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PROFESSOR J. TURKEVICH
Princeton

Responsible Editor

ACADEMICIAN M. A. LEONTOVICH

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L. I. RUDAKOV and R. Z. SAGDEEV

A quasi-hydrodynamic description of a rarefied plasma in a magnetic field+

The Boltzmann equation describing the motion, averaged over the Larmor revolution, of ions in a plasma in a magnetic field, in the absence of collisions [1] is generalised to the non-static case. The set of equations for the lower moments of the electron and ion distribution functions is similar to the equations of two-fluid hydrodynamics. Although the mean free path is formally infinite, it is replaced by the Larmor radius of the ion or electron. Using the equations obtained, the wave solutions corresponding to magnetohydrodynamic and sound oscillations are examined. When the pressure tensor is sufficiently anisotropic, instability arises and the plasma ceases to be homogeneous.

The study of the motion of a fully ionised plasma in strong electromagnetic fields, taking into account the field of the plasma and using the Boltzmann equation, is very difficult. The usual hydrodynamical discussion is valid only for high densities and low temperatures, when the mean free path between collisions is much less than the characteristic dimension. Recently Chew, Goldberger and Low [2] have considered the case of a rarefied plasma in a magnetic field, neglecting collisions. Their work has shown that an expansion of the Boltzmann equation in powers of M/e (M being the mass of the ion and e the charge) leads to a closed system of equations for the lower moments of the distribution function, which is formally analogous to the equations of magnetohydrodynamics. However, this is true only for

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+ Work done in 1957.

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plasma motions across the magnetic field.

This similarity of results in two seemingly opposite cases is explained by the fact that the magnetic field 'turns' the ions and so symmetrises their velocity distribution in the plane perpendicular to the magnetic field. In this sense the action of the magnetic field resembles the action of collisions.

In the present paper we derive the equations of the hydrodynamics of two fluids (the ions and the electrons), starting from the Boltzmann equation which describes the ion (or electron) motion averaged over the Larmor revolution.

The Boltzmann equation for the stationary case has been obtained by S. T. Belyaev [1]. To derive the equations for the quasi-hydrodynamic approximation considered, it is first necessary to generalise this equation to the non-stationary case.

§ 1. The Boltzmann equation

The behaviour of a system of charged particles in electric and magnetic fields, in the absence of collisions, is described by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\text{grad}}\, f + \frac{e}{M}\left(\underset{\sim}{E} + \frac{1}{c}\, \underset{\sim}{v} \times \underset{\sim}{B}\right). \frac{\partial f}{\partial \underset{\sim}{v}} = 0. \qquad (1.1)$$

The solution of this equation is an arbitrary function of the first integrals of the system of characteristic equations

$$dt = \frac{d\underset{\sim}{r}}{\underset{\sim}{v}} = \frac{d\underset{\sim}{v}}{(e/M)(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}/c)}, \qquad (1.2)$$

which gives the equations of motion of a single charged particle in the fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$.

If the fields vary only slowly in space and time, the solution of equations (1.2) can be obtained by expanding in powers of R/L and $1/\omega T$, where R and ω are the Larmor radius and frequency, and L and T are a characteristic length and time of the problem. Such an expansion for the case of static fields, as far as terms of order $1/\omega$, has been made by N. N. Bogolyubov and D. N. Zubarev [3].

For our purposes, however, it is necessary to find a similar expansion of the equations of motion of one particle in variable fields. To do so, we use some results derived by S. I. Braginskiĭ (unpublished).

As the zero-order approximation we take the Larmor revolution about a centre moving with velocity $c\mathbf{E} \times \mathbf{B}/B^2$. We then go to the first ('drift') approximation.

We take a local orthogonal co-ordinate system $(\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2)$ fixed to the magnetic field: $\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_2$, $\boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_2 \times \boldsymbol{\varepsilon}_0$, $\boldsymbol{\varepsilon}_2 = \boldsymbol{\varepsilon}_0 \times \boldsymbol{\varepsilon}_1$, where $\boldsymbol{\varepsilon}_0 = \mathbf{B}/B$ is along the magnetic field. In this system the velocity $\mathbf{v}$ can be written

$$\mathbf{v} = c\mathbf{E} \times \mathbf{B}/B^2 + \boldsymbol{\varepsilon}_0 v_{\parallel} + v_{\perp} (\boldsymbol{\varepsilon}_1 \cos \alpha + \boldsymbol{\varepsilon}_2 \sin \alpha).$$

In (1.2) we make the substitutions

$$\left.\begin{aligned}
\mathbf{r} &= \bar{\mathbf{r}} + \bar{v}_{\perp} (\boldsymbol{\varepsilon}_2 \cos \bar{\alpha} - \boldsymbol{\varepsilon}_1 \sin \bar{\alpha}), \\
\alpha &= \bar{\alpha} + \frac{1}{\omega} (g_1 \cos \bar{\alpha} - f_1 \sin \bar{\alpha}) + \frac{\bar{v}_{\perp}}{\omega^2} (\boldsymbol{\varepsilon}_1 \cos \bar{\alpha} + \boldsymbol{\varepsilon}_2 \sin \bar{\alpha}) \cdot \mathbf{grad}\, \omega + \\
&\quad + \frac{1}{2\omega} (g_2 \cos 2\bar{\alpha} - f_2 \sin 2\bar{\alpha}), \\
v_{\parallel} &= \bar{v}_{\parallel} - \frac{1}{\omega} \sum_{n=1}^{2} \frac{1}{n} \{- G_{4n} \cos n\bar{\alpha} + F_{4n} \sin n\bar{\alpha}\}, \\
v_{\perp} &= \bar{v}_{\perp} - \frac{1}{\omega} \sum_{n=1}^{2} \frac{1}{n} \{- G_{5n} \cos n\bar{\alpha} + F_{5n} \sin n\bar{\alpha}\}.
\end{aligned}\right\} \quad (1.3)$$

The new variables $\bar{v}_{\parallel}$, $\bar{v}_{\perp}$, $\bar{\mathbf{r}}$ and $\bar{\alpha}$ are called drift variables. Omitting some easy calculations, we have the final result

$$
\left.
\begin{aligned}
&\frac{d\bar{v}_\perp}{dt} = \frac{\bar{v}_\perp}{2\omega}\frac{d^o\omega}{dt}, \\
&\frac{d\bar{v}_{\parallel}}{dt} = \varepsilon_o \cdot \left(\frac{e}{m}E - \frac{\bar{v}_\perp^2}{2\omega}\,\mathrm{grad}\,\omega - \frac{d^o w}{dt} - \bar{v}_{\parallel}\frac{d^o\varepsilon_o}{dt}\right), \\
&\frac{d\bar{r}}{dt} = \varepsilon_o \bar{v}_{\parallel} + \frac{1}{\omega}\varepsilon_o \cdot \left(-\frac{e}{m}E + \frac{\bar{v}_\perp^2}{2\omega}\,\mathrm{grad}\,\omega + \frac{d^o w}{dt} + \bar{v}_{\parallel}\frac{d^o\varepsilon_o}{dt}\right), \\
&d\bar{\alpha}/dt = -\omega,
\end{aligned}
\right\}
\tag{1.4}
$$

where $w = cE \times B/B^2$,

$$\frac{d^o}{dt} = \frac{\partial}{\partial t} + \bar{v}_{\parallel}\varepsilon_o \cdot \mathrm{grad} + w\,\mathrm{grad}.$$

These equations have an obvious significance. The first expresses the adiabatic invariance of $\bar{\mu} = \bar{v}_\perp^2/2B$. The second and third describe the motion along and across the magnetic field under the action of the force

$$F = eE - \frac{M\bar{v}_\perp^2}{2\omega}\,\mathrm{grad}\,\omega - M\frac{d^o w}{dt} - M\bar{v}_{\parallel}\frac{d^o\varepsilon_o}{dt}.$$

The force $- Md^o w/dt - M\bar{v}_{\parallel}\, d^o\varepsilon_o/dt$ is the inertia force; it appears because the motion is considered in a moving system of curvilinear co-ordinates.

Instead of making the substitution (1.3) in the Boltzmann equation, we shall use the relation between (1.1) and (1.2). Then, in the new variables $\bar{v}_{\parallel}$, $\bar{\mu} = \bar{v}_\perp^2/2B$, $\bar{\alpha}$, r, t, we have for $\bar{f} = \bar{f}\,(\bar{v}_{\parallel}, \bar{\mu}, \bar{\alpha}, r, t)$ the equation

$$\frac{\partial \bar{f}}{\partial t} + \frac{dr}{dt}\cdot \mathrm{grad}\,\bar{f} + \frac{d\bar{v}_{\parallel}}{dt}\frac{\partial \bar{f}}{\partial \bar{v}_{\parallel}} + \frac{d\bar{\alpha}}{dt}\frac{\partial \bar{f}}{\partial \bar{\alpha}} = 0.$$

Here we have used the fact that $d\bar{\mu}/dt = 0$. The quantities dr/dt, $d\bar{v}_{\parallel}/dt$,

$d\bar\alpha/dt$ are determined from (1.4).

In the first approximation it is reasonable to suppose that $\bar f$ is independent of $\bar\alpha$ (this can be justified by calculating the terms in the first approximation for $d\bar v_\parallel/dt$). The result is

$$\frac{\partial \bar f}{\partial t} + \bar v_\parallel \, \boldsymbol{\varepsilon}_0 \cdot \mathbf{grad}\, \bar f + \mathbf{w}\,\mathbf{grad}\,\bar f + \boldsymbol{\varepsilon}_0 \cdot \left(\frac{e}{M}\mathbf{E} - \bar\mu\, \mathbf{grad}\,\omega - \frac{d^0 \mathbf{w}}{dt}\right) \frac{\partial \bar f}{\partial \bar v_\parallel} = 0. \qquad (1.5)$$

§ 2. The macroscopic equation of motion of the plasma

Multiplying (1.5) by various powers of the velocities and integrating over velocity space, we can obtain an infinite set of 'interlocking' equations for the moments. We put

$$\left.\begin{aligned} n &= B \int \bar f \, d\bar v_\parallel \, d\bar\mu d\bar\alpha, \quad nu = B\int \bar f \bar v_\parallel \, d\bar v_\parallel \, d\bar\mu d\bar\alpha, \\ p &= MB \int \bar f \, (\bar v_\parallel - u)^2 d\bar v_\parallel \, d\bar\mu d\bar\alpha, \\ \mu &= MB \int \bar f \, \bar\mu \, d\bar v_\parallel \, d\bar\mu d\bar\alpha. \end{aligned}\right\} \qquad (2.1)$$

To close the system of equations we assume that

$$\int f \, (\bar v_\parallel - u)^3 \, \bar\mu^n \, d\bar v_\parallel \, d\bar\mu d\bar\alpha = 0,$$

i.e. the distribution functions are symmetrical as regards the local longitudinal velocities.

Integration of equation (1.5) and of the drift equation, using this assumption, gives

$$\frac{\partial n}{\partial t} + \operatorname{div} n(\boldsymbol{\varepsilon}_0 u + \mathbf{w}) = 0, \qquad (2.2)$$

$$M\frac{\overset{o}{d} u}{dt} = -\frac{\omega}{n}\,\boldsymbol{\varepsilon}_0 \cdot \mathbf{grad}\,\frac{p}{\omega} + \boldsymbol{\varepsilon}_0 \cdot \left(e\mathbf{E} - \frac{\mu\,\mathbf{grad}\,B}{n} - M\frac{d^0 \mathbf{w}}{dt}\right), \qquad (2.3)$$

$$\frac{d^o}{dt}\left(\frac{pB^2}{n^3}\right) = 0, \tag{2.4}$$

$$\frac{d^o}{dt}\left(\frac{\mu}{n}\right) = 0, \tag{2.5}$$

$$\frac{d\underset{\sim}{r}}{dt} = u\underset{\sim}{\varepsilon}_o + \frac{1}{\omega}\,\underset{\sim}{\varepsilon}_o\,\underset{\sim}{x}\,(-\frac{e}{M}\underset{\sim}{E} + u\frac{d^o\underset{\sim}{\varepsilon}_o}{dt} + \frac{d^o\underset{\sim}{W}}{dt} + \frac{p}{nM}(\underset{\sim}{\varepsilon}_o\cdot\underset{\sim}{\text{grad}})\underset{\sim}{\varepsilon}_o + \frac{\mu\,\text{grad}\;B}{nM}). \tag{2.6}$$

The equation (2.6) describes the averaged motion of the centres of the circles, and may not be the same as that for the mean velocity of a volume in the plasma, which is by definition, given by $n\underset{\sim}{v} = \int f(r,v,t)\underset{\sim}{v}d^3\underset{\sim}{v}$. In the integrand here we change to the variables $\overline{v}_\parallel$ and $\overline{v}_\perp$ by formulae (1.3) and express $f(\underset{\sim}{v},\underset{\sim}{r})$ in terms of $F(\overline{v}_\parallel, \overline{v}_\perp, \underset{\sim}{r})$; the latter is known to be independent of α. Then we have (cf. [2])

$$n\underset{\sim}{v} = -\,\underset{\sim}{\text{curl}}\,\frac{\mu c\underset{\sim}{\varepsilon}_o}{e} +$$

$$+\, n\,\{u\underset{\sim}{\varepsilon}_o + \frac{1}{\omega}\,\underset{\sim}{\varepsilon}_o\,\underset{\sim}{x}\,(-\frac{e}{M}\underset{\sim}{E} + u\frac{d^o\underset{\sim}{\varepsilon}_o}{dt} + \frac{d^o\underset{\sim}{W}}{dt} + \frac{p}{nM}(\underset{\sim}{\varepsilon}_o\cdot\underset{\sim}{\text{grad}})\,\underset{\sim}{\varepsilon}_o + \frac{\mu}{nM}\,\underset{\sim}{\text{grad}}\;B\}$$

$$= -\,\underset{\sim}{\text{curl}}\;\mu c\underset{\sim}{\varepsilon}_o/e + n\underset{\sim}{v}_{dr}. \tag{2.7}$$

Equations (2.2) - (2.6) for ions and electrons must be supplemented by Maxwell's equations, which in the drift variables are

$$\left.\begin{aligned} &\text{div}\;\underset{\sim}{E} = -4\pi e(n_e - n_i),\\ &\text{div}\;\underset{\sim}{B} = 0,\\ &\underset{\sim}{\text{curl}}\;\underset{\sim}{E} = -\dot{\underset{\sim}{B}}/c,\\ &\underset{\sim}{\text{curl}}\;\underset{\sim}{B} = \frac{4\pi e}{c}\{-c\,\underset{\sim}{\text{curl}}\,\frac{\mu_i+\mu_e}{e}\underset{\sim}{\varepsilon}_o + n_i\underset{\sim}{v}_{dr,i} - n_e\underset{\sim}{v}_{dr,e}\}. \end{aligned}\right\} \tag{2.8}$$

The latter equation is conveniently written

$$\left.\begin{aligned} &\underset{\sim}{\text{curl}}\;\underset{\sim}{H} = 4\pi e\,[n_i\underset{\sim}{v}_{dr,i} - n_e\underset{\sim}{v}_{dr,e}],\\ &\underset{\sim}{B} = \underset{\sim}{H} - 4\pi(\mu_i + \mu_e)\underset{\sim}{\varepsilon}_o. \end{aligned}\right\} \tag{2.9}$$

The physical significance of these equations may be briefly discussed. Equations (2.9) signify that the plasma is diamagnetic, with magnetic permeability $\kappa = 1 + 4\pi(\mu_i + \mu_e)/B$. The motions of the ion and electron 'fluids' (equations (2.3) and (2.6)) are drifts under the action of the forces

$$\mathbf{F} = ne\mathbf{E} - Mnd^{\circ}\mathbf{u}/dt - Mnud^{\circ}\boldsymbol{\varepsilon}_o/dt - p(\boldsymbol{\varepsilon}_o \cdot \mathbf{grad})\varepsilon_o - \mu\,\mathbf{grad}\,B,$$

where $\mu\,\mathbf{grad}\,B$ is the force acting on the magnetic moment in the magnetic field, and $-Mnd^{\circ}\mathbf{u}/dt - Mnud^{\circ}\boldsymbol{\varepsilon}_o/dt - p(\boldsymbol{\varepsilon}_o \cdot \mathbf{grad})\varepsilon_o$ is the inertia force, which has been explained above.

Equations (2.4) and (2.5) give the law of adiabatic compression for the 'longitudinal' ($\gamma = 3$) and 'transverse' ($\gamma = 2$) pressures respectively.

A consideration of the motions along the lines of magnetic force, using these equations, requires the fulfilment of the symmetry condition imposed on the distribution function. When this condition is not satisfied, equation (2.4) cannot be used, and the whole system of equations (2.2), (2.3), (2.5), (2.6) ceases to be meaningful.

§ 3. Instability of a plasma with anisotropic pressure

Using these equations, let us discuss wave motions of the plasma with frequencies much less than the ion Larmor frequency and wavelengths $\lambda \gg R_i$. We assume that the quasi-neutrality condition $n_i = n_e = n$ holds.

The unperturbed state of the plasma is characterised by the following quantities: the density n_o, $u_{i,e} = 0$, the pressure $p_o = p_{o,i} + p_{o,e}$ along the field $B_o = B_{oz}$, the magnetic moment of unit volume $\mu_o = \mu_{o,i} + \mu_{o,e}$, and $\mathbf{E}_o = 0$.

The linearised equations for small perturbations n, $u_{i,e}$, μ, p, $\mathbf{B}$ are

$$\left.\begin{aligned} &\frac{\partial n}{\partial t} - \frac{n_o}{B_o}\frac{\partial B}{\partial t} + n_o\frac{\partial u}{\partial z} = 0, \\ &M\frac{\partial u}{\partial t} + \frac{B_o}{n_o}\frac{\partial}{\partial z}\left(\frac{p}{B_o} - \frac{p_o B}{B_o^{\,2}}\right) + \frac{\mu_o}{n_o}\frac{\partial B}{\partial z} - eE_z = 0, \\ &p = \frac{p_o B_o^{\,2}}{n_o^{\,3}}\left(\frac{3n_o^{\,2}n}{B_o^{\,3}} - \frac{2n_o^{\,3}B}{B_o^{\,3}}\right), \quad \mu = \mu_o n/n_o, \end{aligned}\right\}$$

$$\left.\begin{aligned} &\operatorname{div} \underset{\sim}{B} = 0,\ \underset{\sim}{\operatorname{curl}}\ \underset{\sim}{E} = -\dot{\underset{\sim}{B}}/c,\ n_i = n_e = n, \\ &\underset{\sim}{\operatorname{curl}}\ B \left(1 + \frac{4\pi\mu_o}{B_o} - \frac{4\pi p_o}{B_o^2}\right) = 4\pi \frac{\underset{\sim}{B_o} \times \underset{\sim}{\operatorname{grad}}\ \mu}{B_o} + \\ &\quad + \frac{4\pi p_o}{B_o^2} \frac{\underset{\sim}{B_o} \times \underset{\sim}{\operatorname{grad}}\ B}{B_o} + \\ &\quad + \frac{4\pi n M}{B_o^2} \cdot \underset{\sim}{B_o} \times \frac{d^o \underset{\sim}{v}}{dt}. \end{aligned}\right\} \tag{3.1}$$

We seek the corrections to uniform unperturbed values in the form $\exp(i\omega t + ik_z z + ik_y y)$. Substituting in (3.1) we have, after some simple transformations, the dispersion equation

$$\omega^4 - \left\{3 \frac{p_o}{Mn_o} k_z^2 + v_a^2 k_y^2 \left[\left(1 + \frac{4\pi\mu_o}{B_o} - \frac{4\pi p_o}{B_o^2}\right)\left(1 + \frac{k_z^2}{k_y^2}\right) + \frac{4\pi p_o}{B_o^2} + \frac{4\pi\mu_o}{B_o}\right]\right\}\omega^2 +$$

$$+ 3 \frac{p_o}{Mn_o} v_a^2 k_z^2 k_y^2 \left[\left(1 + \frac{4\pi\mu_o}{B_o} - \frac{4\pi p_o}{B_o^2}\right)\left(1 + \frac{k_z^2}{k_y^2}\right) + \frac{4\pi p_o}{B_o^2} + \frac{4\pi\mu_o}{B_o}\right] -$$

$$- v_a^4 \left(\frac{4\pi\mu_o}{B_o}\right)^2 k_z^2 k_y^2 = 0. \tag{3.1a}$$

Here $v_a = \sqrt{(B_o^2/4\pi M n_o)}$ is the Alfvén velocity. The solution of (3.1a) for ω^2 is

$$\omega^2 = \tfrac{1}{2} k_y^2 v_a^2 \left\{3 \frac{4\pi p_o}{B_o^2} \frac{k_z^2}{k_y^2} + \left[\frac{4\pi p_o}{B_o^2} + \frac{4\pi\mu_o}{B_o} + \left(1 + \frac{4\pi\mu_o}{B_o} - \frac{4\pi p_o}{B_o^2}\right) \times \right.\right.$$

$$\left.\left. \times \left(1 + \frac{k_z^2}{k_y^2}\right)\right]\right\} \pm$$

$$\pm \tfrac{1}{2} k_y^2 v_a^2 \sqrt{\left[\left\{3 \frac{4\pi p_o}{B_o^2} \frac{k_z^2}{k_y^2} + \left[\frac{4\pi p_o}{B_o^2} + \frac{4\pi\mu_o}{B_o} + \left(1 + \frac{4\pi\mu_o}{B_o} - \frac{4\pi p_o}{B_o^2}\right) \times \right.\right.\right.}$$

$$\times \left(1 + \frac{k_z^2}{k_y^2}\right)]\}^2 -$$

$$- 12 \frac{4\pi p_o}{B_o^2} \frac{k_z^2}{k_y^2} \left[\frac{4\pi p_o}{B_o^2} + \frac{4\pi\mu_o}{B_o} + \left(1 + \frac{4\pi\mu_o}{B_o} - \frac{4\pi p_o}{B_o^2}\right)\left(1+\frac{k_z^2}{k_y^2}\right)\right] + 4\pi \left(\frac{4\pi\mu_o}{B_o}\right)^2 \frac{k_z^2}{k_y^2}]. \qquad (3.2)$$

It is seen from (3.1a) that aperiodic solutions ($\omega^2 < 0$) occur in the case where the term independent of ω is less than zero, i.e. when

$$\left(\frac{4\pi\mu_o}{B_o}\right)^2 > 3 \frac{4\pi p_o}{B_o^2} \left[\frac{4\pi p_o}{B_o^2} + \frac{4\pi\mu_o}{B_o} + \left(1+ \frac{4\pi\mu_o}{B_o} - \frac{4\pi p_o}{B_o^2}\right) \times \left(1 + \frac{k_z^2}{k_y^2}\right)\right]. \qquad (3.3)$$

Assuming that k_z/k_y can take any value (as it can for a homogeneous plasma on the above assumptions), we can construct a stability diagram (Fig.1). This shows that the instability of the plasma is due to pressure anisotropy.

To see the physical significance of the lines in Fig. 1, let us consider the limiting cases $k_z/k_y \ll 1$, $k_z/k_y \gg 1$. For the purely oscillatory branch (the plus sign in (3.2)), and $k_z/k_y \gg 1$, we have $\omega^2 = 12\pi p_o v_\alpha^2 k_z^2/B_o^2 = 3p_o k_z^2/n_o M$, which describes one-dimensional ($\gamma = 3$) longitudinal sound. The other branch, where there are imaginary ω (the minus sign in (3.2)), for $k_z/k_y \ll 1$ corresponds to transverse magnetohydrodynamic sound, propagated with velocity

$$\sqrt{[(4\pi p_\perp + B_o^2)/4\pi n_o M]}.$$

We may consider in more detail the physical significance of the instability arising on the branch corresponding to magnetic sound. It follows from the criterion (3.3) that, for $k_z/k_y \ll 1$, the aperiodic solutions lie in the range

$$\frac{4\pi\mu_o}{B_o} < 3\frac{4\pi p_o}{B_o^2} + \sqrt{[9(\frac{4\pi p_o}{B_o^2})^2 + 3\frac{4\pi p_o}{B_o^2}]}.$$

The development of this type of instability can be qualitatively described as follows. Let some perturbation of the uniform density be applied. Since the plasma is diamagnetic, the field $\underline{B}$ is reduced where the density is increased. In a non-uniform field, however, a force $-\mu\,\text{grad}\,B$ acts on the plasma, and this increases the original perturbation.

For $k_z/k_y \gg 1$, the region of instability is given by the inequality $4\pi\mu_o/B_o < (4\pi p_o - B_o^2)/B_o^2$. This case corresponds to a large curvature of the lines of magnetic force. Hence centrifugal forces are important. This is clearly seen from the condition $4\pi\mu_o/B_o \ll 4\pi p_o B_o^2$ for small ω, when the only force acting is $-p\,(\mathcal{E}_o\,.\,\text{grad})\mathcal{E}_o = -p\underline{R}/R^2$, where $\underline{R}$ is the radius of curvature of a line of force.

In this investigation we have used equation (2.4) which is applicable only to perturbations which leave the distribution function symmetrical with respect to longitudinal velocities.

A calculation by means of the Boltzmann equation, which has been done by A. A. Vedenov and R. Z. Sagdeev[4] , gives good agreement with the results obtained above.

In conclusion, we thank Academician M. A. Leontovich, Prof. D. A. Frank-Kamenetskiĭ, and S. I. Braginskiĭ for valuable discussion and continued interest in the work.

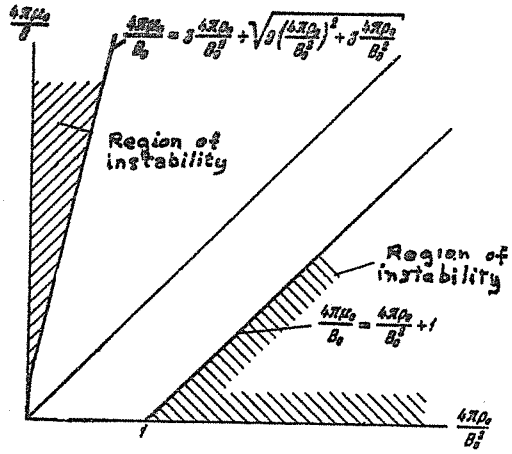

Fig. 1.

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A. A. VEDENOV and R. Z. SAGDEEV

Some properties of a plasma with an anisotropic ion velocity distribution in a magnetic field[+]

The possibility of instability of a plasma in a magnetic field owing to anisotropy of the ion velocity distribution is considered.

In an ordinary gas, with a non-Maxwellian velocity distribution, equilibrium is established after a time of the same order as the collision time $\tau \sim \ell/v$. In a plasma, long-range forces can bring about collective motions, which develop in a time less than τ. An example of such motions is the occurrence of Langmuir oscillations in a plasma with a non-Maxwellian electron velocity distribution [1]. The effect of an external magnetic field on these effects has been studied by E. P. Gross [2], using the Boltzmann equation.

In discussing the electron oscillations of a plasma, Gross naturally neglected the magnetic field of the wave, as being of order v/c. The situation is considerably changed if the ion ('magnetohydrodynamic') oscillations are under discussion: the electric fields in low-frequency oscillations are largely screened by the motion of the electrons, and magnetic effects may be of importance. In [3] the equations for the moments of the distribution function in the drift approximation have been used to obtain the criteria for the appearance of a specific instability when the longitudinal and transverse temperatures are markedly different. The discussion in [3] corresponds to the hydrodynamic derivation of the dispersion equation. In the present paper this problem is studied by using the Boltzmann equation, and criteria are found for instability when collisions are neglected $(1/\tau \gg \nu$, where

[+] Work done in 1957.

τ is the characteristic time and ν the collision frequency). An allowance for the magnetic field of the wave with fairly large anisotropy of the distribution function shows that there is a transfer of energy from transverse to longitudinal motion by the excitation of collective motions pertaining to ion oscillations of the plasma.

We write the Boltzmann equation for ions (or electrons) in the linearised form, retaining only the first-order terms:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{e}{mc}\,\mathbf{v}\times\mathbf{H}_0\cdot\frac{\partial f}{\partial \mathbf{v}} + \frac{e}{m}\left(-\,\mathbf{grad}\,\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right)\cdot\frac{\partial f_0}{\partial \mathbf{v}} +$$

$$+\frac{e}{mc}\,\mathbf{v}\times\mathbf{curl}\,\mathbf{A}\cdot\frac{\partial f_0}{\partial \mathbf{v}} = 0. \qquad (1)$$

Here $f_0\,(\mathbf{v})$ is the unperturbed velocity distribution function, which we take as

$$f_0\,(\mathbf{v}) = \frac{n_0}{\pi^{3/2}}\;\frac{m^{3/2}}{2T_\perp\sqrt{(2T_\parallel)}}\;e^{-m(v_x^2 + v_y^2)/2T_\perp - mv_z^2/2T_\parallel}$$

f is the correction to the distribution function; $\mathbf{H}_0$ is the uniform external magnetic field; Φ and $\mathbf{A}$ are the scalar and vector potentials of the perturbed motion, determined from Maxwell's equations.

Let us assume that the magnetic field is in the z-direction, and take cylindrical co-ordinates in velocity space: $\varrho^2 = v_x^2 + v_y^2$, $v_x = \varrho\cos\delta$, $v_y = \varrho\sin\delta$, $\tan\delta = v_y/v_x$. We shall discuss solutions independent of x. Using a Laplace transformation with respect to time and Fourier transformations with respect to z and y, we can rewrite the initial equation in terms of the Fourier transforms of the distribution function and potentials:

$$(p + ik_z v_z)\bar{f}_k + ik_y \mathcal{g} \sin\delta\, \bar{f}_k - \frac{eH_o}{mc}\frac{\partial \bar{f}_k}{\partial\delta} - i\frac{e}{m}\Phi_k\left(k.\frac{\partial f_o}{\partial v}\right) -$$

$$- \frac{e}{mc}\, p\left(A_k \frac{\partial f_o}{\partial v}\right) + \frac{e}{mc}\, i\, v \times (k \times A_k). \frac{\partial f_o}{\partial v} = f_k(0). \qquad (2)$$

The solution of this equation which is periodic in δ with period 2π is

$$f_k = \frac{e}{m}\sum_n \frac{e^{i(n\delta - [k_y \mathcal{g}/\omega_H]\cos\delta)}}{p + ik_z v_z - in\omega_H}\,\frac{1}{2\pi}\ \times$$

$$\times \int_0^{2\pi} \left\{ ik.\frac{\partial f_e}{\partial v}\left(\bar{\varphi} - \frac{1}{c}\, v.\bar{A}\right) + \frac{1}{c}\bar{A}.\frac{\partial f_e}{\partial v}(p + ikv) + \frac{m}{e} f_k(0)\right\} e^{i(\{k_y \mathcal{g}/\omega_H\}\cos\delta' - n\delta')}\, d\delta'. \qquad (3)$$

The Fourier components of the particle density and current can be obtained by means of (3). To do this, we use the auxiliary parameter τ:

$$\frac{1}{p + ik_z v_z - in\omega_H} = \int_0^\infty \exp\left[(-p - ik_z v_z + in\omega_H)\tau\right] d\tau \quad (\mathrm{re}\ p > 0),$$

and the identity

$$\sum_n e^{in\psi}\,\frac{1}{2\pi}\, f(\phi)\, e^{in\phi}\int_0^{2\pi} g(\phi')e^{-in\phi'}\, d\phi' = f(\phi)g(\phi + \psi).$$

Then, returning to Cartesian variables in velocity space, we have after some easy calculations

$$\int \bar{f}_k dv = -\frac{en_o}{m}\int_0^\infty \exp\left[-p\tau - \frac{k_z^2 T_\parallel}{2m}\tau^2 - 2\frac{k_y^2 T_\perp}{m\omega_H^2}\sin^2 \tfrac{1}{2}\omega_H\tau\right] d\tau\ \times$$

$$\times \left\{\bar{\varphi}\left(\frac{k_y^2}{\omega_H}\sin\omega_H\tau + k_z^2\tau\right) + \frac{i\bar{A}_x}{c}(1 - \cos\omega_H\tau)\frac{k_y}{H}\left[p + k_z^2\frac{\tau}{m}(T_\parallel - T_\perp)\right] + \right.$$

$$\left. + \frac{i\bar{A}_y}{c}\sin\omega_H\tau\,\frac{k_y}{H}\left[-p + k_z^2\frac{\tau}{m}(T_\parallel - T_\perp)\right] + \frac{i\bar{A}_z}{c}k_z\tau\left[-1 - \frac{k_y^2}{\omega_H}\frac{\sin\omega_H\tau}{m}(T_\perp - T_\parallel)\right]\right\} +$$

$$+ F(0), \qquad (4)$$

where

$$F(0) = \int d\underset{\sim}{v} \int_0^\infty d\tau \exp\{ -(p + ik_z v_z)\tau + (ik_y/\omega_H)[v_x(1 - \cos\omega_H\tau) - v_y \sin\omega_H\tau].$$

Expressions can similarly be obtained for $\int v_x \bar{f}_k \, d\underset{\sim}{v}$ and $\int v_z \bar{f}_k \, d\underset{\sim}{v}$, but they are too cumbersome to be given here.

We also use the three Maxwell's equations for the potential:

$$\left(\frac{p^2}{c^2} + k_y^2 + k_z^2\right) \Phi_k = 4\pi e \int (f_{ik} - f_{ek}) \, d\underset{\sim}{v} \text{ etc.}$$

On the right-hand sides we substitute the values found for the Fourier transforms of the charge density and current density components, and eliminate the component A_y of the vector potential by using Lorentz' equation

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} + \operatorname{div} \underset{\sim}{A} = 0. \qquad (5)$$

Equating to zero the determinant of the linear system of equations obtained, we obtain a characteristic equation whose solution gives the growth rate $p = p(\underset{\sim}{k})$. The general investigation of the roots of this equation is quite difficult, however, and we shall take only the limiting case of long waves (and therefore small p):

$$k_y^2 T_\perp/m \ll \omega_H^2, \quad k_z^2 T_\perp/m \ll \omega_H^2, \quad |p| \ll \omega_H. \qquad (6)$$

In the characteristic equation, we retain terms of order up to $1/\omega_H^2$ inclusive. This corresponds to retaining terms of the following orders in the determinant:

	Φ	A_x	A_z
Φ	1	$1/\omega_H$	1
A_y	$1/\omega_H$	$1/\omega_H^2$	$1/\omega_H$
A_z	1	$1/\omega_H$	1

For small $\underline{p}$ (taking only the linear terms) we have

(if $k_z, k_y \neq 0$)

$$\frac{p}{k_z} = \sqrt{\frac{T_\parallel}{M}} \quad \frac{\frac{8\pi n_o T_\perp}{H_o^2}\left(\frac{T_\perp}{T_\parallel} - 1\right) - \left(1 + \frac{8\pi n_o T_\perp}{H_o^2} - \frac{8\pi n_o T_\parallel}{H_o^2}\right)\frac{k_z^2}{k_y^2} - 1}{\sqrt{(\frac{1}{2}\pi)}\; 8\pi n_o T_\perp^2 / H_o^2 T_\parallel} \, . \tag{7}$$

Accordingly, the criterion of instability ($p > 0$) is

$$\frac{8\pi n_o T_\perp}{H_o^2}\;\frac{T_\perp}{T_\parallel} > 1 + \frac{8\pi n_o T_\perp}{H_o^2} \, . \tag{8}$$

In the limit $H_o \longrightarrow 0$, the condition (8) becomes $T_\perp > T_\parallel$.

If $k_y = 0$, the terms linear in p disappear from equation (5), and the quadratic terms give

$$\frac{p^2}{k_y^2} = - \frac{H_o^2}{4\pi n_o M}\left\{1 + \frac{8\pi n_o T_\perp}{H_o^2} - \frac{8\pi n_o T_\parallel}{H_o^2}\right\} . \tag{9}$$

Equation (9) corresponds to the Alfvén magnetohydrodynamic branch. For $8\pi n_o T_\parallel / H_o^2 > 1 + 8\pi n_o T_\perp / H_o^2$ there is instability. For $H_o \longrightarrow 0$ this becomes $T_\parallel > T_\perp$.

These criteria do not coincide quantitatively with those derived in [3] from the hydrodynamical approximation. This situation is entirely analogous to the one for longitudinal Langmuir oscillations of electrons in a plasma, where the correct dispersion equation can be obtained only from the Boltzmann equation.

Although there is no quantitative agreement with [3], the physical interpretation of the appearance of instability given there remains valid.

The whole of the above theory is linear and cannot, of course, give the limiting amplitude of instability of an anisotropic plasma.

An investigation of the second approximation which we have made for some simple limiting cases shows that the appearance of the instability described abve results in a transfer of kinetic energy of the particles (in a plasma volume of dimensions much exceeding the wavelength of the perturbation) from the transverse motion to the longitudinal motion if the instability arises because $T_{\perp} > T_{\parallel}$, and conversely in the opposite case. Hence it is reasonable to suppose that the instability develops until the energies of longitudinal and transverse motion become equal, i.e. until some effective equalisation of the longitudinal and transverse 'temperatures' takes place.

Finally, we shall show that in the magnetic trap (with magnetic stoppers) for a high-temperature plasma proposed by G. I. Budker [4], there may be regions where the distribution function is anisotropic which can lead to the appearance of the instability considered.

Let $f(\varepsilon)$ be the ion (or electron) distribution over energy of the longitudinal motion (along $\underset{\sim}{H}$) in the region where the magnetic field is a minimum. For simplicity we shall assume that the magnetic moment of every particle is the same.

In equilibrium, the function $f(\varepsilon)$ in such a trap must be cut off, i.e. must vanish for $\varepsilon > \varepsilon_{max}$, where ε_{max} is given by the condition $\varepsilon_{max} = \mu(B_{max} - B_{min})$. The ion density varies as a function of B along a line of force, thus:

$$n = \int_{\mu(B - B_{min})}^{\varepsilon_{max}} \frac{f(\varepsilon)\sqrt{\varepsilon}}{\sqrt{[\varepsilon - \mu(B - B_{min})]}}\, d\varepsilon. \qquad (10)$$

The longitudinal temperature on the line of force is

$$T_{\parallel} = \frac{1}{n} \int_{\mu(B - B_{min})}^{\varepsilon_{max}} \frac{\sqrt{\varepsilon}}{\sqrt{[\varepsilon - \mu(B - B_{min})]}} [\varepsilon - \mu(B - B_{min})] f(\varepsilon) d\varepsilon. \qquad (11)$$

We now use the instability criterion (8), which in the present case is

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$$8\pi\mu^2 \left(\int_{\mu(B-B_{min})}^{\varepsilon_{max}} \frac{f(\varepsilon)\sqrt{\varepsilon}}{\sqrt{[\varepsilon-\mu(B-B_{min})]}} d\varepsilon\right)^2 > \int_{\mu(B-B_{min})}^{\varepsilon_{max}} \frac{f(\varepsilon)\sqrt{\varepsilon}\,[\varepsilon-\mu(B-B_{min})]}{\sqrt{[\varepsilon - \mu(B-B_{min})]}} d\,\varepsilon. \quad (12)$$

As $B \to B_{max}$, this condition can be approximately written

$$8\pi\mu^2 f(\varepsilon_{max})\sqrt{\varepsilon_{max}} \left(\int_{\mu(B-B_{min})}^{\varepsilon_{max}} \frac{d\varepsilon}{\sqrt{[\varepsilon-\mu(B-B_{min})]}}\right)^2 >$$

$$> \int_{\mu(B-B_{min})}^{\varepsilon_{max}} \sqrt{[\varepsilon-\mu(B-B_{min})]}d\varepsilon,$$

or

$$48\,\mu^2\,\pi f(\varepsilon_{max}) > \sqrt{[(B_{max} - B)/(B_{max} - B_{min})]}. \quad (13)$$

If the distribution function is cut off sharply, i.e. if $f(\varepsilon_{max}) \not\to 0$, then the inequality (13) must always be satisfied for B sufficiently close to B_{max}.

Of course, the criterion (7), and therefore (8), which have been derived for a uniform initial density, must be applied with caution to such non-uniform initial density distributions as, for instance, that in a system with magnetic stoppers. We should expect, however, that these criteria will remain valid if the characteristic dimension of the inhomogeneity is much greater than the wavelength of the instability considered (the approximation of 'geometrical optics').

One further remark should be made. In the derivation of the relation (7), the quantities $T_\perp$ and $T_\parallel$ signified the true Maxwellian temperatures. For a system with magnetic stoppers, the temperatures are represented by the mean energies of the random motion. This, however, does not affect the qualitative validity of our results

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