NONLINEAR PLASMA THEORY

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EDITOR'S FOREWORD

The problem of communicating in a coherent fashion the recent developments in the most exciting and active fields of physics seems particularly pressing today. The enormous growth in the number of physicists has tended to make the familiar channels of communication considerably less effective. It has become increasingly difficult for experts in a given field to keep up with the current literature; the novice can only be confused. What is needed is both a consistent account of a field and the presentation of a definite "point of view" concerning it. Formal monographs cannot meet such a need in a rapidly developing field, and, perhaps more important, the review article seems to have fallen into disfavor. Indeed, it would seem that the people most actively engaged in developing a given field are the people least likely to write at length about it.

"Frontiers in Physics" has been conceived in an effort to improve the situation in several ways. First, to take advantage of the fact that the leading physicists today frequently give a series of lectures, a graduate seminar, or a graduate course in their special fields of interest. Such lectures serve to summarize the present status of a rapidly developing field and may well constitute the only coherent account available at the time. Often, notes on lectures exist (prepared by the lecturer himself, by graduate students, or by postdoctoral fellows) and have been distributed in mimeographed form on a limited basis. One of the principal purposes of the "Frontiers in Physics" series is to make such notes available to a wider audience of physicists.

It should be emphasized that lecture notes are necessarily rough and informal, both in style and content, and those in the series will prove no exception. This is as it should be. The point of the series is to offer new, rapid, more informal, and, it is hoped, more effective ways for physicists to teach one another. The point is lost if only elegant notes qualify.

A second way to improve communication in very active fields of physics is by the publication of collections of reprints of recent articles. Such collections are themselves useful to people working in the field. The value of the reprints would, however, seem much enhanced if the collection would be accompanied by an introduction of moderate length, which would serve to tie the collection together and, necessarily, constitute a brief survey of the present status of the field. Again, it is appropriate that such an introduction be informal, in keeping with the active character of the field.

A third possibility for the series might be called an informal monograph, to connote the fact that it represents an intermediate step between lecture notes and formal monographs. It would offer the author an opportunity to present his views of a field that has developed to the point at which a summation might prove extraordinarily fruitful, but for which a formal monograph might not be feasible or desirable.

Fourth, there are the contemporary classics—papers or lectures which constitute a particularly valuable approach to the teaching and learning of physics today. Here one thinks of fields that lie at the heart of much of present-day research, but whose essentials are by now well understood, such as quantum electrodynamics or magnetic resonance. In such fields some of the best pedagogical material is not readily available, either because it consists of papers long out of print or lectures that have never been published.

"Frontiers in Physics" is designed to be flexible in editorial format. Authors are encouraged to use as many of the foregoing approaches as seem desirable for the project at hand. The publishing format for the series is in keeping with its intentions. In most cases, both paperbound and clothbound editions of each book are available.

Finally, suggestions from interested readers as to format, contributors, and contributions will be most welcome.

David Pines

Urbana, Illinois
August 1964

PREFACE

This monograph originally derives from a series of lectures given by R. Z. Sagdeev and A. A. Galeev at the International Centre for Theoretical Physics in Trieste, Italy. The editors' notes from these lectures were corrected and revised by Sagdeev and Galeev and then issued as an I.C.T.P. report (IC/66/64). To make this report more useful for students and nonspecialists, the editors have added an introduction and extended and revised those sections of the text dealing with the foundations of the theory.

Although the text presents a self-contained exposition of nonlinear theory, it does assume some knowledge of linear theory. For example, it assumes the reader is familiar with such things as electron plasma waves (or Langmuir waves), ion acoustic waves, Alfvén waves, and Landau damping. A student having completed a graduate level introductory course in plasma theory would certainly be adequately prepared.

As mentioned in the Introduction, the general plan of this monograph is to explain nonlinear plasma theory in terms of three basic interactions: nonlinear wave–wave interaction, wave–particle interaction, and wave–particle nonlinear interaction. One chapter is devoted to each interaction. The basic mechanism and simple properties of each interaction are discussed in the early sections of each chapter, and the application of the interaction to realistic problems, which usually involve considerable mathematical complexity, is relegated to later sections. These later sections may be omitted in a first reading without loss of continuity.

The editors are indebted to Kim In Ku and Wei Chau-Chin for help in recording the original Trieste notes and to Nancy Robeck, Rita Yorio, and Jacqueline Morrow for typing the manuscript. It is also a pleasure to acknowledge the hospitality of the International Centre for Theoretical Physics and the hospitality of Gulf General Atomic, where the final editing was completed.

From a personal point of view, the editors would like to express their warmest thanks to Doctors Sagdeev and Galeev, whose patient explanations of the difficult points have added greatly to the editor's understanding of nonlinear plasma theory.

T. M. O'Neil
D. L. Book

CONTENTS

Introduction

An essential simplification of linear plasma theory derives from the fact that arbitrary perturbations can be expressed as a superposition of eigenmodes, with each eigenmode evolving independently of the others. In this book, we extend this systematic approach to weakly nonlinear plasmas. Arbitrary perturbations are still expressed as a superposition of linear eigenmodes, but the nonlinearity provides a weak interaction between the modes. Consequently, the coefficients in the superposition of modes become slowly varying functions of time and eventually assume values quite different from those predicted by linear theory.

This approach to nonlinear plasma theory is usually referred to as the theory of weak turbulence. It can be justified by a perturbation expansion of the Vlasov (or fluid) equation when the energy in the excited spectrum of modes is small compared with the total plasma energy. Of course, the energy in the excited spectrum must be larger than thermal noise to be of any interest. In other words, the theory of weak turbulence can appropriately describe the evolution of an initially unstable plasma if the free energy liberated by the instability is small compared to the total plasma energy and large compared to thermal noise.

When the energy in the excited modes is of the same order as the total plasma energy, the plasma is said to be strongly turbulent and the weak turbulence perturbation expansion fails. Since there is no satisfactory theory of strong turbulence at the present time, our consideration is limited to weak turbulence.

This theory can be discussed in terms of three basic interactions: the nonlinear wave–wave interaction, the linear (or quasilinear) wave–particle interaction, and the nonlinear wave–particle interaction. The plan of this book is to devote one chapter to each of these basic interactions, and the examples given in each chapter are chosen to illustrate some aspect of the particular interaction under consideration.

The first interaction treated is the nonlinear wave–wave interaction, which is sometimes called resonant wave–wave scattering or the decay instability. The resonance conditions for this interaction can be written as $\omega_3 = \omega_1 \pm \omega_2$ and $\mathbf{k}_3 = \mathbf{k}_1 \pm \mathbf{k}_2$, where $(\omega_1, \omega_2, \omega_3)$ and $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ are the frequencies and wave numbers of the three waves involved in the interaction. As might be guessed from the resonance conditions, the basic mechanism behind this interaction is the strong nonlinear coupling that can occur when two waves beat together such that their sum or difference frequency and wavelength just match the frequency and wavelength of a third wave. Since the interaction does not involve resonant particles, it can be derived from fluid equations (i.e., it is not necessary to use the Vlasov equation). By interpreting ω and $\mathbf{k}$ as the energy and momentum of a single quantum associated with the kth wave, it can be seen that the resonance conditions for this interaction merely guarantee conservation of energy and momentum for the elementary process in which a single quantum decays into two other quanta or two quanta combine and form a third quantum. Consequently, it is not surprising that this interaction conserves the total energy and momentum in the waves.

The second interaction treated is the linear (or quasilinear) wave–particle interaction. It is associated with the resonance condition $\omega = \mathbf{k}\cdot\mathbf{v}$, where v is the velocity of the particle involved in the interaction. When a particle and wave satisfy this resonance condition, the particle maintains a constant phase relative to the wave and is very effectively accelerated by the essentially constant electric field the wave exerts on it. Since the interaction involves resonant particles, it cannot be derived from fluid equations (i.e., it is necessary to use the Vlasov equation). From a quantum point of view, the resonance condition for this interaction is a necessary condition for conservation of energy and momentum in the elementary process in which a particle of velocity v emits or absorbs a quantum of energy ω and momentum k. Consequently, it is not surprising that this interaction conserves the total energy and momentum in the waves and particles, not the energy and momentum in the waves alone. The change in the amplitude of the waves caused by this interaction is called Landau damping (or growth), and the corresponding change in the particle distribution is called quasilinear diffusion. Quasilinear theory, which treats these two effects simultaneously, is a nonlinear theory, in that the rate of change of the wave amplitudes depends on the distribution function, and the rate of change of the distribution function in turn depends on the wave amplitudes.

The third interaction treated is the nonlinear wave–particle interaction, which is often called nonlinear Landau damping. This interaction and the nonlinear wave–wave interaction are also referred to collectively as mode coupling. The resonance condition associated with this interaction is $(\omega_2 \pm \omega_1) = (\mathbf{k}_2 \pm \mathbf{k}_1)\cdot\mathbf{v}$, and the basic mechanism behind it is the same as that behind the linear wave–particle interaction, except that the particle now maintains a constant phase relative to the beats of two waves. Since the interaction involves resonant particles, it can only be derived from the Vlasov equation. Taken with the plus signs, this resonance condition describes the elementary process in which a particle of velocity v simultaneously emits or absorbs two quanta, one of momentum and energy $(\mathbf{k}_1, \omega_1)$ and the other of momentum and energy $(\mathbf{k}_2, \omega_2)$. Taken with minus signs, this resonance condition represents the elementary process in which a particle emits one quanta and absorbs another (i.e., a scattering process). Taken with either choice of signs in the resonance condition, the interaction conserves the total energy and momentum in the waves and particles. In addition, the interaction permits another conservation theorem, when taken with minus signs in the resonance condition. Then, since the interaction can be described as a scattering process, we can also say that the total number of wave quanta will be conserved. Here the number of quanta associated with the kth wave is defined as the energy in the kth wave divided by the frequency of the kth wave (i.e., $n_k \equiv$ action for the kth wave).

Of course, in a real plasma the three interactions occur simultaneously, and it is their combined effect that determines the evolution of the plasma. Also, it is sometimes necessary to take into account ordinary collisions, especially to determine the time-asymptotic state of the plasma.

Chapter I

Nonlinear Wave–Wave Interaction

In this chapter we shall consider the nonlinear wave-wave interaction. As mentioned in the Introduction, the basic mechanism behind this interaction is the strong nonlinear coupling that can occur when the beats between two waves resonantly drive a third wave, or vice versa.

I.1. RESONANT INTERACTION BETWEEN THREE OSCILLATORS

Before considering this interaction for the case of plasma waves, we consider the relatively simple case of three interacting harmonic oscillators. The Hamiltonian for these interacting oscillators can be written as

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2} + \omega_i^2 \frac{x_i^2}{2} + V x_1 x_2 x_3, \tag{I-1}$$

where V is a small coupling constant between the three oscillators. The equations of motion for the three oscillators can be written as

$$\begin{aligned} \ddot{x}_1 + \omega_1^2 x_1 &= -V x_2 x_3 \\ \ddot{x}_2 + \omega_2^2 x_2 &= -V x_1 x_3 \\ \ddot{x}_3 + \omega_3^2 x_3 &= -V x_2 x_1. \end{aligned} \tag{I-2}$$

We want to investigate the conditions that are necessary for effective transfer of energy, from the first oscillator to the other two oscillators, when the first oscillator has been excited initially to a much larger amplitude than the other two oscillators. Consequently, we start by linearizing Eq. (I-2) with respect to x_2 and x_3 but not with respect to x_1:

$$\begin{aligned} \ddot{x}_1 + \omega_1^2 x_1 &= 0 \\ \ddot{x}_2 + \omega_2^2 x_2 &= -V x_1 x_3 \\ \ddot{x}_3 + \omega_3^2 x_3 &= -V x_1 x_2. \end{aligned} \tag{I-3}$$

Since we have assumed that the coupling is small (i.e., $|x_1|\,|V| \ll \omega_2^2, \omega_3^2$), we can also express the oscillator coordinates as the product of a slowly varying amplitude and a rapidly oscillating exponential,

$$x_j = C_j(t)\exp(i\omega_j t) + C_j^*(t)\exp(-i\omega_j t). \tag{I-4}$$

Substituting this expression into Eq. (I.3) yields the equations:

C_1 = constant,

$$\frac{d^2 C_2}{dt^2} + 2i\omega_2 \frac{dC_2}{dt} = -\exp(-2i\omega_2 t)\left(\frac{d^2 C_2^*}{dt^2} - 2i\omega_2 \frac{dC_2^*}{dt}\right)$$
$$-VC_1 C_3 \exp[i(\omega_1 + \omega_3 - \omega_2)t]$$
$$-VC_1^* C_3 \exp[i(\omega_3 - \omega_1 - \omega_2)t]$$
$$-VC_1 C_3^* \exp[i(\omega_1 - \omega_3 - \omega_2)t]$$
$$-VC_1^* C_3^* \exp[-i(\omega_1 + \omega_2 + \omega_3)t] \quad \text{(I-5)}$$
$$\frac{d^2 C_3}{dt^2} + 2i\omega_3 \frac{dC_3}{dt} = -\exp(-2i\omega_3 t)\left(\frac{d^2 C_3^*}{dt^2} - 2i\omega_3 \frac{dC_3^*}{dt}\right)$$
$$-VC_1 C_2 \exp[i(\omega_1 + \omega_2 - \omega_3)t]$$
$$-VC_1^* C_2^* \exp[i(\omega_2 - \omega_1 - \omega_3)t]$$
$$-VC_1 C_2^* \exp[i(\omega_1 - \omega_2 - \omega_3)t]$$
$$-VC_1^* C_2^* \exp[-i(\omega_1 + \omega_2 + \omega_3)t].$$

These equations can be simplified by averaging them over the fast time scale associated with the oscillator frequencies, since the amplitudes C_j may be considered constants in such an average. In general, this averaging procedure will make all of the terms on the right-hand side of Eq. (I-5) vanish and will yield the result C_1, C_2, C_3 = constant. However, when the three oscillator frequencies satisfy a resonance condition that reduces the oscillation rate of one of the exponentials on the right-hand side of Eq. (I-5) to the time scale associated with the amplitudes, this exponential will survive the averaging process. For example, when the frequencies satisfy the resonance condition $\omega_2 = \omega_1 + \omega_3$, averaging Eq. (I-5) yields

$$\frac{d^2 C_2}{dt^2} + 2i\omega_2 \frac{dC_2}{dt} = -VC_1 C_3$$
$$\frac{d^2 C_3}{dt^3} + 2i\omega_3 \frac{dC_3}{dt} = -VC_1^* C_2, \quad \text{(I-6)}$$

where C_1 is to be considered a constant.

To solve these two equations, we try a solution of the form $C_2, C_3 \simeq e^{i\nu t}$. Working only to lowest order in the small parameter ν/ω_j yields the dispersion relation

$$\nu = \pm \frac{|C_1|\,|V|}{(\omega_2 \omega_3 4)^{1/2}}. \quad \text{(I-7)}$$

Consequently, C_2 and C_3 can have growing solutions only when the product $\omega_2\omega_3$ is negative. The condition that $\omega_2\omega_3$ be negative, taken together with the resonance condition $\omega_2 = \omega_1 + \omega_3$ is equivalent to the condition that $|\omega_1| > |\omega_2|, |\omega_3|$. Consequently, we conclude that when one oscillator is initially excited to a much larger energy than two other oscillators it can transfer its energy to the other two oscillators only if it has a higher frequency than the other two oscillators. If we argue that the transfer of energy from the first oscillator to the other two oscillators must occur in elementary quantum steps, then this conclusion has an obvious interpretation in terms of conservation of energy for the elementary quantum process involved.

I-2. RESONANT INTERACTION BETWEEN PLASMA WAVES

In this section we consider the nonlinear wave–wave interaction between three plasma waves. The main difference that we find between this case and the case of three interacting oscillators is that here the interacting plasma waves have to

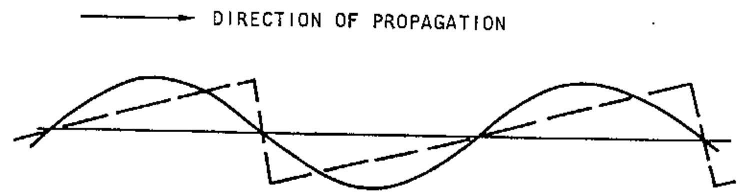

FIGURE I-1. Steepening wave front of a large-amplitude acoustic wave.

satisfy a resonance condition in wave number (i.e., $\mathbf{k}_3 = \mathbf{k}_1 \pm \mathbf{k}_2$) as well as frequency (i.e., $\omega_3 = \omega_1 \pm \omega_2$). [3] Of course, the frequency and wave number for each wave must also satisfy the linear dispersion relation for the system $\omega = \omega(k)$. Obviously, this dispersion relation plays an important role in deciding which nonlinear resonances are possible.

To illustrate this point we draw a distinction between the nonlinear resonances that dominate ordinary gas dynamics and those that dominate plasma physics. When a large-amplitude monochromatic acoustical wave is excited, the theory of ordinary gas dynamics predicts that the main nonlinear distortion of this wave will be a steepening of the wave fronts (see Figure I-1). This steepening can be understood in terms of resonant-harmonic generation. If the initial large-amplitude wave has frequency and wave number (ω, k), then this wave will beat together with itself and drive harmonics at $(2\omega, 2k)$. Since the acoustical dispersion relation is of the form $\omega = c_s k$, the harmonics will also be normal modes of the system, and, being driven resonantly, they will grow in time. Higher harmonics will grow in a similar manner, and the higher k values (i.e., shorter wavelengths) will then permit steepening of the wave front.

On the other hand, plasma dispersion relations are usually highly dispersive (i.e., ω is usually not linearly related to k), so the harmonics of a normal mode are usually not normal modes themselves. Consequently, the harmonics of a large-amplitude plasma wave usually remain at a low amplitude and cause little nonlinear distortion. It is important to note, however, that this does not mean that such a plasma wave can propagate free of any distortion. Even though it cannot interact resonantly with its own harmonics, it may still be able to interact resonantly with two other waves.

Resonant-harmonic generation can also be avoided by choosing the polarization of the large-amplitude wave to be such that the matrix element coupling the wave to its harmonics identically vanishes. For example, it is well known that the circularly polarized Alfvén wave is an exact solution of the nonlinear magnetohydrodynamics (MHD) equations, even though it has the dispersion relation $\omega = k\,V_A$. [1] As mentioned earlier, however, this does not mean that a large-amplitude Alfvén wave can persist in a real plasma.

To illustrate this point, we consider the wave–wave interaction between a large-amplitude Alfvén wave, a small-amplitude Alfvén wave, and a sound wave. [2] The calculation is similar to the one presented in the previous section, since we are again looking for the conditions necessary for effective transfer of energy from the large-amplitude wave to the other two waves.

We express the magnetic field, velocity flow, and density as

$$\begin{aligned} \mathbf{H} &= \mathbf{H}_0 + \delta\mathbf{H}_\perp(z,t) + \mathbf{h}_\perp(z,t), \\ \mathbf{V} &= \delta\mathbf{V}_\perp(z,t) + \mathbf{v}_\perp(z,t) + \mathbf{v}_\parallel(z,t), \\ \rho &= \rho_0 + \rho(z,t), \end{aligned} \tag{I-8}$$

where $\mathbf{H}_0$ is a constant field in the z direction, $\delta\mathbf{H}_\perp(z,t)$ and $\delta\mathbf{V}_\perp(z,t)$ are used to describe the large-amplitude Alfvén wave, $\mathbf{h}_\perp(z,t)$ and $\mathbf{v}_\perp(z,t)$ are used to describe the small-amplitude Alfvén wave, and $\rho(z,t)$ and $v_\parallel(z,t)$ are used to describe the sound wave. Note that all three waves are assumed to propagate in the z direction parallel to the constant field $\mathbf{H}_0$ and that both Alfvén waves are assumed to have transverse polarization.

We will solve the MHD equations by an expansion scheme, treating $\mathbf{H}_0$, ρ_0, $\delta\mathbf{H}_\perp$, and $\delta\mathbf{V}_\perp$ as zero-order quantities and $\mathbf{h}_\perp$, $\mathbf{v}_\perp$, $v_\parallel$, and ρ as first-order quantities. Aware that the large Alfvén wave is circularly polarized, we find the following zero-order equations:

$$\begin{aligned} \frac{\partial}{\partial t}\delta\mathbf{V}_\perp &= \frac{H_0}{\rho_0}\frac{\partial}{\partial z}\frac{\delta\mathbf{H}_\perp}{4\pi} \\ \frac{\partial}{\partial t}\delta\mathbf{H}_\perp &= H_0\frac{\partial}{\partial z}(\delta\mathbf{V}_\perp). \end{aligned} \tag{I-9}$$

Consequently, the large Alfvén wave can be described by a solution of the form

$$\begin{aligned} \delta\mathbf{H}_\perp(z,t) &= \delta\mathbf{H}_\perp \exp[i(k_0 z - \omega_0 t)] + \delta\mathbf{H}_\perp^* \exp[-i(k_0 z - \omega_0 t)] \\ \delta\mathbf{V}_\perp &= -\frac{H_0}{4\pi\rho_0}\frac{k_0}{\omega_0}\delta\mathbf{H}_\perp(z,t), \end{aligned} \tag{I-10}$$

where $\omega_0^2 = (H_0^2/4\pi\rho_0)k_0^2 = V_A^2\,k_0^2$ and $\gamma(\delta\mathbf{H}_\perp\cdot\delta\mathbf{H}_\perp) =$ constant. The first-order MHD equations can be written as

$$\frac{\partial v_\parallel}{\partial t} + \frac{c_s^2}{\rho_0}\frac{\partial\rho}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{\mathbf{h}_\perp\cdot\delta\mathbf{H}_\perp}{4\pi}\right)\frac{1}{\rho_0} \tag{I-11}$$

$$\frac{\partial\rho}{\partial t} + \rho_0\frac{\partial v_\parallel}{\partial z} = 0 \tag{I-12}$$

$$\frac{\partial\mathbf{v}_\perp}{\partial t} - \frac{H_0}{\rho_0}\frac{\partial}{\partial z}\left(\frac{h_\perp}{4\pi}\right) = -v_\parallel\frac{\partial}{\partial z}(\delta\mathbf{V}_\perp) - \frac{H_0\rho}{4\pi\rho_0^2}\frac{\partial}{\partial z}(\delta\mathbf{H}_\perp) \tag{I-13}$$

$$\frac{\partial\mathbf{h}_\perp}{\partial t} - H_0\frac{\partial}{\partial z}\mathbf{v}_\perp = -\frac{\partial}{\partial z}(v_\parallel\,\delta\mathbf{H}_\perp). \tag{I-14}$$

The left-hand side of Eqs. (I-11) and (I-12) describes the sound wave, and the right-hand side of these equations couples this sound wave to the two Alfvén waves. In a similar manner, the left-hand side of Eqs. (I-13) and (I-14) describes the small Alfvén wave and the right-hand side couples this Alfvén wave to the sound wave and the other Alfvén wave.

To simplify Eqs. (I-13) and (I-14) we make the additional assumption that $\beta = c_s^2/V_A^2 \ll 1$. We can estimate the effectiveness of the coupling terms on the right-hand side of these equations by comparing them with the linear terms

$$\begin{aligned} R_1 &= \left[v_\parallel\frac{\partial}{\partial z}(\delta V_\perp)\right]\left[\frac{H_0}{\rho_0}\frac{\partial}{\partial z}\left(\frac{h_\perp}{4\pi}\right)\right]^{-1} \sim \frac{v_\parallel\,\delta V_\perp\,\rho_0}{H_0 h_\perp} \\ R_2 &= \left[\frac{H_0\rho}{\rho_0^2}\frac{\partial}{\partial z}(\delta H_\perp)\right]\left[\frac{H_0}{\rho_0}\frac{\partial}{\partial z}(h_\perp)\right]^{-1} \sim \frac{\rho\,\delta H_\perp}{\rho_0 h_\perp} \\ R_3 &= \left[\frac{\partial}{\partial z}(v_\parallel\,\delta H_\perp)\right]\left(H_0\frac{\partial}{\partial z}v_\perp\right)^{-1} \sim \frac{v_\parallel\,\delta H_\perp}{v_\perp H_0}. \end{aligned}$$

Using the left-hand side of Eqs. (I-11)–(I-14) to express $v_\parallel$ and $v_\perp$ in terms of ρ and $h_\perp$, we can rewrite the foregoing ratios as

$$R_1 \sim R_3 \sim \frac{c_s}{V_A}\frac{\rho\,\delta H_\perp}{\rho_0 h_\perp} = \frac{c_s}{V_A}R_2 \ll R_2.$$

Consequently, in Eqs. (I-13) and (I-14) we need only retain the coupling term corresponding to R_2 [i.e., the term $(H_0\rho/4\pi\rho_0^2)(\partial/\partial z)(\delta\mathbf{H}_\perp)$ in Eq. (I-13)].

If we also assume that the two remaining coupling terms are small compared to the linear terms, then we can express the solutions of Eqs. (I-11)–(I-14) as the product of slowly varying amplitudes times rapidly oscillating exponentials [3]

$$v_\parallel(z,t) = v_\parallel \exp[i(k_s z - \omega_s t)] + v_\parallel^* \exp[-i(k_s t - \omega_s t)]$$
$$\rho(z,t) = \frac{\rho_0 k_s}{\omega_s} v_\parallel(z,t) \qquad \omega_s^2 = c_s^2 k_s^2 \tag{I-15}$$

$$\mathbf{v}_\perp(z,t) = \mathbf{v}_\perp \exp[i(k_A z - \omega_A t)] + \mathbf{v}_\perp^* \exp[-i(k_A z - \omega_A t)]$$
$$\mathbf{h}_\perp = -\frac{H_0 k_A}{\omega_A} \mathbf{v}_\perp(z,t) \qquad \omega_A^2 = V_A^2 k_A^2, \tag{I-16}$$

where $v_\parallel$ and $\mathbf{v}_\perp$ are slowly varying functions of time. Substituting these solutions and the solution for $\delta\mathbf{H}_\perp$ into Eqs. (I-11) and (I-13) yields

$$\begin{aligned}\frac{\partial v_\parallel}{\partial t} = &-\frac{\partial v_\parallel^*}{\partial t}\exp[-2i(k_s z - \omega_s t)] + i(k_0 + k_A)\frac{H_0 k_A}{\rho_0\omega_A}\left(\frac{\delta\mathbf{H}_\perp\cdot\mathbf{v}_\perp}{4\pi}\right)\\ &\times \exp[i(k_0 + k_A - k_s)z - i(\omega_0 + \omega_A - \omega_s)t] + i(k_0 - k_A)\frac{H_0 k_A}{\rho_0\omega_A}\left(\frac{\delta\mathbf{H}_\perp\cdot\mathbf{v}_\perp^*}{4\pi}\right)\\ &\times \exp[i(k_0 + k_A - k_s)z - i(\omega_0 - \omega_A - \omega_s t)] + i(k_A - k_0)\frac{H_0 k_A}{\rho_0\omega_A}\left(\frac{\delta\mathbf{H}_\perp^*\cdot\mathbf{v}_\perp}{4\pi}\right)\\ &\times \exp[i(k_A - k_0 - k_s)z - i(\omega_A - \omega_0 - \omega_s)t] - i(k_A + k_0)\frac{H_0 k_A}{\rho_0\omega_A}\left(\frac{\delta\mathbf{H}_\perp^*\cdot\mathbf{v}_\perp^*}{4\pi}\right)\\ &\times \exp[-i(k_A + k_0 + k_s)z + i(\omega_A - \omega_0 + \omega_s)t],\end{aligned} \tag{I-17}$$

$$\begin{aligned}\frac{\partial \mathbf{v}_\perp}{\partial t} = &-\frac{\partial \mathbf{v}_\perp^*}{\partial t}\exp[-2i(k_A z - \omega_A t)]\\ &- \frac{H_0 k_s}{4\pi\rho_0\omega_s} ik_0\,\delta\mathbf{H}_\perp v_\parallel \exp[i(k_0 + k_s - k_A)z - i(\omega_0 + \omega_s - \omega_A)t]\\ &+ \frac{H_0 k_s}{4\pi\rho_0\omega_s} ik_0\,\delta\mathbf{H}_\perp^* v_\parallel \exp[i(k_s - k_0 - k_A)z - i(\omega_s - \omega_0 - \omega_A)t]\\ &- \frac{H_0 k_s ik_0}{4\pi\rho_0\omega_s}\delta\mathbf{H}_\perp v_\parallel^* \exp[i(k_0 - k_s - k_A)z - i(\omega_s - \omega_0 - \omega_s - \omega_A)t]\\ &+ \frac{H_0 k_s ik_0}{4\pi\rho_0\omega_s}\delta\mathbf{H}_\perp^* v_\parallel^* \exp[-i(k_0 + k_s + k_A)z + i(\omega_0 + \omega_s + \omega_A)t].\end{aligned} \tag{I-18}$$

Averaging these equations over the rapid oscillations in space and time obviously yields $v_\parallel, v_\perp$ = constant (unless the three frequencies and the three wave numbers satisfy resonance conditions).

If the three waves satisfy the resonance conditions $\omega_A = \omega_s + \omega_0$ and $k_A = k_s + k_0$, then the averaging process yields

$$\frac{\partial v_\parallel}{\partial t} = i(k_A - k_0)\frac{H_0 k_A}{\rho_0\omega_A}\left(\frac{\delta\mathbf{H}_\perp^*\mathbf{v}_\perp}{4\pi}\right) \tag{I-19}$$

$$\frac{\partial \mathbf{v}_\perp}{\partial t} = -\frac{iH_0 k_s k_0}{4\pi\rho_0\omega_s} v_\parallel\,\delta\mathbf{H}_\perp. \tag{I-20}$$

Trying a solution of the form $v_\parallel, v_\perp \sim e^{i\nu t}$ yields the dispersion relation

$$\nu^2 = -\frac{(k_A - k_0)H_0^2 k_A k_s k_0}{4\pi\rho_0^2\omega_s\omega_A(4\pi)^2}|\delta\mathbf{H}_\perp|^2$$
$$\nu^2 = -\frac{k_s^2 k_A k_0 V_A^2}{\omega_s\omega_A}\frac{|\delta\mathbf{H}_\perp|^2}{4\pi\rho_0}. \tag{I-21}$$

To interpret this result, we first note that the resonance conditions demand that

$$k_A = k_0 + k_s \tag{I-22}$$

$$\omega_A = \omega_0 + \omega_s, \tag{I-23}$$

where $\omega_0^2 = V_A^2 k_0^2$, $\omega_A^2 = V_A^2 k_A^2$, and $\omega_s^2 = C_s^2 k_s^2$. Because of the form of the dispersion relations, all terms in these equations can be written with either a plus or minus sign. For convenience, let us specify that $\omega_0 = k_0 V_A > 0$. Since we have assumed that $C_s \ll V_A$, it follows that the choice $\omega_A = V_A k_A$ implies $k_s = 0$ and $\nu = 0$. For the choice $\omega_A = -V_A k_A$, the above equations demand that $-k_0 \simeq k_A$ and $k_s \simeq -2k_0$. Consequently, Eq. (I-21) implies $\nu^2 > 0$ (i.e., instability) only when $\omega_s < 0$. In other words, the energy of a quantum of the initial wave must be larger than the energy of a quantum of the perturbing waves (i.e., $\omega_0 > \omega_A$, $|\omega_s|$).

I-3. INTERACTION BETWEEN WAVES OF FINITE AMPLITUDE

The problem of the stability of Alfvén waves of small amplitude was considered in Section I-2. For disturbances in the form of a sum of sound and Alfvén waves, we can write two coupled equations for the perturbed longitudinal velocity and transverse magnetic field. These equations describe the coupling between sound and Alfvén waves. We found that the frequency rule for the decay instability corresponds to the usual energy conservation law of quantum mechanics. Consequently, it can be seen more directly by writing the wave equations in the

Hamiltonian form using quantum parameters, such as the energy of the wave quantum and the number of waves.

To this end, let us rewrite the wave equations. We replace $v_\parallel$ and $h_\perp$ by $[v_\parallel(t), h_\perp(t)] \exp(-i\omega t + ikz)$, where $v_\parallel(t)$ and $h_\perp(t)$ are slowly varying, and ω and k satisfy the sound dispersion relation in the longitudinal equation and the Alfvén dispersion relation in the transverse equation. If we label transverse quantities with an index (1) and longitudinal quantities with an index (2),

$$v_2 \equiv v_\parallel, \qquad k_A \equiv k_1, \qquad \omega_1^2 = k_1^2 V_A^2,$$

$$h_1 \equiv h_\perp, \qquad k_s \equiv k_2, \qquad \omega_2^2 = k_2^2 c_s^2.$$

Then, from Eqs. (I-15), (I-16), (I-19), and (I-20), we find

$$i\frac{\partial v_2}{\partial t} = -\frac{k_2\,\delta H^*}{4\pi\rho_0} h_\perp \exp[-i(\omega_1 - \omega_0 - \omega_2)t], \qquad \text{(I-24)}$$

$$i\frac{\partial h_1}{\partial t} = \frac{k_0 k_1 k_2 V_A^2\,\delta H}{\omega_1\omega_2} v_2 \exp[-i(\omega_2 - \omega_1 + \omega_0)t]. \qquad \text{(I-25)}$$

Let us define the number of quanta n_k as the total energy in a mode divided by the frequency of that mode,

$$n_k = \omega_k^{-1}\left(\frac{\rho_0 v_k^2}{2} + \frac{H_k^2 + E_k^2}{8\pi} + \frac{\rho_k^2 c_s^2}{2\rho_0}\right) \qquad \text{(I-26)}$$

Then we can write Eqs. (I-24) and (I-25) in a symmetric form by introducing probability amplitudes (i.e., $|C_j|^2 = n_j$). For the present case these amplitudes are

$$C_0(t) = \frac{\delta H}{(4\pi|\omega_0|)^{1/2}}, \qquad C_1(t) = \frac{h_1}{(4\pi|\omega_1|)^{1/2}}, \qquad C_2(t) = \frac{v_2}{(|\omega_2|/\rho_0)^{1/2}}. \qquad \text{(I-27)}$$

In terms of these variables, Eqs. (I-24) and (I-25) can be written in a form similar to the Schrödinger equation in the interaction representation [4]

$$i\frac{\partial C_1}{\partial t} = V_{k_1,k_0,k_2} C_0 C_2 \qquad \text{(I-28)}$$

$$i\frac{\partial C_2}{\partial t} = V_{k_2,-k_0,k_2} C_0^* C_1 \qquad \text{(I-29)}$$

where

$$V_{k_2,-k_0,k_1} = V_{k_1,k_0,k_2}\,\mathrm{sign}(\omega_1\omega_2) \equiv -\left(\frac{|\omega_0\omega_1\omega_2|}{\rho_0 c_s^2}\right)^{1/2} \mathrm{sign}\,k_2,$$

$$\omega_1 = \omega_0 + \omega_2.$$

We see that the matrix elements of the interaction operators differ only by a sign for the two modes. This is due to the Hamiltonian form of the hydromagnetic equations. Starting from this point we would like to introduce an important generalization: for any kind of plasma waves (not only for Alfvén waves), if it is possible to represent the three-wave interaction in a Hamiltonian form (as was done here for MHD Alfvén waves), and the same symmetry rules should exist for the decay-type instability. Actually, this holds for any fluid approximation (including multifluid cases). Moreover, in such arbitrary cases we may expect to have, for

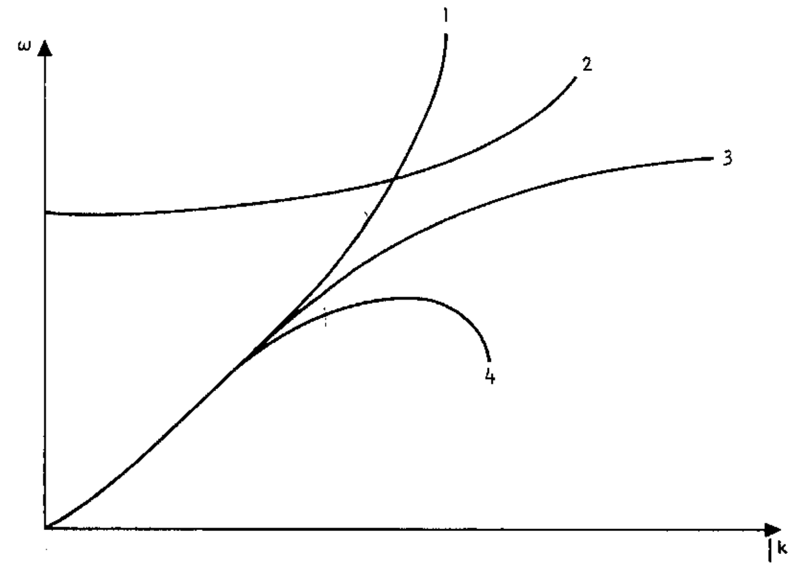

FIGURE I-2. Different types of wave spectra.

the probability amplitude variables, the same type of equations as Eqs. (I-28) and (I-29). Of course, the matrix elements and normalizations should be specified in a different way for the various cases. Besides, in each case it is necessary to fulfill the resonance conditions.

In Figure I-2 various possible forms of spectra are represented. Using the vector inequality $|\mathbf{k}_1 + \mathbf{k}_2| \leqslant |\mathbf{k}_1| + |\mathbf{k}_2|$, we can easily show that the resonance conditions can be satisfied by waves having spectra similar to curves 1 and 4 but that the resonance conditions cannot be satisfied by waves having spectra similar to curves 2 and 3. When the dispersion relation has more than one branch, the resonance conditions may be satisfied by waves corresponding to different branches. In general, the resonance conditions can be satisfied when it is possible to draw a curve similar to 1 or 4 through the three points corresponding to the oscillations ω_0, ω_1, ω_2 (these three points may be on different branches). For waves on different branches, however, prohibitions may arise from the polarizations of the waves. Also, the fulfillment of the resonance conditions does not in itself signify instability with regard to decay.

By solving Eqs. (I-28) and (I-29) we find an exponential behavior [i.e., $|C_j|^2 = \exp(2\nu t)$]. In the general case the growth rate of the perturbing waves is given by

$$\nu^2 = -|V_{k_1,k_0,k_2}|^2 \operatorname{sign}(\omega_1 \omega_2)|C_0|^2.$$

From this expression we find that the decay-type instability occurs only when the frequencies of both perturbing waves are smaller than that of the large-amplitude wave, [4] that is, $|\omega_1|, |\omega_2| < |\omega_0|$.

In the case when the resonance conditions for the frequencies and wave vectors cannot be satisfied among three waves, we may include in our consideration a fourth wave. In order to have a finite growth rate, this fourth wave obviously must be a finite amplitude wave. Therefore, we should consider the stability of the second harmonic of finite amplitude waves. The resonance conditions now have the form

$$2\omega(k_0) = \omega(k_1) + \omega(k_2)$$
$$2\mathrm{k}_0 = \mathrm{k}_1 + \mathrm{k}_2.$$

Using the decay rule $[2|\omega(k_0)| > |\omega(k_1)|, |\omega(k_2)|]$ and these resonance conditions, it may be shown that, in the second order, spectrum 3 of Figure I-2 is unstable and spectrum 2 is stable. The instability of gravity waves on the ocean surface and ion sound waves of the second order was considered as an example of this kind of instability. [5] We expect that the diagram of the unstable regions in the frequency versus wave amplitude plane looks qualitatively like the same kind of diagram for the parametric resonance problem (Figure I-3).

The width of the unstable region near the nth harmonic is of the order of the growth rate and proportional to the nth power of the amplitude. Of course, we must satisfy the decay conditions

$$n\omega(k_0) = \omega(k_1) + \omega(k_2)$$
$$n\mathrm{k}_0 = \mathrm{k}_1 + \mathrm{k}_2.$$

An interesting example of this type of higher-order decay is the instability of a sawtooth Alfvén wave with respect to perturbations of frequency ω_1 and ω_2, where $\max(\omega_1, \omega_2) \gg V_A/\lambda$ and λ is the wavelength associated with the sawtoothed Alfvén wave. [6]

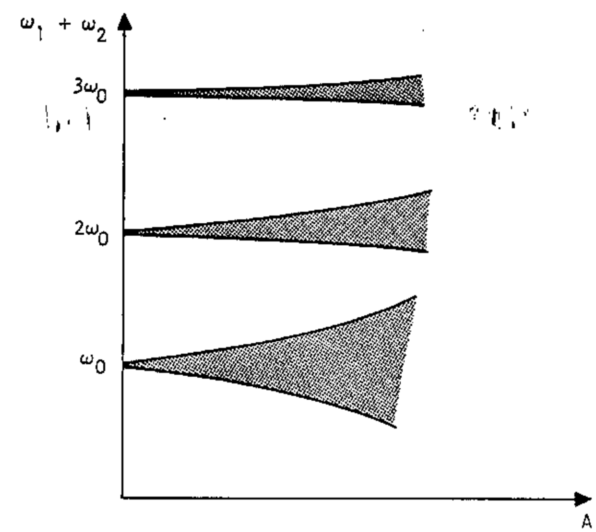

FIGURE I-3. Stability diagram in frequency versus wave amplitude plane. Shaded areas are unstable.

So far we have obtained only equations describing the growth of the disturbing waves and have not considered the reaction of these waves on the primary (or decaying) wave. To describe the relaxation of the primary wave we need to consider the effect of the finite amplitude disturbances in the nonlinear terms of the equation for the primary wave. When C_0, C_1, and C_2 are all of the same order, one would expect to find the following equation:

$$i\frac{\partial C_0}{\partial t} = V_{k_0,k_1,-k_2} C_1 C_2^*. \tag{I-30}$$

Of course, we can derive this equation from the same MHD equations we used to derive Eqs. (I-28) and (I-29).

To solve the coupled set of equations given in Eqs. (I-28)–(I-30), we make the further substitution

$$C_j(t) = a_j(t)\, e^{i\Phi_j(t)}, \qquad \operatorname{Im} a_j = \operatorname{Im} \Phi_j = 0. \tag{I-31}$$

In the case in which $\omega_0 > \omega_1 > |\omega_2|$ and $\omega_2 < 0$, the symmetry rules of the V's imply

$$V_{k_1,k_0,k_2} \equiv -H$$
$$V_{k_2,-k_0,k_1} = V_{k_1,k_0,k_2} \operatorname{sign}(\omega_1, \omega_2) = H$$
$$V_{k_0,k_1,-k_2} = V_{k_1,k_2,k_0} \operatorname{sign}(\omega_0, \omega_1) = -H.$$

Taking the real and imaginary parts of Eqs. (I-28)–(I-30) and using the variables $a_j(t)$ and $\theta \equiv \phi_1 - \phi_2 - \phi_0$ gives us the following equations: [7]

$$\frac{\partial a_1}{\partial t} = H a_0 a_2 \sin\theta$$

$$\frac{\partial a_2}{\partial t} = H a_0 a_1 \sin\theta$$

$$\frac{\partial a_0}{\partial t} = -H a_1 a_2 \sin\theta$$

$$\frac{\partial \theta}{\partial t} = H\left(\frac{a_0 a_2}{a_1} + \frac{a_0 a_1}{a_2} - \frac{a_1 a_2}{a_0}\right)\cos\theta \equiv \operatorname{ctg}\theta \frac{\partial}{\partial t}\ln(a_0 a_1 a_2). \qquad \text{(I-32)}$$

Integrating the last of these equations, we find

$$a_0 a_1 a_2 \cos\theta \equiv \Gamma = \text{const.} \qquad \text{(I-33)}$$

Using the frequency rule, we can easily find a first integral of the remaining equations:

$$a_1^2 \omega_1 + a_2^2 |\omega_2| + a_0^2 \omega_0 = \text{const.} \qquad \text{(I-34)}$$

Integrating $a_0(\partial a_0/\partial t) + a_1(\partial a_1/\partial t)$, etc., we can also find the following constants of the motion:

$$m_1 \equiv n_0 + n_1 = \text{const}$$

$$m_2 \equiv n_0 + n_2 = \text{const}$$

$$m_0 \equiv n_1 - n_2 = \text{const.} \qquad \text{(I-35)}$$

They are known in the theory of parametric amplifiers [8] as the vectorial Manley–Rowe relations taken in the direction of propagation. They may be understood in terms of the diagram for the three-wave decay process (see Figure I-4) by stating that, when a quantum disappears from the ω_{k_0} mode, quanta appear in each of ω_{k_1} and ω_{k_2}; so $\Delta n_0 = -1$, $\Delta n_1 = 1 = \Delta n_2$. Combining relations (I-33) and (I-35) we have

$$\frac{d}{dt} n_0 = 2H[n_0(m_1 - n_0)(m_2 - n_0) - \Gamma^2]^{1/2}. \qquad \text{(I-36)}$$

If the three roots of

$$n_0(m_1 - n_0)(m_2 - n_0) - \Gamma^2 = 0,$$

are labeled as

$$n_c \geqslant n_b \geqslant n_a \geqslant 0,$$

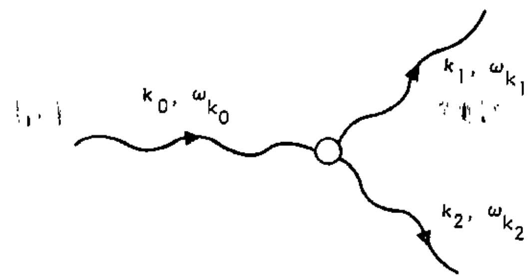

FIGURE I-4. Three-wave decay process.

then Eq. (I-36) can be transformed into the equation

$$H(t - t_0) = \frac{1}{2}\int_{n_0(t_0)}^{n_0(t)} \frac{dn_0}{[(n_0 - n_c)(n_0 - n_b)(n_0 - n_c)]^{1/2}}.$$

This can be transformed to an elliptic integral by the change of variables

$$y(t) \equiv \left[\frac{n_0(t) - n_a}{n_b - n_a}\right]^{1/2}, \qquad \gamma \equiv \left[\frac{n_b - n_a}{n_c - n_a}\right]^{1/2}. \qquad \text{(I-37)}$$

If we define the time t_0 in such a way that $y(t_0) = 0$, then

$$H(t - t_0)\sqrt{n_c - n_a} = \int_0^{y(t)} \frac{dy}{[(1 - y^2)(1 - \gamma^2 y^2)]^{1/2}}. \qquad \text{(I-38)}$$

Therefore,

$$y(t) = sn[H\sqrt{n_c - n_a}(t - t_0), \gamma],$$

and, by definition of y, we have the general solution

$$n_0(t) = n_a + (n_b - n_a)\, sn^2(H\sqrt{n_c - n_a}(t - t_0), \gamma). \qquad \text{(I-39)}$$

Let us consider two simple cases: [7]

Case A: At time $t = 0$, $n_0(0) = 0$, $n_1(0) \gg n_2(0)$. Without loss of generality we can put $\Gamma = 0$ in Eq. (I-36). Then the three roots of Eq. (I-36) are simple:

$$m_1 \equiv n_1(0) = n_c \gg m_2 = n_2(0) = n_b > 0 = n_a.$$

We can simplify the solution by neglecting $\gamma^2 y^2$ in Eq. (I-38), since in this case $\gamma \ll 1$. Thus

$$n_0(t) = n_2(0)\sin^2[Ht\sqrt{n_1(0)}],$$

$$n_1(t) = n_1(0) - n_2(0)\sin^2[Ht\sqrt{n_1(0)}],$$

$$n_2(t) = n_2(0)\cos^2[Ht\sqrt{n_1(0)}].$$

The time variation of the occupation number is shown in Figure I-5. The situation described by this figure is the case in which the frequency of the finite amplitude wave is smaller than the frequency of the perturbations, and, hence, it is stable to the decay-type instability. The small periodic variation in its amplitude is due to the small amount of energy initially in the perturbations.

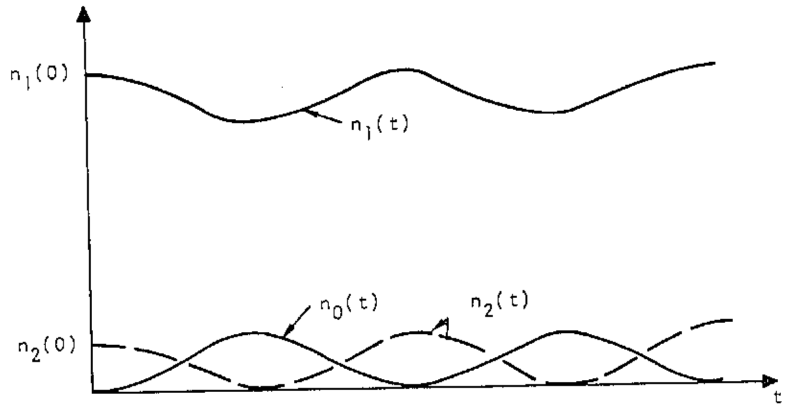

FIGURE I-5. Three waves that are stable to the decay instability.

Case B: Let us now consider the decay of finite amplitude waves, when at time $t = 0$, $n_2(0) = 0$, $n_0(0) \gg n_1(0) > 0$. Putting $\Gamma = 0$ again, we find the constants

$$n_c = m_1 \equiv n_0(0) + n_1(0) > n_b = m_2 \equiv n_0(0) > n_a = 0.$$

Thus from Eqs. (I-39) and (I-35) we obtain

$$\begin{aligned} n_0(t) &= n_0(0)\, sn^2[H(t-t_0)\sqrt{n_c}, \gamma] \\ n_1(t) &= n_1(0) + n_0(0)\{1 - sn^2[H(t-t_0)\sqrt{n_c}, \gamma]\} \\ n_2(t) &= n_0(0)\{1 - sn^2[H(t-t_0)\sqrt{n_c}, \gamma]\}, \end{aligned} \tag{I-40}$$

where

$$1 - \gamma^2 = \frac{n_1(0)}{n_0(0) + n_1(0)} \ll 1. \tag{I-41}$$

Since $n_1 = 0$ at $t = 0$, we can write

$$1 - sn^2(Ht_0\sqrt{n_c}, \gamma) = 0.$$

Therefore, $Ht_0\sqrt{n_c}$ is one-fourth of the period of sn,

$$Ht_0\sqrt{n_c} = K(\gamma). \tag{I-42}$$

With $\gamma' = \sqrt{1-\gamma^2}$, we have

$$K(\gamma) = -[(\gamma'/2)^2 + 0(\gamma'^4)] + \ln(4/\gamma')[1 + (\gamma'/2)^2 + 0(\gamma'^4)]. \tag{I-43}$$

Therefore, from Eqs. (I-41)–(I-45) we find

$$t_0 \approx \frac{1}{2H\sqrt{n_0}} \ln \frac{n_0(0)}{n_1(0)}. \tag{I-44}$$

During this time the amplitude of the initial steady wave n_0 decreases to zero, so we can say that it is the time of decay. The inverse of t_0 differs from the

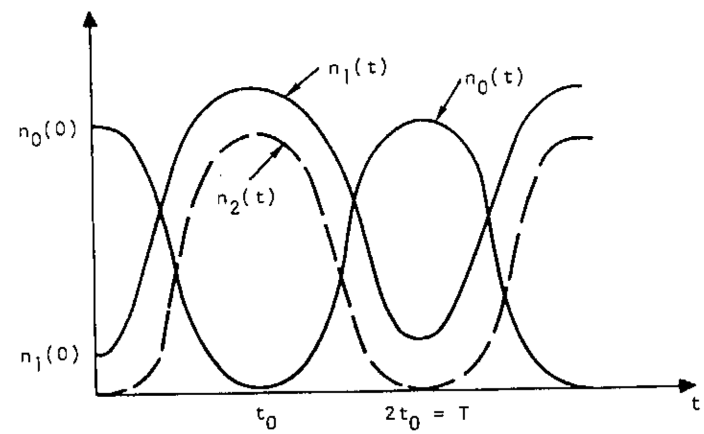

FIGURE I-6. Three waves that are unstable to the decay instability.

linear growth rate $\nu = H\sqrt{n_0}$ only by a logarithmic factor. We need this factor when the amplitudes of the disturbances are comparable to the amplitude of the initial wave. The behavior of the relative number of wave quanta is shown in Figure I-6. It turns out that the amplitude of the two perturbing modes for long times drop to zero again. Thus, the decay found in this problem is not really irreversible.

This type of problem was considered first in 1954 by Pierce. [8] He discussed the use of such resonant three-wave interaction in high-frequency electronics. Now it is a highly developed part of the radio-frequency electronics in which these nonlinear methods are used for the production of mixing frequencies and parametric amplification.

Another illustration of these methods is nonlinear optics. Here the nonlinearity usually comes in terms of the current dependence on the electric field. [9] Also, all optical media have dispersive properties. In the simplest approximation the dielectric function is given in terms of the resonant frequencies—at least far from these resonances—by

$$\epsilon = \sum \alpha_l \left(1 - \frac{\omega_{pl}^2}{\omega^2 - \omega_l^2}\right).$$

Such a dispersion makes it possible to satisfy the resonance relations $\omega_0 = \omega_1 + \omega_2$, $\mathbf{k}_0 = \mathbf{k}_1 + \mathbf{k}_2$. Thus, if a ruby laser beam is transmitted through quartz, a smaller amount of the second harmonic is generated. However, if two lasers are passed through quartz at the appropriate angle with respect to one another in order to satisfy the conditions $\omega_1 + \omega_2 = \omega_3$, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$, a considerable amount of energy can be converted into harmonics.

I-4. MANY-WAVE INTERACTION IN RANDOM PHASE APPROXIMATION

In plasmas we generally do not deal with highly ordered phenomena, since many modes are available to satisfy the resonance conditions. Consequently, the

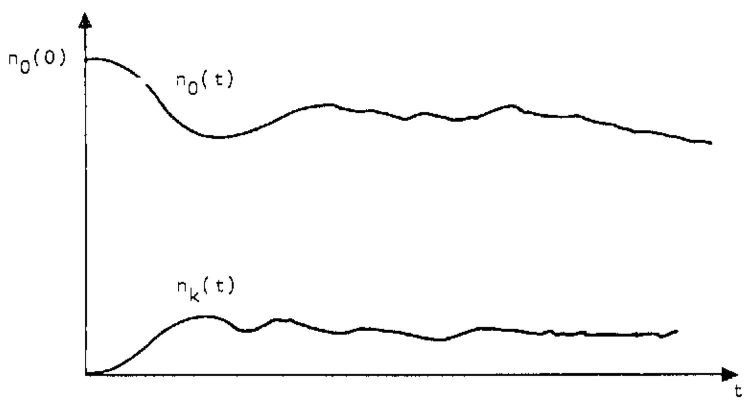

FIGURE I-7. Decay instability for many waves.

Manly–Rowe type of behavior (see Figure I-6) usually does not occur in a plasma. Instead, the recurrent character of the evolution is destroyed (for incommensurable frequencies) (see Figure I-7).

We can treat this situation by using the random phase approximation (RPA) (i.e., we look only at the modulus of the wave amplitude and average over the

phases). There is no way to distinguish the primary wave from the others, and this is reflected in the notation used in the equation for the wave amplitude

$$i\frac{dC_{\mathbf{k}}}{dt} = \sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}'} C_{\mathbf{k}'}(t)\, C_{\mathbf{k}-\mathbf{k}'}(t)\, e^{-i(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'}-\omega_{\mathbf{k}})t}. \tag{I-45}$$

With a particular choice of normalization, the amplitudes $C_{\mathbf{k}}$ play the role of probability amplitudes, so the mode occupation number is equal to the square of this amplitude:

$$n_{\mathbf{k}} = |C_{\mathbf{k}}|^2. \tag{I-46}$$

Within this normalization the matrix elements have the following symmetry properties (compare with the ones for the interaction of the Alfvén and sound waves):

$$\begin{aligned} V_{\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}'} &= V_{\mathbf{k}-\mathbf{k}',-\mathbf{k}',\mathbf{k}}\, \mathrm{sign}\,(\omega_{\mathbf{k}}\omega_{\mathbf{k}-\mathbf{k}'}) \\ V_{\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}'} &= V_{\mathbf{k},\mathbf{k}-\mathbf{k}',\mathbf{k}'} = V_{-\mathbf{k},-\mathbf{k}',-\mathbf{k}+\mathbf{k}'}. \end{aligned} \tag{I-47}$$

Since we are interested only in the time variation of the occupation numbers $n_{\mathbf{k}}$, we seek an equation in terms of $n_{\mathbf{k}}$ alone. To this end we treat the equation for the wave amplitudes by the method of time-dependent perturbation theory from quantum mechanics. [4] Expanding $C_{\mathbf{k}}(t)$ in powers of the interaction $V_{\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}'}$,

$$C_{\mathbf{k}}(t) = C_{\mathbf{k}}^{(0)} + C_{\mathbf{k}}^{(1)} + C_{\mathbf{k}}^{(2)} + \cdots$$

and substituting in Eq. (I-45) yields the solutions

$$\begin{aligned} C_{\mathbf{k}}^{(1)} &= -i \sum_{\mathbf{k}',\mathbf{k}''} C_{\mathbf{k}'}^{(0)} C_{\mathbf{k}''}^{(0)} \int_0^t V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t')\, dt' \\ C_{\mathbf{k}}^{(2)} &= -\sum_{\mathbf{k}',\mathbf{k}'',\mathbf{q}',\mathbf{q}''} C_{\mathbf{k}'}^{(0)} C_{\mathbf{q}'}^{(0)} C_{\mathbf{q}''}^{(0)} \int_0^t dt' \int_0^{t'} dt''\, V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t')\, V_{\mathbf{k}'',\mathbf{q}',\mathbf{q}''}(t'') \\ &\quad - \sum_{\mathbf{k}',\mathbf{k}'',\mathbf{q}',\mathbf{q}''} C_{\mathbf{k}''}^{(0)} C_{\mathbf{q}'}^{(0)} C_{\mathbf{q}''}^{(0)} \int_0^t dt' \int_0^{t'} dt''\, V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t')\, V_{\mathbf{k}',\mathbf{q}',\mathbf{q}''}(t'') \end{aligned} \tag{I-48}$$

$$V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t) = V_{\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}'} \exp[-i(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}-\mathbf{k}'} - \omega_{\mathbf{k}})t]\, \delta_{\mathbf{k}'',\mathbf{k}-\mathbf{k}'},$$

$$\delta_{\mathbf{k},\mathbf{k}'} = \begin{cases} 1, & \mathbf{k} = \mathbf{k}' \\ 0, & \mathbf{k} \neq \mathbf{k}' \end{cases}. \tag{I-49}$$

The quantities $C_{\mathbf{k}}^{(0)}$ are time-independent and correspond to the solutions in the absence of interaction between the modes. They can be written as a positive amplitude times a phase factor $e^{i\phi_{\mathbf{k}}}$. Although the phases $\phi_{\mathbf{k}}$ are fixed by initial conditions in any one experiment, it is reasonable to assume that they are distributed randomly when one considers an ensemble of experiments (i.e., $\langle C_{\mathbf{k}}^{(0)} C_{\mathbf{k}'}^{(0)}\rangle_{av} = |C_{\mathbf{k}}^{(0)}|^2 \delta_{\mathbf{k},-\mathbf{k}'}$). We now use this property to calculate the ensemble average of the change in occupation number [i.e., $|C_{\mathbf{k}}(t)|^2 - |C_{\mathbf{k}}(t_0)|^2$]. That the ensemble average of the change in occupation number is a good approximation of the actual change in any one experiment is, of course, a statistical assumption based on the large number of modes.

To the lowest order we find

$$\langle |C_{\mathbf{k}}(t)|^2\rangle_{av} = |C_{\mathbf{k}}(t_0)|^2 + \langle |C_{\mathbf{k}}^{(1)}|^2\rangle_{av} + \langle |C_{\mathbf{k}}^{(0)} C_{\mathbf{k}}^{(2)*}|\rangle_{av} + \langle C_{\mathbf{k}}^{(0)*} C_{\mathbf{k}}^{(2)}\rangle_{av}. \quad \text{(I-50)}$$

Substituting Eq. (I-48) into Eq. (I-50) yields

$$|C_{\mathbf{k}}(t)|^2 - |C_{\mathbf{k}}(t_0)|^2$$

$$= \sum_{\mathbf{k}',\mathbf{k}'',\mathbf{q}',\mathbf{q}''} \Bigg[C_{\mathbf{k}'}^{(0)} C_{\mathbf{k}''}^{(0)} C_{\mathbf{q}'}^{(0)*} C_{\mathbf{q}''}^{(0)*} \int_0^t V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t')\,dt' \int_0^t V^*_{\mathbf{k},\mathbf{q}',\mathbf{q}''}(t')\,dt'$$

$$- \operatorname{Re} C_{\mathbf{k}}^{(0)*} C_{\mathbf{k}''}^{(0)} C_{\mathbf{q}'}^{(0)} C_{\mathbf{q}''}^{(0)} \cdot 2 \int_0^t V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t')\,dt' \int_0^{t'} V_{\mathbf{k}',\mathbf{q}',\mathbf{q}''}(t'')\,dt''$$

$$- \operatorname{Re} C_{\mathbf{k}}^{(0)*} C_{\mathbf{k}''}^{(0)} C_{\mathbf{q}'}^{(0)} C_{\mathbf{q}''}^{(0)} \cdot 2 \int_0^t dt'\, V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t') \int_0^{t'} dt''\, V_{\mathbf{k}',\mathbf{q}',\mathbf{q}''}(t'') \Bigg]. \quad \text{(I-51)}$$

After averaging this equation over random phases, the product of the four $C_{\mathbf{k}}^{(0)}$'s on the right-hand side can be reduced to the product of two occupation numbers. The two possible combinations of the $C_{\mathbf{k}}^{(0)}$'s are indicated in Eq. (I-51) by dashed and straight lines. In the first term, the $C_{\mathbf{k}}^{(0)}$'s combine as $|C_{\mathbf{k}'}^{(0)}|^2 |C_{\mathbf{k}''}^{(0)}|^2 = n_{\mathbf{k}'} n_{\mathbf{k}''}$ and in the last two terms the $C_{\mathbf{k}}^{(0)}$'s combine as $|C_{\mathbf{k}}^{(0)}|^2 |C_{\mathbf{k}'}^{(0)}| = n_{\mathbf{k}} n_{\mathbf{k}'}$ and $|C_{\mathbf{k}}^{(0)}| C_{\mathbf{k}''}^{(0)}|^2 = n_{\mathbf{k}} n_{\mathbf{k}''}$. Application of the symmetry relations between the matrix elements $V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}$ shows that the product of two matrix elements can always be written as

$$\left| \int_{t_0}^t V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t')\,dt' \right|^2$$

times a sign depending on the signs of ω_k, $\omega_{k'}$, and $\omega_{k''}$. For times longer than a wave period, the time integrals can be evaluated as

$$\left| \int_{t_0}^t V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t')\,dt' \right|^2 = \frac{\{4\sin^2[(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''})(t - t_0)/2]\}}{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''})^2} |V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}|^2 \delta_{\mathbf{k},\mathbf{k}'+\mathbf{k}''}$$

$$\left| \int_{t_0}^t V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(t')\,dt' \right|^2 = 2\pi\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''}) |V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}|^2 (t - t_0)\, \delta_{\mathbf{k},\mathbf{k}'+\mathbf{k}''}.$$

Consequently, the change in occupation number can be written as

$$\Delta n_{\mathbf{k}} = 4\pi \Delta t \sum_{\mathbf{k}',\mathbf{k}''} |V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}|^2 [n_{\mathbf{k}'}^{(0)} n_{\mathbf{k}''}^{(0)} - \operatorname{sign}(\omega_{\mathbf{k}}\omega_{\mathbf{k}''}) n_{\mathbf{k}}^{(0)} n_{\mathbf{k}'}^{(0)}$$
$$- \operatorname{sign}(\omega_{\mathbf{k}}\omega_{\mathbf{k}'}) n_{\mathbf{k}}^{(0)} n_{\mathbf{k}''}^{(0)}]\, \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''})\, \delta_{\mathbf{k},\mathbf{k}'+\mathbf{k}''}. \quad \text{(I-52)}$$

We now make the additional assumption that as the system evolves in time the amplitudes $C_{\mathbf{k}}$ remain uncorrelated in phase, at least to the lowest order. With this assumption in mind we can rewrite Eq. (I-52) as a differential equation, imagining that the above calculation is repeated in many consecutive time intervals Δt. In other words, we set $\Delta n_{\mathbf{k}}/\Delta t = dn_{\mathbf{k}}/dt$ and $n_{\mathbf{k}}^{(0)} = n_{\mathbf{k}}(t)$. This procedure yields the kinetic equation for waves (including only the wave-wave interaction) [4, 10–14]

$$\frac{dn_{\mathbf{k}}}{dt} = 4\pi \sum_{\mathbf{k}',\mathbf{k}''} |V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}|^2 [n_{\mathbf{k}'} n_{\mathbf{k}''} - \operatorname{sign}(\omega_{\mathbf{k}}\omega_{\mathbf{k}''}) n_{\mathbf{k}} n_{\mathbf{k}'}$$
$$- \operatorname{sign}(\omega_{\mathbf{k}}\omega_{\mathbf{k}'}) n_{\mathbf{k}} n_{\mathbf{k}''}]\, \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''})\, \delta_{\mathbf{k},\mathbf{k}'-\mathbf{k}''}. \quad \text{(I-53)}$$

We can go from Eq. (I-45) directly to this result, written in terms of positive frequencies only, by using time-dependent perturbation theory and the golden rule. Let us consider only that part of the nonlinear process with $\omega_{\mathbf{k}} > \omega_{\mathbf{k}'}, \omega_{\mathbf{k}''} > 0$, which corresponds to the decay process of the high-frequency mode $\omega_{\mathbf{k}}$ and the combination process of the two low-frequency modes $\omega_{\mathbf{k}'}$, $\omega_{\mathbf{k}''}$. We can write the rate of change of the occupation number $n_{\mathbf{k}}$ due to these processes as

$$\frac{dn_{\mathbf{k}}}{dt} = -4\pi \sum_{\mathbf{k},\mathbf{k}''} |V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}|^2 [n_{\mathbf{k}}(n_{\mathbf{k}'} + 1)(n_{\mathbf{k}''} + 1)$$
$$- n_{\mathbf{k}'} n_{\mathbf{k}''} (n_{\mathbf{k}} + 1)]\, \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''})\, \delta_{\mathbf{k},\mathbf{k}'+\mathbf{k}''}. \quad \text{(I-54)}$$

In the classical limit, $n_{\mathbf{k}} \gg 1$, we recover the appropriate part of the previous result. From this derivation it follows that the "collisional term" for the four-wave interaction is proportional to the third power of the occupation number. In the absence of resonance interaction among three waves (as, for example, for the electron plasma waves [14] or shallow water waves [15] the mode coupling then is of the third order in the wave energy.

The kinetic equation written in the form given in Eq. (I-54) is used in the physics of solids for the description of the phonon interaction due to lattice irregularities. [9] There is, however, a qualitative difference between applications of this equation to phonon and plasma turbulence. In solids we usually deal with a state not far from thermal equilibrium. Thus, nonlinear phenomena produce small corrections only. In a plasma, however, nonlinear phenomena are very important. The mean free path for a wave in a turbulent plasma can be quite short, and equipartition of energy between modes does not hold for a plasma. We see this in detail in the next section.

I-5. PLASMA TURBULENCE IN TERMS OF THE KINETIC EQUATION FOR WAVES

In the previous section we derived a kinetic equation for waves using the random phase approximation (RPA). Now we investigate some of the properties of this equation and apply it to several problems.

Investigation of plasma stability in magnetic confinement shows that often, under the influence of various small disturbances, the plasma arrives at a state of disordered motion. In general we must describe this motion by means of all of the plasma characteristics—velocity, temperature, etc.—(in the fluid-type description of the plasma) at each point in space and time. If the deviation from steady state is slight (or the total turbulent energy is small), we can represent this turbulent motion by a superposition of linear eigenmodes.

$$v(\mathbf{r}, t) = \sum_{\mathbf{k}} v_{\mathbf{k}} \exp[-i\omega t + i\mathbf{k}\cdot\mathbf{r}], \quad \text{etc.,} \tag{I-55}$$

where the frequencies are determined from the dispersion relation $\omega = \omega(\mathbf{k})$. Now the state may be described in terms of the amplitudes of the eigenmodes. The problem is to determine these amplitudes as a function of k and t.

The distribution of energy between different scales of turbulence can be found on the basis of wave–wave interactions. These interactions are easily treated within the framework of the RPA, which is probably valid for the case of many interacting waves. This kind of approach is called *weak turbulence theory.*

If we wish, we can write the results of this treatment either in terms of mode energy of the number of waves $n_{\mathbf{k}}$ in the kth mode. In the latter representation, the wave kinetic equation becomes

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = 2\gamma_{\mathbf{k}} n_{\mathbf{k}} + \mathrm{St}(n_{\mathbf{k}}, n_{\mathbf{k}}), \tag{I-56}$$

where the form of the collision term, $\mathrm{St}(n_{\mathbf{k}}, n_{\mathbf{k}})$, was derived in the previous lecture. In problems of nonlinear stability, it is necessary to consider the sources

of instability and dissipation. This accounts for the inclusion of the first term on the right-hand side. In steady state, we can drop $\partial n_{\mathbf{k}}/\partial t$ and just consider the right-hand side of Eq. (I-56) equated to zero.

Let us begin a preliminary examination of the relationship of weak plasma turbulence to Kolmogoroff's treatment in conventional hydrodynamic turbulence. It is very difficult to find a rigorous description of the energy transfer among different scales of hydrodynamic turbulence, because we cannot express strong turbulence in terms of eigenoscillations (waves), and we have no rigorous statistical equivalent of the equation corresponding to the wave kinetic equation

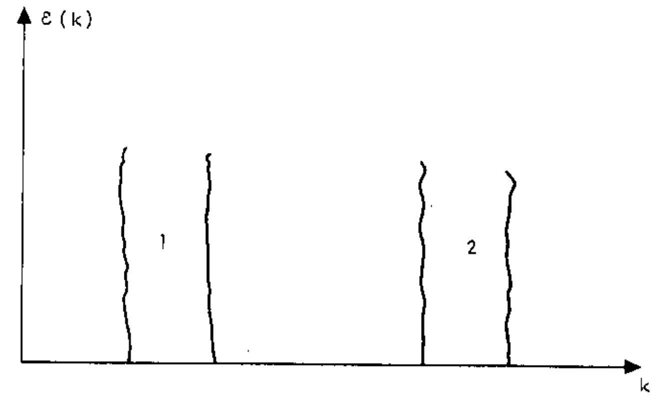

FIGURE I-8. Dissipation of strong turbulence according to the Kolmogoroff–Obukhov theory. Region 1 is the scale on which turbulence is created and region 2 is the scale on which ordinary viscosity dissipates the turbulence.

above. In conventional hydrodynamics the most reliable procedure is the use of dimensional arguments.

Imagine a situation where the source of large-scale (small k) turbulent motion (region 1 of Figure I-8) is separated from the region in which the turbulent motion damps rapidly due to viscosity in small-scale motion (region 2). Energy passes from large scales to small continuously in k space. In the intermediate region (between 1 and 2) the well-known dimensional arguments lead to the spectrum

$$\mathscr{E}_{\mathbf{k}} \sim k^{-5/3} \qquad \text{(Kolmogoroff–Obukhov law)}$$

where $\mathscr{E}_{\mathbf{k}}$ is the energy per wave number per unit volume. In deriving this it was assumed that the turbulence is isotropic and that it may be described in terms of local characteristics. Energy is transferred from small k to large in a resonant manner, passing through each decreasing scale in succession (see, for example, [16]).

It is interesting to find an analogy with Kolmogoroff's spectrum for a weakly turbulent plasma since in this case we have an equation for tha spectral density derived from first principles. In the plasma, however, we can imagine a great variety of different waves, so there is probably no universal type of spectrum with simple power dependence $n_k \sim k^{-s}$. However, for any given type of plasma waves we hope to have a simple spectrum.

Let us consider an idealized model investigated by Zakharov. [17] We deal with the nonlinear dispersive (therefore, plasma-like) media described by the wave equation

$$\frac{\partial^2 u}{\partial t^2} - (\nabla^2 - \epsilon\nabla^2\nabla^2)u = \nabla^2 u^2 \qquad (\epsilon > 0), \tag{I-57}$$

where ∇^2 is the Laplacian.

When we Fourier-transform, we find

$$\frac{\partial^2 u_{\mathbf{k}}}{\partial t^2} - \omega_{\mathbf{k}}^2 u_{\mathbf{k}} = k^2 \int d\mathbf{k}'\, u_{\mathbf{k}'} u_{\mathbf{k}-\mathbf{k}'}\,, \tag{I-58}$$

with

$$\omega_{\mathbf{k}}^2 = k^2 + \epsilon k^4, \tag{I-59}$$

which corresponds to a decay-type spectrum. We can introduce the mode amplitudes $C_{\mathbf{k}}$ defined by

$$u_{\mathbf{k}}(t) = C_{\mathbf{k}}(t)\, \frac{2k}{(|\omega_{\mathbf{k}}|)^{1/2}}\, e^{i\omega_{\mathbf{k}} t}. \tag{I-60}$$

In terms of $C_{\mathbf{k}}$ we can write Eq. (I-58) in a canonical form:

$$i\,\frac{\partial C_{\mathbf{k}}}{\partial t} = \int V_{k,k',k''}\, C_{\mathbf{k}'}(t)\, C_{\mathbf{k}}(t) \exp[i(\omega_k - \omega_{k'} - \omega_{k''})t$$
$$\times\, \delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}'')\, d^3\mathbf{k}'\, d^3\mathbf{k}'', \tag{I-61}$$

where

$$V_{k,k',k''} = -\mathrm{sign}\,\omega_k \left(\frac{k^2 k'^2 k''^2}{|\omega_k \omega_{k'} \omega_{k''}|}\right)^{1/2}.$$

Let us note here that the wave kinetic equation was derived in Section I-3 for the case of a set of discrete modes. In the continuum limit, $V_{k,k',k''} \sim 1/(k^3 C_k t)$ (instead of $1/C_k t$) and the kinetic equation seems to lack a dimensionality factor of k^3 in the right-hand side. Therefore, in the derivation, instead of $\delta_{1,\mathbf{k}}$ we will have δ functions: $\delta_{1,\mathbf{k}} \to (1/V)\,\delta(1\text{-}\mathbf{k})$, where V is a normalization volume. Thus we should really define $n_{\mathbf{k}} = (1/V)\,|C_{\mathbf{k}}|^2$, whereupon everything is correct.

We will take $\omega = k$ for all modes (i.e., $\epsilon \approx 0$) except in evaluating resonant denominators; this imposes a restriction on the magnitude of the amplitudes for which the analysis applies, namely, $|C_{\mathbf{k}}| \ll 4\epsilon k$. All modes would interact strongly if $\epsilon = 0$ exactly.

In terms of $n_{\mathbf{k}}$ we have

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = 2\gamma_{\mathbf{k}} n_{\mathbf{k}} - 4\pi \int \frac{k^2 k'^2 k''^2}{\omega_k \omega_{k'} \omega_{k''}} [\delta(\omega_k - \omega_k - \omega_{k''})$$
$$\times\, \delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}'')(n_{\mathbf{k}'} n_{\mathbf{k}''} - 2n_{\mathbf{k}} n_{\mathbf{k}'}) + 2\delta(\omega_k + \omega_{k'} - \omega_{k''})$$
$$\times\, \delta(\mathbf{k} + \mathbf{k}' - \mathbf{k}'')(n_{\mathbf{k}'} n_{\mathbf{k}''} + n_{\mathbf{k}} n_{\mathbf{k}''} - n_{\mathbf{k}} n_{\mathbf{k}'})]\, d^3\mathbf{k}'\, d^3\mathbf{k}''. \tag{I-62}$$

To avoid difficulty with signs, we have made all frequencies positive ($\omega_k = k \equiv |\mathbf{k}|$). The term $2\gamma_{\mathbf{k}} n_{\mathbf{k}}$ has been added to represent a source and sink of wave energy.

Since we are interested in isotropic solutions, we average the wave equation over angles. First we integrate out the $\mathbf{k}''$ dependence, replacing $\mathbf{k}''$ by $\pm\, \mathbf{k} - \mathbf{k}'$ according to the frequency rule. Then we write

$$d^3\mathbf{k}' = k'^2\, dk' \sin\theta\, d\theta\, d\phi$$

(θ measured from $\mathbf{k}$ direction)

$$\delta(\omega_k - \omega_{k'} - \omega_{k''}) = \delta[k - k' - (k^2 - 2kk'\cos\theta + k'^2)^{1/2}],$$

(in the first term) and integrate with respect to $\cos\theta$ from -1 to $+1$. As a result the δ function is replaced by

$$\left(\frac{kk'}{|k - k'|}\right)^{-1}.$$

Integration with respect to ϕ gives a factor of 2π. The second term can be treated similarly.

In the first term, the condition $\omega_{k'} < \omega_k$ means that the integration over the modulus of $\mathbf{k}'$ is only over the interval $0 \to k$. In the second term, however, this restriction is lifted and the upper limit is ∞.

We can put the wave kinetic equation in a more convenient form by introducing a new variable. If we define

$$N_k = n_k k^2, \tag{I-63}$$

the result is

$$\frac{\partial N_k}{\partial t} = 2\gamma_k N_k + 8\pi^2 k^2 \left(\int_0^k dk' N_{k-k'} N_{k'} - 4N_k \int_0^k N_{k'} dk' + 2\int_0^\infty N_{k'} N_{k+k'} dk' \right.$$

$$\left. - 4\frac{N_k}{k^2}\int_0^k k'^2 N_{k'} dk' - 8\frac{N_k}{k}\int_k^\infty k' N_{k'} dk' \right). \qquad \text{(I-64)}$$

One solution in the form of a power law can be found (for $\gamma_k = 0$) by inspection, in the form

$$N_k = \frac{T}{\omega_k} k^2 \simeq Tk,$$

i.e., $n_k \sim 1/\omega_k$. This is just the Rayleigh–Jeans law for equipartition of energy among all modes. However, it is clear that for the turbulence problem this law is useless, and some of the integrals taken separately in Eq. (I-64) diverge at large k.

It is clear that we must seek a solution of the form $N_k \sim k^{-s}$, $s > 2$, to avoid this ultraviolet catastrophe. The proportionality constant will be determined by requiring that the result be connected with the regions of growth and dissipation.

It might appear that a similar divergence now occurs at $k \to 0$. That this is not so may be seen by combining terms in the previous equation:

$$\int_0^k (N_{k'} N_{k-k'} - 4N_k N_{k'} + 2N_{k'} N_{k+k'})\, dk'$$

$$\sim \int_0^{k/2} (2N_{k-k'} - 4N_k + 2N_{k+k'})\, N_{k'}\, dk'$$

$$\approx 2\int_0^{k/2} dk'\, k'\, N_{k'} \frac{\partial^2 N_k}{\partial k^2}.$$

Note that the first term diverges in the same way at both $k' = 0$ and $k' = k$. The result is convergent if $N_k \sim k^{-s}$, where $s < 3$.

Now substitute $N_k = A/k^s$ into the collision term and look for solutions with $2 < s < 3$. Using the relation

$$\int_0^k k'^{m-1}(k-k')^{n-1}\, dk' = k^{m+n-1} B(m,n),$$

where

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)},$$

we find

$$\mathrm{St}\{n_k\} \equiv A^2 F(s) = 0,$$

$$F(s) = \frac{\Gamma^2(1-s)}{\Gamma(2-2s)} - \frac{4}{1-s} + 2\frac{\Gamma(1-s)\,\Gamma(2s-1)}{\Gamma(s)} - \frac{4}{3-s} - \frac{8}{s-2}. \qquad \text{(I-65)}$$

From the behavior of the Γ functions and the last two terms, it is clear that $F(s) \to \infty$ at $s = 3$ and $-\infty$ at $s = 2$, and so we expect to find a root near the middle of this interval. In fact, $s = 2.5$ is a solution, as may be verified by substitution.

Let us apply this analysis to the problem in which a source of instability and damping are present. Ordinarily we can estimate N_k simply by

$$\gamma/4\pi^2 k_0 \sim N_k, \qquad \text{(I-66)}$$

where k_0 is a characteristic magnitude in k space over which γ varies. This is incorrect if regions of damping and growth are separated by a transparent region. Let the damping be described by

$$\gamma_k = -\gamma k^\alpha/4\pi^2 \qquad (\alpha > \tfrac{1}{2}), \qquad \text{(I-67)}$$

and the instability by Eq. (I-66) with $\gamma = \nu$.

For sufficiently large k, the damping must end. The intermediate region is described (for small enough γ) by

$$N_k \approx Bk^{-2.5}. \qquad \text{(I-68)}$$

The boundary of the damping region is given by k_1, where

$$\gamma k_1^{\alpha-2.5} B \sim B^2 k_1^{-2} \quad \text{or} \quad k_1 \sim \left(\frac{B}{\gamma}\right)^{\frac{1}{\alpha-0.5}}. \qquad \text{(I-69)}$$

Integrating the steady-state equation, we obtain the conservation law

$$\int_0^\infty k\gamma_k N_k\, dk = 0.$$

From this follows

$$\nu k_0^2 \frac{B}{k_0^{2.5}} \sim \gamma B \int_{\sim 0}^{k_1} k^{-1.5+\alpha}\, dk,$$

where the lower bound of the integration of the right-hand side is taken at some $k \gg k_0$. We use Eq. (I-69) to solve for B

$$B \sim \frac{\nu(\alpha - 0.5)}{k_0^{1/2}}. \qquad \text{(I-70)}$$

Thus Eq. (I-68) is approximately valid if the transparent region is sufficiently broad and the damping sets in sufficiently abruptly. Then a spectrum similar in appearance to the Kolmogoroff spectrum holds, although the mechanics involved are quite different. In our case, this spectrum is a direct result of wave–wave processes; in the Kolmogoroff case, the modes are vortices whose statistical interaction we cannot deduce using the weak turbulence approach.

I-6. NEGATIVE ENERGY INSTABILITY

There is much wider scope for dynamical phenomena in a plasma than in fluid mechanics. For example, in a nonequilibrium plasma there can be waves with negative energy. This means that the total of wave kinetic and electromagnetic energy decreases as the wave amplitudes grow. This phenomenon was first investigated by Kadomtsev, Mikhailovskii, and Timofeev. [18]

We start with the well-known formula for the energy of a monochromatic wave of frequency ω in a dispersive isotropic medium ($\langle\rangle$ means an average over a wave period):

$$U = \frac{1}{8\pi}\left[\frac{d}{d\omega}(\epsilon\omega)\langle E^2\rangle + \frac{d}{d\omega}(\mu\omega)\langle H^2\rangle\right], \qquad \text{(I-71)}$$

where ϵ and μ are the dielectric and magnetic permeabilities. If we consider only electrostatic waves, we see that the sign of the energy depends only on that of $\partial\epsilon/\partial\omega$ and can be negative in thermodynamically nonequilibrium media (in equilibrium this is ruled out by the Kramers–Kronig relations). We can consider nonisotropic (having beams present) or nonuniform media.

As an example, we take a plasma with anisotropic ion velocity distribution (with $T_\parallel/T_\perp \to 0$). Near the nth harmonic of the ion gyrofrequency $\Omega_H = eH/Mc$, the dielectric constant is

$$\epsilon = 1 - \frac{\omega_p^2 k_z^2}{\omega^2 k^2} - \frac{\Omega_p^2 \Gamma_n}{(\omega - n\Omega_H)^2}\frac{k_z^2}{k_\perp^2};$$

here

$$\omega_p^2 = \frac{4\pi e^2 n}{m}, \qquad \Omega_p^2 = \omega_p^2\frac{m}{M},$$

$$\Gamma_n = I_n(\alpha_\perp^2)\, e^{-\alpha_\perp^2},$$

where I_n is the modified Bessel function of nth order and $\alpha_\perp^2 = (k_\perp^2\ T_\perp)/(M\Omega_H{}^2)$.

In linear theory such waves already have peculiar properties. Thus, the amplitude A_r of a wave reflected by the boundary of a region with negative dispersion ($d\omega/dk < 0$) is larger than that of the incident wave A_i.

It is well known in hydrodynamics that a wave reflects from a boundary moving with supersonic velocity ($v > C_s$) with increased amplitude in similar fashion. And, for $v > 2C_s$, there exists an angle of incidence such that the incident wave has zero amplitude, but the reflected and transmitted waves are finite. (This, of course, is just the familiar Cherenkov radiation.)

An effect related to this anomalous reflection is the observation that absorption of energy from the wave by the medium in which it is propagating leads to an increase in amplitude.

Another effect which is explicable from the point of view of negative wave energy is a form of instability in turbulent plasmas. If a wave propagates with negative energy, then its amplitude increases as the energy decreases (magnitude of energy increases). If a negative energy wave gives energy up to a positive energy wave, both increase in amplitude with accompanying increases in fields. Likewise, the simple decay of a negative energy wave into two waves, one of each type, results in a final state with a larger-amplitude negative energy wave. We can easily find the appropriate kinetic equation from the one previously derived by the substitution

$$|n_k| = \left|\frac{\partial\epsilon}{\partial\omega}\right|\frac{k^2|\phi_k|^2}{8\pi};$$

$$\operatorname{sign}\omega_k \to \operatorname{sign}\left[\omega_k\frac{\partial\epsilon(\omega,k)}{\partial\omega_k}\,\omega_k\right] = \operatorname{sign}\frac{\partial\epsilon}{\partial\omega_k}.$$

Note that this definition yields positive occupation numbers. So the wave kinetic equation becomes

$$\frac{\partial}{\partial t}|n_k| = \sum_{k',k''}|V_{kk'k''}|^2\,\delta(\omega_k - \omega_{k'} - \omega_{k''})$$
$$\times\,\delta(k - k' - k'')\left[|n_{k'}||n_{k''}| - \operatorname{sign}\left(\frac{\partial\epsilon}{\partial\omega_{k'}}\frac{\partial\epsilon}{\partial\omega_k}\right)|n_k||n_{k''}|\right.$$
$$\left. - \operatorname{sign}\left(\frac{\partial\epsilon}{\partial\omega_{k''}}\frac{\partial\epsilon}{\partial\omega_k}\right)|n_k||n_{k'}|\right].$$

Including the sign of the energy in the definition of the occupation number,

$$n_k = |n_k|\operatorname{sign}\frac{\partial\epsilon}{\partial\omega_k},$$

we have the more symmetrical form

$$\frac{\partial n_k}{\partial t} = \sum_{k',k''}\operatorname{sign}\left(\frac{\partial\epsilon}{\partial\omega_k}\frac{\partial\epsilon}{\partial\omega_{k'}}\frac{\partial\epsilon}{\partial\omega_{k''}}\right)|V_{kk'k''}|^2$$
$$\times\,\delta(\omega_k - \omega_{k'} - \omega_{k''})\,\delta(k - k' - k'')\,(n_{k'}n_{k''} - n_k n_{k''} - n_k n_{k'}). \qquad \text{(I-72)}$$

The negative energy instability can be seen as follows. Suppose the existence of an equilibrium described by the Rayleigh–Jeans law ($\epsilon_k \approx T$, $T > 0$). Take n_k to have one sign, and $n_{k'}$, $n_{k''}$ to have the other (for certainty, choose the signs $+ \cdot \cdot \cdot$). Then all terms in the right-hand side of Eq. (I-72) are the same sign as n_k, i.e., $\partial n_k/\partial t \sim \gamma n_k$, where $\gamma > 0$. Thus the assumed form of the equilibrium is actually unstable.

I-7. THE ADIABATIC APPROXIMATION (COUPLING BETWEEN HIGH-FREQUENCY AND LOW-FREQUENCY MODES)

The simple form of the turbulence spectrum found in Section I-5 corresponds to that of the kernel of the wave–wave collision integral for the idealized model described in Eq. (I-57). In a more realistic situation, this kernel usually has a complicated form. Even the procedure for its derivation is quite cumbersome, especially in the cases of interaction of different modes. It is, therefore, very important to find some additional simplifications of this problem.

Let us show how it can be simplified if the interacting modes have very different dispersive properties. If we have two modes (characterized by frequencies ω_k and Ω_k) such that $\omega_k \gg \Omega_k$, we may treat the problem adiabatically. In this context "adiabatic" means that one mode is treated as propagating in a slowly varying, weakly inhomogeneous background; the changes in the background are just those produced by the passage of the second (low-frequency) mode. The latter in turn experiences a reaction from the high-frequency mode. It sees effects averaged over many periods of the rapid oscillation. For example, let us consider the interaction between high-frequency, short-wave, longitudinal, plasma oscillations

$$\omega_k^2 \approx \omega_{pe}^2 + 3k^2 \frac{T_e}{m},$$

and low-frequency, long-wave, ion-acoustic vibrations

$$\Omega_q^2 = \pm q \cdot C_s, \qquad C_s^2 = \frac{T_e}{M},$$

with $k \gg q$. This problem was first considered by Vedenov and Rudakov. [19]

We start from the Liouville equation for the number of electron plasma oscillations (plasmons) in the six-dimensional space of coordinates and wave vectors:

$$\frac{\partial N_k}{\partial t} + \frac{\partial \omega_k}{\partial \mathbf{k}} \cdot \nabla_r N_k - \frac{\partial \omega_k}{\partial \mathbf{r}} \cdot \nabla_k N_k = 0$$

$$N_k = \frac{|E_k|^2}{4\pi\omega_k}. \tag{I-73}$$

In a uniform plasma, changes in the frequency ω_k as a function of position arise only from the ion-acoustic vibration that we describe by the hydromagnetic equations:

$$nM\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right] = -\frac{\partial p}{\partial \mathbf{r}} - \frac{n_0 e^2}{2m} \nabla \sum_k \frac{|E_k|^2}{\omega_k^2}, \tag{I-74}$$

$$(\partial n/\partial t) + \nabla\cdot(n\mathbf{V}) = 0, \tag{I-75}$$

$$p = MC_s^2 n, \tag{I-76}$$

where the last term on the right-hand side of Eq. (I-74) is obtained by averaging the electron term $-nm(\mathbf{v}\cdot\nabla)\mathbf{v}$ over many plasma oscillation periods, and just represents a gradient of the radiation pressure:

$$\mathbf{v}_k = \frac{e\mathbf{E}_k}{-\omega_k m}$$

$$-n_0 m(\mathbf{v}\cdot\nabla)\mathbf{v} = -n_0 m \sum_{k',k} e^{ik\cdot r}(\mathbf{v}_k\cdot\nabla)\mathbf{v}_{k'} e^{ik'\cdot r}$$

$$= -\frac{n_0 m}{2} \sum_{k,k'} \frac{e^2}{m^2 \omega_k \omega_{k'}} [i(\mathbf{k}'\cdot\mathbf{E}_k)\mathbf{E}_{k'} + i(\mathbf{k}\cdot\mathbf{E}_{k'})\mathbf{E}_k]$$

$$\times \exp[i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}] = -\frac{n_0 e^2}{2m} \frac{\partial}{\partial \mathbf{r}} \sum_k \frac{|E_k|^2}{\omega_k^2}.$$

Eq. (I-76) is just the adiabatic equation of state.

Using the definition of the plasma frequency

$$\omega^2 = \frac{4\pi n e^2}{m}\left(1 + 3\frac{\mathbf{k}^2 T_e}{m\omega_{pe}^2}\right), \tag{I-77}$$

and linearizing Eq. (I-75), we can write Eq. (I-73) in the form

$$\frac{\partial N_k}{\partial t} + \frac{\partial \omega_k}{\partial \mathbf{k}} \cdot \frac{\partial N_k}{\partial \mathbf{r}} + s\nabla(\nabla\cdot\boldsymbol{\xi}) \cdot \frac{\partial N_k}{\partial \mathbf{k}} = 0, \tag{I-78}$$

where we have used $s = \omega_{pe}/2$ and

$$-\frac{\partial \omega_{pe}}{\partial \mathbf{r}} = -\frac{\omega_{pe}}{2}\frac{1}{n_0}\nabla n = s\nabla(\nabla\cdot\xi).$$

In a similar manner, Eq. (I-74) becomes

$$\rho\left[\frac{\partial^2 \xi}{\partial t^2} - C_s^2 \nabla(\nabla\cdot\boldsymbol{\xi})\right] = -s\nabla \sum_k n_k. \tag{I-79}$$

In Eqs. (I-78) and (I-79), $\boldsymbol{\xi}$ is the fluid displacement.

In order to see that this set of equations is useful to work with, let us apply it to some specific problems.

Damping of an ion acoustic wave in a gas of Langmuir plasmons

Let us assume a dependence of $N_{\mathbf{k}}$ and $\boldsymbol{\xi}$ on $\mathbf{r}$ and t, of the form $\exp[-i\Omega t + i\mathbf{q}\cdot\mathbf{r}]$ Linearizing Eq. (I-78), we obtain the correction to the plasmon distribution function:

$$\delta N_{\mathbf{k}} = \frac{is}{\Omega - \mathbf{q}\cdot\dfrac{\partial\omega}{\partial\mathbf{k}}}(\mathbf{q}\cdot\boldsymbol{\xi})\left(\mathbf{q}\cdot\frac{\partial N_{\mathbf{k}}}{\partial\mathbf{k}}\right). \tag{I-80}$$

Substituting in Eq. (I-79) and dotting with q we find the dispersion relation connecting the frequency Ω and wave vector q of an ion-acoustic wave:

$$-\Omega^2 + q^2 C_s^2 = q^2 \frac{s^2}{\rho}\int \frac{d^3\mathbf{k}}{\Omega - \mathbf{q}\cdot\dfrac{\partial\omega}{\partial\mathbf{k}} + i\epsilon}\left(\mathbf{q}\cdot\frac{\partial N_{\mathbf{k}}}{\partial\mathbf{k}}\right). \tag{I-81}$$

For Ω having a small imaginary part $\Gamma_{\mathbf{q}} \ll \Omega$, we find

$$\Omega = \pm q C_s + i\Gamma_{\mathbf{q}}^{\pm},$$

$$\Gamma_{\mathbf{q}}^{\pm} = \pi\frac{s^2 q}{2\rho C_s}\int d^3\mathbf{k}\cdot\left(\mathbf{q}\cdot\frac{\partial N_{\mathbf{k}}}{\partial\mathbf{k}}\right)\delta\left(\pm q C_s - \mathbf{q}\cdot\frac{\partial\omega_{\mathbf{k}}}{\partial\mathbf{k}}\right). \tag{I-82}$$

We can see a direct analogy with the usual Landau damping (growth) of waves on particles. Here, ion sound waves damp on quasiparticles with velocity $\partial\omega_{\mathbf{k}}/\partial\mathbf{k}$ and distribution function $N_{\mathbf{k}}$. And, of course, under the reciprocal influence of the ion sound wave spectrum, this plasmon distribution will relax.

For this relaxation process we have, from Eqs. (I-78) and (I-82),

$$\frac{\partial}{\partial t}\langle N_{\mathbf{k}}\rangle = \frac{\partial}{\partial k_\alpha} D_{\alpha\beta}\frac{\partial}{\partial k_\beta}\langle N_{\mathbf{k}}\rangle,$$

$$D_{\alpha\beta} = \pi\sum_{\mathbf{q}} s^2 q_\alpha q_\beta |\mathbf{q}\cdot\boldsymbol{\xi}|^2\,\delta\left(\Omega - \mathbf{q}\cdot\frac{\partial\omega}{\partial\mathbf{k}}\right), \tag{I-83}$$

where $\langle\rangle$ means an average over the fast (plasma) oscillations.

Instability of a plasmon gas

Another interesting problem which we can solve within this framework is the instability of a gas of plasmons due to coupling with sound waves.

Under suitable conditions, a narrow spectrum of Langmuir oscillations is formed, which we can regard as excitation of a single mode:

$$N_{\mathbf{k}} = k_0^3 N_0\,\delta(\mathbf{k} - \mathbf{k}_0). \tag{I-84}$$

After substitution in Eq. (I-81) and integration by parts, we obtain

$$\Omega^2 = q^2 C_s^2\left[1 + \frac{3}{4n_0 m}\frac{q^2 w}{\left(\Omega - \mathbf{q}\cdot\dfrac{\partial\omega}{\partial\mathbf{k}_0}\right)^2}\right],$$

where $w = \int \omega_{\mathbf{k}} N_{\mathbf{k}} d^3\mathbf{k} = N_0 k_0^3 \omega_{pe}$ is the total energy in the spectrum and the dispersion equation Eq. (I-77) has been used again in calculation $\mathbf{q}\cdot(\partial\omega/\partial\mathbf{k})$. If $\mathbf{q}\cdot(\partial\omega/\partial\mathbf{k}) \gg C_s$, this equation has the solution

$$\Omega = \frac{1}{2}\mathbf{q}\cdot\frac{\partial\omega}{\partial\mathbf{k}_0} + \nu$$

$$\left(\frac{1}{2}\mathbf{q}\cdot\frac{\partial\omega}{\partial\mathbf{k}} + \nu\right)^2\left(\frac{1}{2}\mathbf{q}\cdot\frac{\partial\omega}{\partial\mathbf{k}} - \nu\right)^2 \approx \frac{3wq^2}{4n_0 m}q^2 C_s^2$$

$$\nu^2 = \left(\frac{1}{2}\mathbf{q}\cdot\frac{\partial\omega}{\partial\mathbf{k}_0}\right)^2 \pm \left(\frac{3}{4}\frac{w}{n_0 T_e}\cdot\frac{M}{m}\right)^{1/2} q^2 C_s^2.$$

From this it is clear that the criterion for instability is

$$\frac{w}{n_0 T_e} > \frac{3M}{4m}(k\lambda_{De})^4\left(\frac{\mathbf{k}\cdot\mathbf{q}}{|\mathbf{k}|\,|\mathbf{q}|}\right)^4.$$

The growth rate is

$$\nu \approx q C_s\left(\frac{w}{n_0 T_e}\frac{M}{m}\right)^{1/4}. \tag{I-85}$$

It can also be shown that the plasmon gas is unstable with respect to the decay mode discussed in the second section in connection with Alfvén waves. A steady-state plasma oscillation $(\omega_0, \mathbf{k}_0)$ decays into another plasmon $(\omega, \mathbf{k})$ and an ion sound wave $(\Omega, \mathbf{q})$. [3, 20]

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Chapter II

Wave–Particle Interaction

In this chapter, we consider the linear (or quasilinear) wave–particle interaction that is associated with the resonance condition $\omega = \mathbf{k}\cdot\mathbf{v}$. When a particle and wave satisfy this resonance condition, the particle maintains a constant phase relative to the wave and is very effectively accelerated by the essentially constant electric field it then experiences. Since the interaction involves resonant particles, it cannot be derived from the MHD equations; instead the Vlasov equation and Maxwell's equations must be used.

II-1. WAVE-PARTICLE INTERACTION FOR A SINGLE WAVE

It is convenient to start with the relatively simple problem of the resonant interaction between electrons and a monochromatic Langmuir wave in one dimension. Of course, this system is described by the equations

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = 0 \tag{II-1}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi ne\left(1 - \int dvf\right), \tag{II-2}$$

where $f(x, v, t)$ is the electron distribution function and $\phi(x, t) \equiv \phi \cos(kx\text{-}\omega t)$ is the electric potential associated with the monochromatic wave.

The only nonlinear term in these equations is the third term of the Vlasov equation, $(\partial\phi/\partial x)(\partial f/\partial v)$. One can linearize this term by holding $\partial f/\partial v$ constant (i.e., replacing $\partial f/\partial v$ by $\partial f_0/\partial v$) or by holding the amplitude of the wave constant (i.e., replacing $\phi \cos(kx\text{-}\omega t)$ by $\phi_0 \cos(kx\text{-}\omega t)$. Landau followed the first linearization procedure (i.e., replacing $\partial f/\partial v$ by $\partial f_0/\partial v$) and obtained the well-known result that the amplitude of the wave damps or grows as $e^{\gamma_L t}$, where $\gamma_L = \pi/2\omega\omega_p^2/k^2(\partial f/\partial v)|\omega/k$. In this section we follow the second linearization procedure [i.e., replacing ϕ $\cos(kx\text{-}\omega t)$ by $\phi_0 \cos(kx\text{-}\omega t)$] and find that the distribution function is strongly modified in the resonant region. As might be expected, the time scale for this modification is the oscillation period for a resonant electron in a trough of the wave, $\tau = \sqrt{m/e\phi_0 k^2}$

It is reasonable to follow Landau's linearization procedure when the amplitude of the wave changes much faster than does $\partial f/\partial v$. In terms of the two time scales, γ_L and τ, this condition can be expressed as $|\gamma_L \tau| \gg 1$. Note that this condition requires the initial amplitude of the wave to be smaller than a certain value [i.e., $|\phi_0| \ll (\gamma_L^2 m/ek^2)$]. On the other hand, one should follow the second linearization procedure when $\partial f/\partial v$ changes much faster than the amplitude of the wave. This condition can be expressed as $|\gamma_L \tau| \ll 1$ or $|\phi_0| \gg (\gamma_L^2 m/ek^2)$. Of course, when $|\gamma_L \tau|$ is of the order of unity, the problem is essentially nonlinear and neither linearization procedure can be used.

To investigate the modification in the distribution function, we examine the electron phase space trajectories in a coordinate system moving with the wave,

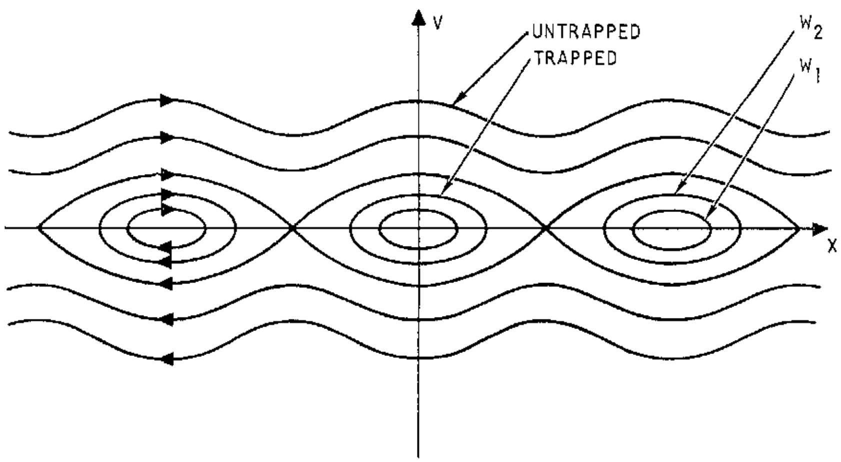

FIGURE II-1. Phase space trajectories in coordinate system moving with the monochromatic wave.

$\phi = \phi_0 \cos(kx-\omega t)$ (see Figure II-1). These trajectories are governed by the equation $W = 1/2mv^2 - |e\phi_0| \cos(kx)$. Electrons on trajectories with $W < 0$ are trapped in a trough of the wave and electrons with $W > 0$ are untrapped.

It is convenient to introduce energy angle variables (W, θ), where W defines a particular trajectory and θ a point along that trajectory, and to express the distribution function in terms of these variables, $f = f(\theta, W)$. As is well-known, f can be time-independent only if it does not depend on θ (i.e., f must be constant along particle trajectories). By making such distributions self-consistent with the electric field, one can construct steady state (i.e., nondamped) wave-like solutions of the Vlasov-Poisson equations. These solutions are called BGK solutions. [1] In the present problem, f is initially a function of both θ and W, but it tends toward a particular BGK solution in the time-asymptotic limit.

To see this, we first consider the evolution of the trapped particles. Two particles on neighboring trajectories (i.e., two particles with slightly different energies W) have slightly different frequencies of rotation in phase space (see Figure II-1):

$$\omega_2 - \omega_1 = \frac{\partial \omega}{\partial W}(W_2 - W_1).$$

If the particles start out with the same phase θ, then after a time $t \simeq 1/(\omega_T \omega_1)$ the particles will be separated in phase by $\Delta\theta \simeq 1$. In this way the phases become scrambled and f becomes constant along a particle trajectory, when looked at from a course grain (or time average) point of view. Similar arguments can be applied to the untrapped particles, so long as f is initially periodic in space.

In a real plasma, collisions will perform the coarse grain average in a natural manner. As the phases become scrambled the actual distribution becomes a very jagged function of W or v. For long enough times, the scale of this jaggedness becomes so fine that it is no longer justified to neglect the $\partial^2 f/\partial v^2$ term in the collision operator, so even in a collisionless plasma, collisions will eventually smooth out f so that it approaches the coarse grain average value. The time scale for this process will be finite and quite insensitive to the collision frequency ν. Landau used this point to show how entropy can change in a collisionless plasma (such an entropy production was discussed in [2, 3] as a dissipation mechanism for collisionless shocks).

This situation is similar to that in MHD turbulence theory, in which turbulence first develops on the scale of large wavelengths and then degrades into smaller wavelengths. For sufficiently small wavelengths, real damping (i.e., viscosity) finally dissipates the turbulence.

To obtain the value of the coarse grain distribution on any trajectory, we need only average the initial distribution over this trajectory. In the resonant region, we may also approximate the initial distribution by

$$f_0(v) \simeq f_0(\omega/k) + \frac{\partial f_0}{\partial v}\bigg|_{V=\omega/k} (v - \omega/k). \qquad \text{(II-3)}$$

Since the average of the second term vanishes for trajectories corresponding to trapped electrons, it is apparent that the coarse grain distribution $f(W)$ has the same value for all these trajectories [i.e., $f = f(\omega/k)$]. In other words, a plateau is formed in the region of phase space corresponding to trapped electrons (see Figure II-2). The formation of such a plateau is a general characteristic of the wave particle interaction and we encounter it again in the many-wave problem.

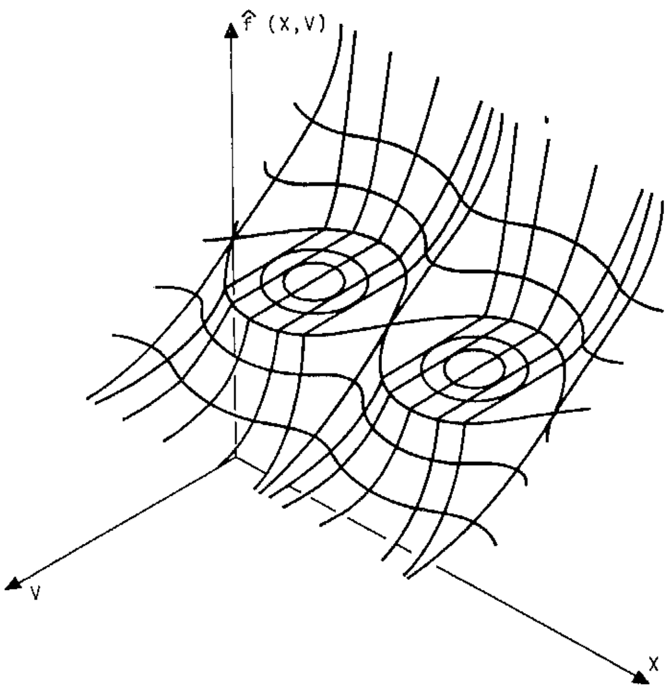

FIGURE II-2. The coarse grain distribution function.

For trajectories corresponding to untrapped electrons, the coarse grain distribution can be written as

$$f = \frac{\int_0^\lambda f_0(v)\,\Delta v(x)\,dx}{\int_0^\lambda \Delta v(x)\,dx}$$

$$f = f_0(\omega/k) + \frac{\partial f_0}{\partial v}\bigg|_{\omega/k} \frac{\int_0^\lambda [v(x) - \omega/k]\,\Delta v(x)\,dx}{\int_0^\lambda \Delta v(x)\,dx}, \qquad \text{(II-4)}$$

where constant $= \Delta W = (\partial W/\partial v)\,\Delta v(x) = m(v-\omega/k)\,\Delta v(x)$. Using the transformations

$$(v - \omega/k) = \left[\frac{2}{m} W + |2e\phi_0/m|(kx)\right]^{1/2},$$

$$\kappa^2 \equiv \frac{|2e\phi_0|}{W + |e\phi_0|} \qquad \xi = \frac{kx}{2}, \qquad \text{(II-5)}$$

Eq. (II-4) can be rewritten as

$$f = f_0(\omega/k) + \frac{\partial f_0}{\partial v}\bigg|_{\omega/k} \frac{\pi}{\kappa\tau k F(\kappa)}, \qquad \text{(II-6)}$$

where $F(\kappa) \equiv \int_0^{\pi/2} d\xi(1-\kappa^2\sin^2\xi)^{-1/2}$ is the complete elliptic integral of the first kind.

So far, we have treated the amplitude of the wave as constant and calculated the change in the distribution produced by that wave. However, we can now use

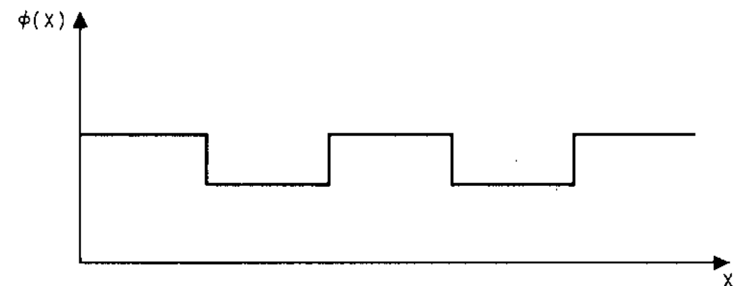

FIGURE II-3. Squaretooth wave.

this change in the distribution to calculate a small correction to the wave amplitude. We find that this correction is of the order of $\Delta\phi_0 \sim (\gamma_L \tau)\phi_0$; so one may consider this procedure to be an expansion in terms of the small parameter $|\gamma_L \tau|$.

It is convenient to introduce a time-dependent damping coefficient

$$\gamma(t) \equiv \frac{1}{2\mathscr{E}(t)} \frac{d}{dt} \mathscr{E}(t), \qquad \text{(II-7)}$$

where

$$\mathscr{E}(t) \equiv \int_0^\lambda dx/\lambda (\partial\phi/\partial x)^2/4\pi.$$

To evaluate $d\mathcal{E}/dt$ we use the conservation of energy equation

$$\frac{d\mathcal{E}}{dt} = ne \int_0^\lambda \frac{dx}{\lambda} \frac{d\phi}{dx} \int_{-\infty}^{+\infty} dv\, vf(x, v, t). \tag{II-8}$$

The time-dependent expression for $f(x, v, t)$, needed to evaluate the above integral, has been found for two forms of the wave potential. In the first case, [3] the potential was assumed to be a squaretooth wave (see Figure II-3).

The damping coefficient for this potential was found to be of the form shown in Figure II-4. The damping coefficient starts at the value predicted by Landau, $\gamma_L = \pi/2\omega\omega_p^2/k^2\, \partial f/\partial v \,|\, \omega/k$, but then oscillates with a period of the order of τ,

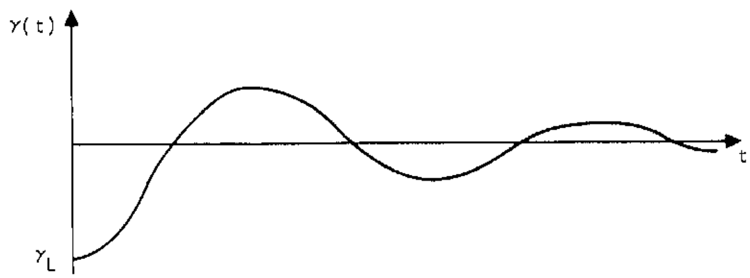

FIGURE II-4. Time-dependent damping coefficient.

the average period of oscillation for an electron in a trough of the wave. Finally the damping coefficient approaches zero, as the phases of the oscillating electrons become scrambled in phase space.

The case of a sinusoidal wave was treated by O'Neil. [4] He obtained the damping coefficient

$$\gamma(t) = \gamma_L \sum_{n=0}^{\infty} \frac{64}{\pi} \int d\kappa \left\{ \frac{2n\pi^2 \sin\left(\frac{\pi n t}{\kappa F \tau}\right)}{\kappa^5 F^2(1+q^{2n})(1+q^{-2n})} + \frac{(2n+1)\pi^2 \kappa \sin\left[\frac{(2n+1)\pi t}{2F\tau}\right]}{F^2(1+q^{2n+1})(1+q^{-2n-1})} \right\}, \tag{II-9}$$

where $F' \equiv F[(1-\kappa^2)^{1/2}, \pi/2]$ and $q \equiv e^{\pi F'/F}$. We easily can show that this damping coefficient has qualitatively the same behavior as the damping coefficient for the squaretooth wave.

The BGK solution predicted by the above theory as the time-asymptotic limit of a large-amplitude wave will eventually be destroyed by effects not included in this theory. For example, the BGK solution might decay (via the nonlinear wave–wave interaction) into another plasma wave and an ion-acoustic wave. The time scale for this process would be the ion plasma frequency.

Alternatively, the BGK solution can be destroyed by ordinary collisions. This possibility was considered by Zakharov and Karpman, [5] who found that collisions make the BGK solution damp away at the rate

$$\gamma = \beta \left(\frac{k v_{th}}{\omega}\right)^2 \frac{\tau_{osc}}{\tau_{coll}} \gamma_M, \tag{II-10}$$

where $\beta \simeq 3$, γ_M is the Landau damping coefficient for a Maxwellian, $\tau_{osc} = \sqrt{m/e\phi k^2}$ is the oscillation period for an electron in a trough of the wave, and τ_{coll} is the time it would take collisions to establish local equilibrium in the resonance region [i.e., τ_{coll} is reduced from the ordinary collision period by the factor $(\Delta v)^2/(\omega/k)^2 \simeq (e\phi_0/m)/(\omega/k)^2$]. Viewing Eq. (II-10) in the light of the Landau damping formula, $\gamma_L = \pi/2\omega\omega_p^2/k^2\, \partial f/\partial v \,|\, \omega/k$, indicates that the BGK solution damps away as if the slope of the distribution were reduced from that of a Maxwellian by the ratio $|\tau_{osc}/\tau_{coll}| \ll 1$. In other words, the slope of the distribution seems to be determined by a competition between collisions that try to make it into a Maxwellian and particle phase mixing that tries to maintain the BGK plateau (at the expense of wave energy).

II-2. THE MANY-WAVES CASE

Let us now turn to a problem with two waves. If the waves have well-separated phase velocities $(\omega/k)_{1,2}$, then they will not interact and we simply can superpose the results for the trapped particles of a single wave. However, if the waves are close together,

$$\Delta\left(\frac{\omega}{k}\right) \sim \sqrt{\frac{e\bar{\phi}}{m}},$$

where $\bar{\phi}$ is the wave potential, then the situation becomes completely different. We expect to find overlapping or collectivization of the trapped particles. We can continue by looking for solutions with 3, 4, etc., waves present, but the analysis is hopelessly complicated and only rough qualitative conclusions can be drawn. However, with a very large number of waves present, we can introduce the RPA and the statistical approach used before in these lectures.

Suppose that there is a velocity interval, $(\omega/k)_{min} < v < (\omega/k)_{max}$, with waves present having phase velocities throughout this interval such that between any two neighboring waves there is a collectivization of the resonant particles.

If the phases of these waves are random, particles undergo Brownian motion in their velocity coordinate. In phase space, this Brownian motion in velocity is superposed on the streaming motion, so that sample particle trajectories appear as shown in Figure II-5.

In the previous section we found that the time-asymptotic distribution function was constant along particle trajectories, at least when the smoothing effect of small angle coulomb collisions had been taken into account. Extending this conclusion to the present case implies that the time-asymptotic distribution is constant (i.e., flattened off) in the strip of phase space between $v = (\omega/k)_{min}$ and $v = (\omega/k)_{max}$, because particle trajectories can wander throughout this region (see Figure II-5). Note that except for the smoothing effect of the small

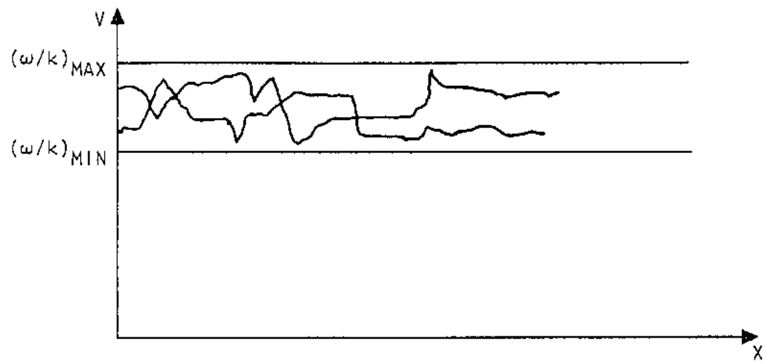

FIGURE II-5. Brownian motion of electrons in phase space.

angle coulomb collisions, the distribution would be a very complicated and jagged function, smooth only in a course grain average (or time average) sense. Of course, the jagged function conserves entropy and the smooth one does not.

The temporal evolution of the smooth (or averaged) distribution is governed by the quasilinear diffusion equation. [6–8] The most rigorous way of obtaining the quasilinear equation, as well as the criterion for overlapping between neighboring monochromatic waves, was given by Al'tshul' and Karpman [9] and by Dupree. [10] We content ourselves here with a simpler derivation.

We start by Fourier-transforming the spatial dependence of the one-dimensional Vlasov equation

$$\frac{\partial f_k}{\partial t} + ivkf_k - \frac{e}{m} ik\phi_k \frac{\partial f_0}{\partial v} = \frac{e}{m} \sum_q{}' i(k-q)\phi_{k-q} \frac{\partial f_q}{\partial v}, \qquad \text{(II-11)}$$

where the prime on the sigma indicates that the $q = 0$ term is not included in the summation. For $k = 0$, this equation can be written as

$$\frac{\partial f_0}{\partial t} = -\frac{e}{m} \sum_q{}' iq\phi_{-q} \frac{\partial f_q}{\partial v}. \qquad \text{(II-12)}$$

For $k \neq 0$, the terms on the right-hand side of Eq. (II-11) presumably represent the coupling between different plasma modes (i.e., nonlinear wave–wave interactions and nonlinear wave–particle interactions) and we need not include them in a theory that considers only the linear (or quasilinear) wave–particle interaction. Consequently, for $k \neq 0$, we use the linear equation

$$\frac{\partial f_k}{\partial t} + ivkf_k - \frac{e}{m} ik\phi_k \frac{\partial f_0}{\partial v} = 0. \qquad \text{(II-13)}$$

The Green's function solution of this equation can be written as

$$f_k(v,t) = \frac{e}{m} ik \int_0^t dt' \exp[ikv(t'-t)]\,\phi_k(t') \frac{\partial f_0}{\partial v}(v,t'). \qquad \text{(II-14)}$$

To calculate $\phi_k(t)$ we substitute this solution into Poisson's equation

$$k^2 \phi_k(t) = 4\pi ne \int dv f_k(v,t), \qquad \text{(II-15)}$$

and use a WKB approximation in time [i.e., we assume $f_0(v, t)$ changes only slightly in one period of oscillation ω_p^{-1}]. This procedure yields

$$\begin{gathered}\phi_k(t) = \phi_k(0) \exp\left\{\int_0^t [-i\omega_k + \gamma_k(t')]\,dt'\right\} \\ \omega_k = \omega_p(1 + 3/2k^2 L_D^2) \quad \gamma_k(t) = \frac{\pi}{2}\omega_k \omega_p^2/k^2 \left(\frac{\partial f_0}{\partial v}\right)_{\omega/k}.\end{gathered} \qquad \text{(II-16)}$$

To obtain the time dependence of $f_0(v, t)$, which determines the time dependence of $\gamma_k(t)$, we substitute Eqs. (II-14) and (II-16) into Eq. (II-12):

$$\begin{aligned}\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\Bigg[\int_0^t dt' \left(\frac{e}{m}\right)^2 \sum_k k^2 |\phi_k(t)|^2 \exp[i(kv-\omega_k)(t'-t)] \\ \times \frac{\partial f_0}{\partial v}(v,t') \exp\left[\int_t^{t'} \gamma_k(t'')\,dt''\right]\Bigg]\end{aligned} \qquad \text{(II-17)}$$

If we assume that the width of the excited spectrum is large enough that $\Delta(kv-\omega_k) \gg \gamma, \tau_R^{-1}$ [where τ_R is the relaxation time for $f_0(v, t)$], then the sum over k,

$$\sum_k k^2|\phi_k(t)|^2 \exp[i(kv-\omega_k)(t'-t)],$$

will phase mix to zero for $(t-t') \gtrsim \gamma^{-1}, \tau_R$, and we may set $f_0(v, t') = f_0(v, t)$ and $\gamma_k(t'') = \gamma_k(t)$ in the above integral. Carrying out the t' integral yields

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left(\frac{e}{m}\right)^2 \sum_k k^2|\phi_k(t)|^2 \, \frac{1-\exp[i(kv-\omega_k)t+\gamma_k t]}{i(kv-\omega_k)+\gamma_k(t)} \frac{\partial f_0}{\partial v}. \tag{II-18}$$

Taking into account the time-asymptotic relationship

$$\frac{1-\exp[i(kv-\omega_k)t+\gamma_k t]}{i(kv-\omega_k)+\gamma_k} = \frac{P}{i(kv-\omega_k)+\gamma_k} + \pi\delta(kv-\omega_k),$$

and the reality conditions $\omega_{-k} = -\omega_k$ and $\gamma_k = \gamma_{-k}$, one can rewrite Eq. (II-18) as

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v} D(v) \frac{\partial f_0}{\partial v}$$

$$D(v) = \left(\frac{e}{m}\right)^2 \sum_k k^2|\phi_k(t)|^2 \left[\frac{P\gamma_k}{(kv-\omega_k)^2+\gamma_k^2} + \pi\delta(kv-\omega_k)\right]. \tag{II-19}$$

Of course, this diffusion equation must be supplemented with the equation for wave growth

$$\frac{\partial}{\partial t}|\phi_k|^2 = 2\gamma_k|\phi_k|^2$$

$$\gamma_k = \frac{\pi}{2}\omega_k \frac{\omega_p^2}{k^2} \frac{\partial f_0}{\partial v}\bigg|_{\frac{\omega_k}{k}} \tag{II-20}$$

The two terms in the diffusion coefficient (i.e., the delta function term and the principle part term) are of very different character. The delta function term is positive definite and describes the flattening of the distribution in the resonant region. As expected, this is an irreversible process. On the other hand, the principle part term describes a reversible process (i.e., $2\gamma_k|\phi_k|^2 = \partial|\phi_k|^2/\partial t$ changes sign under time reversal). This "fake diffusion" (or adiabatic diffusion) describes the adjustment of the nonresonant particles to changes in wave amplitude. As the amplitude of the wave increases, for example, the oscillatory

kinetic energy associated with the wave also increases and the nonresonant particles appear to be heated (see Section II-5). Of course, there is no change in entropy associated with this apparent heating.

The $k = 0$ Fourier component of the distribution $f_0(v, t)$ played a special role in the above theory in that it was used as the zero order approximation to $f(x, v, t)$. This is physically reasonable, since $f_0(v, t)$ is the average of $f(x, v, t)$ taken along unperturbed particle trajectories, and particle streaming actually tends to average $f(x, v, t)$ along these trajectories. Of course, when there is an external electrostatic field present and the unperturbed trajectories are curved lines in the phase space, one must use for the zero order approximation to $f(x, v, t)$ its average along the actual trajectories, not the $k = 0$ Fourier component of the distribution.

Taking into account the Doppler shift in frequency, one can see that the condition on the width of the excited spectrum, $\Delta(kv-\omega_k) \gg \gamma, \tau_R^{-1}$, is equivalent to the condition that the field and motion as seen by each electron be distinguishable into a rapidly varying part and a slowly varying part.[1] An alternative derivation of quasilinear theory, mathematically similar to the van der Pol method, can be based on the existence of these two time scales. We express the distribution as the sum of a slowly varying part and a rapidly varying part

$$f = \bar{f} + f'. \tag{II-21}$$

For the rapidly varying part, we use the linearized form

$$\frac{\partial f'}{\partial t} + v\frac{\partial f'}{\partial x} - \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial \bar{f}}{\partial v} = 0, \tag{II-22}$$

neglecting the product of the two rapidly varying parts $(\partial\phi'/\partial x)(\partial f'/\partial v)$. During a single period, $\bar{f}$ changes only slightly so we can use a WKB approximation in time to solve for f' (see Eq. II-16). For $\bar{f}$, we just average the Vlasov equation over the fast motion

$$\frac{\partial \bar{f}}{\partial t} + v\frac{\partial \bar{f}}{\partial x} - \frac{e}{m}\left\langle\frac{\partial \phi'}{\partial v}\frac{\partial f'}{\partial v}\right\rangle_{\text{av}} = 0. \tag{II-23}$$

[1] Using the diffusion equation to estimate τ_R for an excited spectrum of width $\Delta(\omega/k)$,

$$\tau_R \approx \frac{[\Delta(\omega/k)]^2}{D(\omega/k)} \simeq \frac{[\Delta(\omega/k)]^3 k}{\pi\left(\frac{e}{m}\right)^2 \sum_k k^2|\phi_k|^2},$$

we can also see that the condition $\Delta(kv-\omega_k) \gg \tau_R^{-1}$ is equivalent to the condition that the trapping width be much less than the spread in phase velocity

$$\left[\left(\frac{e}{m}\right)^2 \sum_k |\phi_k|^2\right]^{\frac{1}{4}} \ll \Delta(\omega/k).$$

Substituting for f' and averaging the result over unperturbed trajectories reduces this equation to Eq. (II-19). To avoid confusion between $f_0(v, t)$ and the initial distribution we use the notation $\bar{f}(v, t)$ rather than $f_0(v, t)$

The first problem considered with Eqs. (II-19) and (II-20) was the relaxation of the unstable distribution shown in Figure II-6(a). [6, 7] We assume that the spectrum is initially some smooth function of thermal velocity [see Figure

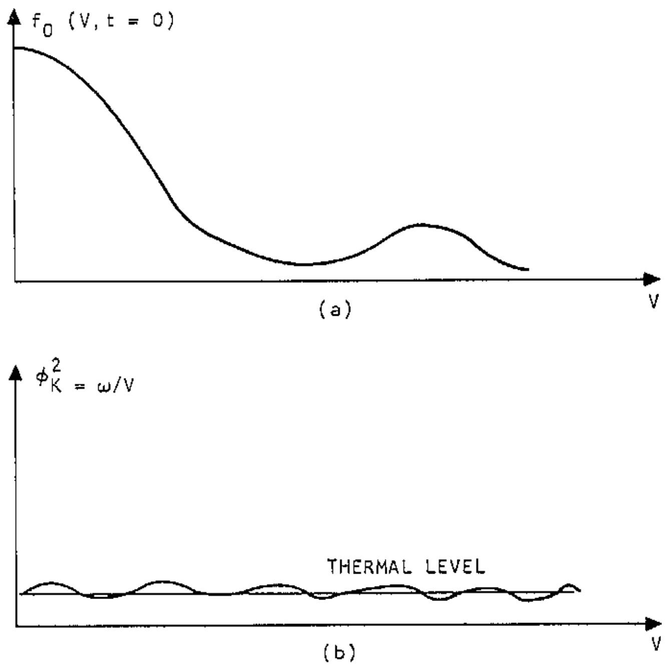

FIGURE II-6. Initial (a) distribution and (b) spectrum.

II-6(b)]. The waves that have phase velocities for which $\partial\bar{f}/\partial v > 0$ will grow and after a few e folding times the spectrum will be large enough to flatten off the distribution in this region [see Figures II-7(a) and II-7(b)]. This process will continue until the distribution is completely flattened off and the spectrum stops growing [see Figures II-(8a) and II-8(b)]. Note that in the time-asymptotic limit, the nonresonant (or main part) or the distribution has been shifted slightly to the right in order to conserve momentum. Momentum initially associated with the gentle bump is now associated with the oscillatory wave motion.

In the resonant region, the time-asymptotic distribution is uniquely defined by conservation of particle number

$$\int_{v_1}^{v_2} \bar{f}(v, t=0)\, dv = \bar{f}(t=\infty)(v_2 - v_1),$$
$$\bar{f}(v_1, t=0) = \bar{f}(v_2, t=0) = \bar{f}(t=\infty). \qquad \text{(II-24)}$$

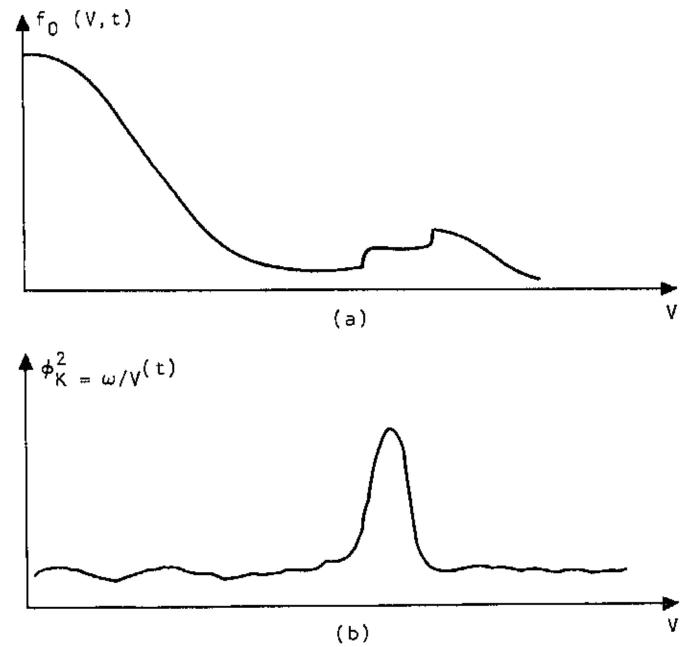

FIGURE II-7. Distribution (a) and spectrum (b) at intermediate time.

To obtain the asymptotic spectrum, we substitute Eq. (II-20) into Eq. (II-19) and retain only the delta function term

$$\frac{\partial}{\partial t}\left[\bar{f}(v, t) - \frac{\partial}{\partial v}\left(\frac{e}{m}\right)^2 \sum_k \frac{k^4}{\omega_p^3}\,\delta(kv - \omega_k)|\phi_k|^2\right]. \qquad \text{(II-25)}$$

Assuming that the initial energy in the spectrum is small compared to the energy in the bump, the integral of this equation yields

$$\sum_k k^4 |\phi_k(t=\infty)|^2\, \delta(kv - \omega_k) = \omega_p^2 \left(\frac{e}{m}\right)^2 \int_{v_1}^{v} dv[\bar{f}(t=\infty) - \bar{f}(v, t=0)]. \qquad \text{(II-26)}$$

Of course, this is not really a steady state spectrum; in deriving the quasilinear equations we neglected mode coupling terms (i.e., the nonlinear wave–wave interaction and the nonlinear wave-particle interaction), and these terms eventually distort the above spectrum. Since the quasilinear relaxation time and the mode coupling time are both inversely proportional to the first power of wave energy, we must order other parameters in the system to ensure that mode coupling is negligible during the quasilinear relaxation.

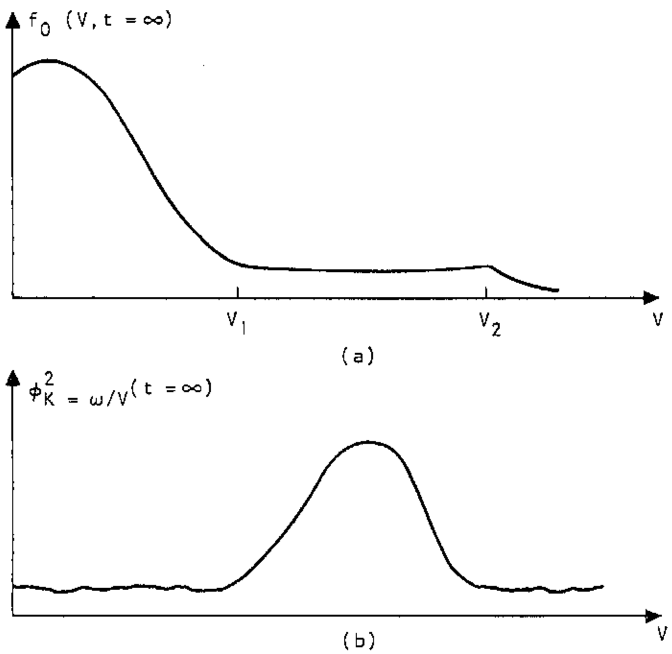

FIGURE II-8. Time-asymptotic (a) distribution and (b) spectrum.

So far we have considered quasilinear theory in only one dimension. In two or three dimensions, the diffusion equation is similar to its one-dimensional cousin,

$$\frac{\partial \bar{f}}{\partial t} = \sum_{\alpha,\beta} \frac{\partial}{\partial v_\alpha} D_{\alpha\beta}(v,t)\,\frac{\partial \bar{f}}{\partial v_\beta} \tag{II-27}$$

$$D_{\alpha\beta} = \sum_{\mathbf{k}} |\phi_{\mathbf{k}}(t)|^2 k_\alpha k_\beta \left[\frac{P\gamma_k}{\gamma_k^2 + (\mathbf{k}\cdot\mathbf{v} - \omega_k)^2} + \pi\delta(\mathbf{k}\cdot\mathbf{v} - \omega_k)\right],$$

but its solution is more complicated. The one-dimensional problem is in a sense degenerate, the resonant particles occupying only a restricted region in velocity space. In higher-dimensional cases, even for a wave packet that is localized in k space, we find a broadening of the resonance region.

Let us consider a two-dimensional problem. The initial level curves of f (characteristics), which are, for example, circles centered on the origin, are plotted in Figure II-9. Let us introduce a fairly narrow wave packet propagating in the v_x direction. Because of the formation of a quasilinear plateau, a new system of level curves parallel to v_x will be established inside this narrow band,

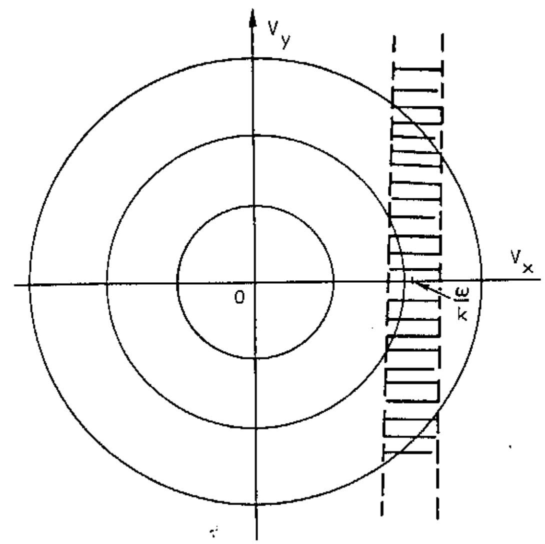

FIGURE II-9. Initial and final curves of $f(v_x, v_y)$.

$v_x \sim \omega/k$. These, of course, connect with the circles in the part of velocity space outside the resonant band. It is easy to see that only a finite amount of energy is needed to reconstruct f in this way.

Suppose we have many such wave packets present propagating in different directions. In Figure II-9 each will be traversed by its own set of level curves. Since these curves will intersect in some domain extending to infinity, an infinite amount of energy is now needed to make $\bar{f}$ constant along all these level curves.[2] If, for example, all directions of propagation are present this domain

[2] Since the quasilinear plateau is no longer an exact steady state in the two-dimensional case (even for quite narrow wave packets) it is necessary to understand its meaning by considering it as an approximation (i.e., a "quasiplateau").

fills all of velocity space outside of the circle, $v_x^2 + v_y^2 = (\omega/k)^2$, because every part of this region is common to at least two different resonance bands. Since f is constant out to infinity, it is obvious that an infinite amount of energy is involved. This means that any steady state, corresponding to finite wave packet energy, is impossible, and the wave spectrum must damp to zero. [11]

To see what will actually happen to the distribution function, let us suppose that we have a two-dimensional wave packet that has cylindrical symmetry in k space. Then f will be isotropic,

$$\bar{f} = \bar{f}(v_x^2 + v_y^2, t),$$

and on substitution in Eq. (II-27) we find

$$\frac{\partial \bar{f}}{\partial t} = \frac{e^2}{m^2} |\phi|^2 \omega^2 \frac{1}{v} \frac{\partial}{\partial v} \frac{1}{[v^2 - (\omega/k)^2]^{1/2}} \frac{1}{v} \frac{\partial \bar{f}}{\partial v} \qquad \text{(II-28)}$$

$$v^2 = v_x^2 + v_y^2,$$

where we have replaced the summation over k by an integral and taken a very narrow spectrum, $|\phi_k|^2 = |\phi|^2 \delta(k - k_0)$. (Quasilinear theory is still valid because this distribution gets smeared out when the spread in angles is taken into account.) For $\omega/k > v$ we have $\partial \bar{f}/\partial t = 0$. Together with Eq. (II-20), Eq. (II-28) has an exact solution if we make one additional simplification. If initially there is a large amount of energy in the wave packet, then we obtain a reconstruction of $\bar{f}$ in the form of an outward expansion, so that finally for most of the distribution function we can neglect ω/k in comparison with v. Now it is quite easy to solve the equations. As an example of a case in which it is valid to neglect ω/k without restrictions, we point to the interaction between electrons and ion sound waves, since

$$\frac{\omega}{k} \sim \sqrt{\frac{T_e}{M}} \ll \sqrt{\frac{T_e}{m}} \sim v.$$

For Langmuir oscillations this will be valid only after a considerable time, when the distribution has spread a long way.

Now let us introduce a new variable in place of time

$$\tau = \frac{25 e^2}{4 m^2} \int_0^t \omega^2 |\phi|^2(t')\, dt' \equiv \int_0^t D(t')\, dt'.$$

Using this variable Eq. (II-28) can be rewritten as

$$\frac{\partial \bar{f}}{\partial \tau} = \frac{4}{25} \frac{\partial}{\partial v^2} \left(\frac{1}{v} \frac{\partial \bar{f}}{\partial v^2} \right). \qquad \text{(II-29)}$$

To solve Eq. (II-29) for a given initial distribution $\bar{f}_0$ we can Laplace-transform and obtain the solution in terms of the Green's function for $\bar{f}$ using modified Bessel functions of the order of $-3/5$.

This is not a very convenient solution to work with, but we can go to $t \to \infty$, where $\bar{f}$ no longer depends on $\bar{f}_0$, and write the similarity solution of Eq. (II-29) as [12]

$$\bar{f} = C \exp\left[- \frac{v^5}{\int_0^t D(t')\, dt'} \right] \cdot \left[\int_0^t D\, dt' \right]^{-2/5}, \qquad \text{(II-30)}$$

where

$$C = \frac{5}{\Gamma(2/5)} \int_{\omega/k}^{\infty} f_0(v)\, v\, dv, \qquad v > \frac{\omega}{k}.$$

To find γ, we use the formula

$$\gamma_{\mathbf{k}} = \frac{\pi}{2} \omega_{\mathbf{k}} \frac{\omega_p^2}{k^2} \int \mathbf{k} \cdot \frac{\partial \bar{f}}{\partial \mathbf{v}} \delta(\omega_{\mathbf{k}} - \mathbf{k} \cdot \mathbf{v})\, d^2 \mathbf{v}.$$

For the asymptotic form Eq. (II-30), we find [12]

$$\gamma_{\mathbf{k}} = - \frac{\beta}{\left[\int_0^t D(t')\, dt' \right]^{3/5}}, \qquad \beta_{\mathbf{k}} \equiv C \frac{\omega_{\mathbf{k}}^2 \omega_{pe}^2 \Gamma(2/5)}{10 k^3},$$

$$\frac{d |\phi_{\mathbf{k}}|^2}{dt} = - \frac{2 \beta_{\mathbf{k}} |\phi_{\mathbf{k}}|^2}{\left[\int_0^t D(t)\, dt \right]^{3/5}}. \qquad \text{(II-31)}$$

Note that ω/k drops out of expression (II-31) for $\gamma_{\mathbf{k}}$, because the main contribution to $\gamma_{\mathbf{k}}$ comes from $v \gg \omega/k$.

The original system of equations has now been reduced to Eq. (II-31) alone, which can be transformed to a second-order nonlinear differential equation and solved, though this is a cumbersome procedure. The qualitative behavior is obvious. Starting with some initial value, the energy will damp and eventually go to zero. Initially we have Landau damping, after some finite time $\mathcal{E}(t) \to 0$. There is no energy available for further reconstruction of $\bar{f}$, and γ tends to a constant value. The results are clearly different from the one-dimensional case in which a plateau is formed.

The picture we have obtained has obvious applications to so-called turbulent heating. Suppose that we have a system in which turbulent heating is being employed; usually this means that the plasma is carrying a current that drives some instability, so we have relative motion of the electrons and ions. If this drift velocity is slightly greater than ω/k, ion sound waves are unstable.[3] As mentioned above, ω/k drops out of the problem and the results can be applied directly. So even without knowing the wave spectrum we know that, after heating, the electron spectrum will not be Maxwellian but will have the form just seen. The actual form of the tail of the distribution will come not from quasilinear theory but from other considerations, perhaps wave–wave interactions.

However, it must be remarked that, in the calculation just completed, we assumed an isotropic wave spectrum; for ion sound turbulent heating this is certainly not satisfied. Suppose a current to be directed perpendicular to the magnetic field (the usual situation): j is in the x direction, H is in the z direction.

For a small but finite value of the magnetic field, this problem is exactly the one just considered. Since particles gyrate about the direction of the field, particles are mixed in the v_x–v_y plane. We can regard this as a rotation of the wave packet instead of a rotation of particles, and the distribution function will depend on (v_x, v_y) through $v_x^2 + v_y^2$ only, even for a one-dimensional wave packet.

We thus find that there is also a difference between the one-dimensional case with and without a magnetic field. Of course, **H** must not be too large, because the simple picture of plasma dynamics used here then gets drastically modified. **H** is used only to contain the particles. We can neglect it in considering longitudinal electron wave properties because $\omega_{pe}^2 \gg \omega_{He}^2$ is usually the case in turbulent heating situations.

II-3. QUASILINEAR THEORY OF THE ELECTROMAGNETIC MODES

In this section we apply quasilinear theory to electromagnetic modes propagating through a plasma immersed in a uniform magnetic field $\mathbf{H}_0$. We consider the simplest cases of these modes, namely, ion and electron whistlers propagating parallel to the field $\mathbf{H}_0$. [6] The limitation to parallel propagation simplifies the algebra but does not significantly change the results.

The basic equations for the problem are the kinetic equation

$$\frac{\partial f_j}{\partial t} + \mathbf{v}\cdot\frac{\partial f_j}{\partial \mathbf{r}} + \frac{e_j}{m_j c}\,\mathbf{v}\times\mathbf{H}_0\cdot\frac{\partial f_j}{\partial \mathbf{v}} + \frac{e_j}{m_j}\left[\mathbf{E}_\perp + \frac{1}{c}(\mathbf{v}\times\mathbf{H}_\perp)\right]\cdot\frac{\partial f_j}{\partial \mathbf{v}} = 0$$

[3] The nonlinear theory of this kind of instability is discussed in Chapter III.

and Maxwell's equations. If we divide the distribution into slowly varying and rapidly varying parts, the equation for the slowly varying part takes the form

$$\frac{\partial \bar f_j}{\partial t} = \left(\frac{e_j}{m_j}\right)^2 \sum_k \left[-\frac{k v_\perp}{\omega_{-k}}\frac{\partial}{\partial v_z} + \left(1 + \frac{k v_z}{\omega_{-k}}\right)\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp\right] \times \frac{|E_k|^2}{(-i\omega_k \pm i\omega_{Hj} + ikv_z)}\left[\left(1 - \frac{kv_z}{\omega_k}\right)\frac{\partial \bar f_j}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial \bar f_j}{\partial v_z}\right], \quad \text{(II-32)}$$

into which we have introduced cylindrical coordinates in velocity space. The ± sign in the resonant denominator refers to right- and left-hand circularly polarized waves. In the resonant region this equation becomes

$$\frac{\partial \bar f_j}{\partial t} = \frac{e_j^2}{m_j^2}\sum_k \left[\left(1 - \frac{kv_z}{\omega_k}\right)\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp + \frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z}\right]|E_k|^2 \times \pi\delta(\omega_k - kv_z \pm \omega_{Hj})\left[\left(1 - \frac{kv_z}{\omega_k}\right)\frac{\partial}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z}\right]\bar f_j, \quad \text{(II-33)}$$

the resonance condition now picking out particles with velocity

$$v_z = \frac{\omega \pm \omega_{Hj}}{k}.$$

The frequency of the field as seen in a coordinate system moving with this velocity has been Doppler-shifted to the gyrofrequency. The resonant particles therefore rotate around the field $\mathbf{H}_0$ at the same rate as the electric vector $\mathbf{E}_\perp$ and they are accelerated very effectively by this field.

As a first application of Eq. (II-33), we assume a large-amplitude wave packet is impressed on a Maxwellian plasma, the resonant region for the wave packet being as shown in Figure II-10. It lies in the left-hand plane because $\omega - \omega_{Hj}$ is negative for the whistler modes. The solid circles in this figure are the level curves for the Maxwellian. As in the quasilinear theory of Langmuir oscillations, the resonant particles diffuse until a quasistatic state is reached, and from Eq. (II-33) one can see that for a sufficiently narrow wave packet [i.e., $\Delta(\omega/k) \ll \omega/k$], this quasistatic state will be such that

$$\left[\left(1 - \frac{k}{\omega_k}v_z\right)\frac{\partial}{\partial v_\perp} + \frac{kv_\perp}{\omega_k}\frac{\partial}{\partial v_z}\right]\bar f_j = 0. \quad \text{(II-34)}$$

This condition is completely equivalent to the condition that $\partial f/\partial v = 0$ in the quasilinear theory of Langmuir oscillations. From Eq. (II-34) it follows that the level curves of the quasistatic distribution are given by

$$\frac{v_\perp^2}{2} + \frac{v_z^2}{2} - \frac{\omega}{k}v_z = \text{const.} \quad \text{(II-35)}$$

These curves are circles but their origin has been displaced to the right by ω/k (see the dashed curves in Figure II-10). Since the marginal stability condition for the whistler mode is just Eq. (II-34) integrated over $v_\perp$,

$$\int dv_\perp v_\perp^2\left[\left(1-\frac{kv_z}{\omega}\right)\frac{\partial \bar{f}_j}{\partial v_\perp}+\frac{kv_\perp}{\omega}\frac{\partial \bar{f}_j}{\partial v_z}\right]\Bigg|_{v_z=\frac{\omega-\omega_{Hj}}{k}}=0, \qquad \text{(II-36)}$$

we find that the damping coefficient goes to zero in the time-asymptotic limit, just as it did in the analogous case for Langmuir oscillations.

In order to continue this analogy, we can reduce the two-dimensional quasilinear diffusion operator in Eq. (II-33) to a one-dimensional form. Indeed, if we introduce

$$w=\frac{v_\perp^2}{2}+\frac{v_z^2}{2}-\frac{\omega}{k}v_z \qquad \text{(II-37)}$$

as one of the new variables, $\partial/\partial w$ derivatives cancel, and Eq. (II-33) with the new variables w and, for example, $v=v_z$ have the form[4]

$$\frac{\partial \bar{f}}{\partial t}=\sum_k\left[\left(1-\frac{kv}{\omega}\right)\frac{1}{v_\perp}+\frac{kv_\perp}{\omega}\frac{\partial}{\partial v}\right]\frac{e^2}{m^2}|E_k|^2\pi\delta(\omega-kv\pm\omega_H)\frac{kv_\perp}{\omega}\frac{\partial \bar{f}}{\partial v}$$

$$\frac{\partial \bar{f}}{\partial t}\equiv\frac{e^2}{m^2}\frac{\partial}{\partial v}\left[v_\perp(w,v)\frac{|H_k|^2(v)}{\left|v-\frac{d\omega}{dk}\right|}\frac{\partial \bar{f}}{\partial v}\right]. \qquad \text{(II-38)}$$

Also integral (II-36), which determines the imaginary part of the frequency, can easily be written in a one-dimensional form, [6, 13]

$$\operatorname{Im}\omega\sim\int dv_\perp\cdot v_\perp^2\left[\left(1-\frac{kv_z}{\omega}\right)\frac{\partial \bar{f}}{\partial v_\perp}+\frac{kv_\perp}{\omega}\frac{\partial \bar{f}}{\partial v_z}\right]\Bigg|_{v_z=\frac{\omega-\omega_H}{k}}$$

$$\equiv\int_{w_{min}(v)}^{\infty} v_\perp^2(w,v)\frac{\partial \bar{f}}{\partial v}\,dw. \qquad \text{(II-39)}$$

In a plasma with a nonisotropic velocity distribution, the integral given in Eqs. (II-36) and (II-39) might be negative, implying instability. One of the most interesting examples of a nonisotropic velocity distribution is the loss-cone distribution. This distribution is found in laboratory plasmas confined by a mirror machine and in space plasmas. The loss-cone distribution is of the form

$$f=f_0(v_\perp^2+v_z^2)\,\eta\,(\alpha v_\perp^2-v_z^2), \qquad \text{(II-40)}$$

[4] In the original paper [6] the one-dimensional form of the quasilinear equation for whistlers was written inaccurately. It was corrected in [13]. But the one-dimensional whistler case is also degenerate in some respects; there is no exact plateau in a multidimensional case. So actually we can have only a quasiplateau. [14]

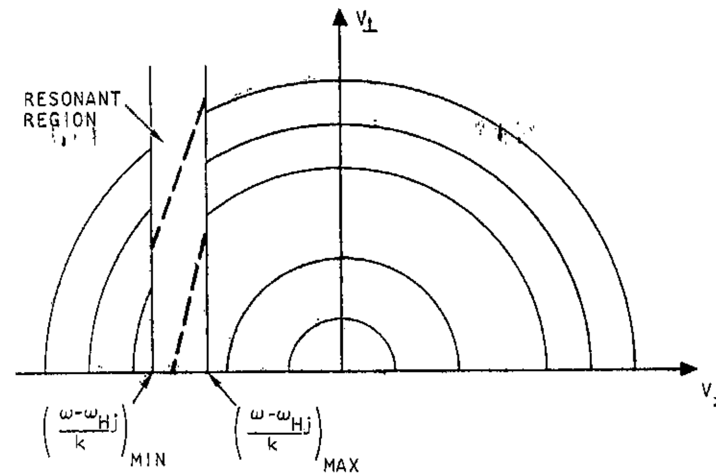

FIGURE II-10. Resonant region for the whistler mode. Solid curves are level curves for Maxwellian and dashed lines are level curves for the quasilinear plateau.

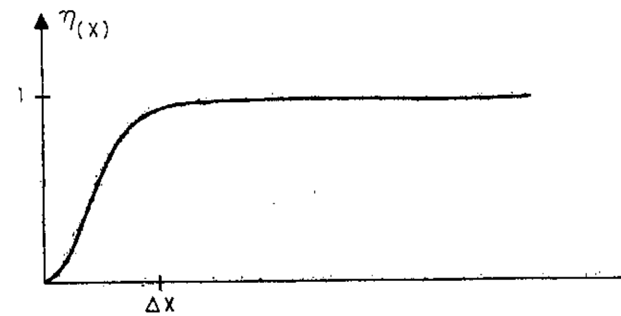

FIGURE II-11. Plot of $\eta(x)$ for loss-cone distribution.

where

$$\alpha\equiv\frac{H_{max}-H}{H}$$

and $\eta(x)$ is shown in Figure II-11. For a given α the level curves appear as shown in Figure II-12. Substituting this distribution into the stability criterion for whistlers gives

$$\int_0^\infty \epsilon_\perp\left[\left(1-\frac{kv_z}{\omega}\right)\frac{\partial}{\partial\epsilon_\perp}(f_0\eta)+\frac{k}{\omega}\frac{\partial}{\partial(mv_z)}(f_0\eta)\right]d\epsilon_\perp<0, \qquad \text{(II-41)}$$

where

$$\epsilon_\perp \equiv mv_\perp^2/2\,.$$

Integrating the first term by parts and carrying out the differentiation on the other two terms gives

$$-\int_0^\infty f_0\,\eta\,d\epsilon_\perp - \frac{2kv_z}{\omega m}\int_0^\infty \epsilon_\perp \eta \left(\frac{\partial f_0}{\partial v_\perp^2} - \frac{\partial f_0}{\partial v_z^2}\right) d\epsilon_\perp$$

$$- \frac{kv_z}{\omega}(\alpha + 1)\int_0^\infty d\epsilon_\perp\,\epsilon_\perp \eta' f_0 < 0. \qquad \text{(II-42)}$$

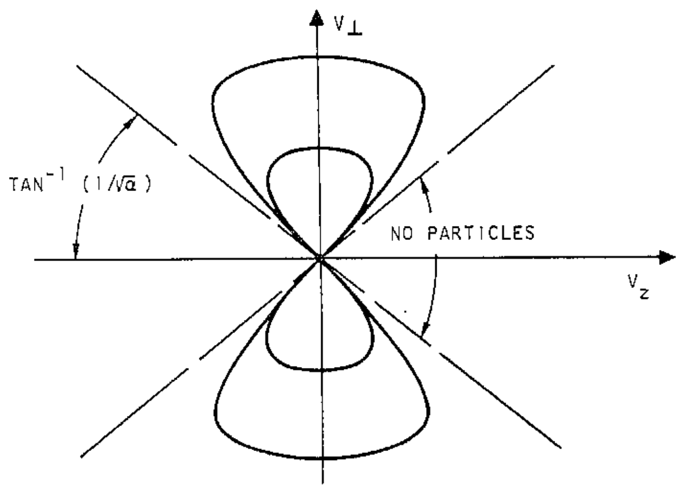

FIGURE II-12. Level curves for a loss-cone distribution.

The second term in this equation vanishes identically, and, provided the range over which η changes is small compared to v_{th}^2, we can replace $\eta'(x)$ by $\delta(x)$ in the third term. Consequently we find [14]

$$-f_0\,v_{thJ}^2\left[1 + \frac{kv_z^3(\alpha+1)}{\omega\alpha^2 v_{thJ}^2}\right] < 0 \qquad \text{(II-43)}$$

where $v_z = (\omega - \omega_{Hj})/k$. Since v_z is negative, this criterion gives instability for large enough $|v_z|$ or small enough α.

At the end of a mirror machine, α goes to zero so there is always a small unstable region. However, the violation of the local stability criterion in a small

region ΔZ does not mean that the plasma is unstable as a whole. This would require the wave to grow through many e foldings before it left the unstable region

$$\int_{\Delta z} dz\,\frac{\text{Im}\,\omega}{\partial\omega/\partial k} \gg 1. \qquad \text{(II-44)}$$

Carrying out this integration gives the following nonlocal criterion for instability:

$$\Delta z \gg \frac{c}{\omega_{pj}} F(\beta_j), \qquad \text{(II-45)}$$

where $\beta_j \equiv 8\pi n T_j/H_0^2$ and $F(\beta_j)$ goes to infinity as β_j goes to zero. In present machines this condition cannot be satisfied by ion whistlers, but it can be satisfied by electron whistlers if β_e is not too small.

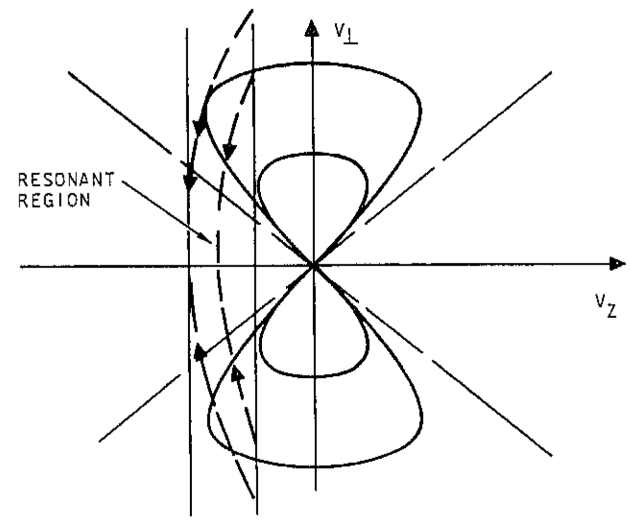

FIGURE II-13. Quasilinear depletion of particles in loss cone.

According to Eq. (II-43) the plasma will also be unstable when the resonant velocity is much larger than the thermal velocity. In any thermalized laboratory plasma there will, of course, be very few particles so far out on the tail of the distribution, and the instability will be slow. Particles therefore are swept into the loss cone (Figure II-13) when the plasma tries to form the quasilinear plateau, but very slowly. Kennell and Petschek [15] successfully applied this basic mechanism of diffusion into the loss cone to the plasma in the magnetosphere in which the necessary high-energy particles exist in large enough numbers to make this instability an important mechanism for particle untrapping.

The importance of the whistler-type loss-cone instability is often decreased by the competition of another instability due to the loss cone—the Post–Rosenbluth instability. This instability is a pure electrostatic mode with a more complicated kind of polarization. Since this is very important we discuss it in detail.

II-4. QUASILINEAR THEORY OF THE POST-ROSENBLUTH LOSS-CONE INSTABILITY

We should like to consider in detail the Post-Rosenbluth instability [16] for the case in which the volume of the loss cone is very small (for example, it may correspond to a large mirror ratio). In this case, we can treat the relative volume of the loss cone as a small expansion parameter. Further, in real traps of finite length L, the time of escape through the magnetic mirrors is finite and is of the order of L/v_z where v_z is the ion velocity along magnetic field lines. Here we consider the highly idealized situation in which this time is much greater than the time of the quasilinear diffusion into the loss cone τ_D (i.e., $L > \tau_D v_z$). If we neglect the particle escape through the mirrors, the loss cone will be filled during the time τ_D and then instability will disappear. The relaxation of the ion distribution function can be described in terms of quasilinear theory. In the case of a large mirror ratio it turns out that the level of the turbulence is sufficiently small that we can neglect mode-mode coupling.

As in [16] we assume that the unstable oscillations are electrostatic and that the perturbation scale is sufficiently small for the plasma to be regarded as homogeneous. Consequently, the electric field potential ϕ can be expanded in a sum of individual harmonic oscillations:

$$\phi = \sum_{\mathbf{k}\omega} \phi_{\mathbf{k}\omega} \exp(-i\omega t + ik_z z + i\mathbf{k}_\perp \cdot \mathbf{r}), \qquad \text{(II-46)}$$

where k_z and $\mathrm{k}_\perp$ are the components of the wave vector along and transverse to the unperturbed magnetic field $\mathrm{H}_0 = \{0, 0, \mathrm{H}\}$. We also assume that the electrons are cold and that the frequency and wavelength of the oscillations are within the intervals

$$\omega_{H_e} \gg \mathrm{Im}(\omega) \gg \omega_{H_i}, \qquad k_\perp r_{H_i} \gg 1 \gg k_\perp r_{H_e}, \qquad \text{(II-47)}$$

where ω_{Hi} is the gyrofrequency of the ith species and r_{Hi} is the average Larmor radius of the ith species. Development of instability within this frequency interval is possible only in a dense plasma,

$$\left(\frac{4\pi e^2 n}{m_i}\right)^{1/2} \equiv \omega_{pi} \gg \omega_{H_i}. \qquad \text{(II-48)}$$

Under assumptions (II-47) we may neglect the influence of the magnetic field on the ion motion and use the drift approximation for description of the electron motion. Consequently, the kinetic equations for the ions and electrons can be written in the form

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla - \frac{e}{m_i}\nabla\phi(\mathbf{r}, t)\cdot\frac{\partial}{\partial \mathbf{v}}\right] f_i(\mathbf{r}, \mathbf{v}, t) = 0, \qquad \text{(II-49)}$$

$$\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z} - c\frac{\nabla\phi \times \mathbf{H}_0}{H_0^2}\cdot\frac{\partial}{\partial \mathbf{r}} + \frac{e}{m_e}\frac{\partial\phi}{\partial z}\cdot\frac{\partial}{\partial v_z}\right) f_e(\mathbf{r}, M, v_z, t) = 0, \qquad \text{(II-50)}$$

where M is the magnetic moment of the electron (i.e., $M = m_e v_\perp{}^2/2\mathrm{H}$ = constant). The complete system of equations contains, besides Eqs. (II-49) and (II-50), the equation for the electrical field potential

$$\Delta\phi = -4\pi \sum_{j=i,e} n_j e_j \int f_j(\mathbf{v})\, d^3\mathbf{v}. \qquad \text{(II-51)}$$

In the linear approximation we can reduce Eqs. (II-49)–(II-51) to the dispersion relation

$$\epsilon(\omega, \mathbf{k}) \equiv 1 + \frac{\omega_{pe}^2}{\omega_{He}^2} + \frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{+\infty} \frac{k_z \dfrac{\partial f_e}{\partial v_z}}{\omega - k_z v_z + i\epsilon}\, dv_z + \frac{\omega_{pi}^2}{k^2}\int \frac{\mathbf{k}\cdot\dfrac{\partial}{\partial \mathbf{v}} f_i}{\omega - \mathbf{k}\cdot\mathbf{v} + i\epsilon} = 0. \qquad \text{(II-52)}$$

In a very dense plasma, $\omega_{pe} \gg \omega_{He}$, we must take into account in Eq. (II-50) the inertial drift of the electrons, which corresponds to the additional term $\omega_{pe}^2/\omega_{He}^2$ in Eq. (II-52). We omit this term in the following calculations because it changes only the definition of the plasma frequencies,

$$\omega_{pi} \rightarrow \omega_{pi}\left(1 + \frac{\omega_{pe}^2}{\omega_{He}^2}\right)^{-1/2}, \text{ etc.}$$

Insofar as $k_z \ll k_\perp$, we neglect the ion motion along magnetic field lines. Then the unperturbed ion distribution does not depend on the azimuthal angle in velocity space, and we can easily integrate over this angle:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} dv_z \int_0^{\infty} v_\perp dv_\perp \int_0^{2\pi} d\phi \frac{k_\perp v_\perp \cos\phi}{\omega - k_\perp v_\perp \cos\phi}\frac{\partial}{v_\perp \partial v_\perp} f_i(v_\perp, v_z)$$

$$\equiv \frac{1}{v_{thi}^2}\left[\Psi(0) + F\left(\frac{\omega}{k_\perp v_{thi}}\right)\right]$$

$$\Psi(W) \equiv v_{thi}^2 \int_{-\infty}^{+\infty} f_i(v_\perp^2, v_z)\, dv_z \tag{II-53}$$

$$F(y) \equiv 2\int_0^{\infty} dW \frac{d\Psi/dW}{(1 - W/y^2)^{1/2}}.$$

Here $\Psi(x)$ is the ion distribution as a function of the dimensionless velocity, $W = v_\perp^2/v_{thi}^2$, and it satisfies the normalization condition. The desired root of the integrand in $F(y)$ corresponds to taking the integrals in Eq. (II-53) for ω in the upper half plane; so that for $y = y_r + i\epsilon$

$$(1 - W/y^2)^{-1/2} = -iy_r(W - y_r^2)^{-1/2}, \qquad W > y_r^2. \tag{II-54}$$

Finally, we neglect the electron thermal motion according to the third assumption and rewrite the dispersion equation in the form

$$\epsilon(\omega, \mathbf{k}) \equiv 1 - \frac{\omega_{pe}^2}{\omega^2}\frac{k_z^2}{k^2} + \frac{\omega_{pi}^2}{k^2 v_{thi}^2}\left[\Psi(0) + F\left(\frac{\omega}{k v_{thi}}\right)\right] = 0. \tag{II-55}$$

When the plasma is near the marginally stable state, one may expand Eq. (II-55) with respect to $\text{Im}\,\omega \equiv \gamma \ll \omega_r$ and show that

$$\gamma = -\frac{\omega_r \,\text{Im}\, F\left(\dfrac{\omega}{k v_{thi}}\right)}{2[k^2\lambda_{Di}^2 + \Psi(0) + F_r(\omega_r/k_{thi})]}$$

$$\omega_r = \pm\omega_{pe}\frac{k_z}{k}\left\{1 + \frac{\left[\Psi(0) + F_r\left(\dfrac{\omega}{k v_{thi}}\right)\right]}{k^2\lambda_{Di}^2}\right\}^{-1/2}. \tag{II-56}$$

The solution with $\text{Im}\,\omega > 0$, corresponding to growing disturbances, appears only if

$$\int_0^{\infty} dW \frac{\partial\Psi}{\partial W} W^{-1/2} > 0. \tag{II-57}$$

By averaging the kinetic equation over the period of the rapid oscillations in coordinate space, we derive the quasilinear equation for the averaged distribution function f_{0i}

$$\frac{\partial f_{0i}}{\partial t} = -\frac{e}{m_i}\sum_{\mathbf{k},\omega} i\phi_{\mathbf{k}\omega}^* \mathbf{k}\frac{\partial f_{\mathbf{k}\omega}^i(\mathbf{v})}{\partial \mathbf{v}} \tag{II-58}$$

Under our first assumption $L > v_z\tau_D$ (τ_D is the diffusion time into the loss cone) no loss of the particles from the trap appears during the relaxation of the ion distribution. Due to the small volume of the loss cone (our second assumption), the wave energy is small. Hence, we can neglect mode–mode coupling and use the rapidly oscillating part of the distribution function $f_{\mathbf{k}\omega}^i$ and the electric field potential $\phi_{\mathbf{k}\omega}$ derived from the linear approximation

$$f_{\mathbf{k}\omega}^i = -\frac{e}{m_i}\frac{\mathbf{k}_\perp\cdot\mathbf{v}_\perp \dfrac{\partial}{v_\perp \partial v_\perp} f_{i0}(v_\perp^2, v_z)}{(\omega - \mathbf{k}_\perp\cdot\mathbf{v}_\perp + i\epsilon)}\phi_{\mathbf{k}}\delta_{\omega,\omega\mathbf{k}}$$

$$\phi_{\mathbf{k}\omega} = \phi_{\mathbf{k}}\delta_{\omega,\omega_\mathbf{k}},$$

where $\omega_\mathbf{k}$ is the frequency of the eigenoscillation with wave vector $\mathbf{k}$ and amplitude $\phi_\mathbf{k}$;

$$\delta_{\omega,\omega_\mathbf{k}} = \begin{cases} 1, & \omega = \omega_\mathbf{k} \\ 0, & \omega \neq \omega_\mathbf{k}. \end{cases}$$

Within this approximation we can reduce Eq. (II-58) to the usual form of the quasilinear diffusion equation (which is valid in this case even for $\gamma \sim \omega$ since $\gamma \ll kv_{thi}$)

$$\frac{\partial\Psi(w,t)}{\partial t} = \frac{\partial}{\partial w}\sum_{y\mathbf{k} > w}\frac{|\omega_\mathbf{k}|\, e^2 |\phi_\mathbf{k}|^2}{m_i^2 v_{thi}^4 (w/y_\mathbf{k}^2 - 1)^{1/2}}\frac{\partial\Psi(wt)}{\partial w} \tag{II-59}$$

$$y_\mathbf{k} \equiv \omega_\mathbf{k}/k_\perp v_{thi}.$$

We used here the axial symmetry of the distribution function in velocity space to average Eq. (II-58) over the azimuthal and longitudinal components of the ion velocity.

The quasilinear diffusion equation (II-59) and the equation for wave growth

$$\frac{\partial|\phi_\mathbf{k}|^2}{\partial t} = 2\gamma_\mathbf{k}|\phi_\mathbf{k}|^2, \tag{II-60}$$

where $\gamma_\mathbf{k}$ is determined in terms of $f_{0i}(\mathbf{v})$ by expression (II-56), comprise the basic system, which we have to solve to find the time evolution of a plasma that initially has a loss-cone distribution. This is actually a very complicated system.

However, we can simplify Eq. (II-59), as we did in Section II-2, with Eq. (II-27). This procedure is right if the main contribution to the integral (II-57) $\int (\partial f/v_\perp \, \partial v_\perp)\, dv_\perp$ comes from that domain of velocity space where $v_\perp \gg \omega/k_\perp$. If we take the initial distribution function in form (a) shown qualitatively in Figure II-14, the condition $v \gg \omega/k$ would mean that $\omega/k \ll v_p$, where v_p is the plateau size. Actually, any kind of initial distribution, even, for example, of type (b) very soon will have form (a), since the quasilinear diffusion is very high for smaller v.

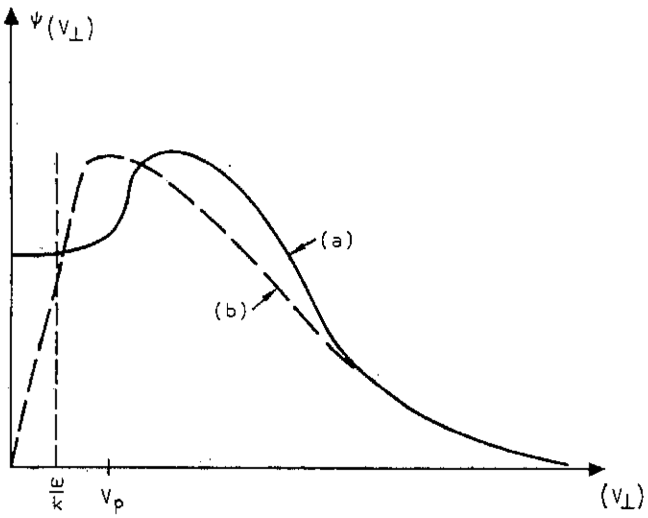

FIGURE II-14. Initial loss-cone distribution.

Thus, Eq. (II-59) may be rewritten in the approximate form [17,18]

$$\frac{\partial \Psi}{\partial D} = \frac{4}{25} \frac{\partial}{\partial w} W^{-1/2} \frac{\partial \Psi}{\partial w}, \tag{II-61}$$

where

$$D = \frac{25}{4} \sum_k \int_0^t \gamma_k \omega_k \frac{e^2 |\phi_k|^2(t)}{m_i^2 v_{thi}^4} \, dt.$$

This equation can be easily solved using the Laplace transform. The result is [18]

$$\Psi(W, D) = \int_0^\infty dW' \Psi_0(W') G(W, W'; D), \tag{II-62}$$

where $\Psi_0(W)$ is the initial ion distribution and

$$G(W, W'; D) \equiv \tfrac{5}{2} \alpha e^{-\alpha(W^{5/2} + W'^{5/2})} I_{-3/5}(2\alpha W^{5/4} W'^{5/4}) W^{3/4} W'^{3/4}$$

is the Green's function of Eq. (II-61) with $\alpha \equiv D^{-1}$. Now we can find the growth rate γ_k in terms of $\Psi(W, D)$, and substitute it in Eq. (II-60):

$$\frac{d|\phi_k|^2}{dt} = \frac{\gamma_k \omega_k}{k^2 \lambda_{Di}^2 + \Psi(0)} \int_0^\infty dW \frac{d\Psi(W, D)}{dW} W^{-1/2} |\phi_k|^2. \tag{II-63}$$

Let us consider an idealized initial ion distribution for which we can evaluate the integrals of Eq. (II-62) exactly:

$$\Psi_0(W) = A\left[1 - \Delta \exp - \left(\frac{W^{5/2}}{\epsilon^{5/2}}\right)\right] e^{-W^{5/2}}, \tag{II-64}$$

where

$$A \equiv \frac{5(1 + \epsilon^{5/2})^{2/5}}{2\Gamma(2/5)\,[(1 + \epsilon^{5/2})^{2/5} - \Delta\epsilon]}$$

is the normalization constant and $1 \geqslant \Delta \geqslant 0$ [$\Delta = 1$ corresponds to the case of empty loss cone: $\Psi_0(0) = 0$]. Then the solution for the distribution function and the equation for the wave amplitude can be written in the form

$$\Psi(W, D) = A \exp - \left(\frac{W^{5/2}}{1 + D}\right)\left\{(1 + D)^{-2/5} - \Delta(1 + D + D/\epsilon^{5/2})^{-2/5} \times \exp\left[-\frac{W^{5/2}}{\epsilon^{5/2}(1 + D)(1 + D + D/\epsilon^{5/2})}\right]\right\} \tag{II-65}$$

$$\frac{d|\phi_k|^2}{dt} = \omega_{pi} \frac{\gamma_k^2 k \lambda_{Di} |\phi_k|^2}{k^2 \lambda_D^2 + \Psi(0, D)} A \Gamma\left(\frac{4}{5}\right)\left[\frac{\Delta\epsilon^{-1/2}}{(1 + D/\epsilon^{5/2})^{3/5}} - \frac{1}{(1 + D)^{3/5}}\right]. \tag{II-66}$$

From the last expression we see that, if initially $\Delta > \sqrt{\epsilon}$, the growing oscillations cause turbulent diffusion of ions into the loss cone. As we can see from expression (II-65), for the particular choice of the initial ion distribution (II-64), the quasilinear relaxation of the turbulent spectrum can be described by changing in time the parameters $\Delta(t)$, $\epsilon(t)$, and the effective "temperature" of the main body of the ion distribution only. This change can be obtained by using formula (II-65):

$$\Delta(t) = \Delta/(1 + D + D/\epsilon^{5/2})$$

$$\epsilon(t) = \epsilon[(1 + D)(1 + D + D/\epsilon^{5/2})]^{2/5} \tag{II-67}$$

$$T_i(t) = T_i[1 + D]^{2/5}.$$

Now it is easy to describe the solution of Eq. (II-66) qualitatively. At some moment t_0 we reach the marginally stable point $\Delta(t_0) = \sqrt{\epsilon(t_0)}$ and the oscillations stop growing. However, the ions continue to diffuse into the loss cone [see Eq. (II-61)] and the oscillations become damped. It is obvious that the relaxation process stops only when the oscillation amplitude reaches the zero level, when $t \to \infty$.

The final ion distribution has a margin of stability with respect to the perturbations considered (i.e., $\gamma < 0$ for arbitrary wave vector **k**). The parameters of this distribution depend on D_∞ only and can be found from the energy conservation law[5]

$$\int_0^\infty \Psi_0(W)\, W\, dW = \int_0^\infty \Psi(W, D_\infty)\, W\, dW, \tag{II-68}$$

where

$$D_\infty = \lim_{t\to\infty} D(t). \tag{II-69}$$

From Eqs. (II-65) and (II-68) we immediately obtain

$$1 + \frac{\Delta\epsilon^2}{(1+\epsilon^{5/2})^{4/5}} = (1+D_\infty)^{2/5}\left\{1 - \frac{\Delta\epsilon^2(1+D_\infty+D_\infty/\epsilon^{5/2})^{2/5}(1+D_\infty)^{2/5}}{[1+\epsilon^{5/2}(1+D_\infty+D_\infty/\epsilon^{5/2}]^{4/5}}\right\}. \tag{II-70}$$

In the limit of large mirror ratio the parameter ϵ is small and the amplitudes of the oscillations remain small during the relaxation process (i.e., $D \ll 1$). On the other hand, by reducing the quasilinear equation (II-59) to the form of Eq. (II-61) we suppose that the width of the sink on the distribution function essentially increases due to the quasilinear diffusion. Hence, we can apply result (II-70) only to the case of initially strong instability

$$\Delta \gg \sqrt{\epsilon}. \tag{II-71}$$

Expanding Eq. (II-70) in the small parameters ϵ, D_∞, $\sqrt{\epsilon/\Delta} \ll 1$, we obtain

$$\frac{D_\infty}{\epsilon^{5/2}} = \left(\frac{5\Delta}{2\sqrt{\epsilon}}\right)^{5/3} \gg 1. \tag{II-72}$$

Using Eqs. (II-67), (II-68), and (II-72) we see directly that the sink on the ion distribution under condition (II-71) becomes wider and less deep after relaxation,

[5] In [18] it was proposed to use for this purpose the equation $\gamma(D_\infty) = 0$ because, after the first state of relaxation to the quasisteady state (II-68), the particle loss through the mirrors supports the instability on a very low level. But in that case we must add to the right-hand side of Eq. (II-61) a term describing this particle loss and change the expression for the growth rate. (This was not done in [18].)

and we can justify approximation (II-61). The main part of the distribution function is only slightly disturbed by quasilinear diffusion.

In this section we introduced many idealizations in order to get soluble equations. Of course, in a realistic situation, the development and nonlinear relaxation of the loss-cone instability have a more complicated nature, due, for example, to the continuous loss of particles through the mirrors, which we did not take into account. In these circumstances, the nonlinear mode–mode coupling becomes very important, as was shown in [18].

II-5. NONRESONANT WAVE-PARTICLE INTERACTION

So far we have considered only the resonant interaction of waves and particles. For their nonresonant (or adiabatic) interaction we must take into account the principal part term in the quasilinear diffusion equation [see Eq. (II-19)]. As mentioned in the discussion following Eq. (II-20), this term in the diffusion equation describes, for Langmuir waves, the participation of the main body of the plasma distribution in the plasma oscillations. For example, the oscillatory kinetic energy associated with a wave increases as the amplitude of the wave increases and the main body of the distribution appears to be heated. To see this quantitatively, we first write the nonresonant part of Eq. (II-19) as

$$\begin{aligned}\frac{\partial f}{\partial t} &= \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial v}\sum_k \frac{\gamma_k E_k^2}{[(kv-\omega_k)^2+\gamma_k^2]}\frac{\partial f}{\partial v}\\ \frac{\partial f}{\partial t} &\simeq \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial v}\sum_k \frac{\gamma_k E_k^2}{\omega_p^2}\frac{\partial f}{\partial v},\end{aligned} \tag{II-73}$$

where we have approximated $[(kv-\omega_k)^2+\gamma_k^2]$ by ω_p^2. Using the equation for wave growth, one can rewrite this equation as

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\frac{1}{nm}\left(\frac{d}{dt}\sum_k \frac{E_k^2}{8\pi}\right)\frac{\partial f}{\partial v}. \tag{II-74}$$

Multiplying both sides of this equation by $mv^2/2$ and integrating over velocity yields

$$\frac{d}{dt}\frac{m}{2}\int_{-\infty}^{+\infty} dv\, v^2 \bar{f}(v,t) = \frac{d}{dt}\sum_k \frac{|E_k|^2}{8\pi}. \tag{II-75}$$

In other words, the kinetic energy associated with electrons in the main body of the distribution changes at the same rate as the electrostatic energy. Of course, this is just a result of the well-known fact that the total energy in a plasma wave is composed of equal parts of electrostatic energy and kinetic energy of oscillation.

To see that this change in kinetic energy can also be looked upon as an effective heating of the plasma, we change coordinates in Eq. (II-73) from t to $\tau = \sum_k |E_k|^2/4\pi n$,

$$\frac{\partial \bar{f}}{\partial \tau} = \frac{1}{2m}\frac{\partial^2 \bar{f}}{\partial v^2}. \qquad \text{(II-76)}$$

For the initial conditions

$$\bar{f}(v, \tau = 0) = \sqrt{\frac{m}{2\pi T}} \exp - \frac{mv^2}{2T}. \qquad \text{(II-77)}$$

This equation has the solution

$$\bar{f}(v, \tau) = \left[\frac{m}{2\pi(T+\tau)}\right]^{1/2} \exp - \frac{mv^2}{2(T+\tau)}. \qquad \text{(II-78)}$$

In other words, the main body of the plasma is effectively heated by the temperature increment $\tau = \sum_k E_k^2/4\pi n$. Of course, quasilinear theory is not valid unless $\tau \ll T$.

In a similar manner, we could show that the main body of the distribution also carries the momentum associated with the waves. However, we would have to retain the velocity dependence of the denominator $1/[(kv-\omega_k)^2 + \gamma_k^2]$ in Eq. (II-73). This velocity dependence would shift the peak of the distribution in the direction of wave propagation and thereby account for wave momentum.

As opposed to those considered thus far in this chapter, many instabilities are algebraic in nature and have nothing to do with a wave-particle resonance. In such cases the nonresonant diffusion does more than account for wave energy and momentum; it describes the relaxation of the instability.

To illustrate this point, we consider the firehose instability [20, 22]. In this instability, ω is zero but γ is given by

$$\gamma_k = kv_{thi}\sqrt{\frac{T_\parallel - T_\perp}{T}}, \qquad \text{(II-79)}$$

when $\beta \gg 1$. If the plasma is not far from marginal stability (i.e., $\Delta T/T \ll 1$) then γ_k will be much less than kv_{thi}, and we can apply quasilinear theory. Note that quasilinear theory does not always require $\gamma \ll \omega$; as mentioned above, ω is in fact zero for this instability. Since $kv_{thj} \ll \omega_{Hj}$ for the firehose instability there will be no resonant particles; the instability itself is algebraic in nature and does not require resonant particles. To get the diffusion equation, we replace ω_k in Eq. (II-32) by $i\gamma_k$ and use the reality condition $\gamma_{-k} = \gamma_k$. This procedure yields

$$\frac{\partial \bar{f}_j}{\partial t} = \frac{e_j^2}{2m_j^2c^2\omega_{Hj}^2}\left(v_\perp^2\frac{\partial^2 \bar{f}_j}{\partial v_z^2} + v_z^2\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp\frac{\partial \bar{f}_j}{\partial v_\perp} - 2v_\perp\frac{\partial}{\partial v_z}v_z\frac{\partial \bar{f}_j}{\partial v_\perp}\right)\frac{d}{dt}\sum_k |H_k|^2, \qquad \text{(II-80)}$$

where we have used

$$|E_k|^2 = \frac{\omega^2}{k^2c^2}|H_k|^2. \qquad \text{(II-81)}$$

We can solve Eq. (II-80) by the following approximate procedure. Since the plasma is near marginal stability (i.e., $\Delta T/T \ll 1$), the distribution can be expressed as

$$f \simeq f_M + \frac{\Delta T}{T} f_1,$$

where f_M is a Maxwellian and f_1 is a corrective term producing the anisotropy. If we neglect this small corrective term when evaluating the right-hand side of Eq. (II-80) we obtain

$$\frac{\partial \bar{f}_j}{\partial t} = \frac{e_j^2}{2m_j^2c^2\omega_{Hj}^2}\cdot\frac{(v_\perp^2 - 2v_z^2)}{v_{thj}^2} f_M \frac{d}{dt}\sum_k |H_k|^2. \qquad \text{(II-82)}$$

The solution of this equation is

$$\bar{f}_i = f_0 + \frac{e_i^2}{2m_i^2c^2\omega_{Hi}^2}\cdot\frac{(v_\perp^2 - 2v_z^2)}{v_{thj}^2} f_M \sum_k |H_k|^2. \qquad \text{(II-83)}$$

This function can now be used to calculate the growth rate

$$\begin{aligned}\gamma_k &= k\left[\int d\mathbf{v}\bar{f}_i(v_z^2 - v_\perp^2/2)\right]^{1/2} \\ &\approx kv_{thi}\left[\left(\frac{\Delta T}{T}\right)_0 - 3\frac{e_i^2}{m_i^2c^2\omega_{Hi}^2}\sum_k |H_k|^2\right]^{1/2}. \end{aligned} \qquad \text{(II-84)}$$

The maximum growth rate in linear theory is found for $k \approx 1/r_{Hi}\sqrt{\Delta T/T}$. [23] If we are willing to use this value of k for the coefficient in Eq. (II-84), we obtain a tractable nonlinear equation for the growth of the waves,

$$\frac{d}{dt}\sum_k |H_k|^2 = \sum_k |H_k|^2\omega_{Hi}\left[\left(\frac{\Delta T}{T}\right)_0 - 3\frac{\sum_k |H_k|^2}{H_0^2}\right]. \qquad \text{(II-85)}$$

The solution to this equation is of the form shown in Figure II-15.

We can give a simple physical interpretation for the removal of the temperature anisotropy. The instability is low frequency so the integral

$$J = \int v_\parallel \, dl \qquad \text{(II-86)}$$

should be constant. At first the field lines are straight, but, as the amplitude of the wave builds up, the field lines become very wiggly. Since the path of integration in Eq. (II-86) becomes longer, $v_\parallel$ must become smaller, and this presumably reduces $T_\parallel$.

Although the nonresonant diffusion describes a wave's interaction with all particles, the strength of this interaction can be quite different for different domains in velocity space. In such situations the quasilinear distribution function can, in the process of relaxation, have a quite peculiar form.

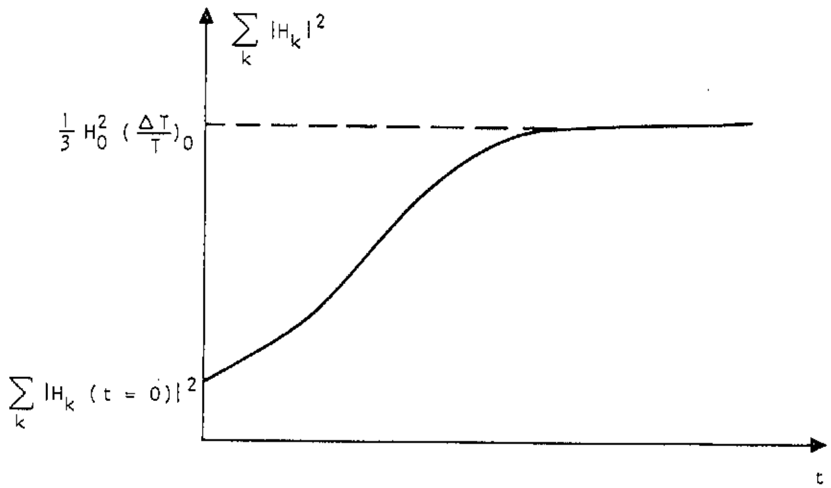

FIGURE II-15. Nonlinear evolution of the firehose instability.

An example of such a situation is the case of the "whistler" mode with no (initial) magnetic field. [24, 25] If we have a nonisotropic distribution function, it is usually unstable against whistler-type perturbations.

Let us consider such an anisotropic distribution $f(v_z^2, v_x^2)$. As indicated in Figure II-16, the effective temperature in the x direction is greater than that in the z direction. We can show easily that a pure transverse perturbation (E⊥k) will be unstable for any arbitrarily small anisotropy. To do this, we use the linearized Vlasov equation, where the first-order quantities contain the factor $e^{i(k_z z - \omega t)}$,

$$-i(\omega - k_z v_z) f_j + \frac{e_j E_x}{m_j}\frac{\partial f_{0j}}{\partial v_x} - \frac{e_j}{m_j c} H_y v_z \frac{\partial f_{0j}}{\partial v_x} + \frac{e_j}{m_j c} H_y v_x \frac{\partial f_{0j}}{\partial v_z} = 0, \qquad j = i, e, \tag{II-87}$$

and two of Maxwell's equations,

$$-ik_z H_y = \frac{4\pi e}{c}\int (f_i - f_e)\, v_x\, d^3 v \tag{II-88}$$

$$ik_z E_x = \frac{i\omega}{c} H_y. \tag{II-89}$$

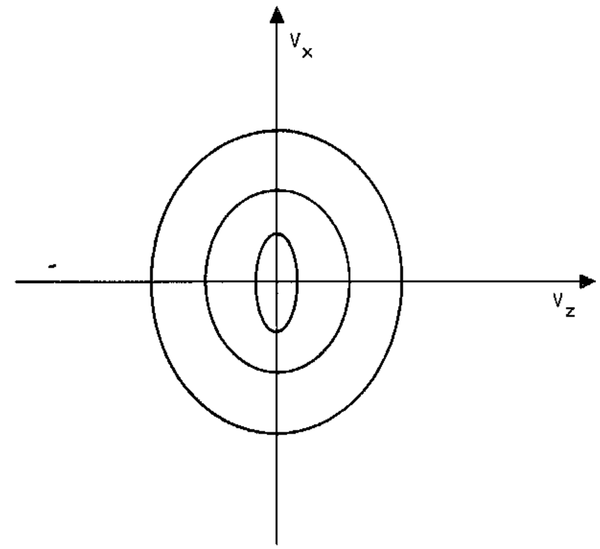

FIGURE II-16. Level curves for an anisotropic velocity distribution.

So we have four equations for f_i, f_e, H_y, E_x. We get (expressing the fields in terms of H_y)

$$f_j = \frac{1}{i(\omega - k_z v_z)}\left[\frac{e_j H_y}{m_j c}\left(v_x \frac{\partial f_{0j}}{\partial v_z} - v_z \frac{\partial f_{0j}}{\partial v_x}\right) + \frac{e_j}{m_j}\frac{\omega}{k_z c} H_y \frac{\partial f_{0j}}{\partial v_x}\right]. \tag{II-90}$$

Suppose now that the ion distribution is anisotropic. (If f_{0i} were isotropic, the magnetic field terms in the Lorentz force would cancel.) Substituting in expression (II-88) for the current, we find

$$ik_z H_y = -\frac{4\pi e}{c}\frac{ieH_y}{m_i c}\int \frac{d^3 \mathbf{v}}{\omega - k_z v_z + i\epsilon}\left(v_z f_i + v_x^2 \frac{\partial f_i}{\partial v_z} - \frac{\omega}{k_z} f_i\right)$$

$$+ \text{similar electron contribution.} \tag{II-91}$$

The dispersion relation now has a simple form

$$k_z^2 = \sum_j \frac{4\pi e^2}{m_j c^2}\left(\int d^3 v \frac{k_z v_x^2}{\omega - k_z v_z + i\epsilon}\frac{\partial f_j}{\partial v_z} - 1\right). \qquad \text{(II-92)}$$

If the plasma is isotropic, this becomes the usual dispersion relation for waves in a plasma in the absence of a magnetic field. When we have an anisotropic distribution, we can look for a new root of this equation. It arises as a very low-frequency mode. Let us suppose that $f \sim \exp[-(mv_x^2/2T_x)-(mv_z^2/2T_z)]$. Taking $\omega \ll k_z v_{thi}$, we find, from Eq. (II-92),

$$\omega = i\frac{c^2}{\pi^{1/2}\omega_{pe}^2}\left[\sum_j \frac{\omega_{pe}^2}{c^2}\left(\frac{T_{xj}}{T_{zj}} - 1\right) - k_z^2\right]\frac{T_{ze}}{T_{xe}}|k_z| v_{the} . \qquad \text{(II-93)}$$

Now we can see that this wave is unstable for sufficiently small k_z^2 if $1-T_x/T_z < 0$. [In terms of a more general distribution function $f(v_x^2, v_z^2)$, the stability condition would be $\int (v_x^2/v_z)(\partial f/\partial v_z)\,dv + 1 < 0$]. There is no real part of ω, and therefore it describes an aperiodic instability.

Suppose we have the opposite situation, $T_x < T_z$. Then instability results if the direction of propagation is rotated by 90° (i.e., by interchanging the roles of x and z). So this situation is absolutely unstable, even for very small anisotropy.

If we look at the imaginary part of ω, we see that it increases with increasing wave number; but we cannot take k_z very large because the additional term becomes significant and gives stabilization. Thus there exists a critical wave number below which waves are unstable:

$$k_z^2 < \frac{\Delta T}{T}\frac{\omega_{pe}^2}{c^2}.$$

Let us see how quasilinear theory may be applied to such a wave. The equation for $\bar{f}$ has the form

$$\frac{\partial \bar{f}}{\partial t} + \left\langle\left[\frac{eE'}{m} + \frac{-e}{mc}\mathbf{H}' \times \mathbf{v}\right]\cdot\frac{\partial f'}{\partial v}\right\rangle = 0 \qquad \text{(II-94)}$$

where the prime denotes rapidly varying functions. Next we substitute the expression found in linear theory for the rapidly varying functions and average the quadratic expression over many cycles. The resulting equation is

$$\frac{\partial \bar{f}}{\partial t} = \mathrm{Im}\frac{e^2}{m^2c^2}\left\langle(\mathbf{v}\times\mathbf{H})\frac{\partial}{\partial \mathbf{v}}\frac{i}{\omega + \mathbf{k}\cdot\mathbf{v}}\cdot\left[\frac{\omega}{kc}E' + \mathbf{v}\times\mathbf{H}\right]\right\rangle\frac{\partial \bar{f}}{\partial \mathbf{v}} \qquad \text{(II-95)}$$

where the bracket ⟨⟩ indicates the average. We can represent $i/(\omega + kv_z)$ as $(d/dt)/(\omega^2 + k_z^2 v_z^2)$ and neglect ω^2 in the denominator since $\omega^2 \ll k^2 {v_{thi}}^2$.

Now we can easily go through the same sort of calculation as that just completed for the firehose instability. Finally, the quasilinear equation becomes

$$\frac{\partial \bar{f}}{\partial t} = \frac{e^2}{m^2c^2}\sum_{\mathbf{k}}\frac{1}{k^2}\frac{d}{dt}|H_{\mathbf{k}}|^2\left(v_x\frac{\partial}{\partial v_z} - v_z\frac{\partial}{\partial v_x}\right)\frac{v_x}{v_z^2}\frac{\partial \bar{f}}{\partial v_z}. \qquad \text{(II-96)}$$

The physical meaning of the quasilinear diffusion is clear. The instability creates magnetic flutes, producing a chaotic magnetic field. This chaotic field influences the motion of the particles, and we have scattering of particles due to small-scale fluctuations in the magnetic field.

Although this represents an adiabatic interaction of modes with all particles, we can see from Eq. (II-96) that quasilinear diffusion for the particles with small v_z is much greater than for the main body of the distribution. Therefore one can expect considerable modification of the particle distribution to occur only for $v_z \ll v_{thi}$.

II-6. QUASILINEAR THEORY OF DRIFT INSTABILITY

One of the most important plasma instabilities is the drift instability of nonuniform plasma, which leads to an increase in the particle and heat loss due to turbulent transport. Therefore, careful attention should be paid to the influence of such instabilities on plasma confinement and to the nonlinear stage of their development.

Linear theory of drift waves

Let us consider in detail the low-frequency (i.e., $\omega \ll \omega_{Hi}$) "universal drift instability" of a plane slab of low β plasma in a strong magnetic field $\mathbf{H} = \{0, 0, H\}$ [26–29]

$$\frac{m_e}{m_i} < \frac{4\pi n_0(T_i + T_e)}{H^2} \equiv \beta \ll 1. \qquad \text{(II-97)}$$

We assume that the plasma has no temperature gradient but has a small density gradient

$$r_{Hi}\frac{\nabla n}{n} < \sqrt{\frac{m_e}{m_i}}. \qquad \text{(II-98)}$$

For simplicity we put $T_i = T_e$ and $\omega_{pi}^2 \gg \omega_{Hi}^2$, so the perturbation can be considered as quasineutral. Then the plasma is unstable with respect to an electrostatic perturbation with phase velocity in the interval

$$v_{thi} < \frac{\omega}{k_z} < v_A = \frac{H_0}{(4\pi n M)^{1/2}}. \qquad \text{(II-99)}$$

The electric field potential ϕ is written here as the sum of propagating plane waves,

$$\phi = \sum_{\mathbf{k},\omega} \phi(\mathbf{k}\omega) \exp[-i(\omega t - \mathbf{k}\cdot\mathbf{r})]. \quad \text{(II-100)}$$

Generally speaking, the equation for the x dependence of the wave disturbance has the form of an integrodifferential equation [30–32] that has, in the WKB approximation, a solution of the usual type:

$$\phi(x) = \exp\left[\pm i \int_{\bar{x}}^{x} k_x(x, \omega, k_y, k_z)\, dx\right]. \quad \text{(II-101)}$$

Here k_x is a complex function, but with a small imaginary part for an instability with a small growth rate. Then the square of this function $-k_x^2(x, \omega, \mathrm{k}) \equiv U_{\omega,\mathbf{k}}(x) + iV_{\omega,\mathbf{k}}(x)$ plays a role analogous to the complex potential of the Schrödinger equation. The turning points of the real part of $-k_x^2$ restrict the region of coordinate space where the wave packet can propagate. The description of an unstable perturbation by wave packets is justified if they grow in the time between reflection from the turning points up to a level at which we cannot neglect the nonlinear mode coupling. [33] Therefore, by choosing a perturbation of the form of Eq. (II-100), we suppose that [6] [18]

$$\gamma_{\mathbf{k}} \Delta x/(\partial\omega/\partial k_x) > \Lambda = \ln\left|\frac{\phi_{\mathbf{k}}^{(0)}}{\phi_{\mathbf{k}th}}\right|, \quad \text{(II-102)}$$

where $\phi_{\mathbf{k}}^{(0)}$ is the amplitude of the wave in the quasistationary turbulent state and $\phi_{\mathbf{k}th}$ is the amplitude of the electric field fluctuation at time $t = 0$ (for a quiet plasma $\phi_{\mathbf{k}th}$ is the amplitude of the thermal fluctuation).

We start by deriving the linear dispersion relation for the drift instability. [35] The equilibrium distribution function depends only on the constants of motion, $v_\perp^2, v_\parallel, X = x + v_y/\omega_{Hj}$.

$$f_{0j} = f_{0j}(v_\perp^2, v_\parallel, X). \quad \text{(II-103)}$$

In the linear approximation we can write the Boltzmann equation with an electrostatic perturbation in the form

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \frac{e_j}{m_j c}(\mathbf{v}\times\mathbf{H}_0)\frac{\partial}{\partial \mathbf{v}}\right]\delta f_j^{(1)} = \frac{e_j}{m_j}\nabla\phi\frac{\partial f_{0j}}{\partial \mathbf{v}}. \quad \text{(II-104)}$$

[6] Convection of the wave packets in the nonuniform plasma was considered in [34]. The stability condition found there is similar to Eq. (II-102).

On the left-hand side we have a derivative taken along the unperturbed particle trajectories in a magnetic field:

$$x_j(t') = x_j(t) - \frac{v_\perp}{\omega_{Hj}}\{\sin[\theta_j(t) - \omega_{Hj}(t'-t)] - \sin\theta_j(t)\}$$

$$y_j(t') = y_j(t) + \frac{v_\perp}{\omega_{Hj}}\{\cos[\theta_j(t) - \omega_{Hj}(t'-t)] - \cos\theta_j(t)\}$$

$$z_j(t') = z_j(t) + v_\parallel(t'-t), \quad \text{(II-105)}$$

where $v_\perp$ and $v_\parallel$ are the velocities of the particle across and along the magnetic field, and θ_j is the angle the velocity vector makes with the v_x axis at the instant $t' = t$. We write these in vector form using the unit vector $\mathrm{h} = \mathbf{H}/H$:

$$\mathbf{r}_\perp = -\frac{(\mathbf{v}\times\mathbf{h})}{\omega_{Hj}}. \quad \text{(II-106)}$$

Integrating Eq. (II-104) along unperturbed trajectories and using Eqs. (II-103) and (II-105), we obtain the first order correction to f_{0j}:

$$\delta f_j^{(1)} = \frac{ie_j}{m_j}\sum_{\mathbf{k}} \phi_{\mathbf{k}} \exp\left[-i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{r}) + i\frac{(\mathbf{k}\cdot\mathbf{v})\cdot h}{\omega_{Hj}}\right]$$

$$\times \int_{-\infty}^{t} dt'\left[\frac{\partial}{v_\perp\partial v_\perp}\mathbf{k}\cdot\mathbf{v}_\perp(t') + k_z\frac{\partial}{\partial v_z(t')} + \frac{k_y}{\omega_{Hj}}\frac{\partial}{\partial x}\right]f_{0j}$$

$$\times \exp\left\{-i(\omega_{\mathbf{k}} - k_z v_z)(t'-t) - i\frac{[\mathbf{k}\times\mathbf{v}(t')]\cdot\mathbf{h}}{\omega_{Hj}}\right\}, \quad \text{(II-107)}$$

where all quantities that do not depend on the time t' following the trajectory have been carried outside the integral.

Now we use the fact that the total time derivative along a particle trajectory is

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} = -i\omega_{\mathbf{k}} + i\mathbf{k}\cdot\mathbf{v}(t) \quad \text{(II-108)}$$

and use the Bessel function expansion

$$\exp\left[i\frac{(\mathbf{k}\times\mathbf{v})\cdot\mathbf{h}}{\omega_{Hj}}\right] = \sum_{l=-\infty}^{+\infty} J_l\left(\frac{k_\perp v_\perp}{\omega_{Hj}}\right)\exp\left[il\left(\frac{\pi}{2} + \theta_j - \omega_{Hj}t - \Psi_{\mathbf{k}}\right)\right], \quad \text{(II-109)}$$

where

$$\mathbf{k} = (-k_\perp \sin\Psi_{\mathbf{k}}, k_\perp\cos\Psi_{\mathbf{k}}, k_z). \quad \text{(II-110)}$$

Then we can do the time integration in Eq. (II-107) and obtain the Fourier component of the perturbed distribution function,

$$\delta f_{kj}^{(1)} = \frac{e_j}{m_j}\left\{\phi_k \frac{\partial}{v_\perp \partial v_\perp} - \sum_{l=-\infty}^{+\infty} \frac{\omega \dfrac{\partial}{v_\perp \partial v_\perp} + k_z v_z\left(\dfrac{\partial}{v_z \partial v_z} - \dfrac{\partial}{v_\perp \partial v_\perp}\right) + \dfrac{k_y}{\omega_{Hj}}\dfrac{\partial}{\partial X}}{\omega - k_z v_z + l\omega_{Hj} - i\epsilon}\right.$$

$$\left.\times J_l\left(\frac{k_\perp v_\perp}{\omega_{Hj}}\right)\exp\left[i\frac{(\mathbf{k}\times\mathbf{v})\cdot\mathbf{h}}{\omega_{Hj}} + il\left(\theta_j + \frac{\pi}{2} - \Psi_k\right)\right]\right\}$$

$$\times f_{0j}(v_\perp^2, v_z, X)\exp\left[-i(\omega t - \mathbf{k}\cdot\mathbf{r})\right]. \qquad \text{(II-111)}$$

Here the small additional term $i\epsilon$ has been inserted to make the integral convergent, and corresponds to the adiabatic switching on of the perturbation at $t = -\infty$.

Let us now consider the stability of the local Maxwellian distribution

$$f_{0j} = n_{0j}X\left(\frac{m_j}{2\pi T_j}\right)^{3/2}\exp\left(-\frac{m_j v^2}{2T_j}\right),$$

where n_{0j}, T_j are the density and temperature of the jth species. For low-frequency waves ($\omega \ll \omega_{Hi}$), we neglect in Eq. (II-111) all terms with $l \neq 0$. Also, by Eq. (II-99) we may drop the $k_z v_z$ term for ions, which describes resonant wave–ion interaction. Then on integrating with respect to velocity we find the perturbed ion density

$$n_{ki} = \frac{e_k}{T}n_0\left[1 - \frac{\omega - k_y v_d^i}{\omega}\Gamma_0(k_\perp r_{Hi})\right]. \qquad \text{(II-112)}$$

Here the function

$$\Gamma_0(\alpha) = I_0(\alpha^2/2)\,e^{-\alpha^2/2}, \qquad \alpha = k_\perp r_{Hi}$$

describes the weakening of the effective electric field experienced by the ions, averaged over their circular Larmor orbits. In the limits of small and large Larmor radius, respectively,

$$\Gamma_0 \approx 1 - \tfrac{1}{2}\alpha^2 + 0(\alpha^4), \qquad \alpha \ll 1$$

$$\Gamma_0 \approx \frac{1}{\sqrt{\pi}\alpha}\left[1 + 0(\alpha^{-2})\right], \qquad \alpha \gg 1$$

$$v_d^j = \frac{cT_j}{e_j H_0}\frac{n_0'}{n_0}. \qquad \text{(II-113)}$$

The Larmor radius of the electrons is very small and for them Eq. (II-111) corresponds to the usual drift approximation. Expanding the electron term in $\omega/k_z v_{the}$ yields

$$n_e \approx \frac{e\phi_k}{T}n_0\left[1 + i\pi^{1/2}\frac{(\omega - k_y v_d^e)}{k_z v_{the}}\right]. \qquad \text{(II-114)}$$

In the limit of quasineutrality ($k^2\lambda_D^2 \ll 1$, $\lambda_D^2 \equiv T/4\pi n_0 e^2$) the frequency and the growth rate are given by

$$\omega_k + i\gamma_k = \frac{k_y v_d^e \Gamma_{0i}}{2 - \Gamma_{0i} + i\pi^{1/2}\left[\dfrac{k_y v_d^e(\Gamma_{0i} - 1)}{|k_\parallel|\, v_{the}}\right]}. \qquad \text{(II-115)}$$

We see that the instability arises as a result of finite ion Larmor radius and that the growth rate goes to zero simultaneously with the ion Larmor radius. In order to see more clearly what can happen near this marginally stable case, we write the expression for the growth rate in terms of the electron distribution

$$\gamma_k \approx -\frac{\pi T\omega_k}{2|k_z|m_e}\left(k_z\frac{\partial}{\partial v_z} + \frac{k_y}{\omega_{Hj}}\frac{\partial}{\partial x}\right)f_e(v_z, x)\bigg|_{v_z = \omega/k_z} \qquad \text{(II-116)}$$

In the approximation of small ion Larmor radius, the two terms on the right-hand side almost cancel for a Maxwellian electron distribution and instability is very weak. (This cancellation takes place also for different temperatures of the species $T_i \neq T_e$.) Therefore, even small deviations from linear theory may strongly change the stability with respect to finite amplitude waves. There are two effects of this type.

If the amplitude of a monochromatic drift wave is large enough the resonance region is broadened by a considerable amount, $\Delta v_{res} \simeq \sqrt{2e\phi/m}$. As we show in the next paragraph this broadening can stabilize a large amplitude drift wave. Subsequently we consider the stabilization accompanying the quasilinear relaxation of the electron distribution.

Nonlinear stability of the monochromatic drift wave

We choose the following form for the electric potential of the monochromatic drift wave,

$$\phi(y, z, t) = -\phi_0\left[\cos(k_y y + k_z z - \omega t) + 0\left(\frac{e\phi_0}{T}\right)\right], \qquad \text{(II-117)}$$

setting $k_x = 0$ for simplicity. If we work in a coordinate system moving with the wave, the rate of increase in resonant electron kinetic energy, which must equal the rate of decrease in wave energy, can be written as

$$\frac{dT}{dt} = \frac{nm_e}{2}\int_{-\lambda_z/2}^{\lambda_z/2}\frac{dz}{\lambda_z}\int_{-\infty}^{+\infty} dv_z(v_z + \omega/k_z)^2\,\frac{\partial f_e}{\partial t}. \qquad \text{(II-118)}$$

In the drift approximation, the general solution for the distribution function can be written as

$$f_e(\mathbf{r}, v_z, t) = f_e[r_0(\mathbf{r}, v_z, t), v_{z0}(\mathbf{r}, v_z, t), 0],$$

where $f_e(\mathbf{r}_0, v_{z_0}, 0)$ is the initial distribution and $(\mathbf{r}_0, v_{z_0})$ is the initial position of the particle. We can divide the distribution into two parts

$$f_e(\mathbf{r}_0, v_{z0}, t) = f_0(x_0, v_{z0}) + f_1(x_0, v_{z0}, 0)\cos(k_y y_0 + k_z z_0), \qquad \text{(II-119)}$$

where the first part is the local Maxwellian and the second part is a perturbation due to the wave. The second part only makes a contribution to the harmonic generation and we can neglect it while we evaluate the rate of change of kinetic energy. Consequently, we find

$$\frac{\partial f_e}{\partial t} = \frac{\partial f_{0e}(x_0, v_{z0}, 0)}{\partial x_0}\frac{\partial x_0}{\partial t} + \frac{\partial f_{0e}(x_0, v_{z0}, 0)}{\partial v_{z0}}\frac{\partial v_{z0}}{\partial t}$$
$$= \frac{e}{m_e}\phi_0 \sin(k_y y_0 + k_z z_0)\left(k_z\frac{\partial}{\partial v_{z0}} + \frac{k_y}{\omega_{He}}\frac{\partial}{\partial x_0}\right) f_{0e}. \qquad \text{(II-120)}$$

The particle equations of motion in the drift approximation,

$$\dot{y} = 0$$
$$\dot{x} = c\frac{k_y \phi_0}{H_0}\sin(k_y y + k_z z) \qquad \text{(II-121)}$$
$$\dot{z} = -\frac{e k_z \phi_0}{m_e}\sin(k_y y + k_z z),$$

have a solution in the form of elliptic integrals. First we introduce a new variable, $k_y y + k_z z = 2\xi$, and rewrite the equation for conservation of energy

$$\frac{m\dot{z}^2}{2} - e\phi_0\cos(k_y y + k_z z) = W$$

in the form

$$\dot{\xi}^2 = 1/\kappa^2\tau^2(1 - \kappa^2\sin^2\xi), \qquad \text{(II-122)}$$

where $\kappa^2 \equiv 2e\phi_0/(W + e\phi_0)$ and $\tau \equiv (m_e/e\phi_0 k_z^2)^{1/2}$. For the case where $\kappa^2 < 1$, we can write the solution of Eq. (II-122) as

$$F(\kappa, \xi_0) = F(\kappa, \xi) - t/\kappa\tau. \qquad \text{(II-123)}$$

When $\kappa^2 > 1$, it is convenient to introduce the new variable ζ, defined as

$$\kappa\sin\xi = \sin\zeta, \qquad \dot{\zeta}^2 = 1/\tau^2(1 - 1/\kappa^2\cdot\sin^2\zeta).$$

This equation has the solution

$$F(1/\kappa, \zeta_0) = F(1/\kappa, \zeta) - t/\tau. \qquad \text{(II-124)}$$

To get the time dependence of the growth rate we need only put these solutions into Eqs. (II-118) and (II-120). This problem was solved by O'Neil [4] in the approximation that reduces to the linear growth rate at $t = 0$ (see Section II-1). We would now like to take into account nonlinear effects that are present initially, so we must include more terms than O'Neil did. After substituting Eq. (II-120) into Eq. (II-118), we obtain

$$\frac{dT}{dt} = \frac{16ne\omega\phi_0}{\pi k_z^2}\int_0^\infty d\dot{\xi}\int_0^{\pi/2} d\xi\dot{\xi}\left[\left(k_z\frac{\partial f_{0e}}{\partial v_{z0}} + \frac{k_y}{\omega_{He}}\frac{\partial f_{0e}}{\partial x_0}\right)\Bigg|_{\substack{x_0 = x\\ v_{z0} = 0}}\right.$$
$$\left. + \frac{\partial^2}{\partial v_z^2}\left(k_z\frac{\partial f_{0e}}{\partial v_{z0}} + \frac{k_y}{\omega_{He}}\frac{\partial f_{0e}}{\partial x_0}\right)\Bigg|_{\substack{x_0 = x\\ v_{z0} = 0}}\dot{\xi}^2/4k_z^2\right]\sin(2\xi_0), \qquad \text{(II-125)}$$

where we have dropped even functions of $\dot{\xi}$ by symmetry arguments.

Carrying out the integration in Eq. (II-125) gives us the following expression for the growth rate:

$$\gamma_k(t) \equiv \frac{T_e}{n_0 e^2|\phi|^2}\frac{dT}{dt}$$
$$= \frac{\pi^{1/2}\omega^2}{k_z v_{the}}\int_0^1 dx\sum_{n=0}^{\infty}\left(\left\{k_\perp^2 r_{Hi}^2 + \frac{8}{k_z^2 v_{the}^2\tau^2}\left[\frac{2}{\kappa^2}\right.\right.\right.$$
$$\left.\left. + \frac{4}{3}\left(\frac{\pi^2 n^2}{4F^2} - 1\right) - \frac{\kappa^2}{3}\right]\right\}\frac{2n\pi^2\sin\dfrac{\pi n t}{\kappa F\tau}}{x^5 F^2(1 + q^{2n})(1 + q^{-2n})} \qquad \text{(II-126)}$$
$$+ \left\{k_\perp^2 r_{Hi}^2 + \frac{8x^2}{k_z^2 v_{the}^2\tau^2}\left[2 + \frac{4}{3}\left(\frac{\pi^2(2n+1)}{16F^2} - 1\right) - \frac{\kappa^2}{3}\right]\right\}$$
$$\left.\times\frac{(2n+1)\pi^2\kappa\sin\left[\dfrac{(2n+1)\pi t}{2F\tau}\right]}{F^2(1 + q^{2n+1})(1 + q^{-2n-1})}\right) e - e\phi_0/T_e,$$

where $F(\kappa\pi/2) \equiv F$, $q \equiv \exp(\pi F'/F)$, and $F' \equiv F[(1-\kappa^2)^{1/2}, \pi/2]$.

In the limit in which $t/\tau \ll 1$, the main contribution to the growth rate comes from the $n = 1$ term in the first sum; so we find

$$\gamma(t) \simeq \pi^{1/2}\frac{\omega^2}{|k_z|v_{the}}\left(k_\perp^2 r_{Hi}^2 - \frac{4e\phi_0}{T_e}\right). \qquad \text{(II-127)}$$

Consequently, a wave with finite amplitude can be stable in the nonlinear regime if the linear growth rate is small enough. Here we took into account only finite Larmor radius effects, but the same idea can also be applied to the current- or gravity-driven drift instabilities in an inhomogeneous plasma. Therefore, we can use this expression to estimate the amplitude of the separate drift waves observed by Buchelnilcova in a potassium plasma.

Since the amplitude we obtained from Eq. (II-127),

$$\frac{e\phi_0}{T_e} \simeq \tfrac{1}{4} k_\perp^2 r_{Hi}^2,$$

is very small, we can obtain the level of harmonic generation by expanding the Vlasov equation in the small parameter $e\phi_0/T_e$. For example, the amplitude of the second harmonic is given by

$$\frac{e\phi_{2\mathbf{k}}^{(2)}}{T_e} = \left(\frac{e\phi_{\mathbf{k}}^{(1)}}{T_e}\right)^2 \Bigg\{1 - \frac{\omega - k_y v_d^i}{\omega} \int_0^\infty dv_\perp \left[1 - J_0\left(\frac{2k_\perp v_\perp}{\omega_{Hi}}\right)\right]$$
$$\times J_0\left(\frac{k_\perp v_\perp}{\omega_{Hi}}\right) \frac{d}{dv_\perp} J_0\left(\frac{k_\perp v_\perp}{\omega_{Hi}}\right) \exp - \left(\frac{m_i v^2}{2T}\right)\Bigg\} \Bigg/ \left[2 - \frac{(\omega - k_y v_d^i)}{\omega} \Gamma_0(2k_\perp r_{Hi})\right],$$

where we have chosen $T_e = T_i$. In the limit of small Larmor radius this can be written as

$$\frac{e\phi_{2\mathbf{k}}^{(2)}}{T} \approx \frac{1}{2}\left[\frac{e\phi_{\mathbf{k}}^{(1)}}{T}\right]^2.$$

We can expect that the amplitude of the nth harmonic will be proportional to the nth power of the main harmonic

$$\frac{e\phi_{n\mathbf{k}}^{(n)}}{T} \sim \left[\frac{e\phi_{\mathbf{k}}^{(1)}}{T}\right]^n.$$

Consequently, we find that the amplitude of the harmonic decreases exponentially as a function of the frequency.

The picture drawn here is valid only for a narrow wave packet in which the phase velocities of the different waves are very close,

$$\Delta\left(\frac{\omega}{k_z}\right) < \left(\frac{e\phi_0}{m_e}\right)^{1/2}.$$

But during an experiment, instability can arise in a wide range of phase velocities and this condition is violated. Then the main stabilization effect comes from the quasilinear relaxation of the electron distribution.

Quasilinear relaxation of the particle distribution and transport processes [36, 37]

As usual, we express the distribution as the sum of a slowly and a rapidly varying part (i.e., $f_j = \bar{f}_j + \delta f_j$) and obtain an equation for the slowly varying part by averaging the Vlasov equation over the rapid oscillations

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} + \frac{e_j}{m_j C}(\mathbf{v} \times \mathbf{H}_0)\cdot\frac{\partial}{\partial \mathbf{v}}\right]\bar{f}_j = \left\langle \frac{e_j}{m_j} \nabla\phi \frac{\partial \delta f_j}{\partial \mathbf{v}} \right\rangle \equiv St(\bar{f}_j). \quad \text{(II-128)}$$

Expanding ϕ and δf_j in Fourier components allows us to express the right-hand side of Eq. (II-128) as

$$St(\bar{f}_j) = \sum_{\mathbf{k}} \frac{e_j}{m_j}\left(\frac{d\phi_{\mathbf{k}}(x)}{dx}\frac{\partial}{\partial v_x} - i\phi_{\mathbf{k}}^* \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}}\right)\delta f_{\mathbf{k}j}(x, \mathbf{v}), \quad \text{(II-129)}$$

where the cross terms have vanished in the averaging process and we explicitly have made allowance for a slow x dependence in $\phi_{\mathbf{k}}(x)$ and $\delta f_{\mathbf{k}l}(x, \mathbf{v})$.

In accord with the philosophy of quasilinear theory, we assume that the distribution can be written in the form

$$\bar{f}_j = \bar{f}_j\left(v_\perp^2, v_\parallel, x + \frac{v_y}{\omega_{Hj}}, \epsilon x, \epsilon t\right),$$

where the dependence on the last two variables is slow enough to be ignored while evaluating $\delta f_{\mathbf{k}j}$. With this point in mind, we can easily generalize expression (II-111) for $\delta f_{\mathbf{k}j}$.

$$\delta f_{\mathbf{k}j}(x, \mathbf{v}) = \frac{e_j}{m_j}\phi_{\mathbf{k}}(x + v_y/\omega_{Hj})\left\{\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} + \exp\left[i\frac{(\mathbf{k}\times\mathbf{v})\mathbf{h}}{\omega_{Hj}}\right]J_0\left(\frac{k_\perp v_\perp}{\omega_{Hj}}\right)\right.$$
$$\left.\times \frac{\left[\dfrac{\omega}{v_\perp}\dfrac{\partial}{\partial v_\perp} + k_z v_z\left(\dfrac{1}{v_z}\dfrac{\partial}{\partial v_z} - \dfrac{1}{v_\perp}\dfrac{\partial}{\partial v_\perp}\right) + \dfrac{k_y}{\omega_{Hj}}\dfrac{\partial}{\partial x}\right]}{(k_z v_z - \omega_{\mathbf{k}})}\right\}$$
$$\times \bar{f}_j\left(v_\perp^2, v, x + \frac{v_y}{\omega_{Hj}}, \epsilon x, \epsilon t\right).$$

We neglect here the contribution to the integral from the second term of the expansion in the slow coordinate,

$$\phi_{\mathbf{k}}[x(t')] = \phi_{\mathbf{k}}(x + v_y/\omega_{Hj}) + \frac{v_y(t')}{\omega_{Hj}}\frac{d\phi_{\mathbf{k}}(x + v_y/\omega_{Hj})}{dx}.$$

Substituting this expression into Eq. (II-129) and averaging the result over the azimuthal angle in velocity space yields

$$St(f_j) = \left(\frac{e_j}{m_j}\right)^2 \sum_k \left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\omega_{Hj}}\frac{\partial}{\partial x}\right) \times \frac{|\phi_k(x)|^2 J_0^2\left(\frac{k_\perp v_\perp}{\omega_{Hj}}\right)}{i(k_z v_z - \omega_k)} \left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\omega_{Hj}}\frac{\partial}{\partial x}\right)\bar{f}_j. \qquad \text{(II-130)}$$

Since $k_z v_{thi} < \omega < k_z v_{the}$, the interaction of the waves with the ions is primarily adiabatic and the interaction with the electrons primarily resonant. Also, we may drop finite Larmor radius effects in the electron equation but not in the ion equation. Consequently, the electron and ion equations become

$$\frac{\partial \bar{f}_e}{\partial t} = St(\bar{f}_e) \equiv \left(\frac{e}{m_e}\right)^2 \sum_k \left(k_z \frac{\partial}{\partial v_z} + \frac{k_x}{\omega_{He}}\frac{\partial}{\partial x}\right)|\phi_k(x)|^2 \times \pi\delta(k_z v_z - \omega_k)\left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\omega_{He}}\frac{\partial}{\partial x}\right)\bar{f}_e \qquad \text{(II-131)}$$

and

$$\frac{\partial \bar{f}_i}{\partial t} = St(\bar{f}_i) \equiv \left(\frac{e}{m_i}\right)^2 \sum_k \left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\omega_{Hi}}\frac{\partial}{\partial x}\right) \times |\phi_k(x)|^2 J_0^2\left(\frac{k_\perp v_\perp}{\omega_{Hi}}\right)\frac{\gamma_k}{\omega_k^2}\left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\omega_{Hi}}\frac{\partial}{\partial x}\right)\bar{f}_i. \qquad \text{(II-132)}$$

Of course, the electron equation could have been derived more simply from the drift approximation.

Next we investigate the formation of the quasilinear "plateau". For this purpose it will be convenient to evaluate the k dependence of the electron diffusion coefficient at that value of k that corresponds to the largest growth rate (i.e., $k = \bar{k}$, where $\bar{k}_z \sim \omega_k/v_A$ and $\bar{k}_y$ can be found by taking into account the competition between quasilinear plateau formation and collisions, as is shown later). The waves with $k_y < \bar{k}_y$ are damped. If we introduce the new variables,

$$\eta = \frac{v_z^2}{2}, \qquad \xi = \frac{v_z^2}{2} + \frac{\omega_{\bar{k}}\omega_{He}}{2\bar{k}_y}x, \qquad \text{(II-133)}$$

we can reduce the differential operators in Eq. (II-131) to

$$\frac{\partial}{\partial \eta} = \left(\frac{1}{v_z}\frac{\partial}{\partial v_z} + \frac{\bar{k}_y}{\omega_{\bar{k}}\omega_{He}}\frac{\partial}{\partial x}\right).$$

Consequently, the level curves of the plateau are described by the equation

$$x - \frac{\bar{k}_y v_z^2}{2\omega_{He}\omega_{\bar{k}}} \equiv \xi = \text{const}$$

Since $v_z \ll v_{the}$ in the resonant region, the level curves of the original Maxwellian can be described by the equation

$$x - \frac{\bar{k}_y v_z^2}{2\omega_{He}\bar{k}_y v_d^e} = \text{const}$$

where $v_d^e = -(cT_e/eH_0)(n'_0/n_0)$. Since $\omega_{\bar{k}} = \bar{k}_y v_d^e\,[\Gamma_0/2-\Gamma_0)]$, it follows that the two sets of curves differ only by finite Larmor radius effects (see Figure

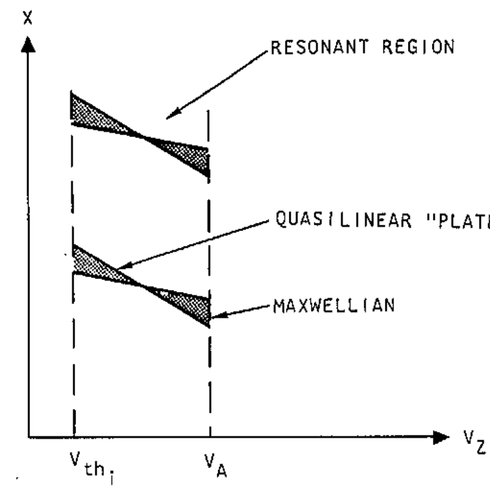

FIGURE II-17. Level curves for Maxwellian and quasilinear plateau.

II-17). The energy gain we receive after relaxation of the electron distribution is consequently small for small ion Larmor radius.

If we plot the distribution as a function of v_z for constant x (see Figure II-18), we find that the slope in the resonant region becomes very steep so that the enhanced Landau damping can stabilize the waves. During this process the electrons lose energy of motion along the field lines. Therefore, we can say that the energy that the unstable waves gain comes from the longitudinal motion of the electrons.

We can also use the constancy of ξ to estimate the change in an electron's position [36]

$$\delta x = \frac{\bar{k}_y}{\omega_{\bar{k}}\omega_{He}}\frac{\delta v_z^2}{2}. \qquad \text{(II-134)}$$

Since $\delta v_z^2 \ll v_A^2$ the displacement of the resonance electrons is much smaller than the plasma radius $r = n/(\partial n/\partial x) = n/n'$:

$$\delta x \simeq \frac{\bar{k}_y}{\omega_{\bar{k}}\,\omega_{He}} \cdot \frac{v_A^2}{2} \sim \frac{n}{n'}\,\frac{v_A^2}{v_{the}^2}.$$

Thus the instability inhibits itself rapidly and the change in electron density is small.

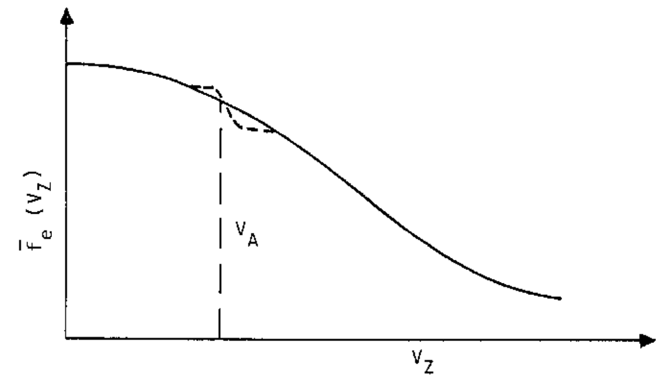

FIGURE II-18. Steepening of the electron velocity distribution.

One easily can see that the electrons and ions diffuse across the magnetic field at the same rate. Integrating Eq. (II-131) over velocity coordinates gives the following diffusion equation for the electrons:

$$\begin{aligned}\frac{\partial n_e}{\partial t} &= \left(\frac{e}{m_e}\right)^2 \sum_{\mathbf{k}} \frac{k_y}{\omega_{He}}\frac{\partial}{\partial x}|\phi_{\mathbf{k}}|^2 \int d\mathbf{v}\,\pi\delta(\omega_{\mathbf{k}} - k_z v_z)\left(k_z\frac{\partial}{\partial v_z} + \frac{k_y}{\omega_{He}}\frac{\partial}{\partial x}\right) f_e \\ &= \left(\frac{e}{m_e}\right)^2 \sum_{\mathbf{k}} \frac{k_y}{\omega_{He}}\frac{\partial}{\partial x}|\phi_{\mathbf{k}}|^2\,\frac{\gamma_{\mathbf{k}}(2-\Gamma_0)\,n_e}{v_{the}^2\,\omega_{\mathbf{k}}(\partial n_e/\partial x)}\,\frac{\partial n_e}{\partial x},\end{aligned} \qquad \text{(II-135)}$$

where we have introduced the growth rate $\gamma_{\mathbf{k}}$ and multiplied and divided by $\partial n/\partial x$. Using Eq. (II-132), we find a similar equation for the ions

$$\frac{\partial n_i}{\partial t} = \left(\frac{e}{m_i}\right)^2 \sum_{\mathbf{k}} \left(\frac{k_y}{\omega_{Hi}}\right)^2 \frac{\partial}{\partial x}|\phi_{\mathbf{k}}|^2\,\frac{\gamma_{\mathbf{k}}\,\Gamma_0}{\omega_{\mathbf{k}}^2}\,\frac{\partial n_i}{\partial x}. \qquad \text{(II-136)}$$

One can easily see from the dispersion relation [i.e., $\omega_{\mathbf{k}} = (n'/n) \times (v_{the}^2/\omega_{He})\,k_y\Gamma_0/(2-\Gamma_0)$] that the diffusion coefficients in these two equations are identical.

Since the collisionless theory gave only negligible diffusion across the magnetic field, we modify the previous work by adding a small collision term to

the right-hand side of Eq. (II-128). The collisions try to make the distribution take the form of a local Maxwellian and thereby prevent the formation of the quasilinear plateau. Consequently, we should find enhanced diffusion in this case. The electron equation now is of the form

$$\frac{\partial \bar{f}_e}{\partial t} = St_{QL}(\bar{f}_e) + St_{coll}(\bar{f}_e),$$

where

$$St_{coll}(\bar{f}_e) \equiv \nu_e\,v_{the}^2\,\frac{\partial}{\partial v_z^2}(\bar{f}_e - f_{Me}).$$

Since we are treating the case where $St_{coll}(\bar{f}_e) \ll St_{QL}(\bar{f}_e)$, we can solve Eq. (II-131) by successive approximations. [37] We express the distribution as $\bar{f}_e = \bar{f}_e^{\,0} + \bar{f}_e^{\,1}$ and then demand that

$$St_{QL}[\bar{f}_e^{(0)}] = 0 \qquad \text{(II-137)}$$

and that

$$St_{QL}[\bar{f}_e^{(1)}] + St_{coll}[\bar{f}_e^{(0)}] = 0. \qquad \text{(II-138)}$$

Equation (II-137) has the solution

$$\frac{1}{v_z}\frac{\partial \bar{f}_e^{(0)}}{\partial v_z} = -\frac{\bar{k}_y}{\omega_{\bar{k}}\,\omega_{He}}\frac{\partial \bar{f}_e^{(0)}}{\partial x}. \qquad \text{(II-139)}$$

To solve Eq. (II-138), we integrate it with respect to v_z

$$\begin{aligned}&\int_0^{v_z} dv_z \left(\frac{e}{m_e}\right)^2 \sum_{\mathbf{k}} \left(k_z\frac{\partial}{\partial v_z} + \frac{k_y}{\omega_{He}}\frac{\partial}{\partial x}\right)|\phi_{\mathbf{k}}|^2\,\pi\delta(\omega_{\mathbf{k}} - k_z v_z) \\ &\quad\times \left(k_z\frac{\partial}{\partial v_z} + \frac{k_y}{\omega_{He}}\frac{\partial}{\partial x}\right)\bar{f}_e^{(1)} = \nu_e\,v_{the}^2\,\frac{\partial}{\partial v_z}(f_{Me} - \bar{f}_e^{(0)}).\end{aligned} \qquad \text{(II-140)}$$

Since $f_e^{\,1}$ is more strongly dependent on v_z than on x, we can neglect the derivative with respect to x in the first parentheses in Eq. (II-140). If we also use Eq. (II-139) to evaluate the second term on the right-hand side of Eq. (II-140) we find

$$\begin{aligned}&\left(\frac{e}{m_e}\right)^2 \sum_{\mathbf{k}} |\phi_{\mathbf{k}}|^2 k_z\,\pi\delta(\omega_{\mathbf{k}} - k_z v_z)\left(k_z\frac{\partial}{\partial v_z} + \frac{k_y}{\omega_{He}}\frac{\partial}{\partial x}\right)\bar{f}^{(1)} \\ &\quad = \nu_e\,v_{the}^2\,v_z\left(\frac{1}{v_z}\frac{\partial}{\partial v_z} + \frac{\bar{k}_y}{\omega_{\bar{k}}\,\omega_{He}}\frac{\partial}{\partial x}\right) f_{Me}.\end{aligned} \qquad \text{(II-141)}$$

Integrating this expression over $v_\perp$ gives

$$\left(\frac{e}{m_e}\right)^2 \sum_{\mathbf{k}} |\phi_{\mathbf{k}}|^2 k_z^2\,\gamma_{\mathbf{k}}^{(1)}\,\pi\delta(k_z v_z - \omega_{\mathbf{k}}) = \nu_e\,v_{the}^2\,\gamma_{\mathbf{k}}^{(M)} \qquad \text{(II-142)}$$

where we have introduced the actual growth rate (γ_k^1) and the growth rate for a Maxwellian plasma (γ_k^M) [see Eq. (II-116)].

Although this expression cannot be used to evaluate either the growth rate or the field amplitude separately, it can be used to evaluate the diffusion coefficient, which depends on the product of these two quantities. From Eq. (II-135) we can see that the diffusion coefficient is given by

$$D_\perp \approx \frac{|e|\, cn_e}{TH_0 n_e'} \sum_k k_y \frac{\gamma_k^{(1)}}{\omega_k} |\phi_k|^2. \tag{II-143}$$

By using Eq. (II-142) we can express this as

$$D_\perp \approx \left| \frac{\nu_e \gamma_k^{(M)} k_y n_e}{\omega_{He} k_z^2 n_e'} \right|, \tag{II-144}$$

where

$$\gamma_k^{(M)} = \pi^{1/2} \omega_k \frac{(\omega_k - k_y v_d^e)}{|k_z|\, v_{the}}.$$

Evaluating this expression at $k_z \sim \omega_k/v_A$ and $k_\perp r_{Hi} \sim 1$, gives [36, 37]

$$D_\perp \simeq \nu_e \left(\frac{m_e}{m_i \beta}\right)^{3/2} \left(n \Big/ \frac{\partial n}{\partial x}\right)^2. \tag{II-145}$$

Of course, this formula is only valid when $St_{QL}(f_{me}) \gg St_{col}[f_e^{(0)}]$. Later we investigate the wave amplitude and rewrite this condition in a concrete way.

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Chapter III

Wave–Particle Nonlinear Interaction

Let us now turn to the last of the three principal mechanisms for nonlinear interaction between waves and plasma. We start with the simple case in which there is no magnetic field.

III-1. ELECTRON-PLASMA OSCILLATION TURBULENCE

If we impress two waves on a plasma, then these waves beat with the mixed frequency $(\omega_1 \pm \omega_2)$ and mixed wavelength $(\mathbf{k}_1 \pm \mathbf{k}_2)$. We already have considered the resonance of this beating with a third wave $(\omega_3, \mathbf{k}_3)$ in the decay-type interaction. But, in analogy with the linear theory, these beats can also resonate with particles moving at the velocity $\mathbf{v}$, where

$$(\mathbf{k}_1 \pm \mathbf{k}_2)\cdot\mathbf{v} = (\omega_1 \pm \omega_2). \qquad \text{(III-1)}$$

This type of process was included for the first time in the theory of weak turbulence by Drummond and Pines [1] for a one-dimensional wave packet, and by Kadomtsev and Petviashvili [2] for the general case (see also [3]). In our derivation we follow the review article. [4] Of course, the rate of energy increase (or decrease) due to this process is proportional to the product of the energy in the two primary waves. Consequently, it is a higher-order correction in the expansion in wave amplitude. But very often the linear growth rate is small, because only a few particles can resonate with the wave, and the nonlinear correction to the damping coefficient can be important.

To illustrate this effect, we consider a wave packet of random-phased Langmuir oscillations. The frequency for Langmuir oscillations is essentially constant [i.e., $\omega^2 = \omega_{pe}^2(1 + 3/2k^2\lambda_D^2)$ where $k^2\lambda_D^2 \ll 1$]. Consequently, it takes at least four waves to satisfy the frequency resonance condition

$$\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} = \omega_{\mathbf{k}_3} + \omega_{\mathbf{k}_4},$$

and wave–wave scattering cannot enter the problem until the third order in the wave energy. On the other hand, in the second order we can satisfy the resonance condition for nonlinear interaction with the particles

$$\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} = (\mathbf{k}_1 - \mathbf{k}_2)\cdot\mathbf{v}. \qquad \text{(III-2)}$$

First we Fourier-transform the potential $\phi(\mathbf{x}, t)$ in both space and time variables. The latter is possible only if ϕ is well behaved for $|t| \to \infty$. Although this assumption does not hold in the linear case if either growth or damping is present, inclusion of nonlinear effects is taken to be sufficient to justify it in the present case. As usual, we expand the distribution in powers of the wave amplitude by using the iteration formula

$$f_j(\mathbf{k}, \omega, \mathbf{v}) = \sum_{n=0}^{\infty} f_j^{(n)}(\mathbf{k}, \omega, \mathbf{v})$$

$$\tilde{f}_j^{(n)}(\mathbf{k}, \omega, \mathbf{v}) = i\frac{e_j}{m_j} \sum_{\substack{\mathbf{k}'+\mathbf{k}''=\mathbf{k} \\ \omega'+\omega''=\omega}} \int_{-\infty}^{t} dt' \, \tilde{\phi}(\mathbf{k}', \omega')\, \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} \tilde{f}_j^{(n-1)}(\mathbf{k}'', \omega'', \mathbf{v})$$

Here $\tilde{f}_j^{(n)}$ and $\tilde{\phi}(\mathbf{k}, \omega)$ are the Fourier transforms multiplied by $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$. Substituting this expression into Poisson's equation gives the dynamic equation for the waves

$$\epsilon_{\mathbf{k}}^{(1)}(\omega)\,\phi(\mathbf{k}, \omega) + \sum_{\substack{\mathbf{k}'+\mathbf{k}''=\mathbf{k} \\ \omega'+\omega''=\omega}} \epsilon_{\mathbf{k}',\mathbf{k}''}^{(2)}(\omega', \omega'')\,\phi(\mathbf{k}', \omega')\,\phi(\mathbf{k}'', \omega'')$$
$$+ \sum_{\substack{\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''=\mathbf{k} \\ \omega'+\omega''+\omega'''=\omega}} \epsilon_{\mathbf{k}',\mathbf{k}'',\mathbf{k}'''}^{(3)}(\omega', \omega'', \omega''')\,\phi(\mathbf{k}', \omega')\,\phi(\mathbf{k}'', \omega'')\,\phi(\mathbf{k}''', \omega''') + \cdots = 0 \tag{III-3}$$

where $\phi(\mathbf{k}, \omega)$ is the transformed potential and

$$\epsilon_{\mathbf{k}}^{(1)}(\omega) = 1 + \sum_j \frac{\omega_{pj}^2}{k^2} \int d\mathbf{v} \frac{\mathbf{k}\cdot(\partial f_{0j}/\partial \mathbf{v})}{\omega - \mathbf{k}\cdot\mathbf{v} + i\nu},$$

$$\epsilon_{\mathbf{k}',\mathbf{k}''}^{(2)}(\omega', \omega'') = -\frac{1}{2}\sum_j \frac{\omega_{pj}^2}{k^2}\frac{e_j}{m_j} \int d\mathbf{v} \frac{1}{\omega' + \omega'' - (\mathbf{k}' + \mathbf{k}'')\cdot\mathbf{v} + i\nu}$$
$$\times \left(\mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} \frac{1}{\omega'' - \mathbf{k}''\cdot\mathbf{v} + i\nu} \mathbf{k}'' \frac{\partial}{\partial \mathbf{v}} + \mathbf{k}'' \frac{\partial}{\partial \mathbf{v}} \frac{1}{\omega' - \mathbf{k}'\cdot\mathbf{v} + i\nu} \mathbf{k}' \frac{\partial}{\partial \mathbf{v}}\right) f_{0j}. \tag{III-4}$$

The positive infinitesimal quantity ν is inserted to provide a prescription for carrying out the velocity integral. It does not arise naturally as in the linear treatment, in which a Laplace transform with respect to time is employed, but is simply invoked to supply a definition of causality, that is, to distinguish $t = +\infty$ from $t = -\infty$. We solve Eq. (III-3) by treating $\gamma_{\mathbf{k}}/\omega \sim |\phi_{\mathbf{k}}|^2$ as a small parameter. It is clear that $\phi(\omega, \mathbf{k})$ is peaked around $\omega = \omega(\mathbf{k})$, the peak having width $\sim \gamma_{\mathbf{k}} \equiv \operatorname{Im} \omega(\mathbf{k})$, so that

$$\phi(\mathbf{k}, \omega) \approx \phi^{(1)}(\mathbf{k}, \omega) = \phi_{\mathbf{k}}^{(1)}\,\delta[\omega - \omega(\mathbf{k})] \tag{III-5}$$

where $\omega(\mathbf{k})$ is the solution of $\operatorname{Re}[\epsilon_{\mathbf{k}}^{(1)}(\omega)] = 0$. One easily can see that the second-order solution is

$$\phi^{(2)}(\mathbf{k}, \omega) = -\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \frac{\epsilon^{(2)}{}_{\mathbf{k}',\mathbf{k}''}(\omega', \omega'')}{\epsilon_{\mathbf{k}}^{(1)}(\omega)}\, \phi_{\mathbf{k}'}^{(1)}\phi_{\mathbf{k}''}^{(1)}\,\delta(\omega - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''}). \tag{III-6}$$

To derive the wave kinetic equation, we first multiply Eq. (III-3) by $\phi^*(k, \tilde{\omega})e^{i(\tilde{\omega} - \omega)t}$ and integrate over $d\omega\, d\tilde{\omega}$. The first term in the resulting expression is of the form

$$\int d\omega \int d\tilde{\omega}\,\epsilon_{\mathbf{k}}^{(1)}(\omega)\,\phi(\mathbf{k}, \omega)\,\phi^*(\mathbf{k}, \tilde{\omega})\,e^{i(\tilde{\omega} - \omega)t} \tag{III-7}$$

Since $\phi(\mathbf{k}, \omega)$ is peaked around $\omega_{\mathbf{k}}$, we can rewrite the imaginary part of this term as

$$\operatorname{Im}\left[\int d\omega \int d\tilde{\omega}\,\epsilon_{\mathbf{k}}^{(1)}(\omega)\,\phi(\mathbf{k}, \omega)\,\phi^*(\mathbf{k}, \tilde{\omega})\,e^{i(\tilde{\omega} - \omega)t}\right]$$
$$\simeq \frac{1}{2}\frac{\partial \epsilon_{\mathbf{k}}}{\partial \omega_{\mathbf{k}}}\frac{d|\phi_{\mathbf{k}}(t)|^2}{dt} + \epsilon_{\mathbf{k}}^{(1)''}(\omega_{\mathbf{k}})|\phi_{\mathbf{k}}|^2,$$

where

$$\epsilon_{\mathbf{k}}^{(1)}(\omega) = \epsilon_{\mathbf{k}}^{(1)'}(\omega) + i\epsilon_{\mathbf{k}}^{(1)''}(\omega). \tag{III-8}$$

If we use Eqs. (III-5) and (III-6) to evaluate the remaining two terms and then ensemble average [i.e., $\langle \phi_{\mathbf{k}}^{(1)}\phi_{\mathbf{k}'}^{(1)*} \rangle = |\phi_{\mathbf{k}}^{(1)}|^2 \delta_{\mathbf{k},\mathbf{k}'}$], we find the well-known wave kinetic equation [2, 4–6], in which the superscript in $\phi^{(1)}$ can be omitted (since all terms in the equation are second order in the small parameter $\sim |\phi|^2$):

$$\frac{1}{2}\frac{\partial \epsilon_{\mathbf{k}}^{(1)'}}{\partial \omega_{\mathbf{k}}}\frac{\partial |\phi_{\mathbf{k}}|^2}{\partial t} = -\epsilon_{\mathbf{k}}^{(1)''}(\omega_{\mathbf{k}})|\phi_{\mathbf{k}}|^2$$
$$+ \operatorname{Im} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \frac{2|\epsilon_{\mathbf{k}',\mathbf{k}''}^{(2)}(\omega_{\mathbf{k}',\mathbf{k}''})|^2 \cdot |\phi_{\mathbf{k}'}|^2 \cdot |\phi_{\mathbf{k}''}|^2}{\epsilon_{\mathbf{k}'+\mathbf{k}''}^{(1)*}(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''})}$$
$$+ \operatorname{Im} \sum_{\mathbf{k}'} \left[\frac{4\epsilon_{\mathbf{k}',\mathbf{k}-\mathbf{k}'}^{(2)}(\omega_{\mathbf{k}'}, \omega_{\mathbf{k}} - \omega_{\mathbf{k}'})\,\epsilon_{\mathbf{k},-\mathbf{k}'}^{(2)}(\omega_{\mathbf{k}}, -\omega_{\mathbf{k}'})}{\epsilon_{\mathbf{k}-\mathbf{k}'}^{(1)}(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})}\right.$$
$$\left. - 3\epsilon_{\mathbf{k}',-\mathbf{k}',\mathbf{k}}^{(3)}(\omega_{\mathbf{k}'}, -\omega_{\mathbf{k}'}, \omega_{\mathbf{k}})\right] |\phi_{\mathbf{k}'}|^2 \cdot |\phi_{\mathbf{k}}|^2, \tag{III-9}$$

where we have dropped terms that are higher than fourth order in the field amplitude.[1] The first term on the right-hand side of this equation is just the

[1] Singularities of type $1/\epsilon^{(1)}(\omega, \mathbf{k})$ near $\omega = \omega(\mathbf{k})$ must be treated in Eq. (III-9) according to the rule (see Chapter I)

$$1/[\epsilon^{(1)}(\omega, k)] = 1/[\epsilon^{(1)'}(\omega, k) + i\nu].$$

quasilinear growth rate. The second term gives the rate at which the modes $\phi_{\mathbf{k}'}$ and $\phi_{\mathbf{k}'}$ decay into $\phi_{\mathbf{k}}$. The third term gives the rate at which $\phi_{\mathbf{k}}$ and $\phi_{\mathbf{k}'}$ decay into $\phi_{\mathbf{k}'}$, but it also gives a contribution due to the nonlinear interaction of the waves and particles.

For Langmuir oscillations the decay-type interactions make no contribution, since we cannot satisfy the frequency condition $\omega_{\mathbf{k}+\mathbf{k}} = \omega_{\mathbf{k}} + \omega_{\mathbf{k}'}$. If we restrict our considerations to the case of a narrow spherical wave packet with width $(\Delta k)\lambda_{De} \ll (v_{thi}/k\lambda_{De}v_{the})^{2/3}$ then we can also drop the nonlinear interaction of the waves with the electrons. With these points in mind, one can easily see that the main contribution to the right-hand side of Eq. (III-9) comes from the third term and that in this term, $\epsilon^{(2)}$, is determined by the electrons and $\epsilon^{(1)}$ by the ions. To evaluate $\epsilon^{(2)}_{\mathbf{k},-\mathbf{k}'}(\omega_{\mathbf{k}},-\omega_{\mathbf{k}'})$ we expand the integral in Eq. (III-4) in terms of the small parameter $(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})/(\mathbf{k}-\mathbf{k}')v_{the}$:

$$\begin{aligned}\epsilon^{(2)}_{\mathbf{k},-\mathbf{k}'}(\omega_{\mathbf{k}},-\omega_{\mathbf{k}'}) &= \frac{1}{2}\frac{\omega_{pe}^2}{k^2}\frac{e}{m_e}\int d^3v\,\frac{1}{\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}}\\ &\times\left[\frac{\mathbf{k}\cdot\mathbf{k}'}{(\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v})^2}\,\mathbf{k}\cdot\frac{\partial f_e}{\partial v}-\frac{\mathbf{k}\cdot\mathbf{k}'}{(\omega_{\mathbf{k}'}-\mathbf{k}'\cdot\mathbf{v})^2}\,\mathbf{k}'\cdot\frac{\partial f_e}{\partial v}\right.\\ &\left.+\frac{\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}}{(\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v})(\omega_{\mathbf{k}'}-\mathbf{k}'\cdot\mathbf{v})}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\left(\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}}\right)f_e\right]\\ &\approx\frac{1}{2}\frac{\omega_{pe}^2}{k^2}\frac{e}{m_e}\int d^3\mathbf{v}\,\frac{(\mathbf{k}-\mathbf{k}')\cdot(\partial f_e/\partial\mathbf{v})}{\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}}\cdot\frac{\mathbf{k}\cdot\mathbf{k}'}{\omega_{pe}^2}\approx\frac{e}{T_e}\frac{\mathbf{k}\cdot\mathbf{k}'}{k^2}.\end{aligned}\tag{III-10}$$

In a similar way, we find that

$$\epsilon^{(2)}_{\mathbf{k}',\mathbf{k}-\mathbf{k}'}\approx\frac{e}{m_e}\frac{\mathbf{k}\cdot\mathbf{k}'}{(\mathbf{k}-\mathbf{k}')^2}\cdot\int\frac{(\mathbf{k}-\mathbf{k}')\cdot(\partial f_e/\partial\mathbf{v})\,d^3\mathbf{v}}{\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}}\approx\frac{e}{T_e}\frac{\mathbf{k}\cdot\mathbf{k}'}{(\mathbf{k}-\mathbf{k}')^2}\tag{III-11}$$

and that

$$\epsilon^{(1)}_{\mathbf{k}-\mathbf{k}'}\approx 1+\frac{1}{(\mathbf{k}-\mathbf{k}')^2\lambda_{De}^2}\,\frac{i\pi^{1/2}}{(\mathbf{k}-\mathbf{k}')^2\lambda_{Di}^2}\cdot\frac{\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}}{|\mathbf{k}-\mathbf{k}'|v_{thi}}W\left(\frac{\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}}{|\mathbf{k}-\mathbf{k}'|v_{thi}}\right)$$

where

$$W(z)\equiv\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{e^{-t^2}dt}{z-t+i\epsilon}.\tag{III-12}$$

Consequently, Eq. (III-9) becomes

$$\begin{aligned}\frac{\partial}{\partial t}\left[\frac{\partial\epsilon^{(1)\prime}_{\mathbf{k}}}{\partial\omega_{\mathbf{k}}}k^2|\phi_{\mathbf{k}}|^2\right]&=\operatorname{Im}\sum_{\mathbf{k}'}\frac{2}{\pi}\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{nT_i}|\phi_{\mathbf{k}}|^2\,|\phi_{\mathbf{k}'}|^2\\ &\times\left[1+\frac{T_i}{T_e}-i\pi^{1/2}\frac{(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})}{|\mathbf{k}-\mathbf{k}'|\,v_{thi}}W\left(\frac{\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}}{|\mathbf{k}-\mathbf{k}'|\,v_{thi}}\right)\right]^{-1}\end{aligned}\tag{III-13}$$

We can see from Eq. (III-13) that the number of waves is conserved in this process:

$$\frac{\partial}{\partial t}\sum_{\mathbf{k}}n_{\mathbf{k}}=\frac{\partial}{\partial t}\sum_{\mathbf{k}}\frac{1}{\omega_{\mathbf{k}}}\frac{\partial}{\partial\omega_{\mathbf{k}}}[\epsilon^{(1)\prime}_{\mathbf{k}}(\omega_{\mathbf{k}})\,\omega_{\mathbf{k}}]\frac{k^2|\phi_{\mathbf{k}}|^2}{8\pi}=0.$$

This is easy to understand if we look at the resonance condition as an equation determining the energy a particle receives when wave ϕ_{k_1} scatters from it into

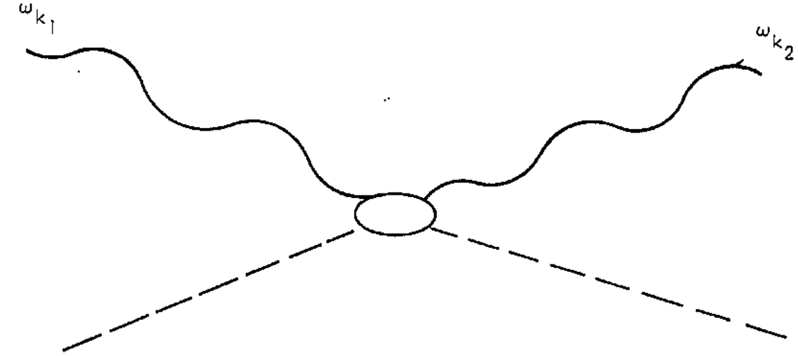

FIGURE III-1. Langmuir plasmon scattering off an electron from state ω_{k_1} to ω_{k_2}.

wave ϕ_{k_2}. A Langmuir plasmon ω_{k_1} scatters off an electron into state ω_{k_2} (Figure III-1) with energy change

$$\Delta E=\frac{\partial E}{\partial p}\cdot\Delta p=(k_1-k_2)\cdot\mathbf{v}\equiv\omega_{k_1}-\omega_{k_2};$$

there is no net change in the number of plasmons, hence the total number of waves must be conserved. Let us note that in the limit in which the phase velocity is much larger than the thermal velocity the condition $\omega_1+\omega_2=(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{v}$ cannot be satisfied and the wave number conservation law can be found for arbitrary waves using the symmetry properties of the coefficients $\epsilon^{(2)}$ and $\epsilon^{(3)}$ in Eq. (III-9). [7]

Equation (III-13) can be solved easily in the case in which the spread of the wave packet is much larger than the ion thermal velocity

$$\frac{\Delta k}{k}\gg\left(\frac{m_e}{m_i}\right)^{1/2}.\tag{III-14}$$

In this limit, Eq. (III-13) can be reduced to the following differential equation (which represents a kind of shock propagation in k space):

$$\frac{\partial N_{\mathbf{k}}}{\partial\tau}-N_{\mathbf{k}}\frac{\partial N_{\mathbf{k}}}{\partial\chi}=-6\alpha^2N_{\mathbf{k}}^2$$

where

$$N_k = \frac{4\pi k^3}{3}\frac{k^2|\phi_k|^2}{8\pi n T_e}, \qquad \tau = \frac{\pi m_e \omega_{pe}}{9\alpha^2 m_i}\cdot\frac{T_e T_i}{(T_e + T_i)} t,$$

$$\alpha = (k_0 \Delta k)^{1/2}\lambda_{De}, \qquad \chi = \frac{k^2}{k_0 \Delta k}. \tag{III-15}$$

The right-hand side of this equation describes decrease in energy due to the scattering of waves to longer wavelengths. The left-hand side of the equation describes the steepening of the wave packet in k space. The equation was derived and analyzed in [4]. The general solution was given there as

$$N_k = e^{\beta\chi} F\{\beta^{-1}[1 - e^{-\beta\chi}(1 - \tau\beta N_k)]\}$$

where $\beta = 6\alpha^2$, and $F[\beta^{-1}(1-e^{-\beta\chi})]$ is the initial energy distribution in the wave packet.

In the time-asymptotic limit the solution can be written in the simpler form [8]

$$N_k = \frac{N_0(\xi)}{1 + \beta N_0(\xi)\tau}, \qquad \xi = \chi + \frac{1}{\beta}\log(1 + \beta N_0 \tau).$$

III-2. CURRENT-DRIVEN ION SOUND TURBULENCE

Now we discuss current-driven ion sound turbulence in the limit in which $T_e \gg T_i$ and $H_0 = 0$. Because the instability is electrostatic in nature, we can use the wave kinetic equation derived in the previous section [i.e., Eq. (III-9)].

For waves with phase velocities between the ion and electron thermal velocities the linear dispersion relation takes the form

$$0 = \epsilon^{(1)}(\mathbf{k}, \omega) \equiv 1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pi}^2}{k^2 C_s^2}\left[1 + i\sqrt{\frac{\pi m_e}{2m_i}}\left(\frac{\omega}{|\mathbf{k}|C_s} - \frac{u}{C_s}\cos\theta\right)\right] \tag{III-16}$$

where $C_s = \sqrt{T_e/m_i}$ is the sound velocity, u is the electron drift velocity, and θ is the angle between **u** and k. In the long wavelength limit, the phase velocity of the oscillation is equal to the ion sound velocity $\omega = |\mathbf{k}|C_s$, and in the short wavelength limit the frequency is equal to the ion plasma frequency (see Figure III-2).

We need

$$u > C_s \tag{III-17}$$

for the existence of the instability in the limit of long wavelength perturbations. For short waves the critical current is less (for small ion temperature, $T_i \ll T_e$).

Since the ion sound instability is a resonance-type weak instability, we may use quasilinear theory for the description of the current relaxation in a

collisionless plasma. To this end we consider the particle distribution in a plasma with a current.

Figure III-3 depicts the equipotentials (characteristics) of the electron and ion velocity distribution functions. The two species are displaced by the electron drift velocity **u**.

Let us take a narrow wave packet propagating parallel to **u** as shown in Figure III-3. The interval between the resonant region and the origin is of the

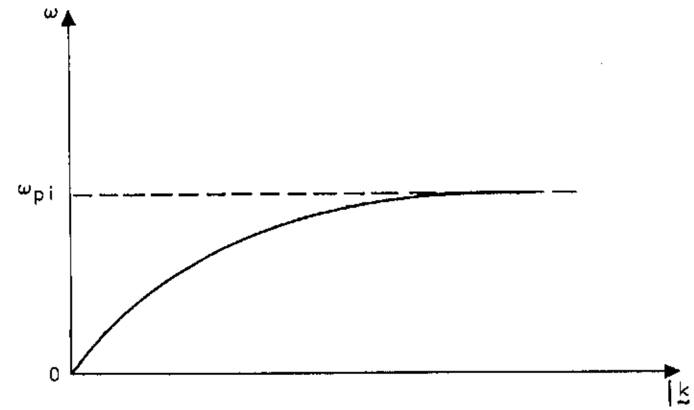

FIGURE III-2. Real part of ion sound dispersion relation.

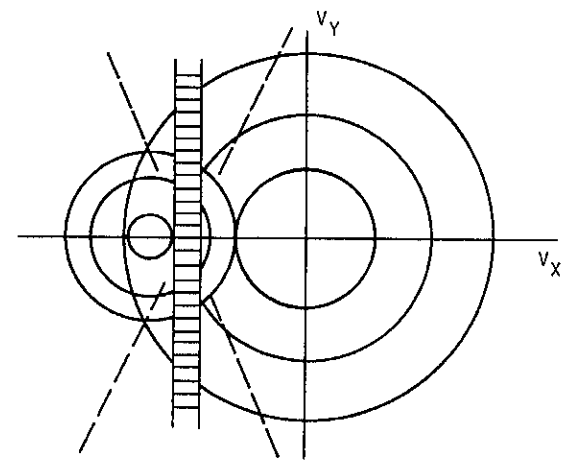

FIGURE III-3. Region of resonant interaction for ion sound waves. The level curves for the ion distribution are shown displaced in the negative v_x direction by the amount **u**.

same order as the ion sound velocity, because the ion sound velocity is calculated in the ion rest frame. We can expect a plateau with electron equipotentials to arise in the resonant region (Figure III-3). If we look for waves with **k** not parallel to the current velocity $\mathbf{u}_x$, then for $u \gg C_s$ we can expect these waves also to be unstable. So we have a resonant region in the form of a cone as shown in Figure III-3 by dashed lines, inside which all waves are unstable.

The angle of the vertex of this cone of course depends on the ratio of the drift velocity to the critical velocity. If the drift velocity is much larger than C_s, almost all angles of propagation will be unstable: only waves within a very small angle with respect to v_y will be stable, and this angle ϕ will approximately satisfy

$$\phi \sim C_s/u.$$

Therefore only electrons moving within a small angle $\sim \phi$ of v_y are not in resonance with the wave spectrum. Some small part of the electrons are always out of resonance with the wave spectrum because the projections of their velocities in the direction of the current are much larger than C_s.

If we now have a magnetic field in the v_z direction, then, as was mentioned in Chapter II, it will play the role of a mixer; all electrons will rotate around the v_z axis, and in this case the small cone of stable electrons has no important effect. So we can consider this problem as if all electrons are resonant with waves.

If we have a magnetic field not of this sort, however (i.e., if we have $\mathbf{H} \parallel v_y$), then the electrons in the stable cone will not be in resonance. The only possibility is that they can resonate with waves (ion sound or other) arising through wave–wave interactions.

If we now look at the simplest experiment, we have a magnetic field in the x direction and apply an electric field parallel to **H**, producing a current velocity which satisfies Eq. (III-17). We find a quasilinear interaction between the ion sound waves and the electrons outside the stable cone. In other words, all electrons outside the stable cone take energy from the electric field, accelerate, and then give up energy to the ion sound waves due to the quasilinear interaction. So there is some kind of balance between the electric field acceleration and the retarding force due to radiation of ion sound phonons.

A fraction of the electrons is accelerated just as in the usual runaway processes when this happens. But if $\phi \ll 1$ (i.e., the electric field is much above the critical value) only a small number of electrons run away. The simplest experimental evidence of such phenomena can be expected to arise as follows: first, if the number of electrons in the stable cone is small, then for such high electric fields we find that the electrical conductivity is much lower than classical values. Secondly, runaway electrons are observed. Both of these effects have been observed already in discharges with high electric fields.

In the opposite case, if we have a magnetic field perpendicular to **E**, all the particles rotate and we have no runaways. Now we expect to observe only the first effect that is, an anomalously low σ. This also has been confirmed by experiment. In this case, the anomalous conductivity should be even smaller, because when **E** was parallel to **H**, some current was carried by the runaway electrons. Every electron is resonant at some phase of its rotation about the magnetic field. This kind of geometry has been produced in the fast θ-pinch experiments at Novosibirsk. Electrical conductivities are found up to five orders of magnitude lower than the classical conductivity (see Figure III-4).

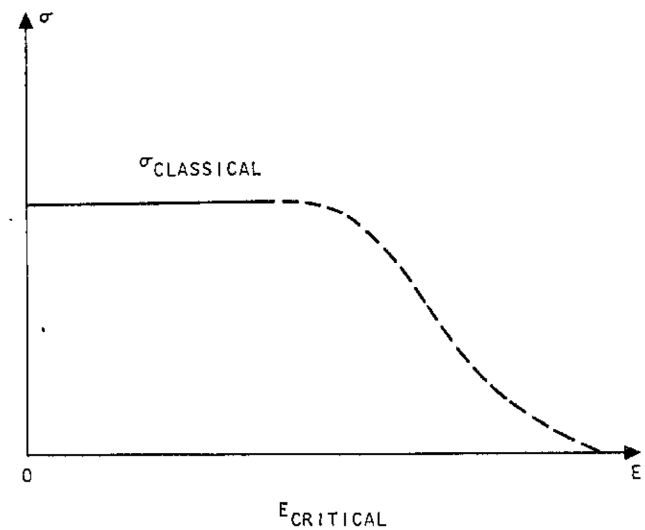

FIGURE III-4. Turbulent resistivity (decrease in conductivity observed in presence of unstable ion sound waves).

Let us look again at the first case, in which $\mathbf{H} \approx 0$. For strong electric fields we can consider almost all electrons resonant, and we find runaway phenomena that are closely analogous to the runaways for the case of a Lorentz gas in which electron–ion interactions are retained. For a Lorentz gas, the collision integral depends dimensionally on velocity as the inverse fifth power for large velocities. We know that if a Lorentz gas is subjected to an electric field, the acceleration due to the field becomes greater than the collisional retarding force for sufficiently high velocities. This problem was solved by Kruskal and Bernstein.

(The Lorentz gas is a soluble model for the runaway problem in a collisional plasma.)

We now recall that the quasilinear collision term that was written in Chapter II has the form

$$\frac{\partial f_0}{\partial t} = \alpha \frac{\partial}{v\,\partial v}\left(\frac{1}{v^2}\frac{\partial f_0}{\partial v}\right)$$

and has the same power dependence on velocity as the Lorentz collisional term. Thus the rather unrealistic Lorentz gas model of Kruskal and Bernstein can now be given the physical interpretation of collisions between electrons and electrostatic waves. (This analogy was suggested by Rudakov [9], who applied the results of Kruskal and Bernstein to this problem and found that for sufficiently large electric fields, even neglecting the cone of stable electrons, runaways result; in other words, over a sufficiently long time, the ion sound instability cannot prevent the runaway process from occurring.) We may inquire what happens after these electrons run away. There exist other instabilities, stronger than this one; for example, two-stream instabilities, with critical velocities of the order of the electron thermal velocity, may play a role.

The following is the situation in two- or three-dimensional quasilinear theory. To calculate any kind of concrete expressions (for example, electrical conductivity) we can use the quasilinear collision term, following the usual procedure starting with the transport equation. To do this, we need to know the quasilinear diffusion coefficient, which is proportional to the square of the electric field. Sometimes this can be done within the framework of pure quasilinear theory, as in the example of a spherically symmetric problem considered in Chapter II [10]; but for the ion wave instability, it is necessary to consider higher order processes like wave–wave scattering, etc. So we now turn to the consideration of mode coupling for this problem.

In Chapter I, it was shown that the dispersion relation of the type drawn in Figure III-1 (concave downward) cannot satisfy the resonance condition for the decay-type interaction. Consequently, the main contribution to the mode coupling comes from the nonlinear wave-particle interaction, which is described by the last two terms in Eq. (III-9). The electron contribution to these terms is negligible, because only a few electrons can satisfy the resonance condition $[\delta n \sim n_0(\omega \pm \omega')/|\mathbf{k}-\mathbf{k}'|v_{the} \ll n_0]$. Since the wave phase velocity is much larger than the ion thermal velocity, the ions only scatter the waves, and the number of waves is conserved.

If we set $\omega'' = \omega_{\mathbf{k}} - \omega_{\mathbf{k}'} \sim kv_{thi}$ and $|\mathbf{k}''| = |\mathbf{k}-\mathbf{k}'| \sim |\mathbf{k}|$, then to lowest order in the small parameter ω''/ω we find

$$\epsilon^{(1)}(\mathbf{k}'', \omega'') = \frac{\omega_{pi}^2}{\mathbf{k}''^2}\int \frac{\mathbf{k}''\cdot(\partial f_i/\partial \mathbf{v})d^3\mathbf{v}}{\omega'' - \mathbf{k}''\cdot\mathbf{v}}$$

$$k^2\,\epsilon^{(2)}_{\mathbf{k}'.\mathbf{k}\quad\mathbf{k}'} = k''^2\epsilon^{(2)}_{\mathbf{k}.\quad\mathbf{k}'} = \frac{k^2 m_i\,\omega_{\mathbf{k}}\,\omega_{\mathbf{k}}}{e(\mathbf{k}\cdot\mathbf{k}')}\,\epsilon^{(3)}_{\mathbf{k}'.\mathbf{k}.\;-\mathbf{k}'} = \frac{e}{m_i}\frac{\mathbf{k}\cdot\mathbf{k}'}{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}\,\epsilon^{(1)}(\omega'', \mathbf{k}''). \quad \text{(III-18)}$$

However, if we substitute these expressions into the wave kinetic equation we see that the nonlinear terms cancel. Consequently, we must take into account terms of higher order in the small parameter $\omega''/\omega \sim \mathbf{k}\cdot\mathbf{v}/\omega$. For $\epsilon^{(3)}$ we find

$$\operatorname{Im}\epsilon^{(3)}_{\mathbf{k}'.\mathbf{k}.\quad\mathbf{k}'} = \frac{\pi\omega_{pi}^2}{k^2}\frac{e^2(\mathbf{k}\cdot\mathbf{k}')^2}{\omega^2}\int d^3\mathbf{v}\left[1 + \frac{4\mathbf{k}\cdot\mathbf{v}}{\omega} + 10\left(\frac{\mathbf{k}\cdot\mathbf{v}}{\omega}\right)^2\right]\mathbf{k}''\cdot\frac{\partial f_i}{\partial \mathbf{v}}\,\delta(\omega'' - \mathbf{k}''\cdot\mathbf{v}). \quad \text{(III-19)}$$

To perform the integration we separate $\mathbf{k}\cdot\mathbf{v}$ into two parts:

$$\mathbf{k}\cdot\mathbf{v} = \frac{(\mathbf{k}''\cdot\mathbf{v})(\mathbf{k}\cdot\mathbf{k}'')}{k''^2} + \frac{[\mathbf{k}''\times\mathbf{v}][\mathbf{k}''\times\mathbf{k}]}{k''^2} \quad \text{(III-20)}$$

The value of the first part is obvious from the argument of the delta function and the second part is independent of the delta function. Consequently, we find

$$\operatorname{Im}\epsilon^{(3)}_{\mathbf{k}'.\mathbf{k}.\quad\mathbf{k}'} \approx -\frac{\omega_{pi}^2}{k^2}\frac{e^2(\mathbf{k}\cdot\mathbf{k}')^2}{\omega^4}\left[1 + \frac{4(\mathbf{k}\cdot\mathbf{k}'')\,\omega''}{k''^2\,\omega} + \frac{10(\mathbf{k}\cdot\mathbf{k}'')^2\,\omega''^2}{k''^2\,\omega^2} + \frac{10(\mathbf{k}\times\mathbf{k}'')^2\,v_{thi}^2}{\omega^2\,k''^2}\right]\operatorname{Im}\epsilon^{(1)}(\mathbf{k}'', \omega''). \quad \text{(III-21)}$$

In an analogous way we find

$$k^2\,\epsilon^{(2)}_{\mathbf{k}'.\mathbf{k}'} = k''^2\,\epsilon^{(2)}_{\mathbf{k}.\quad\mathbf{k}'} = \frac{e}{m_i}\frac{\mathbf{k}\cdot\mathbf{k}'}{\omega_{\mathbf{k}}^2}\,\omega_{pi}^2\left\{\left[1 + \frac{2\mathbf{k}\cdot\mathbf{k}''\,\omega''}{k''^2\,\omega_{\mathbf{k}}} + \frac{3(\mathbf{k}\cdot\mathbf{k})^2\,\omega''^2}{k''^4\,\omega_{\mathbf{k}}^2} + \frac{3(\mathbf{k}\times\mathbf{k}'')^2\,v_{thi}^2}{k''^2\,\omega^2}\right]\int\frac{\mathbf{k}''\cdot(\partial f_i/\partial\mathbf{v})\,d^3\mathbf{v}}{\omega'' - \mathbf{k}''\cdot\mathbf{v}}\right\}. \quad \text{(III-22)}$$

Using Eqs. (III-21) and (III-22) we can write the kinetic equation in the form [11, 12]

$$\frac{\omega_{pi}^2}{\omega_k^2}\frac{\partial|\phi_{\mathbf{k}}|^2}{\partial(\omega_{\mathbf{k}}t)} = \frac{\omega_{pi}^2}{k^2 C_s^2}\sqrt{\frac{\pi m_e}{2m_i}}\left(\frac{u}{C_s}\cos\theta - \frac{\omega_k}{kC_s}\right)|\phi_{\mathbf{k}}|^2 + \frac{16\pi^2 e^4 n}{m_i^3 k^2}\sum_{\mathbf{k}'}\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{\omega_{\mathbf{k}}^3\omega_{\mathbf{k}'}^3}\frac{(\mathbf{k}\times\mathbf{k}')^2\,v_{thi}^2}{k''^2}\int d^3\mathbf{v}\,\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \mathbf{k}''\cdot\mathbf{v}) \times \mathbf{k}''\cdot\frac{\partial f_{0i}}{\partial\mathbf{v}}\,|\phi_{\mathbf{k}}|^2\,|\phi_{\mathbf{k}'}|^2. \quad \text{(III-23)}$$

As expected, the mode coupling term conserves wave number

$$\sum_{\mathbf{k}} n_{\mathbf{k}} = \sum_{\mathbf{k}}\frac{\omega_{pi}^2}{\omega_{\mathbf{k}}^3}\frac{k^2|\phi_{\mathbf{k}}|^2}{4\pi}.$$

Consequently, the mode coupling cannot stop the increase in the number of waves produced by the linear instability, and we need to take into account some additional process if we want to find a stationary spectrum. The energy flows from the high-frequency short-wavelength oscillations to the low-frequency long-wavelength oscillations; so we assume, as Kadomtsev [12] did, that ion–ion collisions produce a turbulence sink in the long-wavelength region. Consequently, we can cut off the spectrum in this region and construct a stationary solution in the rest of phase space by balancing the linear growth rate with the nonlinear flow of energy to long wavelengths.

As an example of this kind of solution, we first assume that the turbulence is limited to two lines in k space and consider the long-wavelength turbulence. After going from discrete to continuous wave vector variables [see discussion following Eq. (I-61)], we have

$$|\phi_{\mathbf{k}}|^2 = I(k)\,\delta(\phi)\,\delta(\cos\theta - \cos\theta_0), \qquad k\lambda_D \ll 1$$

where (k, θ, ϕ) are spherical coordinates in k space with the polar axis along the direction of the current.

When u is only slightly larger than C_s, the linear instability produces waves propagating at small angles with respect to the current (i.e., $\theta_0 \ll 1$). In this case, we can reduce Eq. (III-23) to the form

$$\left(\frac{\pi m_e}{2m_i}\right)^{1/2}\left(\frac{u}{C_s}\cos\theta_0 - 1\right)I(k) \approx -\frac{e^2}{T_e^2}\frac{T_i}{\pi^2}I(k)\,k\frac{\partial}{\partial k}[k^3 I(k)]\cos^2(2\theta_0)\sin^2(2\theta_0) \tag{III-24}$$

where we have put

$$\int d^3\mathrm{V}\mathbf{k}''\cdot\frac{\partial f_{0i}}{\partial \mathbf{v}}\,\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \mathbf{k}''\cdot\mathbf{v}) = -k''^2\frac{\partial}{\partial\omega_{\mathbf{k}}}\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})$$

and replaced the summation over k′ by

$$\sum_{\mathbf{k}'} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} \frac{k^2\, dk}{(2\pi)^3}.$$

The general solution of Eq. (III-24) is

$$\frac{e^2}{\pi^2 T_e^2}I(k) = -\sqrt{\frac{\pi m_e}{2m_i}}\left(\frac{u}{C_s}\cos\theta_0 - 1\right)\frac{T_e}{T_i\theta_0^2 k^3}\ln(kD) \tag{III-25}$$

where we have demanded that I(k) = 0 at some very long wavelength D.

This solution is, of course, not unique. Moreover, the solution is unstable, because any wave propagating along the direction of the current has a larger linear growth rate and smaller mode coupling term than the wave propagating at the same angle θ_0. From this argument we conclude that the spectrum tends to collapse to a line parallel to the current.

Akhiezer [13] has shown that it is possible to construct a self-similar oscillatory solution in the long-wavelength region ($k\lambda_D \ll 1$). Here we only follow his arguments qualitatively. For an axially symmetric solution [i.e., $|\phi_{\mathbf{k}}|^2 = I(k, \theta, t)$], we can write Eq. (III-23) in the form

$$\frac{1}{kC_s}\frac{\partial I(k,\theta,t)}{\partial t} + \sqrt{\frac{\pi m_e}{2m_i}}\left(1 - \frac{u}{C_s}\cos\theta\right)I(k,\theta,t) = \frac{e^2 T_i}{\pi T_e^3}I(k,\theta,t)\,k\frac{\partial}{\partial k}\int_{C_s/u}^{1}(1 - \cos^2\theta\cos^2\theta')k^3 I(k,\theta',t)\,d(\cos\theta). \tag{III-26}$$

One can see that this equation is invariant under the transformation

$$I \to \alpha^{-3} I, \qquad k \to \alpha k, \qquad t \to \alpha^{-1} t, \tag{III-27}$$

where α is an arbitrary parameter. Therefore, we can reduce the number of variables by introducing Ik^3 and $\chi = (\pi m_e/2m_i)^{1/2} ukt$, which are invariant under the above transformations. In the linear stage, the function Ik^3 depends only on χ like $\exp(2\gamma t)$ where $\gamma t \sim \chi$. Consequently, Ik^3 should depend only on χ in the later stages. Regarding the angular dependence of the turbulent spectrum, Akhiezer found that the energy oscillates between the Cherenkov cone [i.e., I(θ) $\sim \delta(\cos\theta - C_s/4)$] and the current line [i.e., $I(\theta) \sim \delta(1-\cos\theta)$] with a period of oscillation proportional to the energy in the spectrum. Consequently, most of the time the energy is distributed throughout the Cherenkov cone (see Figure III-5).

In the short-wavelength limit ($k\lambda_D \gg 1$) we can reduce Eq. (III-23) for the wave amplitude to an integro-differential equation of the same type as Eq. (III-26):

$$\begin{aligned}\frac{\partial I(k,\theta,t)}{\partial(\omega_{pi}t)} &= \frac{\omega_{pi}^2}{k^2C_s^2}\left(\frac{\pi m_e}{2m_i}\right)^{1/2}\left(\frac{u}{C_s}\cos\theta - \frac{\omega_{pi}}{kC_s}\right)I(k,\theta,t)\\ &\quad + \frac{\omega_{pi}^2}{k^2C_s^2}\frac{e^2 I(k,\theta,t)}{4T_e^2}\frac{T_i}{T_e}\lambda_D^{10}k^6\frac{\partial}{\partial k}\Bigg(k^8\int_0^{2\pi}d\phi'\int_{-1}^{+1}d\cos\theta'\\ &\quad \times I(k,\theta',t)[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi' - \phi)]\\ &\quad \times\{1 - [\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi' - \phi)]\}\Bigg).\end{aligned} \tag{III-28}$$

There is no rigorous solution like Eq. (III-27) for this equation. However, we can find, as we did for the wave–wave interaction in Chapter I, a power-type solution, at least for the k dependence. The natural sink in the short-wavelength region is due to the linear ion Landau damping. From Eq. (III-28) we find that the spectral energy of waves very rapidly decreases toward short lengths,

$$\frac{e^2 I(k)}{4T_e^2} \sim \frac{1}{5}\left(\frac{\pi m_e}{2m_i}\right)^{1/2}\left(\frac{u}{C_s}\cos\theta - \frac{5}{6}\frac{\omega_{pi}}{kC_s}\right)\frac{T_e}{k^{13}\lambda_D^{10}\theta_0^2 T_i}. \qquad \text{(III-29)}$$

The power of this solution is completely different from that found in Chapter I. Evidently there is no universal power solution for plasma turbulence like the Kolmogoroff spectrum in hydrodynamic turbulence.

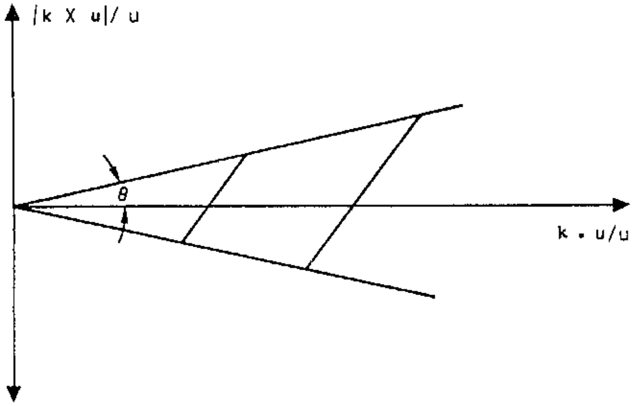

FIGURE III-5. Angular dependence of turbulent ion sound spectrum. Energy oscillates within the limits of the Cherenkov cone (cosθ = c_s/u).

Now with the help of the expressions found for the spectral energy of waves in the limits of large [Eq. (III-25)] and short [Eq. (III-28)] scales, we estimate the turbulent resistivity. Multiplying the quasilinear equation for the electrons,

$$\frac{\partial f_e^{(0)}}{\partial t} = \frac{e^2}{m_e^2}\int \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}}\frac{\gamma_k|\phi_k|^2}{(\omega - \mathbf{k}\cdot\mathbf{v})^2 + \gamma_k^2}\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}}\frac{d^3\mathbf{k}}{(2\pi)^3},$$

by $m_e v$ and integrating the result over velocity space, we obtain the electron momentum loss due to radiation of ion sound waves,

$$\dot{\mathbf{P}}_{\text{rad}} = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{e^2|\phi_k|^2}{m_e}\mathbf{k}\int d^3\mathbf{v}\,\pi\delta(\omega - \mathbf{k}\cdot\mathbf{v})\,\mathbf{k}\cdot\frac{\partial f_e^{(0)}}{\partial \mathbf{v}}.$$

Introducing the effective electron–ion collision frequency as

$$m_e n_0 \nu_{\text{eff}}\mathbf{u}(t) = \frac{d}{dt}[m_e n_0 \mathbf{u}(t)],$$

with the help of Eq. (III-16), we obtain from this expression

$$m_e n_0 \nu_{\text{eff}}\mathbf{u}(t) = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\gamma_k\frac{\partial \epsilon_k^{(1)}(\omega_k)}{\partial \omega_k}\frac{k^2|\phi_k|^2}{8\pi}\mathbf{k}. \qquad \text{(III-30)}$$

On the right-hand side of Eq. (III-30) is the wave momentum density radiated by electrons per unit time (i.e., $-\mathbf{P}_{\text{rad}}$). Equation (III-30) implies that all the waves radiated by electrons are absorbed by ions. The main contribution to the integral on the right-hand side of Eq. (III-30) corresponds to wavelengths of the order of the Debye length, $k\lambda_D \sim 1$. For the case of large current velocity, $u \gg C_s$, we find from Eqs. (III-25) and (III-29)

$$\nu_{\text{eff}} \sim \omega_{pi}\frac{u}{C_s}\frac{T_e}{T_i}. \qquad \text{(III-31)}$$

Therefore the effective collision frequency is larger than the growth rate.

As in the usual Joule heating, the turbulent heating changes mostly the electron temperature, [14] if u is greater than the ion thermal velocity. The heating rate of the ions is always much smaller than the heating rate of the electrons. In fact, the rate of change of energy due to scattering of electrons by ion sound waves is equal to [see Eq. (III-30)]

$$n_0\dot{T}_e = \nu_{\text{eff}}\, m_e n_0 u^2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\gamma_k\mathscr{E}_k\frac{\mathbf{k}\cdot\mathbf{u}}{\omega_k},$$

where $\epsilon_k = \omega_k(\partial/\partial\omega_k)\epsilon^{(1)}(\mathbf{k},\omega)(k^2|\phi_k|^2/8\pi)$ is the wave energy.

Then, even if we suppose that the total energy of the waves is absorbed by ions, the ratio of the heating rates for the two species is limited by the inequality [14]

$$\frac{T_i}{T_e} \leqslant \frac{\int \frac{d^3\mathbf{k}}{(2\pi)^3}\gamma_k\mathscr{E}_k}{\int \frac{d^3\mathbf{k}}{(2\pi)^3}\gamma_k\mathscr{E}_k\frac{\mathbf{k}\cdot\mathbf{u}}{\omega_k}} \sim \frac{C_s}{u}. \qquad \text{(III-32)}$$

This ratio is small if the current velocity is not close to the critical value.

III-3. NONLINEAR THEORY OF THE DRIFT INSTABILITY

In concluding consideration of anomalous transport processes in nonuniform plasma, we try to describe in detail the nonlinear stage of the low-frequency "universal drift instability." [15, 16] The nonlinear Eq. (III-9) for the spectral energy of the electrostatic waves is written in terms of the coefficients of the particle distribution expanded in powers of the wave amplitude. These coefficients are defined by the iteration formula

$$\tilde{f}^{j(n)}_{\mathbf{k}',\mathbf{k}'',\ldots,\mathbf{k}}(n)(\omega',\omega'',\ldots;\mathbf{r},\mathbf{v},t) = i\frac{e_j}{m_j}\int_{-\infty}^{t} dt'\,\tilde{\phi}_{\mathbf{k}'\omega'}(\mathbf{r}',t')\,\mathbf{k}'\frac{\partial}{\partial\mathbf{v}'}\tilde{f}^{j(n-1)}_{\mathbf{k}'',\ldots,\mathbf{k}}(n)(\omega'',\ldots;\mathbf{r}',\mathbf{v}',t'). \quad \text{(III-33)}$$

Here (see Section III-I) the quantities marked ~ satisfy

$$\tilde{\phi}_{\mathbf{k}',\omega'}(\mathbf{r}',t') \equiv \phi_{\mathbf{k}',\omega'}e^{i[\mathbf{k}'\cdot\mathbf{r}(t')\quad\omega t']}$$

$$\tilde{f}^{j(1)}_{\mathbf{k}'}(\omega';\mathbf{r},\mathbf{v},t) \equiv f^{j(1)}_{\mathbf{k},\omega}(\mathbf{v})\,e^{i[\mathbf{k}'\cdot\mathbf{r}(t)\quad\omega t]}$$

$$\tilde{f}^{j(2)}_{\mathbf{k}',\mathbf{k}''}(\omega',\omega'';\mathbf{r},\mathbf{v},t) \equiv f^{j(2)}_{\mathbf{k}',\mathbf{k}'';\omega',\omega''}e^{i[(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{r}(t)-(\omega'+\omega'')t]}, \text{ etc.}$$

and

$$\sum_{\substack{\mathbf{k}'+\cdots+\mathbf{k}^{(n)}=\mathbf{k}\\ \omega'+\cdots+\omega^{(n)}=\omega}} f^{j(n)}_{\mathbf{k}',\ldots,\mathbf{k}^{(n)};\,\omega',\ldots,\omega^{(n)}} = f^{j(n)}_{\mathbf{k};\omega},$$

where $f^{j(n)}_{\mathbf{k};\,\omega}$ is the Fourier transform of $f^{j(n)}(\mathbf{r},\mathbf{v},t)$, and $\phi_{\mathbf{k},\,\omega}$ is the Fourier transform of $\phi(\mathbf{r},t)$

We can describe the mode coupling of drift waves in a nonuniform plasma with the help of this equation so long as

$$\lambda_x\frac{d}{dx}(k_y v_d^e) \ll \gamma_{\mathbf{k}}. \quad \text{(III-34)}$$

This inequality just states that the difference between the eigenfrequencies in a nonuniform plasma is much less than the growth rate. In other words, the difference between the energy levels is much smaller than the energy of interaction.

The dispersion relation in the linear approximation was found in Chapter II. Now let us proceed to the next step: the derivation of the nonlinear equation for the wave energy. Substituting the linear approximation [Eq. (II-107)] in Eq. (III-33), we get

$$f^{j(2)}_{\mathbf{k}',\mathbf{k}''}(\omega',\omega'';\mathbf{r},\mathbf{v},t) = -\frac{e_j^2}{m_jT_j}\int_{-\infty}^{t}dt'\,\phi_{\mathbf{k}'\omega'}\exp\left[-i(\omega'-k_z'v_z)(t'-t)-i\frac{\mathbf{k}\times\mathbf{v}(t')\cdot\mathbf{h}}{\omega_{Hj}}\right]$$
$$\times\exp\left[i\frac{\mathbf{k}\times\mathbf{v}(t)\cdot\mathbf{h}}{\omega_{Hj}}\right]\phi_{\mathbf{k}''\omega''}\exp\left[-i(\omega''-k_zv_z)(t'-t)-i\frac{\mathbf{k}\times\mathbf{v}(t')\cdot\mathbf{h}}{\omega_{Hj}}+i\frac{\mathbf{k}\times\mathbf{v}(t)\cdot\mathbf{h}}{\omega_{Hj}}\right]$$
$$\times\, i\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}(t')}\left\{1-\frac{(\omega''-k_yv_d^j)}{(\omega''-k_z''v_z+i\nu)}J_0\left(\frac{k_\perp v_\perp}{\omega_{Hj}}\right)\exp\left[i\frac{\mathbf{k}''\times\mathbf{v}(t')\cdot\mathbf{h}}{\omega_{Hj}}\right]\right\}f_{0j}. \quad \text{(III-35)}$$

In Eq. (III-3) the velocity derivative gives several terms; however, we recall that in the first-order calculation, the contribution from differentiation of the argument of the Maxwellian cancelled up to a small term, and the same thing happens here; also, if we assume $k_\perp \gg n'/n$, the term arising from differentiation of the density in f_j is negligible. So the main contribution comes from the rapid velocity dependence of the exponential, $\exp[\mathbf{k}\times\mathbf{v}\cdot\mathbf{h}/\omega_{Hj}]$, in the second term of $f_j^{(1)}$. Carrying out the time integration and symmetrizing with respect to k′ and k″, we have

$$f^{j(2)}_{\mathbf{k}',\mathbf{k}''} = -\frac{ie_j}{T_j}\frac{c}{H}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}(\mathbf{k}'\times\mathbf{k}'')\cdot\mathbf{h}J_0\left(\frac{k_\perp'v_\perp}{\omega_{Hj}}\right)J_0\left(\frac{k_\perp''v_\perp}{\omega_{Hj}}\right)$$
$$\times\exp\left(i\frac{\mathbf{k}\times\mathbf{v}\cdot\mathbf{h}}{\omega_{Hj}}\right)\left[\frac{(\omega'-k_y'v_d^j)}{(\omega'-k_z'v_z+i\nu)}-\frac{(\omega'-k_y''v^j)}{(\omega''-k''v_z+i\nu)}\right]f_{0j}$$
$$\times\frac{\phi_{\mathbf{k}'}\phi_{\mathbf{k}''}\exp[-i(\omega'+\omega''-\omega)t]}{[\omega'+\omega''-(k_z'+k_z'')v_z+i\nu]}. \quad \text{(III-36)}$$

Using this expression in the same approximations, we obtain the third-order correction to the distribution function.

$$f^{j(3)}_{\mathbf{k},\omega} = -\frac{ie_j}{T_j}\frac{c^2}{H^2}\sum_{\substack{\mathbf{k}=\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''\\ \omega=\omega'+\omega''+\omega'''}}[\mathbf{k}'''\times(\mathbf{k}'+\mathbf{k}'')]\cdot\mathbf{h}(\mathbf{k}'\times\mathbf{k}'')\cdot\mathbf{h}$$
$$\times J_0\left(\frac{k_\perp'''v_\perp}{\omega_{Hj}}\right)J_0\left(\frac{k_\perp''v_\perp}{\omega_{Hj}}\right)J_0\left(\frac{k_\perp'v_\perp}{\omega_{Hj}}\right)\exp\left[i\frac{(\mathbf{k}\times\mathbf{v})\cdot\mathbf{h}}{\omega_{Hj}}\right]\left[\frac{(\omega'-k_yv_d^j)}{(\omega'-k_z'v_z+i\nu)}\right.$$
$$\left.-\frac{(\omega''-k_y''v_d^j)}{(\omega''-k_z''v_z+i\nu)}\right]\frac{f_{0j}(\mathbf{v})\phi_{\mathbf{k}'}\phi_{\mathbf{k}''}\phi_{\mathbf{k}'''}\exp[-i(\omega'+\omega''+\omega'''-\omega)t]}{[\omega'+\omega''-(k_z'+k_z'')v_z+i\nu][\omega'+\omega''+\omega'''-k_zv_z+i\nu]}. \quad \text{(III-37)}$$

In Eq. (III-9) for the wave amplitude, the main nonlinear contribution comes from the ion terms, until $k_\perp r_{Hi} < \sqrt{m_i\beta/m_e}$. We also use the fact that the phase velocity along the lines is much larger than the thermal velocity for most unstable waves, so the dielectric constant for the beats can be written in the form[2]

$$(\mathbf{k}-\mathbf{k}')^2\lambda_D^2\epsilon^{(1)}(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'},\mathbf{k}-\mathbf{k}') \approx -\Gamma_0(|\mathbf{k}-\mathbf{k}'|\,r_{Hi})\int\frac{[\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-(k_y-k_y')v_d^i]f_{0i}}{\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-(k_z-k_z')v_z+i\nu}d^3\mathbf{v}. \quad \text{(III-38)}$$

[2] The approximation [Eq. (III-38)] breaks down if $k_\perp^2 r_{Hi}^2 < \sqrt{\beta}$. But for such a long wave the interaction through beats is very small and we can neglect it.

If we keep only the ion contribution to $\epsilon^{(2)}$ and $\epsilon^{(3)}$, Eq. (III-9) becomes

$$\frac{\partial|\phi_{\mathbf{k}}|^2}{\partial t}+\frac{\partial\omega}{\partial k_x}\frac{\partial|\phi_{\mathbf{k}}|^2}{\partial x}-\frac{\partial\omega}{\partial x}\frac{\partial|\phi_{\mathbf{k}}|^2}{\partial k_x}=-\pi^{1/2}\frac{\omega_{\mathbf{k}}(\omega_{\mathbf{k}}-k_y v_d^e)}{|k_z|\,v_{the}}|\phi_{\mathbf{k}}|^2$$
$$+\frac{c^2|\phi_{\mathbf{k}}|^2}{H^2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}|\phi_{\mathbf{k}'}|^2\,\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})(k_y-k_y')\,v_d^i\,\frac{[(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{h}]^2}{\omega_{\mathbf{k}'}}$$
$$\times\left[\int_0^\infty e^{-t}J_0^2(\alpha_\perp\sqrt{t})J_0^2(\alpha_\perp'\sqrt{t})\,dt-\frac{1}{\Gamma_0(|\boldsymbol{\alpha}_\perp-\boldsymbol{\alpha}_\perp'|)}\left(\int_0^\infty e^{-t}J_0(|\boldsymbol{\alpha}_\perp-\boldsymbol{\alpha}_\perp'|\sqrt{t})\right.\right.$$
$$\left.\left.\times J_0(\alpha_\perp\sqrt{t})J_0(\alpha_\perp'\sqrt{t})\,dt\right)\right]^2+4\omega_{\mathbf{k}}\frac{c^2}{H^2}\iint\frac{d^3\mathbf{k}'\,d^3\mathbf{k}''}{(2\pi)^3}\left[\int_0^\infty dt\,e^{-t}J_0(\alpha_\perp''\sqrt{t})\right.$$
$$\left.\times J_0(\alpha_\perp'\sqrt{t})J_0(\alpha_\perp\sqrt{t})\right]^2(\mathbf{k}'\times\mathbf{k}''\cdot\mathbf{h})^2\left[\left(\frac{k_y'v_d^i}{\omega_{\mathbf{k}'}}-\frac{k''v_d^i}{\omega_{\mathbf{k}''}}\right)^2|\omega_{\mathbf{k}}|^{-1}|\phi_{\mathbf{k}'}|^2|\phi_{\mathbf{k}''}|^2\right.$$
$$\left.-\left(\frac{k_y v_d}{\omega_{\mathbf{k}}}-\frac{k_y v_d^i}{\omega_{\mathbf{k}'}}\right)\frac{\text{sign}\,\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}{|\omega_{\mathbf{k}''}|}|\phi_{\mathbf{k}}|^2|\phi_{\mathbf{k}''}|^2\right]\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\,\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''}).\quad\text{(III-39)}$$

The additional terms appearing on the left-hand side of Eq. (III-39) are related to the slow time variation and weak spatial dependence in $\phi_{\mathbf{k}}$. We can take account of this behavior by expanding frequency, x component of the wave vector, and x coordinate near the position of the wave packet in the configuration space $(\omega_{\mathbf{k}}, k_x, x)$. This may be done by employing the WKB formalism to describe the wave packet, or by introducing variation on fast and slow time scales. The result is to replace $\text{Im}\,\epsilon^{(1)}\,(\omega_{\mathbf{k}}, \mathbf{k}, x)$ by

$$\epsilon^{(1)''}(\omega_{\mathbf{k}},\mathbf{k},x)\,\phi_{\mathbf{k}}+\frac{\partial\epsilon_{\mathbf{k}}^{(1)'}}{\partial\omega}\frac{\partial\phi_{\mathbf{k}}}{\partial t}-\frac{\partial\epsilon_{\mathbf{k}}^{(1)'}}{\partial k_x}\frac{\partial\phi_{\mathbf{k}}}{\partial x}+\frac{\partial\epsilon_{\mathbf{k}}^{(1)'}}{\partial x}\frac{\partial\phi_{\mathbf{k}}}{\partial k_x}.$$

This we easily reduce to the convective derivative in configuration space with the help of the relations

$$\frac{\partial\omega}{\partial x}=-\frac{\partial\epsilon_{\mathbf{k}}^{(1)}(\omega_{\mathbf{k}},x)}{\partial x}\bigg/\frac{\partial\epsilon_{\mathbf{k}}^{(1)}(\omega_{\mathbf{k}},x)}{\partial\omega_{\mathbf{k}}};\quad\frac{\partial\omega_{\mathbf{k}}}{\partial k_x}=-\frac{\partial\epsilon_{\mathbf{k}}^{(1)}}{\partial k_x}\bigg/\frac{\partial\epsilon_{\mathbf{k}}^{(1)}}{\partial\omega_{\mathbf{k}}}.$$

Another way to see the origin of these terms is to construct a wave packet of general form:

$$\phi(x,t)=\int dk_x\exp[i(k_x x-\omega_{\mathbf{k}}t)]\,\phi_{\mathbf{k}}$$

where k_x and ω_k satisfy the dispersion relation, $\epsilon_{\mathbf{k}}(k_x, \omega_{\mathbf{k}}, x) = 0$. In a homogeneous system (no x dependence), it is easy to see that for a nearly monochromatic packet {for example, $f_k \sim a\exp[-(k_x-k_0)^2a^2]$, $k_0a \gg 1$}, the maximum of the packet moves with the group velocity, $x = \partial\omega_k/\partial k_x$. Similarly, by Fourier-transforming,

$$\phi(x,t)=\int dk_x\exp[i(k_x x-\omega_k t)]\frac{a}{\sqrt{\pi}}\exp[-(k-k_0)^2a^2].$$

In the presence of a weak spatial dependence in ω_k, we find that the characteristic wave number of the packet changes in time according to

$$k_x=-\frac{\partial\omega_k}{\partial x}.$$

The result is that the frequency ω_k characteristic of the packet is unchanged as the packet propagates through space:

$$\frac{d\omega_k}{dt}=\frac{\partial\omega_k}{\partial x}\dot{x}+\frac{\partial\omega_k}{\partial k_x}\dot{k}_x=0,$$

a result that is of course required on physical grounds. We can summarize this discussion by saying that the waves propagate in configuration space as if they satisfy a Hamiltonian formulation with $\omega_k(k, x)$ taking the part of the Hamiltonian. The time variation of $|\phi_k|^2$ is described by Eq. (III-39) as the sum of the Poisson bracket of $|\phi_k|^2$ with $\omega(\mathbf{k}, x)$, plus the nonlinear interaction terms and a linear source term with γ_k, derived from the anti-Hermitian part of $\epsilon^{(1)}$.

The three terms on the right-hand side of Eq. (III-39) describe respectively linear growth, mode coupling due to wave scattering by ions, and decay-type interaction, which can be written in symmetrical form with the proper normalization (see Chapter I).

Let us first consider long-wavelength drift turbulence (i.e., $k_\perp^2 r_{Hi}^2 \ll 1$). In this limit we neglect the wave scattering by ions, which is small by $k_\perp^2 r_{Hi}^2$ in comparison with the decay-type interaction.

As we go to the limit of zero ion Larmor radius, the phase velocity of the wave becomes very close to the electron drift velocity

$$\delta\omega_{\mathbf{k}}\equiv\omega_{\mathbf{k}}-k_y v_d^e\ll\omega_{\mathbf{k}}.\qquad\text{(III-40)}$$

Therefore, the waves are propagating almost in phase with each other for a long time $\Delta t \sim \delta\omega_{\mathbf{k}}^{-1}$, and the interaction becomes stronger. This is reflected in the fact that the integrand of the nonlinear decay-type term goes to zero as $|\delta\omega_{\mathbf{k}}|^2$. But the argument of the δ function in Eq. (III-39) vanishes also. Thus

$$\Delta\omega\equiv\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}\ \ \mathbf{k}}=\delta\omega_{\mathbf{k}}-\delta\omega_{\mathbf{k}'}-\delta\omega_{\mathbf{k}\ \ \mathbf{k}}.$$

Then the nonlinear term is of the order of

$$\frac{c^2|\phi_{\mathbf{k}}|^2}{H^2}\int\frac{d^3\mathbf{k}'}{(2\pi)^2}|\phi_{\mathbf{k}'}|^2(\mathbf{k}\times\mathbf{k}')^2\frac{\delta\omega_{\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\delta(\Delta\omega)\sim\delta\omega_{\mathbf{k}}|\phi_{\mathbf{k}}|^2\times\int\frac{d^3\mathbf{k}'}{(2\pi)^3}\frac{c^2(\mathbf{k}\times\mathbf{k}')^2}{H^2\omega_{\mathbf{k}}^2}|\phi_{\mathbf{k}}|^2.$$

The recoil effect in the quasilinear equation for the distribution function is derived in the following way:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{fluct}} = \left\langle \frac{e\nabla\delta\phi}{m} \cdot \frac{\partial \delta f}{\partial v} \right\rangle \quad , \qquad \text{(III-52)}$$

where $\nabla\delta\phi$ is the contribution to the electric field from the spontaneous fluctuations. Averaging $\delta\phi\ \delta f$, using Eqs. (III-47) and (III-48), we derive this additional term in the kinetic equation:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{fluct}} = -\frac{\partial n e^4}{m^2} \int \frac{d\mathbf{k}}{k^4} \int d\mathbf{v}'\ \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} \frac{\delta(\mathbf{k}\cdot\mathbf{v} - \mathbf{k}\cdot\mathbf{v}')}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} f(\mathbf{v})\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}'}(\mathbf{v}') \qquad \text{(III-53)}$$

These additional terms [in the wave kinetic equation, Eq. (III-51)] and (in the quasilinear equation Eq. (III-53)] are not significant in the case of an unstable plasma. In a stable plasma the competition between term (III-51) and Landau damping determines the equilibrium level of the thermal fluid fluctuation. This equilibrium background of plasma waves, after substitution into the quasilinear collision term gives

$$\left(\frac{\partial f}{\partial t}\right)_{\text{QLA}} = \frac{\partial n e^4}{m^2} \int \frac{d\mathbf{k}}{k^4} \int d\mathbf{v}'\ \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} \frac{\delta(\mathbf{k}\cdot\mathbf{v} - \mathbf{k}\cdot\mathbf{v}')}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} f(\mathbf{v}')\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}}(\mathbf{v}) \qquad \text{(III-54)}$$

Combining terms (III-53) and (III-54) gives the Lenard–Balescu equation. [18, 19]

This might appear to be a surprising result, since the basic equations from which we started were the Vlasov–Poisson system, which includes none of the two-particle effects contained in the binary distribution f_2. However, an additional assumption has been made in Eq. (III-47) and we have invoked the principle of detailed balance in combining Eqs. (III-53) and (III-54). This suffices to reproduce the usual kinetic equation, valid for stable plasmas.

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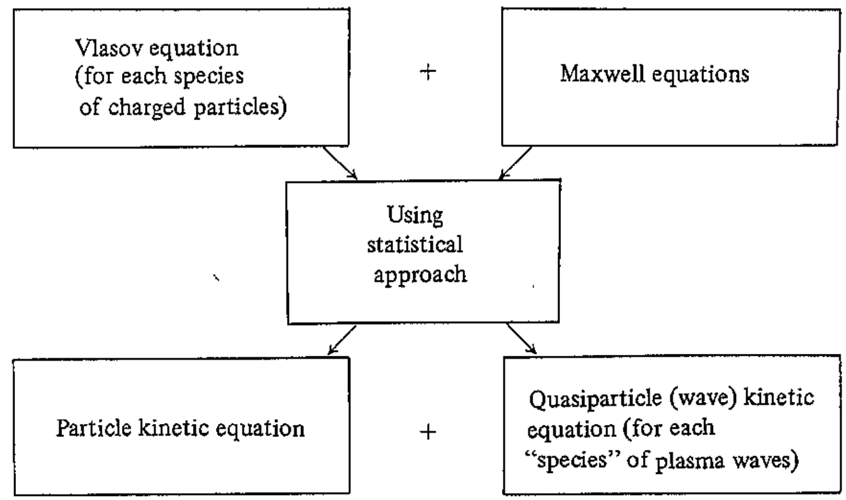
Chapter IV

Summary

In order to establish the relationship among the different parts of the weak turbulence approach, it is useful now to summarize what we have done in the form of a symbolic tree.

IV-1 THE GENERAL SCHEME OF WEAK TURBULENCE THEORY



These kinetic equations have the form

$$\frac{df(\mathbf{v})}{dt} = St[f(\mathbf{v})] \quad + \quad \frac{dN(\mathbf{k})}{dt} = St[N(\mathbf{k})]$$

where the collision terms may be written as follows:

in the 1st approximation

$$St(f) = QLA \qquad St[N(\mathbf{k})] = 2\,\mathrm{Im}\,\omega_k[f(\mathbf{v})]\,N(\mathbf{k})$$

This is the quasilinear approximation (QLA); see Chapter II. Im $\omega_{\mathbf{k}}[f(\mathbf{v})]$ means the imaginary part of $\omega_{\mathbf{k}}$ as a functional of $f(\mathbf{v})$ (from the linear dispersion relation).

This approximation takes into account only the mutual interaction between particle and waves, according to the resonance condition

$$\omega_{\mathbf{k}} - \mathbf{k}\cdot\mathbf{v} = 0;$$

in the 2nd approximation

(a) Wave–wave (three-wave) interaction (Chapter I),

$$\omega_1 + \omega_2 = \omega_3$$
$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3.$$

The collisional term in the particle kinetic equation describes the adiabatic interaction of particles with waves, since the particles participate in the oscillations.	St $(N) = N{\cdot}N$: symbolic notation in order to stress the quadratic character of three-wave interaction term (Chapter I).

(b) Nonlinear wave–particle interaction (Chapter III),

$$\omega_2 - \omega_1 = (\mathbf{k}_2 - \mathbf{k}_1)\cdot\mathbf{v}$$

The contribution of this process to the collision term is cumbersome and we did not consider it in the lectures (see [4, 11] of Chapter II)	$N{\cdot}N{\cdot}f$ in order to show that the particles also participate; see the details in Chapter III.

in the 3rd approximation

$$\left.\begin{aligned}\omega_1 + \omega_2 &= \omega_3 + \omega_4\\ \mathbf{k}_1 + \mathbf{k}_2 &= \mathbf{k}_3 + \mathbf{k}_4\end{aligned}\right\}\text{four-wave interaction}$$

Again, only adiabatic interaction between particles and waves.	$N{\cdot}N{\cdot}N$ Since this collision term is cubic, it can be important only in the decay-free cases. We did not consider it.

as well as higher-order effects, which we also did not consider.

In the case of strong turbulence this approximation procedure breaks down, since higher-order wave interactions yield contributions of the same order of magnitude. In other words, in the strong coupling limit there is no closure of the equations. Here we do not discuss the various attempts to make nonlinear estimates in the strong turbulence cases.

In the case of weak turbulence, the statement of concrete problems in terms of, generally speaking, quite complicated nonlinear integro-differential kinetic equations for particles and waves naturally does not lead to easy interpretation. But, at least, there exist a finite number of equations. In the various limiting cases one can obtain a solution using realistic approximations, as we have tried to demonstrate.

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