_e} \ll \left(\frac{m_e}{m_i}\right)^2, \tag{5.22}$$

to the value when this result is not valid, one gets the wrong answer that the scattering by electrons is effective, that is, $\gamma_e^{\text{n.l.}}/\gamma_k \approx 1$, when

$$\frac{W}{nT_e} \approx \frac{m_e}{m_i}. \tag{5.23}$$

This illustrates the significant role of the turbulent collisions.

5.4. The Wave–Wave Interaction of Ion-sound Oscillations

This kind of interaction was supposed to play a significant role in the formation of ion-sound turbulence by the author (Tsytovich, 1971) in a paper written during his stay in Oxford and Culham; this was the result of discussions on the experimental work done in the Culham Laboratory by Jančarik and Hamberger (1970) and by Daughney *et al.* (1970). We describe here only the qualitative picture of such an interaction. One can see that, if the level of the turbulent energy is high enough, the process (5.3) is allowed, if the correlation width is of the order

$$\frac{\Delta\omega}{\omega} \gtrsim \left(\frac{\omega_{\max}}{\omega_{pi}}\right)^2. \tag{5.24}$$

This means that the necessary conditions can be satisfied if one goes to lower frequencies. If such an interaction is involved, the waves created do not change their angle during the non-linear energy transfer. The interaction (5.3) conserves the plasma energy and does not change the angle. The nature of this interaction is to spread the energy over a larger ω-interval without changing the total energy. Therefore, it creates a broad spectrum. But the condition (5.24) means that in frequency space there exists a wall, because for some frequency greater than the maximum frequency defined by equation (5.24) the process (5.3) is forbidden. Thus the broadening can only be the process of the lowering of the frequency.

One can see that the spectrum created in such a process is in some sense similar to the ion-sound ion interactions considered by Kadomtsev:

$$W_\omega \approx \frac{1}{\omega}\,\frac{8n_0T_e}{\pi}\,\gamma_\xi\Psi(\lambda), \tag{5.25}$$

where the quasi-linear growth-rate is written as $\omega\gamma_\xi$,

$$\xi = \frac{(\boldsymbol{k}\cdot\boldsymbol{E})}{kE}, \tag{5.26}$$

so that γ_ξ describes the non-dimensional growth-rate and $\Psi(\lambda)$ is a logarithmic-type function varying slowly with $\lambda = \omega_{max}/\omega$. The difference from the Kadomtsev interaction, apart from this logarithmic function, is a greater efficiency of the interaction (5.3) and, therefore, a lower value of W needed to balance the growth-rate. However, the correlation width as an effective turbulent collision described in the previous section increases quickly as the frequency in the spectrum decreases. This gives an increase in $\Delta\omega$ and, therefore, the possibility of interactions of waves with different directions of propagation. Indeed, this is allowed by the conservation laws for angles:

$$\Delta\theta \approx \frac{\Delta\omega}{\omega}. \tag{5.27}$$

One can then estimate the time needed to transfer the turbulent energy by a multiple change in the angle, each change having the value (5.27), to the final result of an angle change of the order of unity.

This time can be compared with the time needed to propagate the wave energy into lower frequencies and one finds that the highest frequency does not much differ from ω_{pi} and the lowest frequency into which the energy can go before the angle changes become of the order unity, is of the order of $3\sqrt{(m_e/m_i)}\omega_{pi}$. When the angle is changed in the current-driven instability, it means that the energy goes to the dissipation region and can be absorbed. Of course, in this process the ions are involved and the ions or electrons are heated, but mostly the electrons.The tail of the decay interaction is the ion-sound scattering by ions. It is possible that a multiple change in angle due to correlational broadening has a greater probability than the change due to one step of ion-sound scattering by ions.

5.5. The Anomalous Resistivity of the Plasma

Even if the whole spectrum is not stationary, the anomalous resistivity can obtain a stationary value, because it is determined by the highest frequency in the spectrum. Thus, to have a stationary, or more precisely

quasi-stationary, resistivity it is necessary only to form a spectrum near the maximum frequency. Then one can write down the quasi-linear equation that describes in first approximation only an elastic scattering of particles by waves or only the change of the angular dependence of the distribution function of the electrons. This follows from $\omega/kv_{Te} \ll 1$, which is found for ion-sound waves.

If x is $(\boldsymbol{p}\cdot\boldsymbol{E})/pE$ one finds that the quasi-linear equation is of the form

$$eEx\frac{\partial f_0^{reg}}{\partial p} = \frac{2\pi}{v^3}\frac{v_{Te}^2}{m_e v_s}\frac{\partial}{\partial x}\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} d\xi \int d\omega \frac{\omega W_{\omega,\xi}\xi^2 \dfrac{\partial f_1^{reg}}{\partial x}}{\sqrt{(1-x^2-\xi^2)}}, \tag{5.28}$$

where

$$W = 2\pi \int W_{\omega,\xi}\, d\xi\, d\omega. \tag{5.29}$$

We have split the electron distribution function f^{reg} into an isotropic part f_0^{reg} and an x-dependent (anisotropic) part f_1^{reg}:

$$f^{reg} = f_0^{reg} + f_1^{reg}, \quad f_1^{reg} \ll f_0^{reg}. \tag{5.30}$$

On the other hand, $W_{\omega,\xi}$ is proportional to γ_ξ, which depends also on the function f_1^{reg}:

$$\gamma_\xi = -\frac{2\pi}{n}\xi v_{Te}^2 \int_{-\sqrt{(1-\xi^2)}}^{\sqrt{(1-\xi^2)}} \frac{dx_1}{\sqrt{(1-x_1^2-\xi^2)}}\frac{\partial}{\partial x_1}\int_0^\infty f_1^{reg}(x_1, p)\frac{d^3p}{(2\pi)^3}. \tag{5.31}$$

The self-consistent solution can be written in the form

$$f_1^{reg} = g(x)\frac{v^3}{v_{Te}^3}\frac{\partial f_0^{reg}}{\partial p} v_s, \tag{5.32}$$

where

$$\frac{\partial g(x)}{\partial x} = \frac{\sqrt{\alpha}}{s}\varrho(s), \quad s = \sqrt{(1-x^2)}, \quad \alpha = \frac{\pi e E}{6 m_e v_s \omega_{max}\displaystyle\int_0^1 \Psi(\lambda)d\lambda}, \tag{5.33}$$

while $\varrho(s)$ satisfies the equation

$$\frac{s^3}{\varrho(s)} = \int_0^1 \frac{\varrho(\eta)\, d\eta}{\sqrt{(1-\eta^2)}}\left\{\frac{s^2+\eta^2}{4}\ln\frac{(s+\eta)^2}{(s-\eta)^2} - s\eta\right\}. \tag{5.34}$$

The numerical solution of this equation is shown in Fig. 17.

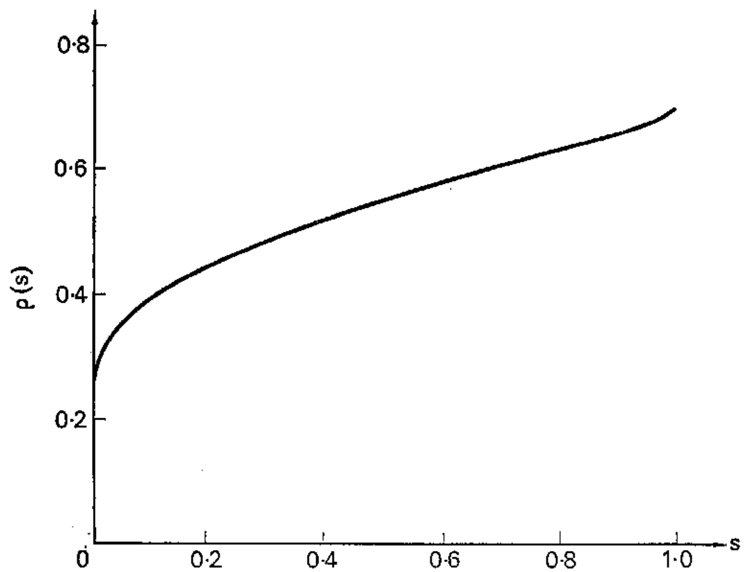

FIG. 17.

The result of the numerical solution of the non-linear equation for $\varrho(s)$.

From $\varrho(s)$ one can easily find the angular distribution of the turbulent spectrum to be proportional to $\lambda(\xi)$, where

$$\lambda(\xi) = \xi \int_{\xi}^{1} \frac{\varrho(s)\, s^2\, ds}{\sqrt{(s^2-\xi^2)}}. \tag{5.35}$$

The result of a numerical computation of $\lambda(\xi)$ is shown in Fig. 18. To find the conductivity it is necessary to find f^{reg} from equation (5.32). One sees that then

$$\sigma \approx \frac{1}{\sqrt{E}}. \tag{5.36}$$

By using the drift velocity, defined by

$$u = \frac{j}{en}, \tag{5.37}$$

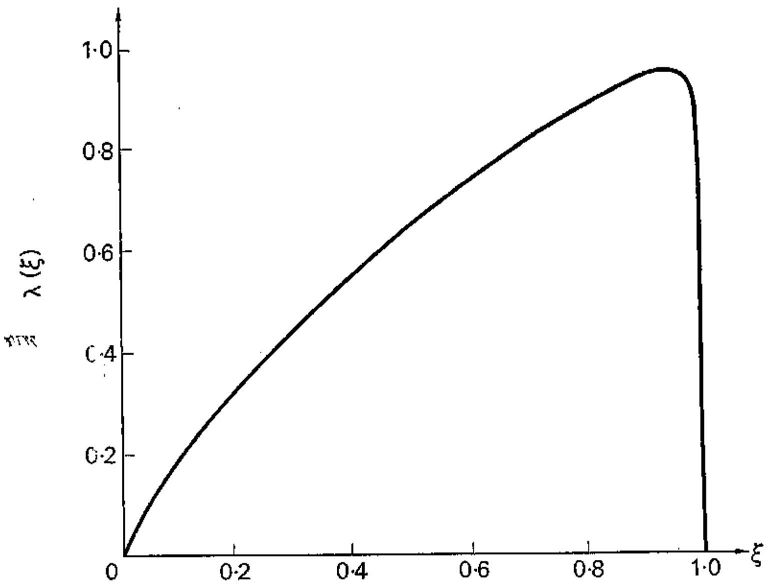

FIG. 18.

The angular dependence of the ion-sound spectrum.

equation (5.36) can be written in the form

$$\sigma = \frac{ne^2\tau}{m}, \tag{5.38}$$

where

$$\tau \approx \frac{100 v_s}{u\omega_{\max}}. \tag{5.39}$$

This value seems to be in agreement with the known data on the thickness of shocks and the measurement of the anomalous conductivity in an electric field. We must bear in mind that $\omega_{\max}$ is determined by the correlation time or by $\Delta\omega$ (see equation (5.24)) and that thus the conductivity depends on $\Delta\omega$ as follows from equation (5.39).†

† The spectrum calculated here extends only to about $\omega_{pi}/2$. To find the conductivity, the spectrum mus tbe extended to ω_{pi} (after that it drops off rapidly). There is no experimental evidence that the spectrum beyond $\omega_{pi}/2$ produces the major part of the conductivity, and it is therefore unlikely that the spectrum will have a large peak between $\omega_{pi}/2$ and ω_{pi}.

6. The Spectrum and Correlation Functions of Langmuir Turbulence

THE Langmuir plasma oscillations are the most typical collective plasma motion. The frequency of these oscillations in a sufficiently weak magnetic field, when $\omega_{He} \ll \omega_{pe}$, is approximately constant, with a weak wavelength variation which is due to the thermal particle motion:

$$\omega_k = \omega_{pe} + \frac{3}{2}\,\frac{k^2 v_{Te}^2}{\omega_{pe}}. \tag{6.1}$$

There exist different types of mechanisms to excite such a motion. One of the best known is excitation by electron beams. This excitation is due to a population inversion of the particles if the beam is present, and the possibility of Cherenkov resonance with the Langmuir waves. The beam does not greatly change the dispersion law (6.1) for the waves provided the density of the beam, n_1, is sufficiently small, and the spread in the particle velocity Δv sufficiently high:

$$\frac{n_1}{n_0} \ll \left(\frac{\Delta v}{v}\right)^3. \tag{6.2}$$

Thus, the presence of such a beam changes only the imaginary part of ω_k, that is, such waves are excited. It is not always necessary to have a well directed beam. Any anisotropy of fast particles with $v \gg v_{Te}$ will produce instability of the Langmuir waves. Analogously to the case of ion-sound waves it is possible to have also an excitation of Langmuir waves by the non-linear decay of waves with higher frequencies. Such waves are the normal transverse waves. Thus one can say that a high frequency field and laser beams can excite these turbulent motions.

Indeed, these two mechanisms of excitation of Langmuir waves by the beam and by transverse waves are similar, as one can see from the diagrams of Fig. 19, which describe the Cherenkov and decay resonance. The mathematical description of these two processes of turbulent excitation is also similar. The essential difference between ion-sound instability and Langmuir instability is that the growth-rate of the latter

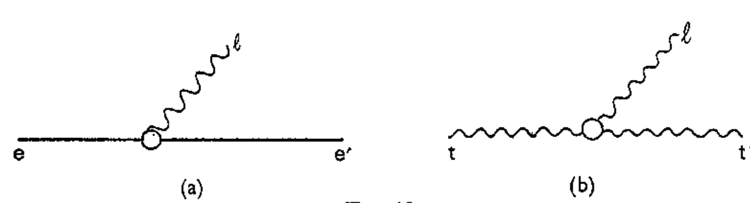

FIG. 19.

Emission of Langmuir waves by particles (a) and by transverse waves (b).

is usually restricted to a particular k-interval, for example, near the phase velocity of the beam. It is obvious that these velocities are less than the velocity of light. In the case of the non-linear excitation of Langmuir waves by a "beam" of transverse waves one finds that the excitation is forbidden for waves with phase velocities larger than the velocity of light. This results from the fact that the group velocity of the transverse waves is less than the light velocity. The higher the beam-particle velocity, the more important the non-linear interactions that can stabilise or lower the growth-rate of the beam–plasma interaction. The excitation of Langmuir waves with relatively small phase velocities is thus the most efficient process. For the non-linear excitation the most efficient process is the excitation of Langmuir waves with a wavelength of the order of magnitude of the wavelength of the transverse wave which produces the excitation, so that, when $\omega \gg \omega_{pe}(c/v_{Te})$, the excitation by non-linear processes becomes inefficient. The stabilisation of this kind of excitation by transverse waves due to the non-linear Langmuir wave interactions is more efficient than for the beam–plasma interaction, because the growth-rate of the non-linear excitation is not very large.

There are the following three possible non-linear processes that can create the turbulence spectrum:

$$e + l \rightleftharpoons e' + l', \tag{6.3}$$
$$i + l \rightleftharpoons i' + l', \tag{6.4}$$
$$l + l_1 \rightleftharpoons l' + l_1'. \tag{6.5}$$

Here l denotes a Langmuir wave and e and i the electrons and ions of the cold plasma, respectively. One can include also the process of the scattering of Langmuir waves by the beam electrons in the case of a beam–plasma instability. However, this comes as a non-linear correction to the quasi-linear growth-rate, which itself is already small. The reason why the process (6.3), that is, the scattering by cold electrons, must be considered is that the cold electrons are not in resonance with the Langmuir waves which are excited.

6.1. The Spectrum of Small-scale Langmuir Turbulence

Estimates show that for high k-numbers (or small dimensions) the effectiveness of the induced scattering by electrons decreases, when k decreases, as k^3, and is most effective for the largest possible k. This value of k cannot be larger than $1/\lambda_D$, where λ_D is the Debye length. The scattering by ions decreases, when k increases, as k^{-2}, and, therefore, one can estimate where the electron scattering is the dominant process and finds

$$\frac{1}{\lambda_D} > k > k_{**} = \frac{1}{\lambda_D}\left(\frac{m_e}{3m_i}\right)^{1/5}. \tag{6.6}$$

The plasmon–plasmon collisions (6.5) are negligible in the range (6.6). Only in a plasma with sufficiently heavy ions is this interval sufficiently broad to be considered as an important region. Also it is necessary that the excitation of turbulence occurs very near $k \sim 1/\lambda_D$ in order to have a region where one can neglect the excitation and to be justified to assume that the spectrum is due only to the balance by non-linear interactions with the electrons. Nevertheless, one can work out the possible turbulent spectrum in the range (6.6) in order to have a complete picture of the Langmuir turbulent spectrum in a plasma. If one supposed that, for example, the distribution of Langmuir waves is isotropic, it is easy to find from the probability for the process (6.3) the non-linear balance equation

$$\frac{\partial W_k}{\partial t} = \beta W_k\left\{\int_k^\infty W_{k_1}\,dk_1\frac{k^2}{k_1^3}(k_1^2-k^2)\left(\frac{1}{3}k_1^2+\frac{4}{7}k^2\right) - \int_0^k W_{k_1}\,dk_1\frac{k_1^2}{k^3}(k^2-k_1^2)\left(\frac{1}{3}k^2+\frac{4}{7}k_1^2\right)\right\},$$

$$\beta = \frac{6v_{Te}\sqrt{(2\pi)}}{5n_0m_e\omega_{pe}^2}. \tag{6.7}$$

After some transformations and differentiations equation (6.7) becomes

$$2\frac{d^3W_k}{dk^3}+\frac{21}{k}\frac{d^2W_k}{dk^2}+\frac{46}{k^2}\frac{dW_k}{dk}+\frac{10W_k}{k^3} = 0. \tag{6.8}$$

There are three solutions of equation (6.8):

$$W_k = \frac{\text{const}}{k^\nu}; \quad \nu = \frac{5}{2}, \quad \nu = \frac{5\pm\sqrt{17}}{2}. \tag{6.9}$$

Only one of them gives non-divergent integrals in equation (6.7) and, therefore, it is the only solution of equation (6.8):

$$\nu = \tfrac{5}{2}. \tag{6.10}$$

In many cases, the beam growth-rate is not zero in the interval considered. In those cases, it is necessary to find the quasi-linear change in the growth-rate which is due to the quasi-linear beam relaxation. This means that in the case of an infinite plasma and an infinite beam interaction the power input is not constant and one must take into account also the change of the turbulent energy that is due to the change in input.

6.2. The Spectrum of Langmuir Turbulence in the Intermediate-scale Region

The intermediate-scale region is defined by the inequalities

$$k_* < k < k_{**}, \tag{6.11}$$

where

$$k_* = \frac{1}{\lambda_D}\cdot\frac{1}{3}\sqrt{\frac{m_e}{m_i}}. \tag{6.12}$$

In this region the scattering by ions is much more efficient than the scattering by electrons. On the other hand, the process (6.5) need not usually be included. The exact non-linear interaction with ions is described by the equation

$$\frac{\partial N_k}{\partial t} = N_k\frac{3}{8}\frac{\omega_{pe}T_e/T_i}{n_0m_ev_{Ti}(1+[T_e/T_i])^2} \times\int\frac{d^3\boldsymbol{k}_1}{(2\pi)^{5/2}}\frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)^2}{k^2k_1^2}\frac{(k_1^2-k^2)}{|\boldsymbol{k}-\boldsymbol{k}_1|}\exp\left\{-\frac{9}{8}\frac{(k^2-k_1^2)v_{Te}^4}{|\boldsymbol{k}-\boldsymbol{k}_1|^2v_{Ti}^2\omega_{pe}^2}\right\}\cdot N_{k_1}, \tag{6.13}$$

which can be found from the probability for ion-plasma scattering. One can see from this equation that the non-linear growth-rate increases with the frequency difference of the interacting waves until this difference is less than $|\boldsymbol{k}-\boldsymbol{k}_1|\, v_{Ti}$. If $|\boldsymbol{k}-\boldsymbol{k}_1|$ is of the order of $|\boldsymbol{k}|$ (which can occur even when $|\boldsymbol{k}_1|$ is equal to $|\boldsymbol{k}|$ in the case when the angle between $\boldsymbol{k}$ and $\boldsymbol{k}_1$ is not small) one finds the maximum $k_1-k = \Delta k$ for which the transfer is most probable

$$\frac{\Delta k}{k} = \frac{1}{3}\sqrt{\left(\frac{m_e}{m_i}\right)}\,\frac{kv_{Te}}{\omega_{pe}}. \tag{6.14}$$

Thus $\Delta k/k \ll 1$ when $k \ll k_*$. If then one considers $\Delta k = k_*$ to be a physically infinitely small value in the interval $k \gg k_*$, one can find a differential equation for the energy transfer.

Indeed, supposing that the turbulence is isotropic and using the approximate description of the function

$$-\frac{\omega}{\sqrt{(2\pi)}\,k^3 v_{Ti}^3}\exp\left[-\frac{\omega^2}{2k^2 v_{Ti}^2}\right] = \delta'(\omega), \tag{6.15}$$

one finds an equation

$$\frac{\partial W_k}{\partial t} = \gamma_k W_k + \alpha W_k \frac{\partial W_k}{\partial k}, \tag{6.16}$$

where

$$\alpha = \frac{\pi\omega_{pe}^3}{27 n_0 m_i v_{Te}^4 [1+(T_e/T_i)]^2}. \tag{6.17}$$

Such an equation can be applied both in the region where the excitation exists and in the absence of excitation and damping, when $\gamma_k = 0$. If $\gamma_k > 0$ and approximately constant in the small k-interval Δk one finds the linear growth of W, as shown in Fig. 20.

Outside Δk equation (6.16) gives $W_k = \text{const}$. The value of this constant is determined by the value reached in the interval Δk. Calculating the power of the turbulent generation

$$Q = \int \gamma_k W_k\, dk, \tag{6.18}$$

where the integral in equation (6.18) is only over the k-interval where

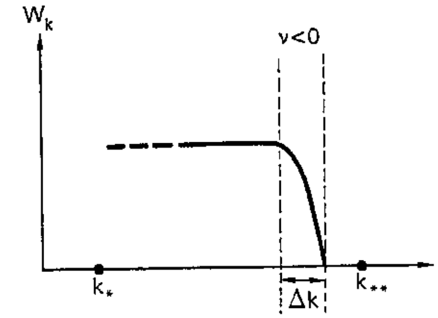

FIG. 20.
The spectrum of Langmuir turbulence created by excitation in the range Δk and scattering by ions.

$\gamma_k \neq 0$, that is, over Δk, one can express the constant in the spectrum of Fig. 20 in terms of Q:

$$W_k = \sqrt{\left(\frac{2Q}{\alpha}\right)}. \tag{6.19}$$

6.3. The Spectrum of Langmuir Turbulence in the Large-scale Region

The large-scale interval is determined by the inequality

$$k < k_*. \tag{6.20}$$

In this region the turbulent energy can be transferred directly to very small k-values. As all the plasmons are pushed into a very small phase-volume, the plasmon–plasmon scattering (6.5) becomes very important. This interaction is accompanied by the scattering of plasmons by ions. Therefore, in the formation of the turbulent spectrum both processes (6.4) and (6.5) are involved. It must be mentioned that the probability for the process (6.5) increases steeply when $k < k_*$. In the region $k < k_*$ a power-type solution for the turbulence spectrum was found by Pikel'ner and Tsytovich (1968):

$$W_k = \frac{\text{const}}{k^\nu}, \tag{6.21}$$

where ν satisfies the equation:

$$F\left(\frac{1}{2}\nu+1,\ -\frac{1}{2}\nu,\ -\frac{1}{2}\nu-1,\ \frac{1}{2}\right) = \Gamma_0+\Gamma_1,\ \Gamma_0 = 2\left(1-\frac{1}{2^\nu}\right). \tag{6.22}$$

where F is the hypergeometric function. If Q is sufficiently large, one can neglect Γ_1 (which describes the contribution from the process (6.4)), and the solution of equation (6.22) is:

$$\nu \approx 2{\cdot}84. \tag{6.23}$$

This solution is found as the intersection when the right-hand side and the left-hand side of equation (6.22) are plotted as functions of ν as shown in Fig. 21. From Fig. 21 one can see that an appreciable change in Γ does not greatly change ν. If Γ_1 is much larger than Γ_0, one sees from Fig. 21 that ν will tend to the value 4.

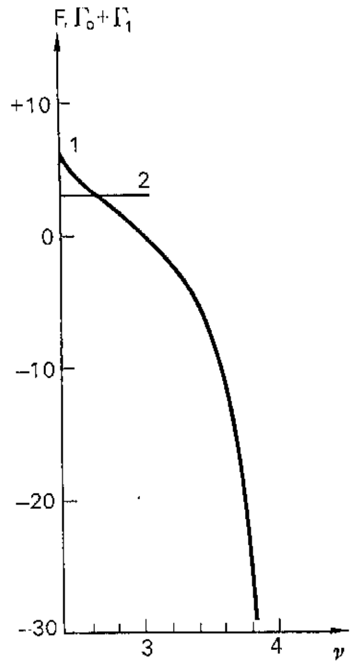

FIG. 21.

The left-hand side (curve 1) and right-hand side (curve 2) of equation (6.22).

The analytical result was checked by computer analysis of the same equation. In the asymptotic region good agreement with the analytical solution is found and this is illustrated in Fig. 21.

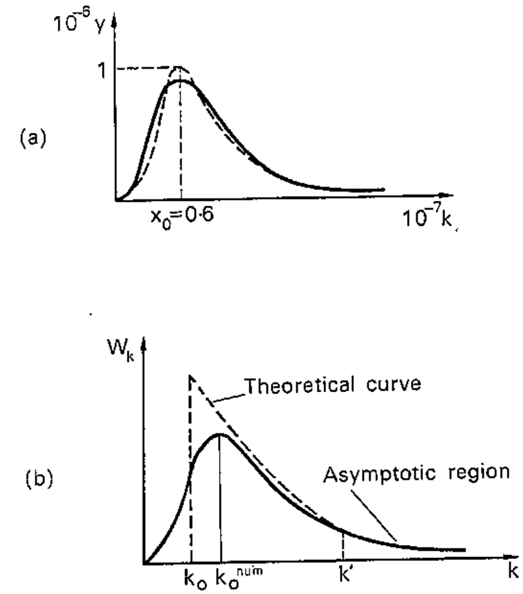

FIG. 22.

(a) The spectrum of Langmuir turbulence in the large-scale region. The solid curve represents the results of numerical calculations, and the dashed curve the approximate solution put on the computer. (b) Comparison of the analytical asymptotic solution (dashed curve) and the numerical solution including the energy-containing region. The coincidence between the analytical and the numerical solution starts at $k \sim (4 \text{ to } 5)k_0$; k_0^{num} is the numerical value of k_0.

For the computed value of the turbulent energy the analytical solution gives the experimental index

$$\nu = 3{\cdot}86, \tag{6.24}$$

while the result of the computer calculations gives

$$\nu = 3{\cdot}89. \tag{6.25}$$

The calculations give the whole curve of the turbulent spectrum, including the energy-containing region $k \approx k_0$. The value k_0 must be much smaller than k_*. The analytical solution makes it possible to estimate k_0 by joining the spectrum in the different k-intervals. In this way one finds that

$$k_0 = k_*\left(\frac{Q'}{Q}\right)^{1/2(\nu-1)}, \quad Q' = \frac{2\nu_e^2 k_*^2}{\alpha}. \qquad (6.26)$$

This result was used for the approximating initial curve in the computer calculation and the resulting curve of the turbulent spectrum has a maximum very close to that of the initial curve (see Fig. 22).

6.4. The Spectrum of Langmuir Turbulence in a Non-isothermal Plasma

If the temperature of electrons is much higher than the ion temperature,

$$T_e \gg T_i, \qquad (6.27)$$

a new non-linear process is involved in the formation of the spectrum:

$$l \rightleftharpoons l' + s. \qquad (6.28)$$

This process is allowed only when

$$k > k_*^s, \qquad (6.29)$$

where

$$k_*^s = k_* \sqrt{\frac{T_e}{T_i}}. \qquad (6.30)$$

The decay process (6.28) dominates over both the plasmon–ion scattering and the electron–plasmon scattering, that is, both the processes (6.3) and (6.4). This is a result of the fact that the scattering processes always are the tails of the resonance decay process, if such a process is allowed.

Therefore, the region $W_k \sim k^{-5/2}$ disappears in the case when $T_e \gg T_i$. If $k \gg k_*^s$, the energy transfer in the process (6.28) again is described by a differential equation, while

$$\frac{\Delta k}{k} \sim \frac{k}{k_*^s}. \qquad (6.31)$$

The equation that describes the energy transfer due to the decay (6.28) is the same as equation (6.16) with a change of the constant α to α', given by the relation

$$\alpha' = \alpha\left(1 + \frac{T_e}{T_i}\right)^2. \qquad (6.32)$$

This means that for the same power Q the constant value on the turbulent spectrum is smaller by a factor 2 in a non-isothermal plasma with $T_e \gg T_i$ than in an isothermal plasma, where $T_e = T_i$.

In the region

$$k_* < k < k_*^s \qquad (6.33)$$

the decay is forbidden and the scattering by ions is the most important. The turbulent energy in this interval thus increases steeply, because of the difference of a factor $(T_e/T_i)^2$ between the constants α and α'.

In the region where $k < k_*$ the same solution as that found for an isothermal plasma occurs. The value of k_0 which describes the position of the maximum of the spectrum is in the case when $T_e \gg T_i$ larger than for the case where $T_e = T_i$. This is illustrated by Fig. 23.

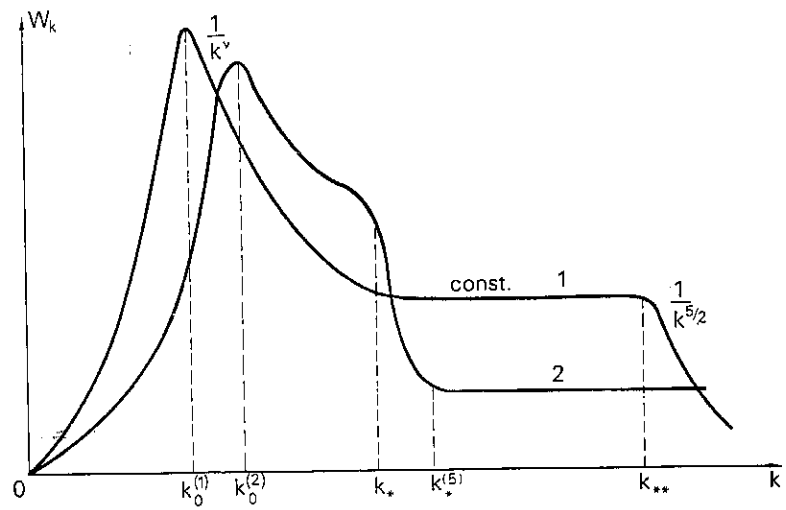

FIG. 23.

Sketch of Langmuir turbulence spectra. Curves 1 and 2 are given for the same Q, but curve 1 corresponds to $T_e = T_i$ and curve 2 to $T_e \gg T_i$.

6.5. The Radiative Type of Langmuir Turbulence Spectra

In the previous considerations the turbulent energy in the energy-containing region was damped by ordinary damping due to binary collisions with the plasma waves. This can be the case when the plasma is (i) optically thin for the radiation processes and they do not play any essential role in energy transfer or (ii) optically thick where the radiation emitted by turbulent motion is reabsorbed and converted back into plasma waves, and, therefore, the energy in the transverse waves is negligible and they do not change the turbulent spectrum.

The optical thickness L_* of the plasma for the non-linear emission

$$\mathrm{l}+\mathrm{l}' = \mathrm{t} \tag{6.34}$$

can be estimated as follows:

$$L_* = \frac{c^2}{v_{Te}\omega_{pe}} \sqrt{\left(\frac{m_e}{m_i}\,\frac{\omega_{pe}n_0T_e}{Q}\right)}. \tag{6.35}$$

If the plasma characteristic dimension L is much larger than L_*, the process (6.34) is unimportant in the formation of Langmuir turbulence. If $L \ll L_*$ and $Q \gg Q_*^{\mathrm{rad}}$, where

$$Q_*^{\mathrm{rad}} \approx \frac{m_e}{m_i}\omega_{pe}n_0T_e\frac{v_e^2}{\omega_{pe}^2}\,\frac{c^6}{v_{Te}^6}, \tag{6.36}$$

the radiation losses dominate over the collision damping. Thus, under such conditions most of the energy that is put into turbulent motion goes into the radiation, and the turbulence is of a radiative type. The change in the spectrum is different in the case where the radiation losses are able to damp the turbulent motion before they reach the maximum of the spectrum in Fig. 22 from the case where this damping becomes essential when the turbulent oscillations are able to reach this maximum. In the first case, the spectrum is flat: $W_k = \text{const}$ for $k > k_{**}$ and exactly the same as if the radiation losses were absent (that is, the expression (6.19)) up to the maximum $k_{\max}$ for which the radiation losses give an exponential cut-off of the spectrum (see Fig. 24; Tsytovich, 1969a) The ratio $k_{\max}$ to k_* is of the order of

$$\frac{k_{\max}}{k_*} \approx 10^2\frac{m_i}{m_e}\,\frac{v_{Te}^4}{c^4}\left(1+\frac{T_e}{T_i}\right)^2. \tag{6.37}$$

The situation shown in Fig. 24 occurs when $k_{\max}/k_* \gg 1$. If $k_{\max}/k_* \ll 1$, the energy reaches a maximum and the exponential cut-off disappears. A maximum is formed which depends weakly on Q. However, the energy loss is due to the emission. One finds that the damping due to emission

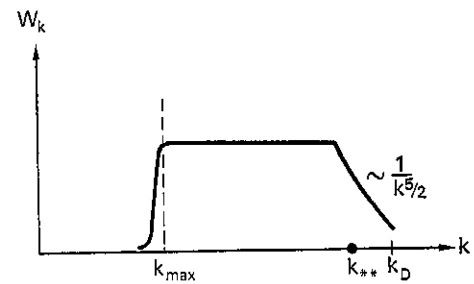

FIG. 24.

The Langmuir turbulence spectrum for the case when the turbulent energy is dissipated into radiation ($T_e = T_i$).

is described by the equation

$$\frac{dW_k}{dt} = -\gamma_{\mathrm{eff}}W_k, \tag{6.38}$$

where

$$\gamma_{\mathrm{eff}} = \frac{3\sqrt{\pi}}{2\sqrt{2}}\omega_{pe}\frac{v_{Te}^3}{c^3}\left(1+\frac{T_e}{T_i}\right)\sqrt{\left(\frac{Qm_i}{\omega_{pe}n_0T_em_e}\right)}. \tag{6.39}$$

Thus, if one neglects the turbulent energy outside the energy-containing region connected with the maximum in the spectrum, one can put

$$W = \frac{Q}{\gamma_{\mathrm{eff}}}. \tag{6.40}$$

The dependence of γ_{eff} on the energy input Q changes the picture of the turbulent spectrum, that is, increasing Q increases γ_{eff} and it, therefore, can be shown that the maximum in the spectrum does not change when Q increases. The position of this maximum is determined by the relation

$$\frac{k_0}{k_*} = \left[\frac{27\left(1+\frac{T_e}{T_i}\right)^4 m_iv_{Te}^6}{8m_ec^6}\right]^{1/2(\nu-1)}. \tag{6.41}$$

If the condition $k_{\max}/k_* \ll 1$ is satisfied, as we have assumed, we have k_0 smaller, but not very much smaller, than k_*. Thus, the position of the maximum is not very far from k_*.

Another possible emission mechanism may be due to the processes

$$\mathrm{l}+\mathrm{i} \rightleftarrows \mathrm{t}+\mathrm{i}', \tag{6.42}$$

or

$$\mathrm{l}+\mathrm{s} \rightleftarrows \mathrm{t}. \tag{6.43}$$

The first process may be essential in an isothermal plasma when $T_e = T_i$, and the second one dominates over the first one in a non-isothermal plasma when $T_e \gg T_i$. Both mechanisms give emission near the plasma frequency ω_{pe}, but the power emitted depends in a very essential way on the plasma inhomogeneity. In a homogeneous plasma both process (6.4) and the processes (6.42) and (6.43) transfer the energy of the l-waves as well as of the t-waves to small k-values, that is, to the region where the dielectric constant is zero and, therefore, such waves have difficulties in getting out of the plasma. For very small k the transverse waves have a frequency

$$\omega^t = \omega_{pe} + \frac{1}{2}\frac{k^2c^2}{\omega_{pe}}, \tag{6.44}$$

which is very close to ω_{pe}, and then can be considered to be transverse plasmons in the sense that they are more like the eigen-oscillations of a plasma than like external electromagnetic waves. The optical depth for the processes (6.42) and (6.43) depends in an essential way on which part of the spectrum the waves come from which take part in the process. At the maximum value of k in the plateau region of the spectrum the optical depth of the processes (6.42) and (6.43) is of the same order of magnitude (we denoted it by $L_*^{\max}$)

$$L_*^{\max} \approx \frac{c}{\omega_{pe}}\left(\frac{m_i}{m_e}\right)^{1/10}\sqrt{\left(\frac{\omega_{pe}n_0T_e}{Q}\right)}, \tag{6.45}$$

and at the minimum k-value on the plateau we have:

$$L_*^{\min} \approx \frac{c}{\omega_{pe}}\left(\frac{m_e}{m_i}\right)^{1/2}\sqrt{\left(\frac{\omega_{pe}n_0T_e}{Q}\right)}. \tag{6.46}$$

Both of them can be less than the value L_*, given by equation (6.33). If the plasma is optically thick for the processes (6.42) and (6.43), the turbulent energy on the plateau is of the same order of magnitude as in the case when these processes are not present, but when there is a subsequent energy exchange between the longitudinal and transverse plasmons. In the region of the maximum this kind of exchange does not change the total energy of longitudinal and transverse plasmons:

$$W = W^l + W^t = \text{constant}. \tag{6.47}$$

6.6. Stochastic Plasma Heating in the Case of Langmuir Turbulence

We established in the general theory that the stochastic heating of a plasma is due to the induced scattering of ions and electrons by the turbulent oscillations. The turbulent heating is due to the energy transfer in the region of damping. In the case of Langmuir turbulence, the energy is transferred from the region of Landau damping. Therefore, there is no turbulent heating in a homogeneous plasma. A change in the direction of the energy transfer is possible, if the plasma particles have an anisotropic distribution. This might be possible in magnetic mirror machines or in Q machines when the longitudinal plasma dimension is larger than the perpendicular dimension and the waves propagating in the perpendicular direction experience a density gradient which can change the direction of energy transfer.

In an isotropic homogeneous plasma only stochastic heating is possible for Langmuir turbulence. Because the scattering by electrons is effective only in a very small k-interval, the ions are mostly heated stochastically. The largest growth-rate of heating is due to the largest k-value in the flat part of the turbulent spectrum. One finds that (Q_{cr} is given below by (6.49))

$$\frac{\partial T_i}{\partial t} \approx 74\left(\frac{m_e}{m_i}\right)^{2/5}\frac{Q}{n_0}\left(1+\frac{T_e}{T_i}\right)^2 \begin{cases} 1, & \text{if} \quad Q \ll Q_{cr}; \\ \sim \frac{1}{10}\sqrt{\left(\frac{Q}{Q_{cr}}\right)}, & \text{if} \quad Q \gg Q_{cr}. \end{cases} \tag{6.48}$$

The result (6.48) is valid for $T_e \lesssim 3T_i$. In the opposite case, the factor

$$\left[1+\frac{T_e}{T_i}\right]^2\left(\frac{m_e}{m_i}\right)^{2/5}$$

is not present when $Q \gg Q_{cr}$: the stochastic heating of ions is much larger than for the case when $Q \ll Q_{cr}$; the quantity Q_{cr} given by the equation

$$Q_{cr} = \frac{\nu_e}{\omega_{pe}} \nu_e n_0 T_e \left(\frac{m_i}{m_e}\right)^{6/5}, \tag{6.49}$$

is rather small and usually $Q \gg Q_{cr}$.

6.7. Stochastic Acceleration of Fast Particles

Particles with velocities much larger than the thermal electron velocity can be resonant with the Langmuir oscillations, and therefore gain energy from them (Tsytovich, 1962). The non-linear effect on these particles is unimportant unless the resonance is allowed. This indeed becomes possible for waves with phase velocities larger than the velocity of light. Mostly they correspond to oscillations in the region of the maximum. Thus, generally speaking, resonant particle acceleration is due to the plateau part of the turbulent plasma spectrum. Sometimes in a sufficiently cold plasma ($T_e < 25$ eV for hydrogen) the tail of the spectral maximum can contribute to the particle accelerations. For non-relativistic particles with a kinetic energy ε_{kin} the acceleration by the plateau region is described by the equation ($\varepsilon_{kin} = \varepsilon - mc^2$)

$$\frac{d\varepsilon_{kin}}{dt} = \frac{\Gamma}{\sqrt{\varepsilon_{kin}}}, \tag{6.50}$$

where

$$\Gamma = \frac{3\sqrt{(3\pi)}}{2} \omega_{pe} v_{Te}^3 \frac{m_e^{3/2} m_i^{1/2} Z^2}{\sqrt{(m)}} \left(1 + \frac{T_e}{T_i}\right) \sqrt{\left(\frac{Q}{\omega_{pe} n_0 T_e}\right)} \tag{6.51}$$

(m is the mass of the particle which is accelerated). In the region of the tail of the maximum, one finds an acceleration which increases rapidly with increasing energy. When the energy of the particle reaches the rest mass energy (that is, when the particle reaches relativistic energies), the diffusion coefficient in energy space becomes constant, and the acceleration follows from the equation $\varepsilon^2 = 4Dt$, or $\varepsilon/t = 4D/\varepsilon$; one finds that the energy growth-rate $d\varepsilon/dt$ decreases with increasing energy as ε^{-1}. The characteristic ε-dependence of $d\varepsilon/dt$ is shown in Fig. 25.

If the energy gain given by equation (6.50) is balanced by the collisions of the fast particles with the cold plasma, one finds for fast particles a

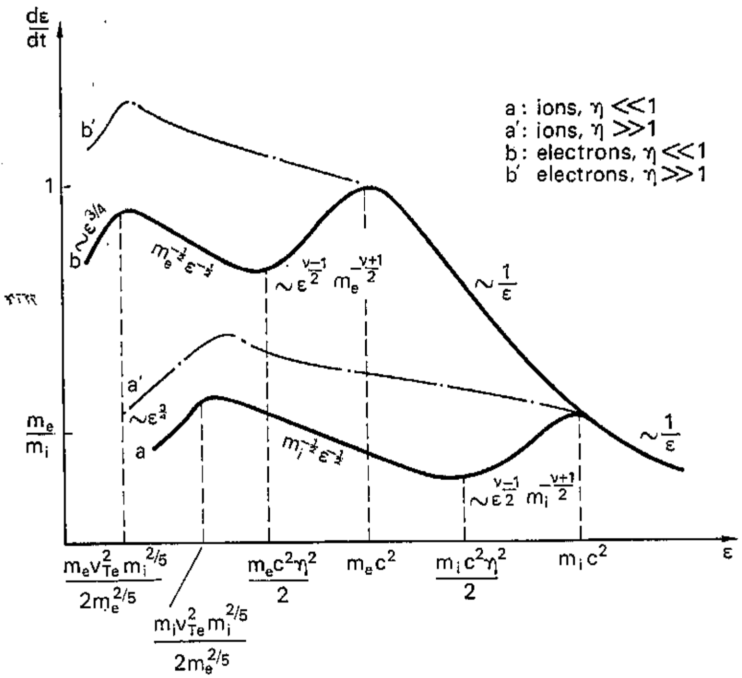

FIG. 25.

Sketch of the acceleration rate $d\varepsilon/dt$ as a function of ε for electrons and for ions in Langmuir turbulence ($\eta = 3v_{Te}^2/cv_{Ti}$). The part proportional to $\varepsilon^{3/4}$ is due to the spectrum $k^{-5/2}$. The part proportional $\varepsilon^{-1/2}$ is due to the plateau region of the spectrum, and the part proportional to $\varepsilon^{(\nu-1)/2}$ is due to the tail of the main maximum in the spectrum. If $\varepsilon \gg mc^2$, the acceleration rate does not depend on the particle mass and is proportional to ε^{-1}. The region $d\varepsilon/dt \sim \varepsilon^{(\nu-1)/2}$ disappears when $\eta > 1$ (dashed curves a′ and b′) and the region $d\varepsilon/dt \sim \varepsilon^{3/4}$ when $T_e \gg T_i$ (curves a and b).

relativistic Maxwellian distribution with a temperature

$$T_* = mc^2 \sqrt{\left(\frac{Q}{Q_{**}}\right)}, \tag{6.52}$$

where

$$Q_{**} = \frac{8m^2 v_e^2 n_0 T_e}{27\pi[1 + (T_e/T_i)]^2 m_e m_i \omega_{pe}}. \tag{6.53}$$

Usually $Q \gg Q_{**}$, which means that $T_* \gg mc^2$. Thus the Langmuir turbulence acting for a long enough time creates relativistic particles, which in cosmic conditions can be identified as cosmic rays. In laboratory conditions the maximum particle energy is determined by the possibility of their confinement by magnetic fields, or, in other words, the maximum energy is of the order of the energy for which the gyro-radius of the accelerated particles is of the order of the characteristic installation dimensions.

The number of particles accelerated is determined by the injection that can take place from the tails of the Maxwellian distribution. The most important role is played by low-frequency magnetic-type oscillations as injection mechanisms which can give preferential heavy-ion injection. When the number of accelerated particles is sufficiently high, the damping of the turbulent motion due to the acceleration can be important. This damping can exist approximately only on the plateau region, or more precisely for oscillations with phase velocities less than c. The turbulence can come to a stage in which practically the whole power is dissipated by fast particles. Under these conditions the maximum of the turbulent spectrum must disappear, because the turbulent energy must be absorbed by fast particles until it comes to the region where such absorption is not possible. The subsequent changes in the turbulent spectrum when the density of the accelerated particles is increased is shown in Fig. 26.

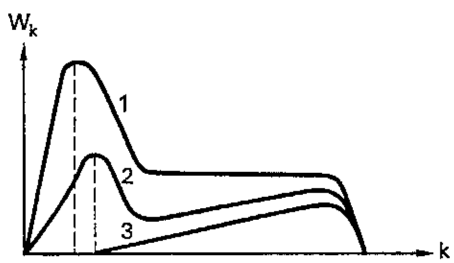

FIG. 26.

Change in the Langmuir spectrum as the density n^A of accelerated particles increases; $n_1^A < n_2^A < n_3^A$.

6.8. Correlation Effects for Langmuir Oscillations

If there is a maximum in the spectrum, the correlation effects for Langmuir oscillations are mainly determined by the plasmon–plasmon scattering and the correlation width is given by the estimate

$$\Delta\omega \approx \frac{1}{30}\,\omega_{pe}\,\frac{\omega_{pe}^2}{k_0^2 v_{Te}^2}\left(\frac{W}{nT_e}\right)^2. \tag{6.54}$$

This width is the half-width of the correlation curve near resonance. Very far from the resonance the correlation tail has the form shown in Fig. 27.

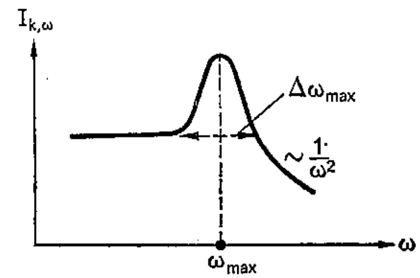

FIG. 27.

The correlations in Langmuir turbulence at low frequencies; $\omega_{max} = 3k^2v_{Te}^2/2\omega_{pe}$.

In this spectrum there is no relation between ω and k, but the maximum frequency in the plateau region of this spectrum is equal to

$$\omega_{max} = \frac{3k^2 v_{Te}^2}{3\omega_{pe}}, \tag{6.55}$$

that is, the difference of two Langmuir frequencies. Such an effect was observed by Fainberg and coworkers (see Fainberg, 1967) in the beam–plasma interaction.

The relation

$$\frac{\Delta\omega_{max}}{\omega_{max}} \approx \frac{k_0}{k}, \tag{6.56}$$

and the existence of a maximum in the curve of the correlation function is due to the maximum in the turbulent spectrum. This maximum disappears when there is no maximum on the spectrum.

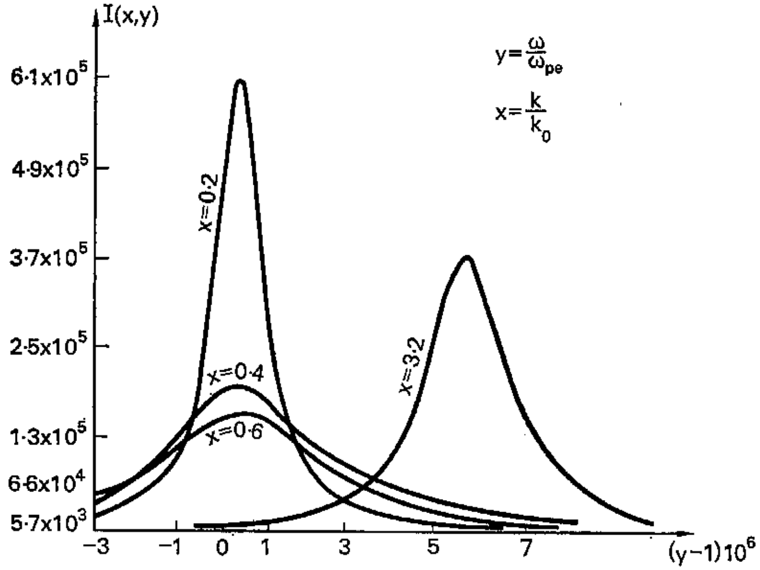

FIG. 28.

Results of a numerical computation of the correlation curve near $\omega = \omega_k$. The different curves correspond to different k-values in the spectrum; $k \ll k_*$; the parameters used in the computation are the same as for Fig. 22. The non-linear shift in ω of the curves is small compared to the thermal one described by equation (6.1).

In Fig. 28 we show the results of numerical computation of the correlation curve near $\omega = \omega_k$ for Langmuir turbulence. This figure represents the computation by Makhankov and Shchinov (1970) of the non-linear integral equation for the correlation function given above in Chapter 4.

7. Electromagnetic Properties of a Turbulent Plasma

THE presence of turbulent collisions has an important influence on the electromagnetic properties of the turbulent plasma response, when a weak field acts on the plasma. This means also that the growth-rate of several plasma instabilities can be changed by exciting turbulence in a plasma. This kind of change may lead to some stabilisation properties, as well as to new kinds of instabilities. One expects that this change in the electromagnetic properties can be essential for perturbations or for an external field with frequencies much less than the characteristic frequency of the turbulent collisions. This effect can be described by a dielectric constant depending on the turbulent energy. If the frequency is less than the turbulent collision frequency, one may expect that the dielectric constant cannot be expanded in terms of the turbulent energy.

To show how such a theory can be developed, let us return to the general description of the turbulent state, introducing the stochastic and regular variables

$$\frac{\partial f^{\text{reg}}}{\partial t} + (\mathbf{v}\cdot\nabla f^{\text{reg}}) + e\left(\mathbf{E}^{\text{reg}}\cdot\frac{\partial f^{\text{reg}}}{\partial \mathbf{p}}\right) = -e\left\langle\left(\mathbf{E}^{\text{stoch}}\cdot\frac{\partial f^{\text{stoch}}}{\partial \mathbf{p}}\right)\right\rangle, \tag{7.1}$$

$$\frac{\partial f^{\text{stoch}}}{\partial t} + (\mathbf{v}\cdot\nabla f^{\text{stoch}}) + e\left(\mathbf{E}^{\text{stoch}}\cdot\frac{\partial f^{\text{reg}}}{\partial \mathbf{p}}\right) + e\left(\mathbf{E}^{\text{reg}}\cdot\frac{\partial f^{\text{stoch}}}{\partial \mathbf{p}}\right) + e\left(\mathbf{E}^{\text{stoch}}\cdot\frac{\partial f^{\text{stoch}}}{\partial \mathbf{p}}\right) - e\left\langle\left(\mathbf{E}^{\text{stoch}}\cdot\frac{\partial f^{\text{stoch}}}{\partial \mathbf{p}}\right)\right\rangle = 0. \tag{7.2}$$

Let us assume that in the stationary turbulent state the regular field $\mathbf{E}^{\text{reg}}$ is absent. Denote all quantities for this state by an index "0". Thus we have

$$\frac{\partial f_0^{\text{reg}}}{\partial t} + (\mathbf{v}\cdot\nabla f_0^{\text{reg}}) = -e\left\langle\left(\mathbf{E}_0^{\text{stoch}}\cdot\frac{\partial f_0^{\text{stoch}}}{\partial \mathbf{p}}\right)\right\rangle, \tag{7.3}$$

$$\frac{\partial f_0^{\text{stoch}}}{\partial t} + (\mathbf{v}\cdot\nabla f_0^{\text{stoch}}) + e\left(\mathbf{E}_0^{\text{stoch}}\cdot\frac{\partial f_0^{\text{reg}}}{\partial \mathbf{p}}\right) + e\left(\mathbf{E}_0^{\text{stoch}}\cdot\frac{\partial f_0^{\text{stoch}}}{\partial \mathbf{p}}\right) - e\left\langle\left(\mathbf{E}_0^{\text{stoch}}\cdot\frac{\partial f_0^{\text{stoch}}}{\partial \mathbf{p}}\right)\right\rangle = 0. \tag{7.4}$$

Suppose now that there exists a small perturbation connected with the field $\boldsymbol{E}^{\text{reg}}$, that is,

$$f^{\text{reg}} = f_0^{\text{reg}} + f_1^{\text{reg}}, \quad f^{\text{stoch}} = f_0^{\text{stoch}} + f_1^{\text{stoch}},$$
$$\boldsymbol{E}^{\text{stoch}} = \boldsymbol{E}_0^{\text{stoch}} + \boldsymbol{E}_1^{\text{stoch}}, \tag{7.5}$$

where $f_1^{\text{reg}} \ll f_0^{\text{reg}}$; $f_1^{\text{stoch}} \ll f_0^{\text{stoch}}$; $E_1^{\text{stoch}} \ll E_0^{\text{stoch}}$ and all quantities with index "1" are proportional to E^{reg}.

From equations (7.1) and (7.2) we get the system of equations

$$\frac{\partial f_1^{\text{reg}}}{\partial t} + (\boldsymbol{v}\cdot\nabla f_1^{\text{reg}}) + e\left(\boldsymbol{E}^{\text{reg}} \cdot \frac{\partial f_0^{\text{reg}}}{\partial \boldsymbol{p}}\right)$$
$$= -\left(\frac{\partial}{\partial \boldsymbol{p}} \cdot \langle \boldsymbol{E}_1^{\text{stoch}} f_0^{\text{stoch}} + \boldsymbol{E}_0^{\text{stoch}} f_1^{\text{stoch}} \rangle\right) \equiv I_1, \tag{7.6}$$

$$\frac{\partial f_1^{\text{stoch}}}{\partial t} + (\boldsymbol{v}\cdot\nabla f_1^{\text{stoch}}) + e\left(\boldsymbol{E}^{\text{reg}} \cdot \frac{\partial f_0^{\text{stoch}}}{\partial \boldsymbol{p}}\right) + e\left(\boldsymbol{E}_0^{\text{stoch}} \cdot \frac{\partial f_1^{\text{reg}}}{\partial \boldsymbol{p}}\right)$$
$$+ e\left(\boldsymbol{E}_1^{\text{stoch}} \cdot \frac{\partial f_0^{\text{reg}}}{\partial \boldsymbol{p}}\right) + e\left(\frac{\partial}{\partial \boldsymbol{p}} \cdot \{\boldsymbol{E}_1^{\text{stoch}} f_0^{\text{stoch}} + \boldsymbol{E}_0^{\text{stoch}} f_1^{\text{stoch}}\right.$$
$$\left. - \langle \boldsymbol{E}_1^{\text{stoch}} f_0^{\text{stoch}} + \boldsymbol{E}_0^{\text{stoch}} f_1^{\text{stoch}} \rangle\}\right) = 0. \tag{7.7}$$

One must add to these equations the Maxwell equations (for the sake of simplicity we have assumed that the fields are longitudinal)

$$\operatorname{div} \boldsymbol{E}^{\text{reg}} = 4\pi \sum_\alpha e_\alpha \int f_1^{\text{reg},\alpha} \frac{d^3 \boldsymbol{p}}{(2\pi)^3}, \tag{7.8}$$

$$\operatorname{div} \boldsymbol{E}_1^{\text{stoch}} = 4\pi \sum_\alpha e_\alpha \int f_1^{\text{stoch},\alpha} \frac{d^3 \boldsymbol{p}}{(2\pi)^3}. \tag{7.9}$$

These equations are exact equations which describe the perturbations from the stationary state. Various approaches similar to the ones which we used earlier to describe the stationary state can be used here for the perturbations.

7.1. Expansion of Turbulent Collision Integrals in Terms of the Turbulent Energy

The collisions with the turbulent oscillations are, for the perturbation, represented by the right-hand side of equation (7.6), which we denote by I_1. Let us use an expansion in the turbulent field $E_\varkappa^{\text{stoch}}$ and calculate the first term of this integral which is proportional to $\langle E_\varkappa^{\text{stoch}} E_{\varkappa'}^{\text{stoch}} \rangle = I_\varkappa \delta(\varkappa + \varkappa')$. Thus in this case we consider only the first-order turbulent collisions, which are proportional to the turbulent energy, that is, the first term in the expansion in terms of W/nT. To calculate this integral one needs only the first-order term, that is, the term which is proportional to the turbulent field in E_1^{stoch}, f_0^{stoch}, and f_1^{stoch}.

Firstly, from equation (4.10) it follows that f_0^{stoch} can be written in the form

$$f_{0,\varkappa}^{\text{stoch}} = \frac{e}{i[\omega - (\boldsymbol{k}\cdot\boldsymbol{v})]}\left(\boldsymbol{E}_{0,\varkappa}^{\text{stoch}} \cdot \frac{\partial f_0^{\text{reg}}}{\partial \boldsymbol{p}}\right), \tag{7.10}$$

where $f_{0,\varkappa}^{\text{stoch}}$ is the Fourier transform of f_1^{stoch}, and $\varkappa = \langle \boldsymbol{k}, \omega \rangle$.

Secondly, neglecting the non-linearity in the stochastic field term in equation (7.7), one finds for the Fourier transform of f_1^{stoch} the equation

$$f_{1,\varkappa}^{\text{stoch}} = \frac{e}{i[\omega - (\boldsymbol{k}\cdot\boldsymbol{v})]}\left(\boldsymbol{E}_{1,\varkappa}^{\text{stoch}} \cdot \frac{\partial f_0^{\text{reg}}}{\partial \boldsymbol{p}}\right) + \frac{e}{i[\omega - (\boldsymbol{k}\cdot\boldsymbol{v})]}$$
$$\times \left(\frac{\partial}{\partial \boldsymbol{p}} \cdot \int d^4\varkappa_1\, d^4\varkappa_2\, \delta(\varkappa - \varkappa_1 - \varkappa_2)\left\{\boldsymbol{E}_{\varkappa_1}^{\text{reg}} f_{0,\varkappa_2}^{\text{stoch}} + \boldsymbol{E}_{0,\varkappa_1}^{\text{stoch}} f_{1,\varkappa_2}^{\text{reg}}\right\}\right). \tag{7.11}$$

On the other hand, from equation (7.9) we have

$$E_{1,\varkappa}^{\text{stoch}} = \frac{4\pi e}{ik} \int f_{1,\varkappa}^{\text{stoch}} \frac{d^3 \boldsymbol{p}}{(2\pi)^3}. \tag{7.12}$$

We have, for the sake of simplicity, taken into account only one kind of charge (say, the electrons) assuming that the average charge density is compensated by a neutralising background (of ions) and that this constant component does not contribute to the higher Fourier components considered in equation (7.12). It is easy to take other charge species into account in a similar way.

Inserting expression (7.11) into equation (7.12) one finds that

$$E^{\text{stoch}}_{1,\varkappa} = \frac{4\pi e^2}{ik\varepsilon_\varkappa}\int d^4\varkappa_1\, d^4\varkappa_2\,\delta(\varkappa-\varkappa_1-\varkappa_2)\frac{d^3p}{(2\pi)^3} \times\frac{1}{\omega-(\boldsymbol{k}\cdot\boldsymbol{v})}\left(\frac{\partial}{\partial \boldsymbol{p}}\cdot[\boldsymbol{E}^{\text{reg}}_{\varkappa_1} f^{\text{stoch}}_{0,\varkappa_2}+\boldsymbol{E}^{\text{stoch}}_{0,\varkappa_1} f^{\text{reg}}_{1,\varkappa_2}]\right), \qquad (7.13)$$

where $\varepsilon_\varkappa$ is determined by equation (4.21). Two points need to be improved in this analysis. Firstly, one cannot in general divide by $\omega-(\boldsymbol{k}\cdot\boldsymbol{v})$, if it is possible that ω can be equal to $(\boldsymbol{k}\cdot\boldsymbol{v})$. Secondly, one cannot divide by $\varepsilon_\varkappa$, if $\varepsilon_\varkappa$ is close to zero. These are the same singularities as in the description of the usual turbulent state.

Nevertheless, if one introduces expressions (7.10) and (7.11) into the right-hand side one finds that

$$I_{1,\varkappa} = -e\left(\frac{\partial}{\partial \boldsymbol{p}}\cdot\int d^4\varkappa'\, d^4\varkappa''\,\delta(\varkappa-\varkappa'-\varkappa'')\left\langle \boldsymbol{E}^{\text{stoch}}_{1,\varkappa'} f^{\text{stoch}}_{0,\varkappa''}+\boldsymbol{E}^{\text{stoch}}_{0,\varkappa'} f^{\text{stoch}}_{1,\varkappa''}\right\rangle\right)$$
$$= -e^2\sum_{i,j}\frac{\partial}{\partial p_j}\int d^4\varkappa'\, d^4\varkappa''\,\delta(\varkappa-\varkappa'-\varkappa'')\Bigg\{\Bigg\langle (E^{\text{stoch}}_{1,\varkappa'})_i\frac{1}{i[\omega''-(\boldsymbol{k}''\cdot\boldsymbol{v})]}$$
$$\times(E^{\text{stoch}}_{0,\varkappa''})_j\frac{\partial f_0^{\text{reg}}}{\partial p_j}+(E^{\text{stoch}}_{0,\varkappa'})_i\frac{1}{i[\omega''-(\boldsymbol{k}''\cdot\boldsymbol{v})]}(E^{\text{stoch}}_{1,\varkappa''})_j\frac{\partial f_0^{\text{reg}}}{\partial p_j}$$
$$+(E^{\text{stoch}}_{0,\varkappa'})_i\frac{1}{i[\omega''-(\boldsymbol{k}''\cdot\boldsymbol{v})]}\frac{\partial}{\partial p_j}\int d^4\varkappa_1\, d^4\varkappa_2\,\delta(\varkappa''-\varkappa_1-\varkappa_2)$$
$$\times\left[(E^{\text{reg}}_{\varkappa_1})_j\frac{1}{i[\omega_2-(\boldsymbol{k}_2\cdot\boldsymbol{v})]}\sum_l(E^{\text{stoch}}_{0,\varkappa_2})_l\frac{\partial f_0^{\text{reg}}}{\partial p_l}+(E^{\text{stoch}}_{0,\varkappa_1})_j f^{\text{reg}}_{1,\varkappa_2}\right]\Bigg\rangle\Bigg\}. \qquad (7.14)$$

Bearing in mind that

$$\left\langle(E^{\text{stoch}}_{0,\varkappa})_i(E^{\text{stoch}}_{0,\varkappa''})_j\right\rangle = \frac{k_ik_j}{k^2}I_\varkappa\,\delta(\varkappa''+\varkappa), \qquad (7.15)$$

and defining $I^{(1)}_{\varkappa,\varkappa''}$ by the equation

$$\left\langle(E^{\text{stoch}}_{1,\varkappa})_i(E^{\text{stoch}}_{0,\varkappa''})_j\right\rangle = \frac{k_ik''_j}{kk''}I^{(1)}_{\varkappa,\varkappa''}, \qquad (7.16)$$

we can rewrite the collision integral (7.14) in the form

$$I_{1,\varkappa} = \sum_{i,j}\frac{\partial}{\partial p_i}D^{(1)}_{ij,\varkappa}\frac{\partial f_0^{\text{reg}}}{\partial p_j}+\sum_{i,j,l}E^{\text{reg}}_{\varkappa,l}\frac{\partial}{\partial p_i}\hat{D}_{ijl}\frac{\partial f_0^{\text{reg}}}{\partial p}+\sum_{i,j}\frac{\partial}{\partial p_i}D^{(0)}_{ij,\varkappa}\frac{\partial f^{\text{reg}}_{1,\varkappa}}{\partial p_j}, \qquad (7.17)$$

where

$$D^{(1)}_{ij,\varkappa} = ie^2\int\frac{\delta(\varkappa-\varkappa'-\varkappa'')}{k'k''}I^{(1)}_{\varkappa',\varkappa''}\left[\frac{k'_ik''_j}{\omega''-(\boldsymbol{k}''\cdot\boldsymbol{v})}+\frac{k'_jk''_i}{\omega'-(\boldsymbol{k}'\cdot\boldsymbol{v})}\right]d^4\varkappa'\,d^4\varkappa'', \qquad (7.18)$$

$$D^{(0)}_{ij,\varkappa} = ie^2\int\frac{k_{1i}k_{1j}}{k_1^2}I_{\varkappa_1}\frac{d^4\varkappa_1}{\omega+\omega_1-([\boldsymbol{k}+\boldsymbol{k}_1]\cdot\boldsymbol{v})+i\delta}, \qquad (7.19)$$

$$\hat{D}_{ijl} = e^2\int\frac{k_{1i}k_{1j}}{k_1^2}I_{\varkappa_1}\frac{d^4\varkappa_1}{\omega+\omega_1-([\boldsymbol{k}+\boldsymbol{k}_1]\cdot\boldsymbol{v})+i\delta}\frac{\partial}{\partial p_l}\frac{1}{[\omega_1-(\boldsymbol{k}_1\cdot\boldsymbol{v})-i\delta]}. \qquad (7.20)$$

To finish these calculations, it is necessary to find $I^{(1)}_{\varkappa',\varkappa''}$. Multiplying equation (7.13) by $E^{\text{stoch}}_{0,\varkappa}$ and using equation (7.10), one finds that

$$I^{(1)}_{\varkappa',\varkappa''} = \frac{4\pi e^2 I_{\varkappa'}}{ik'\varepsilon_{\varkappa'}}\int\frac{d^3p}{(2\pi)^3}\delta(\varkappa'-\varkappa_1+\varkappa'')\frac{1}{\omega'-(\boldsymbol{k}'\cdot\boldsymbol{v})} \times\left(\frac{\partial}{\partial\boldsymbol{p}}\cdot\left[\frac{e\boldsymbol{E}^{\text{reg}}_{\varkappa_1}}{i[\omega''-(\boldsymbol{k}''\cdot\boldsymbol{v})-i\delta]}\left(\frac{\boldsymbol{k}''}{k''}\cdot\frac{\partial f_0^{\text{reg}}}{\partial\boldsymbol{p}}\right)+\frac{\boldsymbol{k}''}{k''}f^{\text{reg}}_{1\varkappa_1}\right]\right)d^4\varkappa_1. \qquad (7.21)$$

Introducing expression (7.21) into equation (7.19), one finds that

$$D^{(1)}_{ij,\varkappa} = \sum_l(E^{\text{reg}}_\varkappa)_l D^{(1)''}_{ijl,\varkappa}+D^{(1)'}_{ij,\varkappa}, \qquad (7.22)$$

where

$$D^{(1)'}_{ij,\varkappa} = 4\pi ie^2\int\frac{I_{\varkappa_1}\,d^4\varkappa_1}{\varepsilon_{\varkappa-\varkappa_1}}\left[\frac{k_{1j}(k_i-k_{1i})}{k_1^2|\boldsymbol{k}_1-\boldsymbol{k}|^2}\frac{1}{\omega_1-(\boldsymbol{k}_1\cdot\boldsymbol{v})+i\delta}+\frac{k_{1i}(k_j-k_{1j})}{k_1^2|\boldsymbol{k}-\boldsymbol{k}_1|^2}\times\frac{1}{\omega-\omega_1-([\boldsymbol{k}-\boldsymbol{k}_1]\cdot\boldsymbol{v})+i\delta}\right]\int\frac{d^3p}{(2\pi)^3}\frac{\left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)f^{\text{reg}}_{1,\varkappa}}{\omega-\omega_1-([\boldsymbol{k}-\boldsymbol{k}_1]\cdot\boldsymbol{v})}, \qquad (7.23)$$

$$D^{(1)''}_{ijl,\varkappa} = -4\pi e^2\int\frac{I_{\varkappa_1}\,d^4\varkappa_1}{\varepsilon_{\varkappa-\varkappa_1}}\left[\frac{k_{1i}(k_{1j}-k_j)}{\omega-\omega_1-([\boldsymbol{k}-\boldsymbol{k}_1]\cdot\boldsymbol{v})}-\frac{k_{1j}(k_{1i}-k_i)}{\omega_1-(\boldsymbol{k}_1\cdot\boldsymbol{v})+i\delta}\right] \times\frac{1}{k_1^2|\boldsymbol{k}-\boldsymbol{k}_1|^2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{[\omega-\omega_1-([\boldsymbol{k}-\boldsymbol{k}_1]\cdot\boldsymbol{v})]} \times\frac{\partial}{\partial p_l}\frac{1}{[\omega_1-(\boldsymbol{k}_1\cdot\boldsymbol{v})]}\left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)f_0^{\text{reg}}. \qquad (7.24)$$

Finally the equation which describes the perturbation (7.6) can be written in the form

$$-i[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]f_{1,\varkappa}^{\rm reg}+e\left(\boldsymbol{E}_{\varkappa}^{\rm reg}\cdot\frac{\partial f_0^{\rm reg}}{\partial \boldsymbol{p}}\right)=I_{1,\varkappa}, \tag{7.25}$$

where the collision integral has the form

$$I_{1,\varkappa}=\sum_{i,j}\left[\frac{\partial}{\partial p_i}D_{ij,\varkappa}^{(0)}\frac{\partial f_{1,\varkappa}^{\rm reg}}{\partial p_j}+\frac{\partial}{\partial p_j}D_{ij,\varkappa}^{(1)'}\frac{\partial f_0^{\rm reg}}{\partial p_j}+\sum_l (E_{\varkappa}^{\rm reg})_l \times\frac{\partial}{\partial p_i}\{\hat{D}_{ijl}+D_{ijl,\varkappa}^{(1)''}\}\frac{\partial f_0^{\rm reg}}{\partial p_j}\right]. \tag{7.26}$$

This equation is not as simple as it seems, because the $D_{ij,\varkappa}^{(1)}$ contain the perturbation $f_{1,\varkappa}^{\rm reg}$. Therefore, equation (7.25) is an integro-differential equation for $f_{1,\varkappa}^{\rm reg}$. The physical meaning of each term in equation (7.26) can be easily understood: the first term describes the change of the quasi-linear interaction, if such an interaction is possible. The other terms are negligible if the resonance $\omega=(\boldsymbol{k}\cdot\boldsymbol{v})$ is possible. If $\omega\neq(\boldsymbol{k}\cdot\boldsymbol{v})$ the third term describes the change in the non-linear scattering and in the decay process.

Because the ω and $\boldsymbol{k}$ of the wave considered are now arbitrary, one can say that $\varepsilon_{\varkappa-\varkappa_1}$ is close to zero, that is, one can consider the frequencies which are close to differences in the frequencies of the turbulent oscillations. This shows that the contribution from the non-linear and Compton scattering is small. Thus one must consider mainly the case of resonant turbulence with coefficients $D^{(0)}$ and non-resonant turbulence with coefficients $D^{(1)'}$.

Before starting on a detailed analysis of equation (7.25) let us emphasise one important feature of this equation, that is, that even if we expand the $I_\varkappa$ in terms of the turbulent energy we find that the dielectric constant of a turbulent plasma cannot be expanded in terms of the turbulent energy. Indeed, if the frequency ω (or more correctly $\omega-(\boldsymbol{k}\cdot\boldsymbol{v})$) is less than the turbulent collision frequency, which can be estimated from the right-hand side of equation (7.25), the plasma response to the field is determined by the turbulent collisions which appear in the denominator in the dielectric constant. This also can be illustrated by the well-known dielectric constant due to the ordinary collisions in the region where the perturbing frequency is less than the collision frequency:

$$\varepsilon\approx 1-\frac{\omega_{\rm pe}^2}{i\omega\nu_{\rm coll}}. \tag{7.27}$$

Very roughly speaking, an equation like equation (7.27) occurs now where instead of $\nu_{\rm coll}$ an effective turbulent collision frequency appears, which depends on W, so that equation (7.25) is proportional to the turbulent energy. Thus, ε of a turbulent plasma cannot be expanded in terms of the turbulent energy.

7.2. Effects of Turbulent Resonance Broadening

The broadening due to the wave-particle interactions in $f_0^{\rm stoch}$ has already been discussed and thus instead of equation (7.10) one finds

$$f_0^{\rm stoch}=-ieg(\omega-(\boldsymbol{k}\cdot\boldsymbol{v}))\left(\boldsymbol{E}_0^{\rm stoch}\cdot\frac{\partial f_0^{\rm reg}}{\partial \boldsymbol{p}}\right). \tag{7.28}$$

A similar technique can be applied to equation (7.7). Because the non-linear terms occur only in $e([\partial/\partial\boldsymbol{p}]\cdot[\boldsymbol{E}_0^{\rm stoch}f_1^{\rm stoch}-\langle\boldsymbol{E}_0^{\rm stoch}f_1^{\rm stoch}\rangle])$ the interaction procedure leads to the same substitution of g for $[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]^{-1}$ in equation (7.13). Other interaction terms are of higher order in the stochastic fields. For small ω and k the factor $\varepsilon_{\varkappa-\varkappa_1}^{-1}$ in equation (7.23) becomes large because $\varepsilon_{\varkappa_1}\approx 0$. One needs to take into account the non-linear corrections proportional to $W_{\boldsymbol{k}}$. This was done by the author (Tsytovich, 1969b). Roughly speaking, the result of such a consideration is to change $\varepsilon_\varkappa$ to $\varepsilon_\varkappa+\varepsilon_\varkappa^{\rm n.l.}$.

7.3. The Dielectric Constant for Ion-sound Turbulence

The most important interaction for the case of ion-sound turbulence comes from $D_{ij,\varkappa}^{(0)}$ (see equation (7.19)). One can consider two limits. First, the case when $\omega\ll\omega_1$ and $k\ll k_1$, that is, the perturbation with frequencies much less than the turbulent frequency, and wavelengths much longer than the turbulent wavelength. In that case, the diffusion coefficient,

$$D_{ij}^{(0)}=\pi e^2\int\frac{k_{1i}k_{1j}}{k_1^2}I_{\varkappa_1}\,\delta(\omega_1-(\boldsymbol{k}_1\cdot\boldsymbol{v}))\,d^4\varkappa_1, \tag{7.29}$$

does not depend on ω and $\boldsymbol{k}$ and is the usual quasi-linear diffusion coefficient. This means that at low frequencies the turbulent collisions act as real collisions.

Secondly, the case of high frequencies and short wavelengths, $\omega \gg \omega_1$, $k \gg k_1$; in that case, the diffusion coefficient is proportional to $\delta(\omega-(\boldsymbol{k}\cdot\boldsymbol{v}))$ which means that, if for certain values of ω and $\boldsymbol{k}$ the resonance condition is not fulfilled, that is, $\omega \neq (\boldsymbol{k}\cdot\boldsymbol{v})$, as for example happens for the transverse waves, the diffusion coefficient is zero. (Of course for transverse waves it is necessary to generalise equation (7.19) to take into account the transverse field, but the conclusion is the same.) Thus the high-frequency waves do not "feel" these turbulent collisions. This shows the difference between the turbulent and ordinary collisions.†

Equation (7.25) can be written for the case of small ω and k in the form

$$-i[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]f_{1,\varkappa}^{\text{reg}}+e\left(\boldsymbol{E}_{\varkappa}^{\text{reg}}\cdot\frac{\partial f_0^{\text{reg}}}{\partial \boldsymbol{p}}\right) = \sum_{i,j}\frac{\partial}{\partial p_i}D_{ij}^{(0)}\frac{\partial f_{1,\varkappa}^{\text{reg}}}{\partial p_j}. \qquad (7.30)$$

The δ-function in equation (7.29) is only an approximation of the g-function which broadened the resonance.

The presence of the collision term in equation (7.30) rather changes the electromagnetic properties of a plasma in the low-frequency region. One can say the turbulent Ohmic heating changes the properties of the drift waves. If the spectrum of the turbulence depends only on $|k_1|$ and on x_1, which is the cosine of the angle $\boldsymbol{k}_1$ makes with some chosen direction (for example, the direction of the electric field), then the diffusion coefficients $D_{ij}^{(0)}$ can depend only on $|\boldsymbol{p}|$ and on ξ which is the cosine of the angle $\boldsymbol{p}$ makes with the same direction. Because angular scattering is the most important effect for ion-sound turbulence, in the right-hand side of equation (7.30) only $\partial/\partial\xi$ appears:

$$-i[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]\,f_{1,\varkappa}^{\text{reg}}+e\left(\boldsymbol{E}_{\varkappa}^{\text{reg}}\cdot\frac{\partial f_0^{\text{reg}}}{\partial \boldsymbol{p}}\right)$$
$$= \frac{1}{p^2}\frac{\partial}{\partial\xi}\pi e^2\int x_1^2 I_{\varkappa_1}\,\delta((\boldsymbol{k}_1\cdot\boldsymbol{v}))\,d^4\varkappa_1\,\frac{\partial f_{1,\varkappa}^{\text{reg}}}{\partial\xi}. \qquad (7.31)$$

Equation (7.31) can be rewritten as follows:

$$-i[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]f_{1,\varkappa}^{\text{reg}}+e\left(\boldsymbol{E}_{\varkappa}^{\text{reg}}\cdot\frac{\partial f_0^{\text{reg}}}{\partial \boldsymbol{p}}\right)$$
$$= \frac{2\pi e^2 v_s}{p^2 v\omega_{\text{pi}}}\frac{\partial}{\partial\xi}\int_0^{\sqrt{(1-\xi^2)}}\frac{x^2\,dx}{\sqrt{(1-\xi^2-x^2)}}\int W_{\omega,x}\,\omega\,d\omega\,\frac{\partial f_{1,\varkappa}^{\text{reg}}}{\partial\xi}. \qquad (7.32)$$

One finds thus that the collision integral is just the same as the one which

† For high frequency waves there still exist anomalous collisions, but they are less important than the quasi-linear ones.

describes the collisions of the unperturbed electrons with the ion-sound waves. The difference lies in the possible polarisation of the field $\boldsymbol{E}_{\varkappa}^{\text{reg}}$ which can be non-longitudinal. Because f_0^{reg} is approximately isotropic we have

$$\frac{\partial f_0^{\text{reg}}}{\partial \boldsymbol{p}} = \boldsymbol{v}\,\frac{\partial f_0^{\text{reg}}}{\partial \varepsilon}. \qquad (7.33)$$

If the turbulent collisions occur often enough, one can neglect $\omega-(\boldsymbol{k}\cdot\boldsymbol{v})$ on the left-hand side, and then by averaging over the angles perpendicular to the z-axis one finds clearly for E_z^{reg} the same anomalous conductivity as for the longitudinal field. Thus we have

$$(j_{\varkappa}^{(1)})_z = \int e v_z f_{1,\varkappa}^{\text{reg}}\frac{d^3p}{(2\pi)^3} = \sigma^{\text{turb}}(E_{\varkappa}^{\text{reg}})_z, \qquad (7.34)$$

where σ^{turb} is the turbulent conductivity. The other components are slightly different, but of the same order of magnitude. One obtains the following estimate for the skin-depth for an electromagnetic wave entering the turbulent plasma:

$$\delta = \frac{c}{\sqrt{(4\pi\sigma^{\text{turb}}\omega)}}. \qquad (7.35)$$

7.4. Electromagnetic Properties of Langmuir Turbulence

Neglecting the renormalisation of the plasma Green function one finds the dielectric constant which describes the plasma response to the field $E_{\varkappa}^{\text{reg}}$ as follows:

$$\varepsilon_{\varkappa} = \varepsilon_{\varkappa}^{(l)}+\frac{\omega_{\text{pe}}^2}{k^2 v_{\text{Te}}^2[1+(n_0/v_{\text{Te}}^2)d]}, \qquad (7.36)$$

where $\omega \ll k v_{Te}$, $\omega \ll \omega_1$, $k \ll k_1$, while $\varepsilon_{\varkappa}^{(l)}$ is a normal linear response. The coefficient d has the following form:

$$d = -\frac{e^2}{4m_e^2\omega_{\text{pe}}n_0}\int d^3k_1\frac{4\pi}{\omega-(\boldsymbol{k}\cdot\boldsymbol{v}_{\text{gr}})}\left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{k}_1}\right)W_{k_1}, \qquad (7.37)$$

where $v_{\text{gr}} = d\omega_1/dk_1$ is the group velocity of the Langmuir waves. The instability found from this dielectric constant corresponds to the result first found by Rudakov and Vedenov (1964) and by Gailitis (1966). Such an instability can exist only if one can neglect the non-linear renormalisation of the plasma Green function. Analysis shows that it is usually difficult to satisfy the conditions for the neglect of this non-linear renormalisation. For example, the total turbulent energy must be concentrated in a very narrow interval of the turbulent energy. For the distribution found above for the Langmuir turbulence spectrum, it is very difficult to satisfy these conditions in practice (Tsytovich, 1969b). There still exists the possibility to have a transverse perturbation described by the transverse dielectric constant ε^t.

If $|\varepsilon^t| \gg 1$, we have $|H| \approx \sqrt{(|\varepsilon^t|)}|E| \gg |E|$ which means that the magnetic field is excited, if the plasma becomes unstable.

This kind of instability is similar to that of magnetic field excitation by a conducting turbulent liquid. The growth-rate γ for the excitation of a magnetic type perturbation by Langmuir turbulence was found to be (Tsytovich, 1968b):

$$\gamma = \frac{k^3 v_{Te}}{2\omega_{pe}} \sqrt{\left(\frac{2}{\pi}\right)} \left\{ \sqrt{\left(1+4\sqrt{\left(\frac{\pi}{2}\right)} \frac{v_* \omega_{pe}^2}{v_{Te} k^2 c^2}\right)} - 1 \right\}, \tag{7.38}$$

where

$$v_* = \frac{1}{12} \sqrt{\left(\frac{\pi}{2}\right)} \frac{v_{Te}}{n_0 m_e c^2} \int W_{k_1}\, d^3k_1 \frac{\omega_{pe}^2}{k_1^2 v_{Te}^2}. \tag{7.39}$$

Some inequalities must also be satisfied which give the range of parameters for which this instability can be developed.

One can find also the skin-depth for a low-frequency wave penetrating into a semi-infinite plasma which in the case of $\omega \ll \nu_{eff}$ is described by

$$\delta \approx \frac{c}{\omega_{pe}} \sqrt{\frac{\nu_{eff}}{\omega}}, \tag{7.40}$$

where

$$\nu_{eff} = \omega_{pe} \sqrt{\left(\int \frac{W_{k_1}\, d^3k_1 \omega_{pe}^2}{12 k_1^2 v_{Te}^2 n_0 T_e}\right)}. \tag{7.41}$$

Usually the increase of δ by turbulence is more efficient for ion-sound turbulence than for Langmuir turbulence.

One can thus see that the turbulent collisions give plasma characteristics, which can easily be measured.

The Langmuir turbulence oscillations were shown earlier to be converted into transverse waves with frequencies very close to the plasma

frequency. This looks like a transverse plasmon. The presence of a transverse plasmon can give rise to an instability which is similar to second sound (Tsytovich, 1969c):

$$\gamma \approx k v'_*, \tag{7.42}$$

$$v'_* \approx v_{Te} \sqrt{\left(\frac{W}{4 n_0 T_e}\right)}. \tag{7.43}$$

This type of instability can be excited also by external transverse waves, if their spectrum is sufficiently broad. Usually, some frequency broadening arises even if the initial signal is narrow in frequency. This can be the case when the wave propagates in the direction in which the mean plasma density is increasing until it reaches ω_{pe}. On the other hand, a high-frequency field excites Langmuir waves. When its frequency reaches the region where it can be equal to the Langmuir frequency, $\omega_{pe} + \frac{3}{2} k^2 v_{Te}^2/\omega_{pe}$, the excitation starts from k close to λ_D^{-1}, when the Landau damping becomes unimportant. This is the frequency which is approximately equal to $(1+\frac{1}{3})\omega_{pe}$. Then, if the development of the instability is faster than the characteristic time for a frequency change due to the wave propagation in the direction of the density gradient, an appropriate amount of the wave energy can be transferred into Langmuir waves. This energy then comes back into transverse waves, and so on. This process gives a subsequent non-linear lowering of the electromagnetic wave frequency. In order that transverse plasmons are present, it is necessary that the plasma is optically thick. The presence of Langmuir waves and especially transverse plasmons can change the properties of drift waves (Krivorutskii, Makhankov, and Tsytovich, 1969). This can lead to some kind of drift-wave stabilisation. For example, one can find that kinetic drift instabilities can disappear when

$$\eta = \frac{\partial \ln T_e}{\partial \ln n} > 2. \tag{7.44}$$

This type of stabilisation is different from the high-frequency stabilisation only in the sense that the high-frequency field has a stochastic nature, and, therefore, a broad spectrum, and also the distribution of this field is not arbitrary, but corresponds to the self-consistent picture of the plasma turbulent spectrum.

It is also possible to stabilise some of the hydrodynamic drift-wave instabilities.

For ion-sound turbulence the presence of turbulent collisions plays a role similar to ordinary Coulomb collisions, as they have the same velocity dependence. But the corresponding $\nu_{\rm eff}$ is, of course, very much larger then the Coulomb collision frequency and is also anisotropic, if the ion-sound turbulence is anisotropic (as in the case of a current-driven instability). As a rough approximation, one can estimate the drift instability growth-rate in the presence of ion-sound turbulence by putting $\nu_{\rm eff}$ instead of $\nu_{\rm coll}$. Usually dissipative drift instabilities develop when ion-sound turbulence is present.

We thus see that the presence of turbulence can have an essential influence on plasma confinement.

8. The Cosmic-ray Spectrum

THE plasma that can be found in cosmic conditions is often turbulent, and the turbulent plasma state may be assumed to be a natural one in the universe. The great efforts which are being applied to trying to stabilise plasmas only emphasise this statement. One can then ask the question: what kind of consequences can be deduced from this point of view? The most important of these seems to be a production (or, rather acceleration) of fast particles. From the theoretical treatment given above, it follows that the presence of fast particles in the turbulent regime is a consequence of the presence of turbulence. This means from the point of view of energetics that the creation of cosmic rays (or relativistic particles) by a turbulent plasma does not meet with any essential difficulties. It is also well known that in galaxies there are enough turbulent energetic sources which can give energy to the cosmic rays (see Ginzburg and Syrovatskii, 1964). Such sources may be supernova explosions, active nuclei of galaxies, and so on. It is also known that cosmic rays are produced by solar flares, which represent an example of an explosion on a small scale. The problem of cosmic rays is thus not so much the problem of energy being transferred to fast particles, but the problem of the distribution of this energy between them, or the problem of the cosmic-ray spectrum. The spectrum of cosmic rays produced by solar flares is different from that produced by radio sources such as supernova remnants and radio galaxies. The solar flares have a very steep spectrum which can be represented by $\varepsilon^{-\gamma}$ where γ ($\sim$ 7 to 8) is a large number, or by an exponential spectrum $e^{-\varepsilon/\varepsilon_0}$. The last spectrum exactly corresponds to the predictions from Langmuir turbulence, which creates a relativistic Maxwell distribution of fast particles. The radio sources have a power spectrum $\varepsilon^{-\gamma}$ where γ usually lies in the range $1 < \gamma < 3$ with an average value of γ of 2·5 to 2·6. These are the cosmic electrons. The cosmic ions—mostly protons—have a power-type spectrum near the earth, with $\gamma \sim 2{\cdot}7$. Thus the mean energy of the particles lies near mc^2, but there are a large number of very energetic particles. The question is then how the energy of the particles can be distributed in such a way that a power-type spectrum is created and why the spectrum of solar-flare cosmic rays is so different from the spectrum of radio sources. One may assume that the difference is connected with the dimensions of the sources, and we shall see later that it could be connected with the optical thickness for the plasma emission mechanisms.

8.1. Energy-dependence of the Acceleration Rate

Let us consider the energy dependence of the acceleration rate. Suppose for simplicity that

$$\frac{d\varepsilon}{dt} = \beta\varepsilon^{\mu}. \tag{8.1}$$

The three different cases, $\mu > 1$, $\mu = 1$, and $\mu < 1$, are shown in Fig. 29.

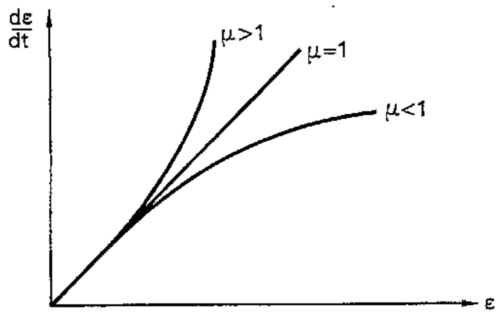

FIG. 29.
The acceleration rate as function of the particle energy.

The characteristic acceleration time τ can be defined as

$$\tau = \frac{\varepsilon}{d\varepsilon/dt} = \frac{1}{\beta}\,\varepsilon^{1-\mu}. \tag{8.2}$$

If $\mu < 1$, the larger the energy, the longer the time needed for acceleration. Thus, a particle injected with $\varepsilon_{\min}$ into the acceleration regime and reaching $\varepsilon_{\max}$ needs to be accelerated in a time that is determined by $\varepsilon_{\max}$.

The opposite occurs for $\mu > 1$. In this case, the larger the energy, the shorter the time needed for acceleration. This means that the maximum possible energy is reached in a time determined by the injection energy.

The case $\mu = 1$—which, for example, corresponds to the Fermi acceleration—is intermediate. In this case, the time is independent of the energy (more precisely, logarithmically dependent).

The difference between the case $\mu < 1$ and $\mu > 1$ plays an important role in the formation of the particle spectrum. When $\mu < 1$ the particle spectrum, that is, the dependence of the particle distribution function f_ε on the energy ε, must have a very important cut-off at sufficiently high energies. For example, in the case of the Langmuir turbulence considered above, the law for the acceleration of relativistic particles is $d\varepsilon/dt = \alpha/\varepsilon$, and if this acceleration is compensated by ionisation losses, the distribution function f_ε has an exponential cut-off. Indeed, as was shown above, the distribution of fast particles forms a relativistic Maxwell distribution with a temperature

$$T_{\text{eff}} \gg mc^2. \tag{8.3}$$

This T_{eff} plays the role of a cut-off factor in the distribution function. We shall normalise the distribution function of the relativistic particles as follows:

$$\int_0^\infty f_\varepsilon\, d\varepsilon = n, \tag{8.4}$$

where n is the total density of fast particles. Because these particles have a maximum energy much higher than mc^2 the non-relativistic part of the distribution function is sometimes of no interest. In this case, $\varepsilon = cp$ and the phase volume $p^2\,dp$ is proportional to $\varepsilon^2\,d\varepsilon$. Thus, the distribution function with $T = T_{\text{eff}}$ can be written in the form:

$$f_\varepsilon \sim \varepsilon^2 e^{-\varepsilon/T_{\text{eff}}}. \tag{8.5}$$

This distribution function seems to be different from that which corresponds to the radio-emission in radio sources. This is the case, because the power-type emission $I_\omega \sim \omega^{-\nu}$ seems to be possible only for a power-type particle distribution. To form a power-type distribution from the distribution (8.5) is difficult. Indeed, the conditions in different parts of the source must be different. The radiation observed is the superposition of the radiation arising in different conditions. Thus, one can average expression (8.5) over a distribution of T_{eff}-values. In order to have a power-type distribution one needs to have a power-type distribution for T_{eff} which seems to be quite an artificial assumption.

This is one example of the consequences of the case when $\mu < 1$. In other examples which can be worked out, the distribution function cannot be exactly of the type (8.5), but it is difficult to believe that in every case with a strong cut-off the distribution of these will be such as to give a power-type spectrum.

The sharp cut-off is perhaps consistent with the cosmic-ray spectrum produced by solar flares, but not with the cosmic electron and cosmic ion spectrum. It is also necessary to emphasise that in the case of Langmuir turbulence the spectrum (8.5) arises only in rather dense plasmas with rather small values of T_{eff}. As we shall see later, Langmuir turbulence creates a power-type spectrum for sufficiently high particle energies.

In the case when $\mu = 1$, the well-known example of the Fermi acceleration leads to similar difficulties. Indeed, it is known that if the time of acceleration is τ, Fermi acceleration leads to a power-type spectrum with

$$\gamma = 1+\frac{1}{2\tau\alpha}, \tag{8.6}$$

where

$$\alpha = \frac{1}{\varepsilon}\,\frac{d\varepsilon}{dt}. \tag{8.7}$$

Both the parameters α and τ can vary in broad intervals so that

$$1 < \gamma < \infty. \tag{8.8}$$

This contradicts observations which give approximately

$$1 < \gamma < 3. \tag{8.9}$$

This fact is normally the basis for rejecting the use of the Fermi acceleration to explain the cosmic-ray spectrum. We want to emphasise here that the greatest difficulty is that the average spectrum will not be a power-type one at all. Indeed, the real spectrum is due to the superposition of different types of spectrum produced in different regions with different α and τ. Thus

$$\bar{f}_\varepsilon = \int \frac{\text{const}}{\varepsilon^\gamma} F(\gamma)\, d\gamma, \tag{8.10}$$

where $F(\gamma)$ is determined by the distribution of α and τ. Only for very special choices of $F(\gamma)$ will $\bar{f}_\varepsilon$ be a power-type function. Thus the Fermi acceleration, as well as any other process with $\mu = 1$, meets a difficulty similar to the $\mu < 1$ case.

One of the reasons that γ varies in such a broad interval as given by equation (8.8) is that the acceleration and energy-loss mechanisms are quite different.

Thus we see that neither $\mu < 1$ nor $\mu = 1$ are appropriate for explaining the power-type spectrum. Thus only the $\mu > 1$ case seems to be possible. Of course, $\mu < 1$ is very effective for small energies, and can, therefore, be considered as a good mechanism for the injection. We shall see that the case $\mu > 1$ can start only at rather high energies, $\varepsilon > \varepsilon_*$, where ε_* can be called the injection energy. If this happens the lowest acceleration rate is for $\varepsilon \approx \varepsilon_*$. One of the interesting points of the acceleration for the case when $\mu > 1$ and $\varepsilon > \varepsilon_*$ is that the highest energies of the cosmic rays are created very rapidly, and one can expect even now to have an acceleration of particles to the highest energies. This question is connected with the problem of forming the highest energies in the cosmic spectrum, which are of the order of 10^{20} eV. One can estimate that if the particles are protons the friction produced by the 3°K black-body radiation essentially reduces their energy on a time scale of 10^9 years. The question is then whether such particles can be accelerated in some regions on a very short time-scale. To have an answer to this question, one does not need a precise model of the source of the cosmic rays. If $\mu > 1$, one need only assume that cosmic-ray creation exists at all, that is, that the rate of acceleration is sufficiently high to compete with the energy losses at ε_*. If this is the case, one can also ask what is the highest possible energy ε_{**} for which μ remains larger than unity. One can ask what the plasma density, temperature, the dimensions of the source, and so on, will be, such that ε_{**} is very high. After this one can ask whether in cosmic conditions these requirements can be met near very energetic sources such as pulsars, quasars or galactic nuclei. Thus the problem of the cosmic ray spectrum becomes of a kind such that it can in principle be answered simply by theoretical considerations.

8.2. Energy-dependence of Resonant Wave-particle Interactions

The rapid rise with energy of the acceleration rate can only come from the rapid rise of the particle-energy diffusion coefficient. This takes place if the larger the particle energy, the larger the spectral interval of turbulent waves interacting with the particles. Thus, for $\mu > 1$ it is necessary first of all to have a broad spectrum of turbulent waves, and secondly to have an essential dependence on particle-energy of the resonant wave-particle interaction. We are now interested in the interaction of ultra-relativistic particles with turbulent oscillations. When

$$\varepsilon \gg mc^2, \tag{8.11}$$

it is necessary to have an energy-dependence of the wave-particle interaction. This provides two important possibilities for finding such a dependence. One comes from the energy dependence of the gyro-frequency:

$$\omega_H = \frac{eHc}{\varepsilon}. \tag{8.12}$$

The resonance condition,

$$\omega - k_z v_z = \omega_H, \tag{8.13}$$

can be considered in two limits: $\omega \gg k_z v_z$ and $\omega \ll k_z v_z$. In the first case, equation (8.13) is reduced to $\omega = \omega_H$ and only one frequency interacts with the particles. If the particle energy increases, ω_H decreases; it is thus only possible to have this resonance over a broad energy interval for a branch of the turbulent oscillations in which the frequency varies over a broad interval. These are mainly only the magnetodynamic and Alfvén waves. For them though, $\omega \gg k_z v_z$ only, if $v \ll c$ and if the angles the particles and waves make with the magnetic field lines are of the order of unity.

In the opposite case, $\omega \ll k_z v_z$, we have

$$k_z v_z = \omega_H, \tag{8.14}$$

or

$$\varepsilon > eH/k, \tag{8.15}$$

and this means that the higher the energy, the smaller the k which can interact with the particles. Therefore, if the turbulent energy increases when k is lowered, one can find the growth of the acceleration rate as function of the particle energy.

Another possibility for the energy-dependence of the wave-particle resonance is an interaction due to the scattering of turbulent oscillations by particles, leading to conversion in a relatively small magnetic field. These resonance conditions have the form

$$\omega - (\boldsymbol{k} \cdot \boldsymbol{v}) = \omega^\sigma - (\boldsymbol{k}^\sigma \cdot \boldsymbol{v}), \tag{8.16}$$

where σ denotes the turbulent wave. In the high-frequency region, one can write $k = \omega/c$, and

$$\omega = \frac{\omega^\sigma - (\boldsymbol{k}^\sigma \cdot \boldsymbol{v})}{1 - (v/c)\cos\theta} < \omega_{\max} = \frac{\omega^\sigma - (\boldsymbol{k}^\sigma \cdot \boldsymbol{v})}{1 - (v/c)}$$

$$= \frac{2\varepsilon^2}{(mc^2)^2}\, \omega^\sigma - (\boldsymbol{k}^\sigma \cdot \boldsymbol{v}), \quad \text{if} \quad \varepsilon \gg mc^2. \tag{8.17}$$

There are also two possibilities as in the case considered above, namely $\omega^\sigma \gg k^\sigma c$ or $\omega^\sigma \ll k^\sigma c$. The first case means that the phase velocity of the turbulent oscillations is larger than c, while it is less than c in the second case. All turbulent oscillations, except Langmuir waves, have a phase velocity less than c.

In this case, only the first-order resonance wave-particle interaction is of interest (see equation (8.13)), as it can be proved that the second-order effect is always small, if the broadening of the resonance is taken into account.

On the other hand, it was shown that the energy of the Langmuir oscillations is mainly concentrated in the region where the phase velocity is larger than c.

The condition (8.17) then has the form:

$$\omega < 2\left(\frac{\varepsilon}{mc^2}\right)^2 \omega^\sigma. \tag{8.18}$$

This means that the higher the particle energy, the higher the frequency of the transverse waves that can interact with the particles. In order to have $\mu > 1$, it is necessary that the energy of the transverse waves increases rapidly with frequency. We shall see that this naturally arises because of a re-absorption of high-frequency radiation by fast particles.

8.3. Acceleration by Low-frequency Turbulence

Let us start by analysing the lowest possible frequencies of turbulent oscillations, which are due to magnetohydrodynamic and Alfvén oscillations. Most of the energy of such oscillations is contained in the cold plasma, but not in the electric field. This leads to the conclusion that only magnetic types of turbulent motion can give an effective acceleration.

Indeed, the diffusion coefficient for the sound wave $\omega \approx kv_s \cos\theta$ ($\omega \ll \omega_{Hi}$) is approximately constant, if the particle velocity is near to c. Therefore, $\varepsilon^2 = Dt$ = const t or ε/t = const/ε, and $\mu = -1$. Thus, the most interesting case is $v_A \gg v_s$. The two types of turbulent waves (Alfvén waves, $\omega \approx kv_A \cos\theta$, and fast magnetohydrodynamic waves, $\omega \approx kv_A$) accelerate in the same way. The distribution of these waves in k or in $\omega = kv_A$ is shown in Fig. 30.

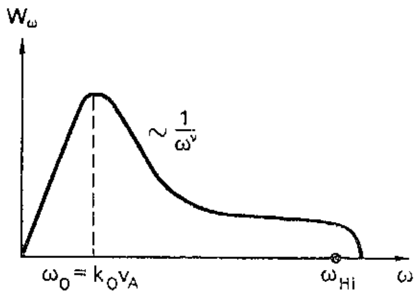

FIG. 30.

The spectrum of collisionless magnetohydrodynamic turbulence; $k_0 \sim 1/L_0$, where L_0 corresponds to the characteristic scale in the energy-containing region. Here k_0v_A is of the order the ion–ion collision frequency ν_i.

The value k_0 corresponds to the energy-containing region. The drop near ω_{Hi} is due to the ion cyclotron-resonance absorption. In the collisionless region the energy flows in the direction of lower frequencies. The value k_0 may correspond to $1/L_0^p$, where L_0^p is a dimension of the plasma, or to $\omega_0 = \nu_i$, where ν_i is the ion–ion binary collision frequency, provided $v_A/L_0^p \ll \nu_i$. In the region $\omega \ll \nu_i$ the ordinary collision-dominated magnetohydrodynamics lead to energy transfer to the highest frequencies. Therefore, one can expect for sufficiently large astrophysical objects that $k_0 \approx \nu_i/v_A$. If v_A is of the order of the ion thermal velocity, k_0^{-1} is of the order of the mean free path of cold ions in the plasma. The acceleration of particles due to the spectrum of Fig. 30 is shown in Fig. 31. Here ε_{**} is of the order of the energy for which the gyro-radius of the accelerated particle reaches L_0; we find thus from equation (8.15):

$$\varepsilon_{**} = \frac{eH}{\nu_i} v_A . \tag{8.19}$$

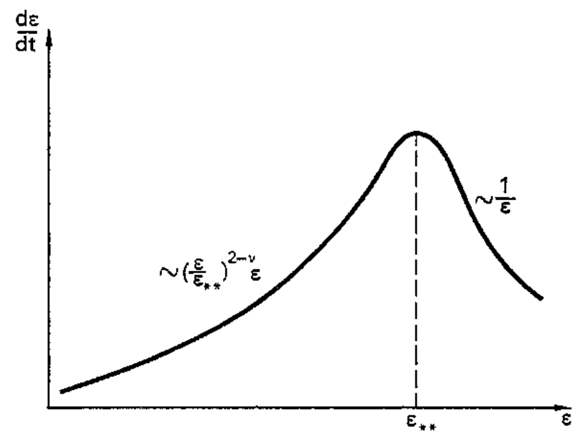

FIG. 31.

The acceleration rate of particles which are accelerated by collisionless magnetohydrodynamic turbulence.

The rise of the acceleration rate with increasing particle energy is due to the fact that the resonant condition (8.15) raises the number of waves interacting with the particles. The decrease of $d\varepsilon/dt$ when $\varepsilon > \varepsilon_{**}$ is due to the fact that all the waves in the spectrum are already included in the acceleration of the particles. The coefficient α is of the order of

$$\alpha = \frac{v_A^2}{L_0 c} \frac{W}{H^2/8\pi}, \tag{8.20}$$

where W is the turbulent energy and is of the order of $\langle H_\sim^2/8\pi \rangle$ where $H_\sim$ is the turbulent magnetic field associated with the Alfvén and magnetohydrodynamic waves.

One can compare equation (8.20) with the Fermi acceleration coefficient:

$$\alpha_F = u^2/Lc, \tag{8.21}$$

where u is the velocity of the magnetic clouds and L the distance between them. One can see that v_A plays the role of u, and L_0 the role of L in this comparison, and there is an extra factor $\langle H_\sim^2 \rangle / H^2$ which is usually $\ll 1$.

The acceleration by magnetohydrodynamic and Alfvén waves can, therefore, be considered equivalent to the Fermi acceleration in modern turbulent plasma theory. One concludes then that this type of acceleration is always weaker than the Fermi type. This is so because of the factor $\langle H_{\sim}^2\rangle/H^2$ and also the factor $(\varepsilon/\varepsilon_*)^{2-\nu}$, if $\nu < 2$. If $\nu > 2$, we have $\mu < 1$ and this does not seem to be a case of interest. The case $\nu < 1$ can be shown to be unstable for the formation of a turbulent spectrum (Lifshitz and Tsytovich, 1970). If $1 < \nu < 2$, we have $2 > \mu > 1$. If $\mu > 1$, that is, $\nu < 2$, the factor $(\varepsilon/\varepsilon_*)^{2-\nu}$ is always small, except when $\varepsilon \sim \varepsilon_*$, and the acceleration rate is less than for the Fermi case. The essential point is that L_0 is very large, of the order of v_A/v_i, and, therefore, the α of equation (8.20) is rather small. There exists some speculation in the literature about the possibility of raising α in collisionless regions by decreasing L_0. In all these estimates, it is necessary to consider more carefully the possible time that such a spectrum with small L_0 can exist. This time can be determined by: (i) non-linear spreading of the turbulent energy which raises L_0, (ii) absorption of the turbulent energy by cold plasma particles, (iii) absorption due to fast particle acceleration. The last possibility was considered by Tverskoi (1968) and can only occur when the first two possibilities do not work. One can estimate the characteristic time of the non-linear interaction to be of the order of

$$\frac{1}{\tau^{\text{n.l.}}} \sim \frac{H_{\sim}^2}{H^2} k v_A . \tag{8.22}$$

If one substitutes here for k the minimum value which corresponds to equation (8.15), one finds

$$\frac{1}{\tau^{\text{n.l.}}} = \omega_{pi} \frac{v_A^2}{c^2} \frac{m_i c^2}{\varepsilon} \frac{H_{\sim}^2}{H^2} . \tag{8.23}$$

The non-linear interaction which raises L_0 is unimportant only if the characteristic time of the non-linear interaction is greater than the acceleration time. Overestimating the last one as $1/\alpha$ (for $\varepsilon = \varepsilon_*$) we find the following inequality, which shows when one can neglect the non-linear interactions:

$$L_0 \ll \frac{c}{\omega_{pi}} \frac{\varepsilon}{m_i c^2} . \tag{8.24}$$

However, the minimum L_0 is c/ω_{pi} ($\omega \sim \omega_{Hi}$) and this means $\varepsilon/m_i c^2 \gg 1$. The last condition can be satisfied only for ultra-relativistic or heavy ions.

On the other hand, ε of the accelerated particles cannot be very high, because their Larmor radius must not exceed L_0 if they are to be efficiently accelerated. This gives:

$$\frac{\varepsilon}{m_i c^2} < \frac{L_0 \omega_{Hi}}{c} , \tag{8.25}$$

which together with equation (8.24) gives

$$\frac{\omega_{Hi}}{\omega_{pi}} = \frac{V_A}{c} \gg 1. \tag{8.26}$$

This last relation needs extremely high magnetic fields. This means that one can find a turbulent spectrum where L_0 is determined by the fast particle acceleration, only in the extremely peculiar cases of very high magnetic fields, and high particle energies. Even under these conditions, efficient acceleration can exist only in a very narrow particle energy interval, and the spectrum produced is not a power-type spectrum. Therefore, one cannot apply the results found from the theory of a spectrum determined by fast-particle acceleration to explain both the acceleration of non-relativistic particles in the interplanetary space and the cosmic particles in the galaxy.

To be certain that the other processes cannot diminish L_0, one needs also to estimate the damping of the oscillations by the cold particles. This occurs mainly for the magnetohydrodynamic oscillations. However, the non-linear interactions very rapidly exchange the energy between the magnetohydrodynamic and the Alfvén oscillations. Both the damping and the non-linear interactions are proportional to ω and this means that if the non-linear transfer exceeds the damping for one ω, it exceeds it also for others. Thus the damping cannot form a cut-off of the turbulent spectrum at a particular value of ω or be responsible for shortening L_0. The inequality which shows when the non-linear interactions are stronger than the damping has approximately the following form:

$$\frac{H_{\sim}^2}{H^2} > \sqrt{\frac{m_e}{m_i} \frac{v_s^2}{v_A^2}} . \tag{8.27}$$

If equation (8.17) is satisfied one can compare the fast-particle acceleration rate with the non-linear interactions and this gives relation (8.24). If equation (8.27) is not satisfied, the damping exceeds the non-linear interactions and the turbulent spectrum is not formed. One can, of course, have an instability in a broad frequency interval, but then the energy of the turbulent oscillations is increased to the value for which the non-linear interactions become important.

These estimates show that the spectrum of turbulence is mainly determined by the non-linear interactions and not by the particle acceleration. This means that L_0 is rather large and excludes any speculations about diminishing L_0 in the collisionless region. This means that we return to the results found in the paper by Ginzburg, Pikelner, and Shklovskii (1955) who estimated the acceleration of particles in supernova remnants; in this paper the L_0 in the Fermi acceleration was taken as the mean free path of the cold particles ($v_A \sim v_{Ti}$ in this case). There is no possibility of applying the results of low-frequency magnetohydrodynamic turbulence to cosmic-ray acceleration in the interstellar medium. Thus, from the modern theory of plasma turbulence it looks as if Fermi-type acceleration mechanisms are rather unlikely. The conclusion that this acceleration is impossible in the interstellar medium, which one can find in the book by Ginzburg and Syrovatskii (1964), is even more conclusive.

However, these types of turbulence, regulated by non-linearity, can give an efficient injection especially of heavy ions (that is, they give preferential ion acceleration) as was shown by Melrose (1969).

The spectrum of the accelerated particles can be shown never to be a power-type spectrum, except in the peculiar case where $\nu = 2$. In this case, the acceleration is of the Fermi type and all the difficulties connected with it were already discussed.

The reason why the spectrum is not a power-type one can be understood if we mention that the acceleration is due to the induced emission and absorption of magnetohydrodynamic waves, as the spontaneous emission is always negligible compared to the fast particle ionisation energy losses. Thus the acceleration and energy losses are due to quite different physical processes.

If the acceleration by magnetohydrodynamic motion is not very efficient, one can ask what happens if one considers the different branches, or raises an effective L_0 by considering the turbulent excitations of higher frequencies. The first of them is the whistler mode. Indeed such motions are known to be excited in the magnetosphere of the Earth, or in mirror magnetic confinement installations. One can find that at least these motions can give effective accelerations of non-relativistic particles, and for the ultra-relativistic particles the diffusion coefficient is approximately constant (Tsytovich and Chikhachev, 1969) and therefore $\mu = -1$.

8.4. Acceleration by High-frequency Turbulent Oscillations

As was already mentioned, even the $v_{ph} \ll c$ part of the Langmuir turbulence spectrum gives an efficient acceleration of particles. The effective temperature of the particles easily reaches ultra-relativistic values. However, the spectrum of the accelerated particles is not a power-type one for $\varepsilon \ll T_{eff}$:

$$f_\varepsilon \approx \text{const}\cdot\varepsilon^2; \qquad \exp(-\varepsilon/T_{eff}) \approx 1. \tag{8.28}$$

The normalisation of expression (8.28) is

$$\int f_\varepsilon \, d\varepsilon = n. \tag{8.29}$$

The cut-off of the spectrum at T_{eff} is due to ionisation losses. There exist conditions for which the change of the spectrum of the fast particles arises at energies much less than T_{eff}, so that the ionisation losses do not become essential at all. We shall see that this change leads to a power-type spectrum.

First of all, we mention that the high acceleration rate by Langmuir turbulence oscillations is due to the fact that their frequency is rather high. We have already shown that the most effective acceleration by magnetohydrodynamic oscillations occurs at a frequency ω of the order of ν_i. On the other hand, the plasma frequency ω_{pe} is larger than that by at least a factor $\sqrt{(m_i/m_e)}\,N_D$, where N_D is the number of particles in the Debye sphere, which in astrophysical conditions is very high. An idea of the order of magnitude of N_D is given by the values in Table 8.1. The Fermi acceleration is due to the reflection of the particles from the magnetic clouds. The acceleration by magnetohydrodynamic waves is due to the change of the particle energy in the case when the particle is not reflected, but propagated through a layer with an increasing magnetic field. This effect is smaller than that due to reflection by a factor $H_\sim^2/H^2$. The less the distances between these layers, the more frequent are the "collisions" of particles with the magnetic waves and the larger the acceleration rate. The same is true for Langmuir waves, which can be considered as consecutive charged layers with opposite signs of the charge. As in the case of magnetohydrodynamic turbulence, the non-linear effects increase the wavelength and decrease the acceleration. There exists,

TABLE 8.1. NUMBER OF PARTICLES IN A DEBYE SPHERE FOR VARIOUS PHYSICAL SYSTEMS

Type of plasma	n in cm^{-3}	T_e in eV	N_D
Thermonuclear	10^{14}	5×10^4	10^9
Tokomak	2×10^{13}	1500	$1{\cdot}3\times10^7$
Laser	10^{22}	10^4	10^4
Stellarator	10^{13}	100	3×10^5
Radiation belt	10^3	4	$2{\cdot}6\times10^8$
Solar corona	10^8	100	10^8
Interstellar medium	0·03	1	5×10^{10}
Photosphere	10^{15}	0·5	10
Cosmic rays	10^{-7}	10^9	10^{26}
Pulsars	10^{10}	10^9	3×10^{17}

however, an essential difference between Langmuir and magnetohydrodynamic turbulence in that this acceleration does not decrease indefinitely. This is the result of increasing the phase-velocity of the waves which becomes greater than the light velocity. If this happens, the turbulent oscillations become non-resonant with the particles or, more exactly, Cherenkov resonance is not possible for them. In the region $v_{ph} \gg c$, where most of the turbulent energy is concentrated, another process of scattering which is described by equation (8.17) becomes important. For the resonance Cherenkov interaction the minimum wavelength is thus the order of

$$L_{\min} \approx \frac{c}{\omega_{pe}}, \tag{8.30}$$

compared to

$$L_0 = \frac{v_A}{v_i} \approx \frac{v_A}{c}\sqrt{\frac{m_i}{m_e}}\,\frac{c}{\omega_{pe}} N_D \tag{8.31}$$

for the magnetohydrodynamic turbulence. The increase of the power input Q in the Langmuir spectrum raises the L_0 of this spectrum that determines the position of the maximum $k_0 \sim 1/L_0$ (sometimes k_0 is not changed, as was already noticed). This does not affect the resonance Cherenkov interaction, though, because in most cases the phase-velocity of Langmuir waves which corresponds to k_0 is much higher than the velocity of light.

One must mention also that if the plasma has very large dimensions so that it is optically thick for the non-linear transformation of Langmuir waves into transverse waves, the conversion of longitudinal into transverse waves becomes important. This mainly happens when the frequencies of the transverse and of the longitudinal waves are equal, that is, $\omega^t = \omega^l$, or

$$\omega_{pe} + \frac{k^2c^2}{2\omega_{pe}} = \omega_{pe} + \frac{3}{2}\,\frac{k^2 v_{Te}^2}{\omega_{pe}}, \tag{8.32}$$

so that the frequency of the transverse waves are very close to the plasma frequency. In this process of conversion the frequency is slightly lowered. The energy converted into transverse waves is again converted into longitudinal waves. This subsequent oscillation of the energy between transverse and longitudinal oscillations for $v_{ph} \ll c$ was investigated numerically by Kaplan and Tsytovich (1967). An essential point is that these transverse motions usually have frequencies which are lower than those of the longitudinal modes, and they thus take part in the process of the frequency lowering. Therefore, they are so closely connected with the plasma properties that one needs to consider them as special transverse plasmons which are present in the plasma. The Compton scattering by longitudinal and by transverse plasmons does not depend on their distribution in $\boldsymbol{k}$-space, as is obvious from the fact that the resonance condition (8.18) is independent of k for $k \ll \omega_{pe}/c$. Moreover, the probabilities for both Compton effects are equal and the result is independent of the subsequent transformation of the energy from longitudinal to transverse plasmons and back, in the region $v_{ph} \gg c$, and depends on the sum of their energies,

$$W = W^l + W^t, \tag{8.33}$$

only, and this sum is constant. The relativistic particles suffering Compton scattering while condition (8.33) holds are limited to high frequencies $\omega \gg \omega_{pe}$ as follows from relation (8.17). Their emissivity was found for power-type particle spectra, $f_\varepsilon = \text{const}\ \varepsilon^{-\gamma}$, in a paper by Tsytovich and Chikhachev (1970). If Q_ω is the spectral density of the power emitted, Q,

$$Q = \int Q_\omega\, d\omega, \tag{8.34}$$

and if

$$f_\varepsilon = \frac{n(\gamma-1)}{\varepsilon_*}\left(\frac{\varepsilon_*}{\varepsilon}\right)^{\gamma}, \tag{8.35}$$

we have

$$Q_\omega = \frac{\omega_{pe}^3}{6\pi}\left(\frac{2\omega_{pe}}{\omega}\right)^{(\gamma-1)/2}\frac{n_1}{n_0^2}(\gamma-1)\left(\frac{m_e c^2}{\varepsilon_*}\right)^{\gamma-1}\frac{W(\gamma^2+4\gamma+11)}{(\gamma+1)(\gamma+3)(\gamma+5)}. \tag{8.36}$$

This emissivity has the same dependence on the frequency as in the case of synchrotron radiation. Both effects are similar—synchrotron radiation is due to the curvature of the particle in an external magnetic field and this gives an emission with a frequency of the order of

$$\omega = \frac{eH}{mc}\left(\frac{\varepsilon}{mc^2}\right)^2, \tag{8.37}$$

while the plasma mechanism gives an oscillation with ω_{pe} and emission at

$$\omega = \omega_{pe}\left(\frac{\varepsilon}{mc^2}\right)^2. \tag{8.38}$$

In the general case, both mechanisms must be taken into account. The curvature of the particle is not essential in the plasma mechanism if the wavelength is less than the gyro-radius, or

$$k_0 \gg \frac{eH}{\varepsilon}. \tag{8.39}$$

Only at this point does the L_0 of Langmuir turbulence appear. The condition (8.39) can be written in the form

$$\frac{\varepsilon}{mc^2} > \xi\frac{\omega_{pe}}{ck_0}\frac{m_e}{m}. \tag{8.40}$$

$$\xi = \frac{eH}{m_e c}\frac{1}{\omega_{pe}}. \tag{8.41}$$

As we shall see, this condition is usually fulfilled if $\xi < 1$, as is normally the case under astrophysical conditions. This results also from the fact that a power-type spectrum is formed at rather high energies.

The next problem is that of the re-absorption coefficient for this high-frequency radiation which is emitted. The emission considered above is

due to spontaneous emission. The balance of the induced emission and absorption leads, for an isotropic particle distribution, to a positive re-absorption coefficient:

$$\gamma_\omega = \frac{\pi}{24}\left(\frac{2\omega_{pe}}{\omega}\right)^{(\gamma+4)/2}\omega_{pe}\frac{n_1(\gamma-1)W(\gamma^2+6\gamma+16)}{n_0^2\varepsilon_*(\gamma+4)(\gamma+6)}\left(\frac{mc^2}{\varepsilon_*}\right)^{\gamma-2}. \tag{8.42}$$

If the synchrotron emission is also taken into account, one must add the two re-absorption coefficients which are due to the plasma and to the synchrotron mechanism. If W is of the order of $H^2/8\pi$ the plasma mechanism gives much larger re-absorption, if $\xi < 1$. The equation of radiation transfer has in stationary conditions the form:

$$\frac{dI_\omega}{dt}+c(\boldsymbol{n}\cdot\nabla I_\omega) = Q_\omega-\gamma_\omega I_\omega, \tag{8.43}$$

where I_ω is the spectral density of the radiation. Therefore, we find (see Tsytovich and Chikhachev, 1969b) as a stationary and homogeneous solution

$$I_\omega = \alpha\omega^{5/2}, \tag{8.44}$$

where

$$\alpha = m_e\omega_{pe}^2\frac{4}{\pi^3}\left(\frac{1}{2\omega_{pe}}\right)^{5/2}\frac{W(\gamma^2+4\gamma+11)+\xi^{(\gamma-3)/2}a_1(\gamma)\dfrac{H^2}{8\pi}}{W(\gamma^2+6\gamma+16)+\xi^{(\gamma-2)/2}a_2(\gamma)\dfrac{H^2}{8\pi}}. \tag{8.45}$$

This kind of spectrum in the re-absorption region is well known for synchrotron mechanisms of emission and arises when both the plasma and synchrotron mechanisms are taken into account. The meaning of this is very simple. One can write $I_\omega \sim \omega^2 T_{eff}$; $T_{eff} \sim \varepsilon \sim \sqrt{\omega}$. Thus intensive transverse waves with frequencies much higher than ω_{pe} are created. As we have seen, the shorter the wavelength, the larger the acceleration rate. One can then consider the transverse-waves acceleration. The wavelength of the transverse wave is

$$\lambda = c/\omega. \tag{8.46}$$

It is obvious that the higher the frequency the smaller λ, but the spectrum (8.44) cannot be true up to infinitely high ω because the total energy of the transverse waves then diverges. There are three possibilities for changing equation (8.44). Firstly, relation (8.44) is not satisfied for

$\omega > \omega_*$, if the region becomes optically thin. If L is a characteristic dimension of the system, this ω_* can be found from equation (8.42):

$$c/\gamma_{\omega *} = L. \tag{8.47}$$

The scattering of the transverse waves by the turbulent waves decreases the optical depth and leads to the system being optically thick for higher frequencies. The γ_{ω_*} of equation (8.47) must be replaced by (see Kaplan and Tsytovich, 1967)

$$\gamma_{\omega *}^{\text{eff}} = \sqrt{(\gamma_{\omega *}\sigma_{\omega *})}, \tag{8.48}$$

for the case when $\sigma_{\omega_*} \gg \gamma_{\omega_*}$. Here σ_{ω_*} is a scattering cross-section which is approximately constant up to $\omega \approx (c/v_{Te})\omega_{pe}$ and which is of the order of

$$\sigma = 4\sqrt{\frac{m_i}{m_e}\,\frac{v_{Te}^3}{c^3}\,\omega_{pe}}\sqrt{\left(\frac{Q}{\omega_{pe}n_0T_e}\right)}, \tag{8.49}$$

while it decreases with increasing frequency as ω^{-4}, when $\omega > (c/v_{Te})\omega_{pe}$. The second possibility for changing equation (8.44) at high frequencies lies in quantum corrections. These are important when $\hbar\omega$ is of the order of ε, or when

$$\omega \sim \omega_{pe}\left(\frac{\varepsilon}{mc^2}\right)^2, \quad \text{if} \quad \frac{\varepsilon}{mc^2} > \frac{mc^2}{\hbar\omega_{pe}}, \tag{8.50}$$

or

$$\omega > \omega_q = \omega_{pe}\left(\frac{mc^2}{\hbar\omega_{pe}}\right)^2. \tag{8.51}$$

The quantum corrections are essential only if $\omega_q < \omega_*$, which can happen only in a very dense, large-sized and turbulent region. Usually the restrictions on equation (8.44) are under astrophysical conditions due to the effect of the optical thickness; however, one cannot exclude the other possibility in the central core of galaxies or quasars.

The third possibility is the non-linear interaction of transverse waves with one another which may become important as the spectral density I_ω increases with frequency. This non-linear interaction has the form

$$\frac{\partial I_\omega}{\partial t} = \gamma_\omega^{\text{n.l.}} I_\omega = \pi\frac{\omega_{pe}^4}{nm} I_\omega \frac{\partial}{\partial\omega}\frac{I_\omega}{\omega}. \tag{8.52}$$

Introducing into $\gamma_\omega^{\text{n.l.}}$ the I_ω from equation (8.44), we have

$$\gamma_\omega^{\text{n.l.}} \sim \omega_{pe}\frac{\omega_{pe}^3}{c^3 n_0}\left(\frac{\omega}{\omega_{pe}}\right)^{1/2}. \tag{8.53}$$

Comparing this relation with equation (8.42), we find the critical $\omega = \omega^{\text{n.l}}$ for which expression (8.53) dominates:

$$\omega^{\text{n.l.}} \approx \omega_{pe}\left[\frac{n_1}{n_0}\,\frac{W}{n_0T_e}\,\frac{c}{v_{Te}}N_D\right]^{2/(\gamma+5)}\left(\frac{mc^2}{\varepsilon_*}\right)^{2(\gamma-1)/(\gamma+5)}. \tag{8.54}$$

This quantity $\omega^{\text{n.l.}}$ is rather large, because of the factor N_D which is usually very large. When $\omega > \omega^{\text{n.l.}}$ we have $I_\omega \sim \omega$, and there is no longer a rapid acceleration.

The very rapid increase of energy of the transverse waves with frequency given by equation (8.44) in regions where it is valid leads to a very rapid increase of the acceleration rate with particle energy. Indeed, corresponding to relation (8.18) the higher the particle energy, the higher the frequency interval of the spectrum (8.44) which is involved in the acceleration process. This acceleration is due to the gain of energy by particles when the waves are re-absorbed.

The spontaneous process and induced process of scattering of particles by the turbulence W must be taken into account; this was done already for the emitted radiation in equation (8.43). The same effect changes the distribution function of the particles:

$$\frac{\partial f_\varepsilon}{\partial t} + (v\cdot\nabla f_\varepsilon) = \frac{\partial}{\partial\varepsilon}D\varepsilon^2\frac{\partial}{\partial\varepsilon}\frac{f_\varepsilon}{\varepsilon^2} + \frac{\partial}{\partial\varepsilon}Af = 0. \tag{8.55}$$

The second term on the right-hand side of equation (8.52) corresponds to the spontaneous emission. One can calculate the energy change of the particles due to this term:

$$\frac{\partial\bar{\varepsilon}}{\partial t} = \frac{\int \varepsilon\frac{\partial f_\varepsilon}{\partial t}\,d\varepsilon}{\int f_\varepsilon\,d\varepsilon} = \frac{\int \varepsilon\frac{\partial}{\partial\varepsilon}Af_\varepsilon\,d\varepsilon}{\int f_\varepsilon\,d\varepsilon} = -\frac{\int Af_\varepsilon\,d\varepsilon}{\int f_\varepsilon\,d\varepsilon} = \bar{A} \tag{8.56}$$

Thus we have

$$A_{pl} = \frac{d\langle\varepsilon\rangle}{dt} = \frac{8\pi}{9}\,\frac{e^4}{m^2}\,\frac{\varepsilon^2}{m^2}W. \tag{8.57}$$

These losses have the same energy-dependence as the synchrotron losses and must be considered together with them:

$$A_{\mathrm{pl}}+A_{\mathrm{synch}} = A = \frac{8\pi}{9}\frac{e^4}{m^2}\frac{\varepsilon^2}{m^2}\left(W+6\frac{H^2}{8\pi}\right). \tag{8.58}$$

This term is balanced by spontaneous emission, that is, by the first term on the right-hand side of equation (8.43):

$$A+\int Q_\omega\, d\omega = 0. \tag{8.59}$$

The induced processes which in equation (8.43) are represented by the second term are balanced by the diffusion term in equation (8.52). If one calculates the quantity D in equation (8.52), one finds it proportional to ε^3, which results from the rapid increase in frequency of I_ω given by equation (8.44). The quantity D is proportional to $\varepsilon^{-2}\int I_\omega\, d\omega/\omega$, which is proportional to ε^3, if one takes into account that $\omega \sim \varepsilon^2$. This ε-dependence of D leads to an acceleration rate which is proportional to ε^2:

$$\frac{d\varepsilon}{dt} = \beta\varepsilon^2. \tag{8.60}$$

The same type of acceleration arises for synchrotron re-absorption, which was not considered properly earlier (more precisely speaking, the influence of the re-absorption on the particle distribution). Both the synchrotron and the plasma effects must be taken into account in equation (8.60). The stationary solution of equation (8.52) for the particle distribution function gives a spectrum of particles. If $D \sim \varepsilon^3$ and $A \sim \varepsilon^2$, this spectrum has the form $\varepsilon^{-\gamma}$. Because all the coefficients included depend on γ (see, for example, equation (8.45)) γ must satisfy an equation of the form

$$F(\xi, \gamma, x) = 0, \tag{8.61}$$

where $x = -\ln (W/nmc^2)$. The solution of this gives γ as a function of ξ and γ. The result is shown in Fig. 32.

All values of γ lie in the interval $0{\cdot}9 < \gamma < 3$ which corresponds to the observed values in radio-sources. There exists a special value of γ which is of the order of 2·6 to 2·7, for which the change of W over 2 or 3 orders of magnitude does not appreciably change γ.

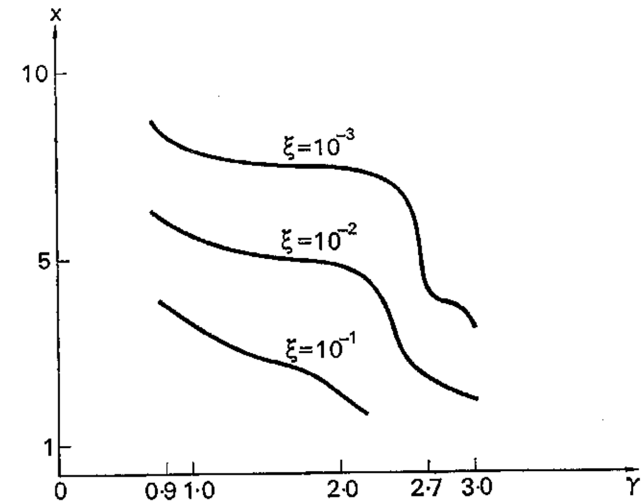

FIG. 32.

Results of a numerical solution giving γ as a function of x and ξ.

The reason why γ is found to lie in a small interval is that both the energy losses and the particle acceleration are due to the same (spontaneous and induced) process. One can expect under these conditions that the super-position of the spectra from different regions leaves the power-type character of the spectrum largely unchanged. In these calculations, the results were averaged over the directions of the magnetic field, assuming that it changes over distances which are large compared to the gyro-radius. The change of γ from 0·9 to 3 corresponds to a greater and greater domination of the plasma mechanisms. Nevertheless, the γ produced by the mechanism considered lie in the range corresponding to observation.

Many problems must still be solved in order that the results can be applied to constructing radio-source models. The first of these is to take into account the particle anisotropy and to find a self-consistent solution in the presence of regular magnetic fields. This may also give a maser effect and polarisation of the emitted radiation.

The second problem is to take into account the resonance diffusion coefficient due to particle acceleration by waves with $\omega/k < c$. This gives the connection of the distribution function (8.28) with a power-type distribution function. The energy corresponding to this junction can be called an injection energy ε_* in this acceleration mechanism. An estimate for ε_* can be found from the intersection of the curves for the acceleration rates $d\varepsilon/dt$ due to resonance ($\omega/k \ll c$) and due to non-resonance plasmons, as shown in Fig. 33.

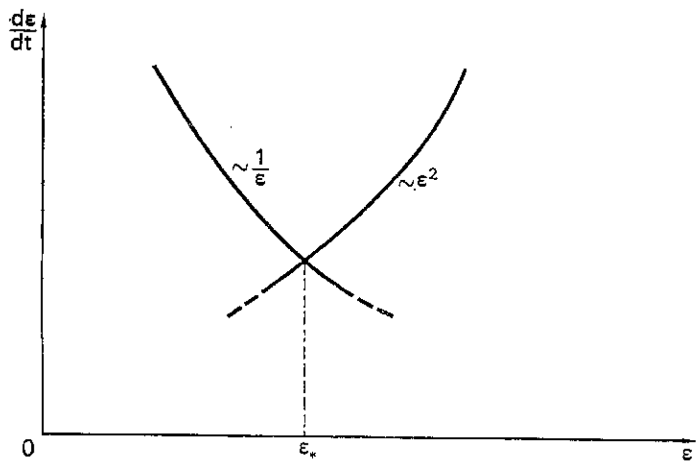

FIG. 33.

Acceleration rates caused by resonant ($\sim\varepsilon^{-1}$) and by non-resonant ($\sim\varepsilon^2$) plasma oscillations. The intersection point $\varepsilon = \varepsilon_*$ corresponds to the injection energy into the fast-acceleration region ($\sim \varepsilon^2$).

Usually $\varepsilon_* \gg mc^2$ so that condition (8.40) is satisfied. The power-type spectrum disappears if $\varepsilon_* > T_{\text{eff}}$, or if $\varepsilon_* > \min\{\varepsilon_{**}, \varepsilon_q, \varepsilon^{\text{n.l.}}\}$, as we saw earlier. Thus one can find conditions for which a particle spectrum with a fast cut-off can be produced. In any case, to produce a power-type spectrum it seems necessary to have the system optically thick for relativistic-particle emission. This is not the case for solar flares and this may lead to a possibility for an explanation why the cosmic-ray spectrum produced by solar flares is very steep.

The third problem is to take into account the effect that the refractive index $n = [1-(\omega_{pe}^2/\omega^2)]^{1/2}$ is not equal to unity for the synchrotron mechanisms (the plasma mechanisms are not affected by $n \neq 1$, if $\omega \gg \omega_{pe}$).

The fourth problem is to take into account the exchange of energy

between cosmic electrons and cosmic ions. If only heavy ions are injected, by low-frequency oscillations, they can in a dense and turbulent plasma have the same power-type spectrum if $\varepsilon \gg (m_i c^2)(m_i/m_e)$.

The last problem, which is the most important one for astrophysical applications, is to construct a model in which the particle and turbulence distributions are spatially dependent. Some of these models can include in the centre a very turbulent and dense region in which high-energy electrons and ions with power-type spectra are formed. They diffuse into the outer regions, excite the turbulent motions, then scatter off them, change the distribution function and penetrate to larger distances from the central regions. This picture, in principle, can give an answer to the spectrum of particles and waves emitted by this region and connect the spectrum of particles produced in the inner, optically thick, regions with the spectrum of particles leaving the source. One can expect that through this diffusion to outer regions the ε_* of the particle distribution is lowered and forms a distribution with relatively small ε_*. If this picture is constructed, one must face the problem of the energy source of the central region, and this, of course, poses other purely astrophysical and more general questions. We must emphasise one point. We can say that we know that it seems that an acceleration by the induced Compton effect can lead to a narrow γ-distribution of the particles accelerated in the range that corresponds to the values observed. Further development of the theory must produce answers to the questions which we have just stated.

Conclusions

The preceding text can only represent the present state of the development of the theory of plasma turbulence. The author has tried to follow the logic, but not the history of this field and, therefore, the references are very fragmented and not by any means complete. The essential physical statements that the author wants to emphasise finally are:

1. The plasma properties in the turbulent region are mostly non-linear. This raises the possibility of universal plasma properties like a universal spectrum that can be independent of the type of instability.

2. Nevertheless, the turbulence is often weak: $W/nT \ll 1$, and when describing the properties of the turbulent oscillation interactions it is not possible to expand the non-linear interactions in terms of the turbulent energy. The elementary excitations such as plasmons and "dressed" particles have thus a finite lifetime which is connected with their non-linear interactions.

3. The small low-frequency perturbations in a turbulent plasma have quite a different nature because of the frequent turbulent collisions, and the dielectric constant that describes such perturbations cannot be expanded in terms of the turbulent energy.

4. The development of a turbulent state is very probable for a plasma as a result of the fact that the energy applied has a tendency to disperse to the greatest possible degree of freedom. Innumerable numbers of different plasma instabilities can bring the plasma to a turbulent state. The plasmas in astrophysical conditions must, therefore, often be turbulent. This can lead to a way of explaining cosmic-ray origins with a universal power-type spectrum.

5. The development of plasma turbulence can occur as a result of development, firstly, of one or a small number of collective modes, with a subsequent spread of the energy to other modes by non-linear interactions as well as by the excitation of many modes at the first stage. For the case of the excitation of one mode, the first stage is not turbulent

and the turbulence develops as the energy is spread, if the system is ergodic. The plasma collective motions seem to be the best test for an investigation of the general problems of the development of the randomisation process, as well as of the general problems of the possibility of a statistical description of a system.

Further experimental and theoretical developments of this problem seem to be one of the future interests in this field.

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