LAGRANGIAN AND HAMILTONIAN METHODS IN MAGNETOHYDRODYNAMICS*

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The Lagrangian and Hamiltonian formulations of classical dynamics are applied to the motion of an infinitely conductive plasma with a frozen-in magnetic field. Two separate cases are considered, one with a scalar plasma pressure and one with a tensor. The treatment of the former is based on the conventional hydromagnetic equations, and that of the latter on the modified hydromagnetic equations of Chew, Goldberger, and Low. In each case the plasma equation of motion is derived, in either the Lagrangian or the Eulerian form, from a variational principle, with the other hydromagnetic equations functioning as holonomic constraints.

The general formalism is applied, for purposes of illustration, to the solution of hydromagnetic stability problems. A simple and concise derivation of the hydromagnetic energy principle is given, and then the energy principle is adapted to steady flows of a certain special type. (It was originally designed only for static equilibria.) Specifically, it is adapted to purely azimuthal steady flows around a symmetry axis, with the frozen-in field either purely toroidal or purely poloidal; and for these it will give the necessary and sufficient condition for $m=0$ stability (stability against small perturbations that do not destroy the symmetry). This type of steady flow is a hydromagnetic analogue of Couette flow, and the stability conditions are generalizations of Rayleigh's condition.

1. Introduction

As is well known, the nonviscous hydrodynamic equations of motion can be brought into the general scheme of classical mechanics by deriving them from a variational principle [1]. This subject has recently been treated in some detail by ECKART [2], and it has been extended by KATZ [3] to the case of a charged fluid interacting with an electromagnetic field. In Katz's treatment there is no conduction current, but only a displacement current $\partial \mathbf{E}/\partial t$, where $\mathbf{E}$ is the electric field, and a convection current $\eta \mathbf{v}$, where η is the charge density and $\mathbf{v}$ is the fluid velocity. Here we shall treat the opposite case of a perfectly-conducting fluid governed by the conventional hydromagnetic equations, a case in which the displacement and convection currents are negligible under most conditions of interest. We shall first develop the general theory from the Lagrangian and Hamiltonian points of view and then illustrate the usefulness of this approach by solving a few representative problems in hydromagnetic stability theory.

In doing stability problems one examines the behavior of small perturbations away from some specified time-independent state of the system, either a static equilibrium state or a steady flow. For static equilibria the stability criterion is already known: It is the familiar hydromagnetic energy principle [4, 5]. For steady flows, on the other hand, the energy principle is not applicable in general. In fact, we do not know of any general condition that is both necessary and sufficient, although FRIEMAN and ROTENBERG have found one that is sufficient but not necessary [6]. (It reduces to the energy principle when the flow velocity vanishes.) There are, however, special cases where the necessary and sufficient condition is given by some simple modification of the energy principle. One example of this has long been known in the field of hydrodynamics—the Couette flow of an incompressible fluid without viscosity, for which the stability condition was first given by Lord RAYLEIGH [7].

What we shall do is consider steady hydromagnetic flows around an axis of symmetry, assuming that the unperturbed magnetic field is either purely toroidal ($B_r = B_z = 0$) or purely poloidal ($B_\theta = 0$), and ask whether they are stable against those perturbations that maintain the azimuthal symmetry. As we shall see, the answer is given in each case by a modified energy principle. In the toroidal case the energy principle will be derived by eliminating an ignorable coordinate, which will reduce the steady-flow problem to one of static equilibrium; and in the poloidal case the same result will be achieved by means of an appropriate contact transformation.

Returning to the general theory, we shall now state our physical assumptions and write down the basic equations. First, we shall assume that the dissipative effects (viscosity, heat conduction, and finite electrical conductivity) may be neglected. Second, we shall neglect all terms of order v^2/c^2, where v may be either the fluid velocity, the speed of sound, or the Alfvén speed, and where c is the speed of light. Finally, we shall assume that the fluid is a plasma governed by the ideal-gas law with a scalar plasma pressure. (The scalar-pressure requirement will be relaxed later on.) These assumptions lead to the following system of hydromagnetic equations, in which the electromagnetic quantities are expressed in rationalized units with $\mu_0 = \varepsilon_0 = 1$:

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$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{v}) = 0\,, \tag{1.1}$$

$$\frac{\partial P}{\partial t} + \mathbf{v} \cdot \nabla P + \gamma P \nabla \cdot \mathbf{v} = 0\,, \tag{1.2}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0\,, \tag{1.3}$$

$$\nabla \cdot \mathbf{B} = 0\,, \tag{1.4}$$

$$\varrho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \varphi \right) + \nabla \left(P + \frac{1}{2} B^2 \right) - \mathbf{B} \cdot \nabla \mathbf{B} = 0\,, \tag{1.5}$$

where ϱ is the mass density, $\mathbf{v}$ is the plasma velocity, φ is the gravitational potential, P is the plasma pressure, $\mathbf{B}$ is the magnetic field, and γ is the adiabatic index (the ratio of specific heats). Equation (1.2) may also be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} (P \varrho^{-\gamma}) = 0\,, \tag{1.6}$$

which is simply the adiabatic ideal-gas law. The time derivative $\mathrm{d}/\mathrm{d}t$ refers to a point moving with the plasma and is related as follows to the time derivative at a fixed point:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\,. \tag{1.7}$$

Equation (1.3) is a direct consequence of Ohm's law with infinite conductivity, as we can see by writing

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0\,, \tag{1.8}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0\,. \tag{1.9}$$

(Note that $\mathbf{E} + \mathbf{v} \times \mathbf{B}$ is simply the electric field in a frame of reference moving with the plasma.) The physical content of this equation is that the plasma motion is flux-preserving. That is to say, if C is any closed curve moving with the plasma, then the magnetic flux through C is a constant of the motion. This fact enables us to picture the magnetic field lines as moving with the fluid [8, 9]. We note, incidentally, that the other magnetic-field equation, Eq. (1.4), is not entirely independent of Eq. (1.3), since it must obviously be satisfied for all time whenever it is satisfied initially.

The electrostatic force $\eta \mathbf{E}$ has been neglected in Eq. (1.5), the equation of motion, and so has the displacement current since the magnetic force was evaluated by substituting $\nabla \times \mathbf{B}$ for the plasma current $\mathbf{J}$:

$$\mathbf{J} \times \mathbf{B} = (\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \left(\frac{1}{2} B^2 \right) + \mathbf{B} \cdot \nabla \mathbf{B}\,. \tag{1.10}$$

To justify the neglect of these terms, let us write down the complete expression for the electromagnetic force. (At this point it is convenient to introduce the tensor notation, which we shall use from now on.) Observing the usual convention whereby repeated indices are summed, and making use of all four of Maxwell's equations, we obtain

$$(\mathbf{J} \times \mathbf{B})_i + \eta E_i = -\frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B})_i - \frac{\partial}{\partial x_i} \left(\frac{1}{2} E^2 + \frac{1}{2} B^2 \right) + \frac{\partial}{\partial x_j} (E_i E_j + B_i B_j)\,, \tag{1.11}$$

which is simply the law of conservation of electromagnetic momentum. The terms involving $\mathbf{E}$, since $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$, are of order B^2/ϱ in comparison with the inertial term $\varrho\,(\partial \mathbf{v}/\partial t + \mathbf{v} \cdot \nabla \mathbf{v})$ in Eq. (1.5); and the remaining terms, since $\nabla \cdot \mathbf{B} = 0$, reduce to the expression (1.10). But since B^2/ϱ is simply the Alfvén speed squared, it follows that our system of hydromagnetic equations is correct to lowest order in the ratio of Alfvén speed to light speed.

The assumption of a scalar plasma pressure requires a collision frequency large enough to maintain isotropy, which requirement might appear at first sight to be incompatible with that of infinite conductivity. It can be shown, however, that a physically interesting regime exists in which the collision frequency is large enough to maintain a scalar pressure, and at the same time so small that the electrical resistance is negligible [10]. The best examples of this regime are in the field of astrophysics, where characteristic lengths are much larger than the collision mean free path and characteristic times much shorter than the time scale for resistive diffusion [8]. On the other hand, there are many important terrestrial applications, especially in thermonuclear-reactor technology, where the mean free path is extremely large, so that collisions are negligible and a tensor plasma pressure is allowed. But if, at the same time, the mean Larmor radius is small compared with the characteristic dimensions, the situation is again relatively simple: The pressure tensor has only two independent components. In fact, it has the simple form

$$P_{ij} = P_\perp (\delta_{ij} - \tau_i \tau_j) + P_\parallel \tau_i \tau_j\,, \tag{1.12}$$

where $\boldsymbol{\tau}$ is a unit vector along the field, and where δ_{ij} is the unit tensor:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j\,. \end{cases} \tag{1.13}$$

CHEW, GOLDBERGER, and LOW, the authors of this last result [11], were able, by making the further assumption of zero heat flow, to derive a pair of equations analogous to the adiabatic law (1.6):

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{P_\perp}{\varrho B} \right) = 0\,, \tag{1.14a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{B^2 P_\parallel}{\varrho^3} \right) = 0\,. \tag{1.14b}$$

By using these equations instead of Eq. (1.6) one obtains another complete system of hydromagnetic equations. At the same time, of course, one must replace the pressure gradient in Eq. (1.5) with the divergence of the tensor pressure:

$$\frac{\partial P}{\partial x_i} \to \frac{\partial P_{ij}}{\partial x_j} = \frac{\partial P_\perp}{\partial x_i} + \frac{\partial}{\partial x_j} [\tau_i \tau_j (P_\parallel - P_\perp)]\,. \tag{1.15}$$

There is one more qualification to be made: The assumption of zero heat flow is not a very good one, except when the heat flow is required to vanish on grounds of symmetry, because there is nothing to hinder the free transport of thermal energy along the field [9]. It has nonetheless been possible, without making this assumption, to construct an essentially complete theory of plasma dynamics in the regime we have just described, that of small Larmor radius and large mean free path [10, 12, 13, 14, 15].

Our first task will be to derive Eq. (1.5), the hydromagnetic equation of motion, from a variational principle. Now there are many different variational principles in classical mechanics, and the one we have chosen is Hamilton's principle: The time integral of the Lagrangian is stationary with respect to all variations in the path leaving the initial and final configurations fixed. The equation of motion will appear as the condition for stationarity, and the remaining equations, Eqs. (1.1)—(1.4), will function as holonomic constraints. It will also be possible to treat the tensor-pressure case in accordance with the Chew-Goldberger-Low theory.

2. The variational principle in its Lagrangian form

In the Lagrangian description of a fluid, one considers the position vector $\mathbf{x}$ of a fluid element as a function of time t and initial position $\mathbf{x}_0$:

$$x_i = x_i(\mathbf{x}_0, t); \tag{2.1}$$

and one specifies the initial configuration of the system by giving the mass density, the pressure, and the magnetic field as functions of $\mathbf{x}_0$:

$$\varrho_0 = \varrho_0(\mathbf{x}_0), \tag{2.2}$$

$$P_0 = P_0(\mathbf{x}_0), \tag{2.3}$$

$$B_{0i} = B_{0i}(\mathbf{x}_0). \tag{2.4}$$

These functions are completely arbitrary, except that $\mathbf{B}_0$ must be divergence-free:

$$\frac{\partial B_{0i}}{\partial x_{0i}} = 0. \tag{2.5}$$

We shall write $\dot{x}_i$ for the generalized velocity of the system in configuration space, which is simply the partial derivative of x_i with respect to t. The derivative is taken, of course, with $\mathbf{x}_0$ held fixed and should therefore be identified with the time derivative $\mathrm{d}/\mathrm{d}t$ of Section 1:

$$\dot{x}_i(\mathbf{x}_0, t) = \mathrm{d}x_i/\mathrm{d}t. \tag{2.6}$$

Before proceeding to the variational principle, we must go through a few geometrical preliminaries. Let x_{ij} be an abbreviation for $\partial x_i/\partial x_{0j}$, let J be the Jacobian determinant of $\mathbf{x}$ with respect to $\mathbf{x}_0$, let ε_{ijk} be the unit alternating tensor, and let A_{ij} be the cofactor of x_{ij} in J:

$$x_{ij}(\mathbf{x}_0, t) = \frac{\partial x_i}{\partial x_{0j}}; \tag{2.7}$$

$$J = \det(x_{ij}); \tag{2.8}$$

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if the subscripts form an even permutation of } 1, 2, 3 \\ -1, & \text{if they form an odd permutation} \\ 0, & \text{if any two of them are equal;} \end{cases} \tag{2.9}$$

$$A_{ij} = \frac{1}{2}\varepsilon_{ikl}\,\varepsilon_{jmn}\,x_{km}\,x_{ln}. \tag{2.10}$$

We then have the following identities:

$$J\,\delta_{ij} = A_{ki}\,x_{kj}, \tag{2.11}$$

which is simply the standard rule for the expansion of determinants,

$$\frac{\partial J}{\partial x_{ij}} = A_{ij}, \tag{2.12}$$

which follows from the preceding equation because x_{ij} is not contained in any cofactor A_{kj} of the j^{th} column, and

$$\frac{\partial A_{ij}}{\partial x_{0j}} = 0, \tag{2.13}$$

which is obtained by differentiating Eq. (2.10) and making use of the antisymmetry of ε_{ijk}. Now let d^3x, $\mathrm{d}\sigma_i$, and $\mathrm{d}x_i$ be elements of volume, area, and length moving with the fluid. (The vector $\mathrm{d}\sigma_i$ is equal in magnitude to the area of the surface element, and its direction is along the normal.) The time-dependence of these elements is given by

$$\mathrm{d}^3x = J\mathrm{d}^3x_0, \tag{2.14}$$

$$\mathrm{d}\sigma_i = A_{ij}\,\mathrm{d}\sigma_{0j}, \tag{2.15}$$

$$\mathrm{d}x_i = x_{ij}\,\mathrm{d}x_{0j}. \tag{2.16}$$

The first and third of these relations are obvious, and the second follows from them because of Eq. (2.10): One simply considers the cylindrical volume element $\mathrm{d}^3x = \mathrm{d}\sigma_i\,\mathrm{d}x_i$.

As was pointed out in Section 1, the system is subject to holonomic constraints given by Eqs. (1.1)—(1.3). We can easily integrate these constraints by observing that the following quantities are constants of the motion: the mass enclosed within an arbitrary element of volume, the quantity $P\varrho^{-\gamma}$, and the magnetic flux through an arbitrary element of area. We may therefore write

$$\varrho\,\mathrm{d}^3x = \varrho_0\,\mathrm{d}^3x_0, \tag{2.17}$$

$$P\varrho^{-\gamma} = P_0\,\varrho_0^{-\gamma}, \tag{2.18}$$

$$B_i\,\mathrm{d}\sigma_i = B_{0i}\,\mathrm{d}\sigma_{0i}. \tag{2.19}$$

Using Eqs. (2.11), (2.14), and (2.15), we obtain

$$\varrho = \varrho_0/J, \tag{2.20}$$

$$P = P_0/J^\gamma, \tag{2.21}$$

$$B_i = x_{ij}\,B_{0j}/J. \tag{2.22}$$

We have thus succeeded in expressing ϱ, P, and $\mathbf{B}$ as functionals of $\mathbf{x}(\mathbf{x}_0)$. (The integration of the magnetic-field constraint was first carried out by WALÉN [16].)

Hamilton's variational principle states that a certain integral is stationary for all variations in the path leaving the initial and final configurations fixed:

$$\delta\int_{t_1}^{t_2}\mathrm{d}t\int L\mathrm{d}^3x_0 = 0, \tag{2.23}$$

where the variation satisfies

$$\delta x_i(\mathbf{x}_0, t_1) = \delta x_i(\mathbf{x}_0, t_2) = 0, \tag{2.24}$$

and where L, the Lagrangian density, is some function of the generalized coordinates and velocities. We shall guess the form of L by analogy with classical mechanics, and then prove it to be the correct form by showing that the variational principle leads to the correct equation of motion. In classical mechanics the Lagrangian has the form

$$L = T - V, \tag{2.25}$$

where T and V are generalized kinetic and potential energies. Here T is simply the kinetic energy of mass flow, whereas V is the sum of three terms—the gravitational energy, the internal thermodynamic energy of the fluid, and the magnetic energy:

$$T = \frac{1}{2} \varrho_0 \dot{x}^2, \tag{2.26}$$

$$V = \varrho_0 \left[\varphi + \frac{P}{(\gamma-1)\varrho} + \frac{B^2}{2\varrho} \right]. \tag{2.27}$$

Using Eqs. (2.20)—(2.22), we express the Lagrangian density in terms of x_i, x_{ij}, and $\dot{x}_i$:

$$L = \varrho_0 \left[\frac{1}{2} \dot{x}^2 - \varphi(\mathbf{x}) \right] - \frac{P_0}{(\gamma-1) J^{\gamma-1}} - \frac{1}{2J} x_{ij} x_{ik} B_{0j} B_{0k}. \tag{2.28}$$

Note that L has an explicit dependence on $\mathbf{x}_0$ through the functions ϱ_0, P_0, and $\mathbf{B}_0$.

Carrying out the variation in Eqs. (2.23), integrating by parts, and using the initial and final conditions (2.24), we obtain

$$\int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3 x_0 \, \delta x_i(\mathbf{x}_0, t) \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}_i} \right) + \frac{\partial}{\partial x_{0j}} \left(\frac{\partial L}{\partial x_{ij}} \right) - \frac{\partial L}{\partial x_i} \right] = 0. \tag{2.29}$$

(We are assuming the boundary conditions to be such as to make the surface term vanish in the integration by parts with respect to $\mathbf{x}_0$.) Since δx_i is arbitrary except for the condition (2.24), it is clear that the bracketed expression must vanish identically. In this manner we obtain the hydromagnetic equation of motion in Lagrangian form,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}_i} \right) + \frac{\partial}{\partial x_{0j}} \left(\frac{\partial L}{\partial x_{ij}} \right) - \frac{\partial L}{\partial x_i} = 0, \tag{2.30}$$

or more explicitly,

$$\varrho_0 \left(\ddot{x}_i + \frac{\partial \varphi}{\partial x_i} \right) - B_{0j} \frac{\partial}{\partial x_{0j}} \left(\frac{1}{J} x_{ik} B_{0k} \right) + A_{ij} \frac{\partial}{\partial x_{0j}} \left(P_0 J^{-\gamma} + \frac{1}{2J^2} x_{kl} x_{km} B_{0l} B_{0m} \right) = 0, \tag{2.31}$$

where use has been made of Eqs. (2.5), (2.12), and (2.13). An alternative form of this equation is readily obtained with the help of Eq. (2.11):

$$\left[\varrho_0 \left(\ddot{x}_j + \frac{\partial \varphi}{\partial x_j} \right) - B_{0k} \frac{\partial}{\partial x_{0k}} \left(\frac{1}{J} x_{jl} B_{0l} \right) \right] x_{ji} + J \frac{\partial}{\partial x_{0i}} \left(P_0 J^{-\gamma} + \frac{1}{2J^2} x_{jk} x_{jl} B_{0k} B_{0l} \right) = 0. \tag{2.32}$$

In the more usual Eulerian description of a fluid, the one that was used in Section 1, we look at a fixed point in space instead of following a fluid element. We can therefore transform the equation of motion from Lagrangian to Eulerian form by changing the independent variable from $\mathbf{x}_0$ to $\mathbf{x}$. For this it will be convenient to have another symbol for the fluid velocity when it is expressed as a function of $\mathbf{x}$. We thus distingu[ish] between the Eulerian velocity $\mathbf{v}$ and the Lagrang[ian] velocity $\dot{\mathbf{x}}$:

$$v_i(\mathbf{x}, t) = \dot{x}_i(\mathbf{x}_0, t). \tag{2.33}$$

Let us write down the identity

$$\delta_{ij} = \frac{\partial x_{0i}}{\partial x_k} \frac{\partial x_k}{\partial x_{0j}} = \frac{\partial x_{0i}}{\partial x_k} x_{kj}, \tag{2.34}$$

and compare it with Eq. (2.11). In this manner we obtain the expression for $\partial x_{0i}/\partial x_k$:

$$\frac{\partial x_{0i}}{\partial x_k} = \frac{1}{J} A_{ki}. \tag{2.35}$$

The Eulerian gradient operator is then given by

$$\frac{\partial}{\partial x_k} = \frac{\partial x_{0i}}{\partial x_k} \frac{\partial}{\partial x_{0i}} = \frac{1}{J} A_{ki} \frac{\partial}{\partial x_{0i}}. \tag{2.36}$$

Another useful operator is obtained by combining Eqs. (2.22) and (2.36), and again making use of Eq. (2.11):

$$B_i \frac{\partial}{\partial x_i} = \frac{1}{J^2} x_{ij} A_{ik} B_{0j} \frac{\partial}{\partial x_{0k}} = \frac{1}{J} B_{0k} \frac{\partial}{\partial x_{0k}}. \tag{2.37}$$

With the help of these relations we can easily put Eq. (2.31) into the Eulerian form:

$$\varrho \left(\frac{\mathrm{d}v_i}{\mathrm{d}t} + \frac{\partial \varphi}{\partial x_i} \right) - B_j \frac{\partial B_i}{\partial x_j} + \frac{\partial}{\partial x_i} \left(P + \frac{1}{2} B^2 \right) = 0, \tag{2.38}$$

which is the same as Eq. (1.5). This completes t[he] proof that the variational principle (2.23), with the Lagrangian density (2.28), leads to the correct equation of motion.

3. Extension of the variational principle to fluids of other types

In this section we shall extend the work of the pr[e]ceding section to fluids of various types that do n[ot] obey the ideal-gas law (1.6). Not every possibility will be considered, but only these three: (1) an arbitrary compressible fluid with a scalar pressure, (2) an incompressible fluid, and (3) a collisionless plasma with a tensor pressure governed by the Chew-Goldberger-Low equations. As before, dissipative processes will be neglected.

Starting with the compressible fluid, let us suppose that the flow takes place adiabatically, in which case we have

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0, \quad \text{or } s(\mathbf{x}_0, t) = s_0(\mathbf{x}_0), \tag{3.1}$$

where s is the entropy per unit mass. (This reduces, of course, to Eq. (2.18) when the fluid is an ideal gas.) Now let U be the internal energy per unit mass expressed as a function of s and ϱ^{-1}:

$$U = U(s, \varrho^{-1}). \tag{3.2}$$

The corresponding terms in the Lagrangian density (2.28) and in the Lagrangian equation of motion (2.31) are then modified as follows:

$$-\frac{P_0}{(\gamma-1)J^{\gamma-1}} \rightarrow -\varrho_0 U(s_0, J/\varrho_0), \tag{3.3}$$

$$A_{ij}\frac{\partial}{\partial x_{0j}}(P_0 J^{-\gamma}) \rightarrow -A_{ij}\frac{\partial}{\partial x_{0j}}\left[\frac{\partial U}{\partial (J/\varrho_0)}\right]. \tag{3.4}$$

This is still the pressure gradient, as is evident from the thermodynamic relation

$$P = -\frac{\partial U}{\partial(\varrho^{-1})}. \tag{3.5}$$

Equations (3.1)—(3.5) are applicable, incidentally, to isothermal as well as adiabatic flow, provided that U and s are reinterpreted as the Helmholtz free energy and the temperature.

The above procedure breaks down when the fluid is incompressible, since the partial derivative in Eq. (3.5) is no longer meaningful; but it can easily be fixed up. We have only to drop the internal-energy term from the Lagrangian and treat the incompressibility as an added constraint. The variational principle is then restricted to variations that satisfy this new constraint condition as well as the initial and final conditions (2.24). The constraint condition is simply

$$J = 1, \tag{3.6}$$

and the variational principle may be written as follows:

$$\delta\int_{t_1}^{t_2} dt \int d^3x_0 [L_1 + \lambda(\mathbf{x}_0, t) J] = 0, \tag{3.7}$$

where λ is an undetermined multiplier and L_1 is the Lagrangian density without the internal-energy term. The Lagrangian equation of motion is then

$$\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{x}_i}\right) + \frac{\partial}{\partial x_{0j}}\left(\frac{\partial L_1}{\partial x_{ij}}\right) - \frac{\partial L_1}{\partial x_i} + A_{ij}\frac{\partial \lambda}{\partial x_{0j}} = 0, \tag{3.8}$$

from which we see that λ is simply the pressure. Equations (3.6) and (3.8) together form a complete system.

We shall now treat the tensor-pressure case. Let $\boldsymbol{\tau}$ be the unit vector along $\mathbf{B}$, and let Δ be defined as follows:

$$\Delta = x_{ij}\, x_{ik}\, \tau_{0j}\, \tau_{0k}. \tag{3.9}$$

Using Eqs. (2.20) and (2.22), we can solve Eqs. (1.14) for $P_\perp$ and $P_\parallel$, obtaining two equations in the place of Eq. (2.21):

$$P_\perp = \frac{\sqrt{\Delta}}{J^2} P_{\perp 0}, \tag{3.10a}$$

$$P_\parallel = \frac{1}{J\Delta} P_{\parallel 0}. \tag{3.10b}$$

It will also be convenient to have an expression for the vector $\boldsymbol{\tau}$:

$$\tau_i = \frac{1}{\sqrt{\Delta}} x_{ij}\, \tau_{0j}, \tag{3.11}$$

which is readily obtained from Eq. (2.22). The internal energy per unit mass is given by

$$U = \frac{1}{\varrho}\left(P_\perp + \frac{1}{2}P_\parallel\right), \tag{3.12}$$

from which we obtain the Lagrangian density

$$L = \varrho_0\left[\frac{1}{2}\dot{x}^2 - \varphi(\mathbf{x})\right] - \frac{1}{2J} x_{ij}\, x_{ik}\, B_{0j}\, B_{0k} - \frac{\sqrt{\Delta}}{J} P_{\perp 0} - \frac{1}{2\Delta} P_{\parallel 0}. \tag{3.13}$$

The Lagrangian equation of motion again has the general form (2.30), which reduces in the present case to

$$\varrho_0\left(\ddot{x}_i + \frac{\partial\varphi}{\partial x_i}\right) - B_{0j}\frac{\partial}{\partial x_{0j}}\left(\frac{1}{J} x_{ik} B_{0k}\right) + A_{ij}\frac{\partial}{\partial x_{0j}}\left[\frac{\sqrt{\Delta}}{J^2} P_{\perp 0} + \frac{\Delta}{2J^2} B_0^2\right] + \frac{\partial}{\partial x_{0j}}\left[x_{ik}\, \tau_{0j}\, \tau_{0k}\left(\frac{1}{\Delta^2} P_{\parallel 0} - \frac{1}{J\sqrt{\Delta}} P_{\perp 0}\right)\right] = 0. \tag{3.14}$$

Let us make use of Eqs. (2.11) and (2.13) to write the last term in the more symmetrical form

$$A_{jm}\frac{\partial}{\partial x_{0m}}\left[\frac{1}{\Delta} x_{ik}\, x_{jl}\, \tau_{0k}\, \tau_{0l}\left(\frac{1}{J\Delta} P_{\parallel 0} - \frac{\sqrt{\Delta}}{J^2} P_{\perp 0}\right)\right], \tag{3.15}$$

from which the Eulerian equation of motion is obtainable by inspection. It is simply

$$\varrho\left(\frac{dv_i}{dt} + \frac{\partial\varphi}{\partial x_i}\right) - B_j\frac{\partial B_i}{\partial x_j} + \frac{\partial}{\partial x_i}\left(P_\perp + \frac{1}{2}B^2\right) + \frac{\partial}{\partial x_j}\left[\tau_i\, \tau_j (P_\parallel - P_\perp)\right] = 0, \tag{3.16}$$

which agrees with Eq. (1.15).

4. The variational principle in its Eulerian form

In this section we shall state the variational principle in a form that will lead directly to the Eulerian equation of motion, restricting ourselves to the scalar-pressure case treated in Section 2. By analogy with Eq. (2.33), we introduce the Eulerian virtual displacement $\boldsymbol{\varepsilon}$:

$$\varepsilon_i(\mathbf{x}, t) = \delta x_i(\mathbf{x}_0, t). \tag{4.1}$$

We change the variables of integration from $\mathbf{x}_0$, t to $\mathbf{x}$, t; and we require the integral of the Lagrangian to be stationary for every Eulerian displacement leaving the initial and final configurations fixed. Thus,

$$\delta\int_{t_1}^{t_2} dt \int d^3x \left(\frac{1}{2}\varrho v^2 - \varrho\varphi - \frac{P}{\gamma-1} - \frac{1}{2}B^2\right) = 0, \tag{4.2}$$

where the variation satisfies

$$\varepsilon_i(\mathbf{x}, t_1) = \varepsilon_i(\mathbf{x}, t_2) = 0. \tag{4.3}$$

The next step, of course, is to express the variations in ϱ, P, $\mathbf{B}$ and $\mathbf{v}$ in terms of the Eulerian displacement.

Holding $\mathbf{x}_0$ fixed, let us take the variation of

Eq. (2.33) and the time derivative of Eq. (4.1). The results are

$$\delta v_i + \frac{\partial v_i}{\partial x_j} \varepsilon_j = \delta \dot{x}_i , \tag{4.4}$$

$$\frac{\partial \varepsilon_i}{\partial t} + v_j \frac{\partial \varepsilon_i}{\partial x_j} = \delta \dot{x}_i . \tag{4.5}$$

Returning now to the vector notation, we obtain the expression for $\delta \mathbf{v}$ as a functional of $\boldsymbol{\epsilon}$:

$$\delta \mathbf{v} = \frac{\partial \boldsymbol{\epsilon}}{\partial t} + \mathbf{v} \cdot \nabla \boldsymbol{\epsilon} - \boldsymbol{\epsilon} \cdot \nabla \mathbf{v} . \tag{4.6}$$

The corresponding expressions for $\delta \varrho$, δP, and $\delta \mathbf{B}$ are obvious:

$$\delta \varrho = -\nabla \cdot (\varrho \boldsymbol{\epsilon}) , \tag{4.7}$$

$$\delta P = -\gamma P \nabla \cdot \boldsymbol{\epsilon} - \boldsymbol{\epsilon} \cdot \nabla P , \tag{4.8}$$

$$\delta \mathbf{B} = \nabla \times (\boldsymbol{\epsilon} \times \mathbf{B}) . \tag{4.9}$$

The variational principle now takes on the explicit form

$$\begin{aligned}\int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3 x \Big[\varrho \mathbf{v} \cdot \delta \mathbf{v} + \left(\tfrac{1}{2} v^2 - \varphi\right) \delta \varrho - \frac{\delta P}{\gamma - 1} - \mathbf{B} \cdot \delta \mathbf{B}\Big] \\ = \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3 x \Big[\varrho \mathbf{v} \cdot \frac{\partial \boldsymbol{\epsilon}}{\partial t} + \varrho \mathbf{v} \cdot \nabla \boldsymbol{\epsilon} \cdot \mathbf{v} - \varrho \boldsymbol{\epsilon} \cdot \nabla \left(\tfrac{1}{2} v^2\right) \\ - \left(\tfrac{1}{2} v^2 - \varphi\right) \nabla \cdot (\varrho \boldsymbol{\epsilon}) + \frac{\gamma P}{\gamma - 1} \nabla \cdot \boldsymbol{\epsilon} \\ + \frac{1}{\gamma - 1} \boldsymbol{\epsilon} \cdot \nabla P - \mathbf{B} \cdot \nabla \times (\boldsymbol{\epsilon} \times \mathbf{B})\Big] = 0 . \end{aligned} \tag{4.10}$$

Because of the initial and final conditions (4.3), we can integrate by parts to obtain

$$\begin{aligned}\int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3 x \, \boldsymbol{\epsilon}(\mathbf{x}, t) \cdot \Big[-\frac{\partial}{\partial t} (\varrho \mathbf{v}) - \nabla \cdot (\varrho \mathbf{v} \mathbf{v}) \\ - \varrho \nabla \varphi - \nabla P + (\nabla \times \mathbf{B}) \times \mathbf{B}\Big] = 0 . \end{aligned} \tag{4.11}$$

The integral vanishes for arbitrary $\boldsymbol{\epsilon}$ if and only if the bracketed expression vanishes identically. We thus obtain the Eulerian equation of motion in the form

$$\frac{\partial}{\partial t} (\varrho \mathbf{v}) + \nabla \cdot (\varrho \mathbf{v} \mathbf{v}) + \varrho \nabla \varphi + \nabla P - (\nabla \times \mathbf{B}) \times \mathbf{B} = 0 , \tag{4.12}$$

which is obviously equivalent to Eq. (1.5).

5. The Hamiltonian equations of motion

We now return to the Lagrangian description and introduce the generalized momentum conjugate to $\mathbf{x}$:

$$\pi_i (\mathbf{x}_0, t) = \frac{\partial L}{\partial \dot{x}_i} = \varrho_0 \dot{x}_i . \tag{5.1}$$

The Hamiltonian density, a functional of $\mathbf{x}$ and $\boldsymbol{\pi}$, is then defined in the usual manner:

$$H = \pi_i \dot{x}_i - L . \tag{5.2}$$

This reduces with a scalar pressure to

$$H = \frac{\pi^2}{2 \varrho_0} + \varrho_0 \varphi(\mathbf{x}) + \frac{P_0}{(\gamma - 1) J^{\gamma - 1}} + \frac{1}{2J} x_{ij} x_{ik} B_{0j} B_{0k} , \tag{5.3}$$

and with a tensor pressure to

$$\begin{aligned} H = \frac{\pi^2}{2 \varrho_0} + \varrho_0 \varphi(\mathbf{x}) + \frac{\sqrt{\Delta}}{J} P_{\perp 0} + \frac{1}{2 \Delta} P_{\parallel 0} \\ + \frac{1}{2J} x_{ij} x_{ik} B_{0j} B_{0k} . \end{aligned} \tag{5.4}$$

The Hamiltonian equations of motions are simply

$$\dot{\pi}_i = -\frac{\partial H}{\partial x_i} + \frac{\partial}{\partial x_{0j}} \left(\frac{\partial H}{\partial x_{ij}}\right) , \tag{5.5a}$$

$$\dot{x}_i = \frac{\partial H}{\partial \pi_i} . \tag{5.5b}$$

(They would, of course, be completely symmetrical if H depended on the gradient of $\boldsymbol{\pi}$.) We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int H \, \mathrm{d}^3 x_0 &= \int \mathrm{d}^3 x_0 \left(\frac{\partial H}{\partial x_i} \dot{x}_i + \frac{\partial H}{\partial x_{ij}} \dot{x}_{ij} + \frac{\partial H}{\partial \pi_i} \dot{\pi}_i\right) \\ &= \int \mathrm{d}^3 x_0 \left\{\left[\frac{\partial H}{\partial x_i} - \frac{\partial}{\partial x_{0j}} \left(\frac{\partial H}{\partial x_{ij}}\right)\right] \dot{x}_i + \frac{\partial H}{\partial \pi_i} \dot{\pi}_i\right\} \\ &= \int \mathrm{d}^3 x_0 (-\dot{\pi}_i \dot{x}_i + \dot{x}_i \dot{\pi}_i) = 0 . \end{aligned} \tag{ }$$

Thus the integrated Hamiltonian is a constant of motion.

6. The hydromagnetic energy principle

Let us now suppose that the fluid is in a static equilibrium state (one where $\dot{\mathbf{x}}$ and $\dot{\boldsymbol{\pi}}$ vanish identically) and ask whether the equilibrium is stable. The conditions for equilibrium are obtained directly from the Hamiltonian equations of motion. The first of these equations gives

$$\left[\frac{\partial H}{\partial x_i} - \frac{\partial}{\partial x_{0j}} \left(\frac{\partial H}{\partial x_{ij}}\right)\right]_{\mathbf{x} = \mathbf{x}_0} = 0 , \tag{6.1}$$

which reduces with a scalar pressure to

$$\varrho_0 \left(\frac{\partial \varphi}{\partial x_i}\right)_{\mathbf{x} = \mathbf{x}_0} - B_{0j} \frac{\partial B_{0i}}{\partial x_{0j}} + \frac{\partial}{\partial x_{0i}} \left(P_0 + \tfrac{1}{2} B_0^2\right) = 0 , \tag{6.2}$$

and with a tensor pressure to

$$\begin{aligned} \varrho_0 \left(\frac{\partial \varphi}{\partial x_i}\right)_{\mathbf{x} = \mathbf{x}_0} - B_{0j} \frac{\partial B_{0i}}{\partial x_{0j}} + \frac{\partial}{\partial x_{0i}} \left(P_{\perp 0} + \tfrac{1}{2} B_0^2\right) \\ + \frac{\partial}{\partial x_{0j}} [\tau_{0i} \tau_{0j} (P_{\parallel 0} - P_{\perp 0})] = 0 . \end{aligned} \tag{6.3}$$

The other Hamiltonian equation gives only a trivial result, $\boldsymbol{\pi} = 0$.

To get a stability criterion it is necessary to examine the behavior of small displacements from the equilibrium state. We accordingly write

$$x_i = x_{0i} + \xi_i , \tag{6.4}$$

treat $\boldsymbol{\pi}$ and $\boldsymbol{\xi}$ as small quantities, and expand the Hamiltonian out to second order. The zero-order part will be dropped, since it has no effect on the Hamiltonian equations of motion; and the first-order part, because of the equilibrium condition, will vanish when integrated over $\mathrm{d}^3 x_0$. The second-order part is obviously the sum of two terms, a kinetic energy depending only on π, and a potential energy depending only on $\boldsymbol{\xi}$:

$$\int H\,\mathrm{d}^3 x_0 = \frac{1}{2}\int \frac{\pi^2}{\varrho_0}\,\mathrm{d}^3 x_0 + W(\boldsymbol{\xi},\boldsymbol{\xi})\,. \tag{6.5}$$

Each term is a quadratic form in its respective variables, and the kinetic energy is obviously positive-definite. Under these conditions, the equilibrium is stable if and only if the potential energy is also positive-definite (i.e., positive for any nontrivial choice of $\boldsymbol{\xi}$ as a function of $\mathbf{x}_0$.) This result, which is due to BERNSTEIN, FRIEMAN, KRUSKAL and KULSRUD [4], is known as the hydromagnetic energy principle.

To find specific expressions for W we write

$$x_{ij} = \delta_{ij} + \xi_{ij}\,, \tag{6.6}$$

$$J = 1 + \xi_{ii} + \frac{1}{2}(\xi_{ii})^2 - \frac{1}{2}\xi_{ij}\xi_{ji}\,, \tag{6.7}$$

$$\Delta = 1 + 2\tau_{0i}\tau_{0j}\xi_{ij} + \tau_{0j}\tau_{0k}\xi_{ij}\xi_{ik}\,. \tag{6.8}$$

Substituting these formulas into Eqs. (5.3) and (5.4) we obtain, in the case of a scalar pressure.

$$W = \frac{1}{2}\int \mathrm{d}^3 x_0 \Big\{ \varrho_0 \Big(\frac{\partial^2 \varphi}{\partial x_i\,\partial x_j}\Big)_{\mathbf{x}=\mathbf{x}_0} \xi_i \xi_j + \Big(P_0 + \frac{1}{2}B_0^2\Big)[\xi_{ij}\xi_{ji} - (\xi_{ii})^2] + (\gamma P_0 + B_0^2)(\xi_{ii})^2 + B_{0j}B_{0k}(\xi_{ij}\xi_{ik} - 2\xi_{ii}\xi_{jk}) \Big\}\,, \tag{6.9}$$

and in that of a tensor pressure,

$$W = \frac{1}{2}\int \mathrm{d}^3 x_0 \Big\{ \varrho_0 \Big(\frac{\partial^2 \varphi}{\partial x_i\,\partial x_j}\Big)_{\mathbf{x}=\mathbf{x}_0} \xi_i \xi_j + \Big(P_{\perp 0} + \frac{1}{2}B_0^2\Big)[\xi_{ij}\xi_{ji} + (\xi_{ii})^2] + (B_0^2 + P_{\perp 0} - P_{\parallel 0})\tau_{0j}\tau_{0k}\xi_{ij}\xi_{ik} - 2(P_{\perp 0} + B_0^2) \times \tau_{0j}\tau_{0k}\xi_{jk}\xi_{ii} + (4P_{\parallel 0} - P_{\perp 0})(\tau_{0i}\tau_{0k}\xi_{ik})^2 \Big\}. \tag{6.10}$$

Expressions of this type for the energy integrals were first given by HAIN, LÜST, and SCHLÜTER [5], but without gravity, and with $P_{\perp 0} = P_{\parallel 0}$. (There is a mistake, though, in their version of Eq. (6.10): They have, as the coefficient of $(\xi_{ii})^2$, $\frac{1}{2}B_0^2$ alone instead of $P_{\perp 0} + \frac{1}{2}B_0^2$.) One can, by a rather long and tedious calculation, transform the expressions (6.9) and (6.10) into the more familiar ones given by BERNSTEIN *et al.* [4].

Suppose that the plasma pressure is isotropic in the equilibrium state ($P_{\perp 0} = P_{\parallel 0}$). Unless the collision frequency is large, this does not mean that the isotropy will be maintained by a small displacement away from equilibrium; and for this reason we need not expect the two energy integrals (6.9) and (6.10) to be equal in this case. In fact, if we set γ equal to 5/3, the appropriate value for a monoatomic gas, then the tensor-pressure integral will always be larger, and by the amount

$$\frac{1}{2}\int \mathrm{d}^3 x_0 \frac{1}{3} P_0 (\xi_{ii} - 3\tau_{0i}\tau_{0j}\xi_{ij})^2\,. \tag{6.11}$$

Thus the tensor-pressure energy principle gives a less stringent stability criterion, when the equilibrium pressure is a scalar, than the scalar-pressure energy principle. This result, which again is due to BERNSTEIN *et al.* [4], may be interpreted as follows: Take two equilibrium states, each with a scalar pressure, and differing only in scale. Suppose that their characteristic dimensions are, respectively, large and small compared with the collision mean free path. Then the large system will conform to the scalar-pressure theory, and the small one to the Chew-Goldberger-Low. We may conclude that if the large system is stable, then so is the small. This conclusion, by the way, does not depend on the assumption of zero heat flow in the small system; it remains valid when the Chew-Goldberger-Low theory is replaced by the more elaborate one mentioned in Section 1 [10, 14, 15].

The question of stability is not restricted to static equilibria, but arises in connection with steady flows also. Now in the Eulerian description a steady flow is obviously independent of time; but in the Lagrangian description this is not so, since the position of each fluid element is changing. For this reason the energy principle is not usually applicable to steady flows, and the stability can only be determined by a calculation of the characteristic frequencies. (The flow is unstable if any of these frequencies have imaginary parts corresponding to exponential growth.) We shall see, however, that there are cases in which a steady-flow problem can be reduced to one of static equilibrium; and in those cases the stability criterion will be given by a modified energy principle.

7. Flow with azimuthal symmetry

From now on we shall devote our attention to flows with an axis of symmetry, neglecting gravity ($\varphi = 0$) and considering only the scalar-pressure theory. Introducing cylindrical coordinates r, θ, z, we have

$$\frac{\partial r}{\partial \theta_0} = \frac{\partial z}{\partial \theta_0} = 0\,; \quad \frac{\partial \theta}{\partial \theta_0} = 1\,. \tag{7.1}$$

With this type of flow we can think of the fluid as composed of a doubly infinite set of ring elements encircling the z axis. Each of these elements is identified by its initial coordinates r_0 and z_0, and its instantaneous position is specified by giving the three coordinates r, $\theta - \theta_0$, z as functions of r_0 and z_0. (In order to keep the notation simple, let us write θ instead of $\theta - \theta_0$ from now on, so that θ will be the angular displacement of the ring element as a whole.)

For the initial volume element we now have

$$\mathrm{d}^3 x_0 = 2\pi r_0\,\mathrm{d}r_0\,\mathrm{d}z_0\,. \tag{7.2}$$

Consequently, if we again define J as the ratio of $\mathrm{d}^3 x$ to $\mathrm{d}^3 x_0$, we obtain

$$J = \frac{r}{r_0}\Big(\frac{\partial r}{\partial r_0}\frac{\partial z}{\partial z_0} - \frac{\partial r}{\partial z_0}\frac{\partial z}{\partial r_0}\Big)\,. \tag{7.3}$$

The mass density ϱ and the pressure P are still determined by Eqs. (2.20) and (2.21), but with the new expression for J. The magnetic-field components are as follows:

$$B_r = \frac{1}{J}\left(\frac{\partial r}{\partial r_0} B_{0r} + \frac{\partial r}{\partial z_0} B_{0z}\right), \tag{7.4 a}$$

$$B_\theta = \frac{r}{J}\left(\frac{\partial \theta}{\partial r_0} B_{0r} + \frac{1}{r_0} B_{0\theta} + \frac{\partial \theta}{\partial z_0} B_{0z}\right), \tag{7.4 b}$$

$$B_z = \frac{1}{J}\left(\frac{\partial z}{\partial r_0} B_{0r} + \frac{\partial z}{\partial z_0} B_{0z}\right), \tag{7.4 c}$$

in which the initial values must satisfy the divergence relation

$$\frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0 B_{0r}) + \frac{\partial B_{0z}}{\partial z_0} = 0. \tag{7.5}$$

Next we have the potential-energy density

$$V = \frac{P_0}{(\gamma-1)J^{\gamma-1}} + \frac{1}{2J}\left[\left(\frac{\partial r}{\partial r_0} B_{0r} + \frac{\partial r}{\partial z_0} B_{0z}\right)^2 + r^2\left(\frac{\partial \theta}{\partial r_0} B_{0r} + \frac{1}{r_0} B_{0\theta} + \frac{\partial \theta}{\partial z_0} B_{0z}\right)^2 + \left(\frac{\partial z}{\partial r_0} B_{0r} + \frac{\partial z}{\partial z_0} B_{0z}\right)^2\right], \tag{7.6}$$

the Lagrangian density

$$L = \frac{1}{2}\varrho_0(\dot r^2 + r^2\dot\theta^2 + \dot z^2) - V, \tag{7.7}$$

and the Lagrangian equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) + \frac{1}{r_0}\frac{\partial}{\partial r_0}\left(r_0 \frac{\partial L}{\partial(\partial q/\partial r_0)}\right) + \frac{\partial}{\partial z_0}\left(\frac{\partial L}{\partial(\partial q/\partial z_0)}\right) - \frac{\partial L}{\partial q} = 0, \tag{7.8}$$

where q is any one of the three coordinates r, θ, z. Finally, we have the Hamiltonian density

$$H = \frac{1}{2\varrho_0}\left(\pi_r^2 + \frac{1}{r^2}\pi_\theta^2 + \pi_z^2\right) + V, \tag{7.9}$$

in which the canonical momenta are given by

$$\pi_r = \partial L/\partial \dot r = \varrho_0 \dot r, \tag{7.10 a}$$

$$\pi_\theta = \partial L/\partial \dot\theta = \varrho_0 r^2 \dot\theta, \tag{7.10 b}$$

$$\pi_z = \partial L/\partial \dot z = \varrho_0 \dot z. \tag{7.10 c}$$

In the remaining sections we shall consider purely azimuthal steady flows with the magnetic field either purely toroidal or purely poloidal, and investigate their stability against displacements that preserve the azimuthal symmetry ($m=0$ displacements in the usual notation). In both cases (the toroidal and the poloidal) it will be shown that the stability criterion is given by a modified energy principle.

8. Stability of steady azimuthal flow with a toroidal field

Let us now consider an azimuthally symmetric flow, not necessarily a steady one, with $B_{0r} = B_{0z} = 0$, i.e., with a purely toroidal initial field. (Of course, the field will also be purely toroidal at all future times.) For the Lagrangian density in this case we have simply

$$L = \frac{1}{2}\varrho_0(\dot r^2 + r^2\dot\theta^2 + \dot z^2) - \frac{P_0}{(\gamma-1)J^{\gamma-1}} - \frac{r^2}{2r_0^2 J} B_{0\theta}^2, \tag{8.1}$$

which does not contain θ or any of its spatial derivatives. This means that θ is an ignorable coordinate, and that its conjugate momentum is a constant of the motion:

$$\dot\pi_\theta = \frac{d}{dt}(\varrho_0 r^2 \dot\theta) = 0, \tag{8.2}$$

which is simply the conservation of angular momentum for the individual ring elements.

We can eliminate the ignorable coordinate θ with the help of a new function R, the Routhian:

$$R = L - \pi_\theta\dot\theta = \frac{1}{2}\varrho_0(\dot r^2 + \dot z^2) - \frac{1}{2}\frac{\pi_\theta^2}{\varrho_0 r^2} - \frac{P_0}{(\gamma-1)J^{\gamma-1}} - \frac{r^2 B_{0\theta}^2}{2r_0^2 J}. \tag{8.3}$$

By using R as a Lagrangian density (ROUTH's procedure [17]) one obtains equations of motion for the remaining coordinates r and z, the θ motion entering only through the constant π_θ. (Note that in effect we are simply counting the azimuthal kinetic energy as part of the potential energy.)

It is now easy to derive the stability criterion for a steady flow with $v_r = v_z = 0$. With this type of steady flow the only time-dependent coordinate is θ; and since θ has been eliminated, the motion is effectively independent of time in the Lagrangian as well as in the Eulerian description. Furthermore, the Routhian function (8.3) is the sum of two terms, one positive-definite and quadratic in the velocities, and the other depending only on the coordinates. In other words, the Routhian for small displacements from the steady flow has the same form as the Lagrangian for small displacements from a static equilibrium. The steady-flow problem has thus been reduced to one of static equilibrium, and the stability criterion will necessarily be given by an energy principle. The only difference is that the energy integral will now include a centrifugal potential energy.

The procedure described above is an extension of one that is familiar in classical mechanics [18], where it is used to investigate the stability of such steady motions as the regular precession of a top. It should be pointed out, though, that a reduction to static equilibrium cannot always be achieved, since the elimination of ignorable coordinates will often lead to a Routhian with some of its terms linear in the velocities. This occurs, for example, in the problem of the sleeping top.

The equilibrium conditions (which are readily obtained from Eq. (7.8) with R in the place of L) may be reduced to the following:

$$P_0 + \frac{1}{2}B_{0\theta}^2 = F(r_0), \tag{8.4 a}$$

$$B_{0\theta}^2 - \frac{\pi_\theta^2}{\varrho_0 r_0^2} = G(r_0), \tag{8.4 b}$$

where F and G must satisfy

$$G(r_0) + r_0 F'(r_0) = 0. \tag{8.5}$$

Although the combinations F and G can only depend on r_0, we note that the individual terms may depend on z_0 as well.

To describe the small displacements we write

$$r = r_0 + \xi, \tag{8.6 a}$$

$$z = z_0 + \varsigma, \tag{8.6 b}$$

and, as in the static-equilibrium case, we expand the potential energy out to second order. The result, if the subscript zeroes are dropped, is

$$W = \pi \int r\,\mathrm{d}r\,\mathrm{d}z \Big[\frac{3\pi_\theta^2}{\varrho}\frac{\xi^2}{r^4} + (\gamma P + B_\theta^2)\,\mathrm{div}^2 - 2\left(P + \frac{1}{2}B_\theta^2\right)\left(\frac{\xi}{r}\frac{\partial \xi}{\partial r} + \frac{\xi}{r}\frac{\partial \zeta}{\partial z} + \frac{\partial \xi}{\partial r}\frac{\partial \zeta}{\partial z} - \frac{\partial \xi}{\partial z}\frac{\partial \zeta}{\partial r}\right) - B_\theta^2\left(\frac{2\xi}{r}\frac{\partial \xi}{\partial r} + \frac{2\xi}{r}\frac{\partial \zeta}{\partial z} + \frac{\xi^2}{r^2}\right)\Big], \tag{8.7}$$

where div is simply the divergence of the vector displacement:

$$\mathrm{div} = \frac{\xi}{r} + \frac{\partial \xi}{\partial r} + \frac{\partial \zeta}{\partial z}. \tag{8.8}$$

The necessary and sufficient condition for stability against azimuthally symmetric displacements is that this integral be positive for every ξ and ζ.

After integrating various terms by parts, and making use of the equilibrium conditions (8.4) and (8.5), we obtain

$$W = \pi \int r\,\mathrm{d}r\,\mathrm{d}z \Big\{ A\,\xi^2 + (\gamma P + B_\theta^2) \Big[\mathrm{div} - \frac{(2r^2 B_\theta^2 - \pi_\theta^2/\varrho)\,\xi}{(\gamma P + B_\theta^2)\,r^3}\Big]^2 \Big\}, \tag{8.9}$$

where A is a certain function of the equilibrium quantities and their derivatives. We shall prove that W is positive-definite if and only if A is positive for all values of r and z. The "if" part of this statement is obvious; and to prove the "only if" part, let us assume that A is negative at some point and construct a displacement for which W is negative. If at some point we have $A<0$, then, by continuity, that point is surrounded by an entire region Ω in the r, z plane such that A is negative at every point of Ω. Let us pick a ξ that vanishes outside of Ω and that satisfies the following condition for all values of r:

$$\int_{-\infty}^{\infty} \mathrm{d}z \Big[\frac{\partial \xi}{\partial r} + \frac{\xi}{r} - \frac{[2r^2 B_\theta^2 - \pi_\theta^2/\varrho]\,\xi}{(\gamma P + B_\theta^2)\,r^3}\Big] = 0. \tag{8.10}$$

(It is clear that these two conditions on ξ are compatible.) Comparing Eqs. (8.8) and (8.10), we see that ζ can be chosen so as to make the bracketed term in W vanish. But then, since ξ^2 vanishes wherever A is non-negative, we have $W<0$ for this particular choice of ξ and ζ, which is what we set out to prove. The stability criterion, then, is given by the following inequality, the left-hand side of which is simply the function A written out explicitly:

$$\frac{1}{r^3}\frac{\partial}{\partial r}\left(\frac{\pi_\theta^2}{\varrho}\right) - \frac{2}{r}B_\theta\frac{\partial B_\theta}{\partial r} + \frac{2B_\theta^2}{r^2} - \frac{(2r^2 B_\theta^2 - \pi_\theta^2/\varrho)^2}{r^6(\gamma P + B_\theta^2)} > 0. \tag{8.11}$$

The flow is stable if and only if this inequality holds at every point in the fluid.

Let us briefly consider two special cases, that of magnetostatic equilibrium, where $\pi_\theta = 0$, and that of pure hydrodynamics, where $B_\theta = 0$. (Note that the equilibrium quantities, although they may have a z dependence in the general case, can only depend on r in these special cases.) In the first case ($\pi_\theta = 0$) the stability criterion reduces to

$$\frac{\mathrm{d}\log B_\theta}{\mathrm{d}\log r} < \frac{\gamma P - B_\theta^2}{\gamma P + B_\theta^2}, \tag{8.12}$$

which was first derived by Tserkovnikov [19]; and in the second ($B_\theta = 0$) it reduces to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\pi_\theta^2}{\varrho}\right) > \frac{\pi_\theta^4}{\gamma P \varrho^2 r^3}. \tag{8.13}$$

For an incompressible fluid we may take γ infinite and obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\pi_\theta^2}{\varrho}\right) > 0, \tag{8.14}$$

which will be recognized as Rayleigh's condition for the stability of Couette flow [7]. In its essentials, Lord Rayleigh's derivation of this stability condition was equivalent to the use of an energy principle; and for this reason the material presented here may be regarded as an extension of his work.

9. Stability of steady azimuthal flow with a poloidal field

We shall now consider flows in which the magnetic field is initially poloidal ($B_{0\theta} = 0$). (Note that the field will not, in general, remain poloidal unless the flow is steady, since a nonvanishing B_θ is induced whenever the angular velocity varies from point to point on a single field line.) It will again be possible to reduce the steady-flow problem to one of static equilibrium, but this time the reduction will be carried through by means of a contact transformation. It is therefore appropriate to start with the Hamiltonian, which is now given by

$$H = \frac{1}{2\varrho_0}\left[\pi_r^2 + \frac{1}{r^2}\pi_\theta^2 + \pi_z^2\right] + \frac{P_0}{(\gamma - 1)J^{\gamma-1}} + \frac{1}{2J}\left[\left(\frac{\partial r}{\partial r_0}B_{0r} + \frac{\partial r}{\partial z_0}B_{0z}\right)^2 + r^2\left(\frac{\partial \theta}{\partial r_0}B_{0r} + \frac{\partial \theta}{\partial z_0}B_{0z}\right)^2 + \left(\frac{\partial z}{\partial r_0}B_{0r} + \frac{\partial z}{\partial z_0}B_{0z}\right)^2\right]. \tag{9.1}$$

The coordinate θ is no longer ignorable, since it enters the Hamiltonian through its derivatives with respect to r_0 and z_0; and the conjugate momentum π_θ is no longer a constant of the motion. There is, however, an integral of π_θ that is constant. Let B_0

be the magnitude of the initial field, and let $\mathrm{d}l_0$ be the element of arc length in the r_0, z_0 plane. Then, taking the integral of π_θ/B_0 along a magnetic field line (which we assume to be closed), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint \frac{\pi_\theta\,\mathrm{d}l_0}{B_0} = \oint \frac{\mathrm{d}l_0}{B_0}\left[\frac{1}{r_0}\frac{\partial}{\partial r_0}\left(r_0\frac{\partial H}{\partial(\partial\theta/\partial r_0)}\right) + \frac{\partial}{\partial z_0}\left(\frac{\partial H}{\partial(\partial\theta/\partial z_0)}\right)\right]$$

$$= \oint \mathrm{d}\left[\frac{r^2}{J}\left(\frac{\partial\theta}{\partial r_0}B_{0r} + \frac{\partial\theta}{\partial z_0}B_{0z}\right)\right] = 0, \quad (9.2)$$

where use has been made of Eq. (7.5). Let us consider the infinitely thin shell between two neighboring flux surfaces. Then, according to Eq. (9.2), the total angular momentum of the fluid within the shell is a constant of the motion. The angular momentum can, however, be transferred by the field from one ring element to another in the same shell, so that an individual ring element need not have a constant angular momentum.

From our present viewpoint there are two respects in which a purely azimuthal steady flow differs from a static equilibrium: The coordinate θ depends on time, and the momentum π_θ does not vanish. What we shall try to do, therefore, is find a contact transformation from the variables θ, π_θ to new variables β, π_β such that $\dot\beta$ and π_β both vanish when the flow is steady. Let us first note that the field-lines in a steady flow must rotate as rigid bodies (the law of isorotation [20, 21]), which enables us to write

$$B_{0r}\frac{\partial\theta}{\partial r_0} + B_{0z}\frac{\partial\theta}{\partial z_0} = 0. \quad (9.3)$$

Now the Hamiltonian has a term proportional to the square of this expression, an expression that has just been seen to vanish when the flow is steady. It is strongly suggested, therefore, that π_β is proportional to that expression. As for the conjugate variable β, we may expect it to be some functional of π_θ, since π_β is a functional of θ; and, since $\dot\pi_\theta$ vanishes when the flow is steady, so will $\dot\beta$.

We now give the generating function of a contact transformation that will lead to variables β, π_β with the desired properties:

$$S(\theta,\beta) = -\beta\, r_0\sqrt{\varrho_0}\left(B_{0r}\frac{\partial\theta}{\partial r_0} + B_{0z}\frac{\partial\theta}{\partial z_0}\right). \quad (9.4)$$

The conjugate momenta are derived from the generating function in the usual manner:

$$\pi_\theta = -\frac{1}{r_0}\frac{\partial}{\partial r_0}\left(r_0\frac{\partial S}{\partial(\partial\theta/\partial r_0)}\right) - \frac{\partial}{\partial z_0}\left(\frac{\partial S}{\partial(\partial\theta/\partial z_0)}\right)$$

$$= B_{0r}\frac{\partial}{\partial r_0}(\beta r_0\sqrt{\varrho_0}) + B_{0z}\frac{\partial}{\partial z_0}(\beta r_0\sqrt{\varrho_0}), \quad (9.5\mathrm{a})$$

$$\pi_\beta = -\frac{\partial S}{\partial\beta} = r_0\sqrt{\varrho_0}\left(B_{0r}\frac{\partial\theta}{\partial r_0} + B_{0z}\frac{\partial\theta}{\partial z_0}\right), \quad (9.5\mathrm{b})$$

where use has been made of Eq. (7.5) in the evaluation of π_θ. Let us imagine that Eq. (9.5a) has been solved for β as a functional of π_θ. (The constant of integration must, of course, be chosen independently of time on each field line, or else $\dot\beta$ will not vanish in a steady flow.) If the field lines are closed, then β will generally be multi-valued. It follows from Eq. (9.2), however, that $\dot\beta$ is single-valued; the jump in β is a constant of the motion.

It is easily verified that the new variables β, π_β ob[ey] a Hamiltonian system of dynamical equations, or in other words, that the transformation from θ, π_θ to β, π_β is actually a contact transformation. The Hamiltonian of this new system of equations is easily obtained by substituting Eqs. (9.5) into Eq. (9.1):

$$H = \frac{1}{2\varrho_0}\left(\pi_r^2 + \frac{r^2}{J r_0^2}\pi_\beta^2 + \pi_z^2\right) + \frac{P_0}{(\gamma-1)J^{\gamma-1}}$$
$$+ \frac{1}{2J}\left[\left(\frac{\partial r}{\partial r_0}B_{0r} + \frac{\partial r}{\partial z_0}B_{0z}\right)^2 + \left(\frac{\partial z}{\partial r_0}B_{0r} + \frac{\partial z}{\partial z_0}B_{0z}\right)^2\right]$$
$$+ \frac{1}{2\varrho_0 r^2}\left[B_{0r}\frac{\partial}{\partial r_0}(\beta r_0\sqrt{\varrho_0}) + B_{0z}\frac{\partial}{\partial z_0}(\beta r_0\sqrt{\varrho_0})\right]^2. \quad (9.6)$$

Note that part of the kinetic energy has been counted as potential, and part of the potential energy as ki[n]etic. Specifically, the θ components of the kinetic and magnetic-field energies have been counted, resp[ec]tively, as potential and kinetic.

We now have a system of canonical variables such that the generalized velocities and momenta all vanish when the flow is steady. Furthermore, the Hamiltonian is the sum of two terms, one positive-definite and quadratic in the momenta, and the other depending only on the coordinates. As before, this means that the steady flow has been reduced to a static equilibrium, and that the stability criterion for azimuthally symmetric displacements will be given by an energy principle.

The conditions for equilibrium are readily obtain[ed] by setting $\dot\pi_r$, $\dot\pi_\beta$, and $\dot\pi_z$ equal to zero in the Ham[il]tonian equations of motion. They are

$$-\frac{\pi_{0\theta}^2}{\varrho_0 r_0^3} + \frac{\partial P_0}{\partial r_0} + B_{0z}\left(\frac{\partial B_{0z}}{\partial r_0} - \frac{\partial B_{0r}}{\partial z_0}\right) = 0, \quad (9.7\mathrm{a})$$

$$\frac{\partial P_0}{\partial z_0} + B_{0r}\left(\frac{\partial B_{0r}}{\partial z_0} - \frac{\partial B_{0z}}{\partial r_0}\right) = 0, \quad (9.7\mathrm{b})$$

$$B_{0r}\frac{\partial}{\partial r_0}\left(\frac{\pi_{0\theta}}{\varrho_0 r_0^2}\right) + B_{0z}\frac{\partial}{\partial z_0}\left(\frac{\pi_{0\theta}}{\varrho_0 r_0^2}\right) = 0, \quad (9.7\mathrm{c})$$

in which $\pi_{0\theta}$ is not regarded as a canonical momentum, but only as an abbreviation for the expression (9.5a) with $\beta=\beta_0$. Note that Eq. (9.7c) is simply the condition for rigid-body rotation of the field lines.

As before, we expand the potential energy to second order in the displacement, which now includes a third component $\eta=\beta-\beta_0$. (Note that η is single-valued even though β itself is not.) The result, if the subscript zeroes are again dropped, is

$$W = \pi\int r\,\mathrm{d}r\,\mathrm{d}z\left\{\frac{1}{\varrho r^2}\left[\mathbf{B}\cdot\nabla(\eta r\sqrt{\varrho}) - \frac{2\pi_\theta\xi}{r}\right]^2 - \frac{\pi_\theta^2\xi^2}{\varrho r^4}\right.$$
$$+ (\gamma P + B^2)\,\mathrm{div}^2 - 2\left(P + \frac{1}{2}B^2\right)\left(\frac{\xi}{r}\frac{\partial\xi}{\partial r} + \frac{\xi}{r}\frac{\partial\zeta}{\partial z}\right.$$
$$\left. + \frac{\partial\xi}{\partial r}\frac{\partial\zeta}{\partial z} - \frac{\partial\xi}{\partial z}\frac{\partial\zeta}{\partial r}\right) + (\mathbf{B}\cdot\nabla\xi)^2 + (\mathbf{B}\cdot\nabla\zeta)^2$$
$$\left. - 2(B_r\mathbf{B}\cdot\nabla\xi + B_z\mathbf{B}\cdot\nabla\zeta)\,\mathrm{div}\right\}, \quad (9.8)$$

where

$$\mathbf{B}\cdot\nabla = B_r\frac{\partial}{\partial r} + B_z\frac{\partial}{\partial z}, \qquad (9.9)$$

and where div is defined as before. The stability criterion is that W be positive for all ξ, η, ζ.

Let us now minimize W with respect to η, holding ξ and ζ fixed:

$$W_{\min}(\xi,\zeta) = \mathrm{Min}_\eta\, W(\xi,\eta,\zeta). \qquad (9.10)$$

Then the flow is stable if and only if $W_{\min}$ is positive for all ξ, ζ. The minimization condition is

$$\mathbf{B}\cdot\nabla\left[\frac{1}{\varrho r^2}\mathbf{B}\cdot\nabla(\eta r\sqrt{\varrho}) - \frac{2\pi_\theta\,\xi}{\varrho r^3}\right] = 0, \quad (9.11)$$

or

$$\mathbf{B}\cdot\nabla(\eta r\sqrt{\varrho}) = \varrho r^2 C + 2\pi_\theta\,\xi/r, \qquad (9.12)$$

where C is constant along the field lines. Integrating over the infinitesimal shell between two magnetic surfaces, we obtain

$$0 = C\langle\varrho r^2\rangle + \langle 2\pi_\theta\,\xi/r\rangle, \qquad (9.13)$$

where the brackets indicate a volume average over the shell. Note that the volume average can also be written as follows in terms of line integrals:

$$\langle F\rangle = \frac{\oint F\,\mathrm{d}l/B}{\oint \mathrm{d}l/B}. \qquad (9.14)$$

Substituting Eqs. (9.12) and (9.13) into Eq. (9.8), we finally obtain

$$\begin{aligned} W_{\min} = \pi\int r\,\mathrm{d}r\,\mathrm{d}z\Big\{&\varrho r^2\frac{\langle 2\xi\,\pi_\theta/r\rangle^2}{\langle\varrho r^2\rangle^2} - \frac{\pi_\theta^2\,\xi^2}{\varrho r^4} \\ &+ (\gamma P + B^2)\,\mathrm{div}^2 - 2\left(P + \frac{1}{2}B^2\right) \\ &\times\left(\frac{\xi}{r}\frac{\partial\xi}{\partial r} + \frac{\xi}{r}\frac{\partial\zeta}{\partial z} + \frac{\partial\xi}{\partial r}\frac{\partial\zeta}{\partial z} - \frac{\partial\xi}{\partial z}\frac{\partial\zeta}{\partial r}\right) \\ &+ (\mathbf{B}\cdot\nabla\xi)^2 + (\mathbf{B}\cdot\nabla\zeta)^2 - 2(B_r\mathbf{B}\cdot\nabla\xi + B_z\mathbf{B}\cdot\nabla\zeta)\,\mathrm{div}\Big\}. \end{aligned} \qquad (9.15)$$

We shall not go any further into the general case of a poloidal field, but in the next section we shall consider the special case of a purely axial field in detail.

10. The special case of a purely axial field

Let us now suppose that the unperturbed magnetic field has no radial component. Then, if we continue to suppress the subscript zeroes, the equilibrium condition (9.7) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(P + \frac{1}{2}B_z^2\right) = \frac{\pi_\theta^2}{\varrho r^3}, \qquad (10.1)$$

in which the individual terms can no longer depend on z but only on r. (The stability of a purely azimuthal steady flow in the presence of an axial field has also been treated by CHANDRASEKHAR [22, 23], but under a different, and considerably more difficult, set of conditions. He has taken account of viscosity and finite electrical conductivity, but with an incompressible fluid and a uniform field. We, on the other hand, are neglecting viscosity and assuming infinite conductivity, but we are taking account of compressibility and allowing the field intensity to depend on r.)

Setting B_r equal to zero in Eq. (9.15), integrating various terms by parts, and making use of the equilibrium condition (10.1), we obtain

$$\begin{aligned} W_{\min} = \pi\int r\,\mathrm{d}r\,\mathrm{d}z\Big\{&B_z^2\left(\frac{\partial\xi}{\partial r} + \frac{\xi}{r}\right)^2 + \Big[\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\pi_\theta^2}{\varrho}\right) \\ &- \frac{4\pi_\theta^2}{\varrho r^4} - \frac{\pi_\theta^4}{\gamma P\varrho^2 r^6}\Big]\xi^2 + B_z^2\left(\frac{\partial\xi}{\partial z}\right)^2 \\ &+ \frac{4\pi_\theta^2}{\varrho r^4}\langle\xi\rangle^2 + \gamma P\left(\mathrm{div} + \frac{\pi_\theta^2\,\xi}{\gamma P\varrho r^3}\right)^2\Big\}. \end{aligned} \qquad (10.2)$$

Let us define a new integral W_0, in which r is the only variable of integration:

$$W_0 = \pi\int r\,\mathrm{d}r\Big\{B_z^2\left(\frac{\partial\xi}{\partial r} + \frac{\xi}{r}\right)^2 + \left[\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\pi_\theta^2}{\varrho}\right) - \frac{4\pi_\theta^2}{\varrho r^4} - \frac{\pi_\theta^4}{\gamma P\varrho^2 r^6}\right]\xi^2\Big\}. \qquad (10.3)$$

We can easily show that $W_{\min}$ is positive for all ξ, ζ if and only if W_0 is positive for all ξ. The "if" part is obvious, since the last three terms of Eq. (10.2) are necessarily positive; and to prove the "only if" part, let $f(r)$ be some function of r such that W_0 is negative when $\xi = f(r)$. Then $W_{\min}$ will be negative when ξ and ζ are chosen as follows:

$$\xi(r,z) = f(r)\cos kz, \qquad (10.3\,\mathrm{a})$$

$$\zeta(r,z) = -\left[f'(r) + \frac{1}{r}f(r) + \frac{\pi_\theta^2}{\gamma P\varrho r^3}f(r)\right]\frac{\sin kz}{k}, \qquad (10.3\,\mathrm{b})$$

where k is very small; for the term $B_z^2(\partial\xi/\partial z)^2$ is negligible when k is small, the next term vanishes because $\langle\xi\rangle = 0$ whenever $k \neq 0$, and the last term vanishes because of the choice of ζ. The necessary and sufficient condition for stability, then, is that W_0 be positive for every $\xi(r)$.

If the magnetic field is so small that we can drop it from the integrand of W_0, then the stability condition is simply

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\pi_\theta^2}{\varrho}\right) > \frac{4\pi_\theta^2}{\varrho r} + \frac{\pi_\theta^4}{\gamma P\varrho^2 r^3}. \qquad (10.4)$$

Now this condition does not agree with the corresponding condition (8.13) for stability in the presence of a small B_θ. In fact, it is possible for a flow to satisfy the latter condition without satisfying the former; and whenever this happens the flow will be stable in the absence of a field, or in the presence of a small B_θ, but it will be made unstable by an arbitrarily small B_z. (Of course, the growth rate of the instability will be correspondingly small.) What the axial field does is to remove a constraint, the constancy of π_θ, and by so doing it enlarges the class of allowed motions. Suppose, for example, that a particular ring element on a particular flux tube starts to move outward.

Then its angular velocity will decrease; and the axial field, because of its distortion by the nonuniform rotation of the flux tube, will transfer angular momentum from the other ring elements on the flux tube to the one that is moving outward. That element will then be subjected to an increased centrifugal force, tending to push it still further outward and thus to destabilize its radial motion.

If the magnetic field is not small, then the simple condition (10.4) is still sufficient for stability but no longer necessary. To determine the stability when that condition is not satisfied, we may have recourse to a method that has been worked out in connection with the diffuse linear pinch. Here we shall simply give the results, referring to the pinch work for details [24].

We consider the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r B_z^2 \frac{\mathrm{d}\xi}{\mathrm{d}r}\right) - g\xi = 0\,, \tag{10.5}$$

where

$$g(r) = \frac{1}{r^2}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\pi_\theta^2}{\varrho}\right) - \frac{4\pi_\theta^2}{\varrho r} - \frac{\pi_\theta^4}{\gamma P \varrho^2 r^3}\right] + \frac{B_z^2}{r} - \frac{\mathrm{d}B_z^2}{\mathrm{d}r}\,. \tag{10.6}$$

(This equation is simply the Euler-Lagrange condition for stationary values of W_0). Singularities occur at every value of r for which B_z changes sign, and the solutions $\xi(r)$ may exhibit either of two types of behavior: If r_s is the singular point, then the solutions may have an infinite number of oscillations in the neighborhood of r_s, or they may behave like real powers of $r - r_s$:

$$\xi(r) \sim \begin{cases} (r - r_s)^{-n_1} \\ (r - r_s)^{-n_2}, \quad \text{where } n_1 + n_2 = 1\,. \end{cases} \tag{10.7}$$

The condition for nonoscillatory solutions is that the following inequality be satisfied at the singular point:

$$\left(\frac{\mathrm{d}B_z}{\mathrm{d}r}\right)^2 + \frac{4}{r^3}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\pi_\theta^2}{\varrho}\right) - \frac{4\pi_\theta^2}{\varrho r} - \frac{\pi_\theta^4}{\gamma P \varrho^2 r^3}\right] > 0. \tag{10.8}$$

It can be shown (Corollary 9—1 of Ref. [24]) that instability occurs whenever the solutions are oscillatory; hence the condition (10.8) is necessary for stability.

Now suppose that the flow takes place between concentric cylinders of radius a and b, and that the cylinders are rigid and perfectly conducting. We then have the boundary condition

$$\xi(a) = \xi(b) = 0\,. \tag{10.9}$$

Let the singular points, if any, be r_{s1}, r_{s2}, etc., and divide the interval $a < r < b$ into subintervals $a < r < r_{s1}$, $r_{s1} < r < r_{s2}$, etc. We may assume that the inequality (10.8) holds at each singular point, in which case the solutions of Eq. (10.5) will have the form (10.7), since we already know what happens when it does not hold. We pick out solutions in each subinterval as follows: (1) In the first subinterval let $\xi(r)$ vanish at the left endpoint $r = a$. (2) In every other subinterval let $\xi(r)$, in the neighborhood of the left endpoint, be the smaller of the two expressions (10.7). Except for a multiplicative constant, a unique solution has now been defined in each subinterval, and we can ask whether there are any interior points of the subinterval at which the solution vanishes. It can be shown (Theorem 10 of Ref. [24]) that the flow is stab[le if] and only if there are no such points in any of the s[ub]intervals. This result enables us, in any particular case, to determine the stability by solving a second-order differential equation numerically.

Erratum

Consider an upright glass of water, and let the water be subjected to the virtual displacement $\xi_\theta = r$, $\xi_r = \xi_z = 0$, a rigid-body rotation. For this displacement, since $\xi_{ii} = 0$ and $\xi_{ij}\,\xi_{ji} = -2$, the energy integral (6.9) reduces in the absence of a field to

$$W = -\int P_0\,\mathrm{d}^3 x_0 < 0. \tag{E1}$$

Thus, according to our version of the hydromagn[etic] energy principle, a glass of water is rotationally [un]stable. It is clear that some mistake has been made.

The trouble arises from an improper definition of $\boldsymbol{\xi}$. Suppose for the sake of definiteness that the fluid is enclosed by a rigid wall, with the magnetic field purely tangential. Then the boundary conditions are

$$B_{0i}\,\mathrm{d}\sigma_{0i} = 0, \tag{E2}$$

$$\xi_i\,\mathrm{d}\sigma_{0i} = 0, \tag{E3}$$

where $\mathrm{d}\boldsymbol{\sigma}_0$ is a surface element of the fluid boundary. These boundary conditions are used in proving that the first-order part of W vanishes. Restricting [our]selves temporarily to the scalar-pressure case, we h[ave]

$$W^{(1)} = \frac{1}{2}\int \mathrm{d}^3x_0 \left[\varrho_0\left(\frac{\partial\varphi}{\partial x_i}\right)_{\mathbf{x}=\mathbf{x}_0}\xi_i - \left(P_0 + \frac{1}{2}B_0^2\right)\xi_{ii} + B_{0i}B_{0j}\xi_{ij}\right]$$
$$= \frac{1}{2}\int \mathrm{d}^3x_0\,\xi_i\left[\varrho_0\left(\frac{\partial\varphi}{\partial x_i}\right)_{\mathbf{x}=\mathbf{x}_0} + \frac{\partial}{\partial x_{0i}}\left(P_0 + \frac{1}{2}B_0^2\right) - B_{0j}\frac{\partial}{\partial x_{0j}}\ldots\right]$$
$$+ \frac{1}{2}\int\left[-\mathrm{d}\sigma_{0i}\,\xi_i\left(P_0 + \frac{1}{2}B_0^2\right) + \mathrm{d}\sigma_{0j}\,B_{0j}B_{0i}\,\xi_i\right]. \tag{E4}$$

The volume integral vanishes because of the equilibrium condition, and the surface integral because of the boundary conditions. But, if ξ_i is defined as $x_i - x_{0i}$, then $\xi_i\,\mathrm{d}\sigma_{0i}$ vanishes only to lowest order; and the surface integral, instead of vanishing exactly, reduces to a non-negligible second-order term. What we must do is write

$$x_i = x_{0i} + \xi_i + \frac{1}{2}\eta_i, \tag{E5}$$

where η_i is a second-order term chosen to make ξ_i satisfy the boundary condition (E3) exactly. It is easily shown that $\boldsymbol{\eta}$ must then satisfy the boundary condition

$$\eta_i\,\mathrm{d}\sigma_{0i} = \mathrm{d}\sigma_{0i}\,\xi_{ij}\,\xi_j. \tag{E6}$$

The energy integral will now have a contribution from $\boldsymbol{\eta}$.

From now on, as in Sections 8—10, we shall drop the subscript zeroes from the equilibrium quantities. Proceeding as in Eq. (E4), we find that the contribution of $\boldsymbol{\eta}$ to W is equal to the surface integral

$$-\frac{1}{2}\int \mathrm{d}\sigma_i\, \eta_i \left(P+\frac{1}{2}B^2\right) = -\frac{1}{2}\int \mathrm{d}\sigma_i\, \xi_{ij}\, \xi_j \left(P+\frac{1}{2}B^2\right). \tag{E7}$$

By adding on a term that vanishes as a result of the first-order boundary condition (E3), we can reduce this expression to a volume integral not involving any second derivatives of $\boldsymbol{\xi}$:

$$\begin{aligned}\frac{1}{2}\int \mathrm{d}^3 x\, \frac{\partial}{\partial x_i}\left\{\xi_i \frac{\partial}{\partial x_i}\left[\left(P+\frac{1}{2}B^2\right)\xi_j\right] - \left(P+\frac{1}{2}B^2\right)\xi_{ij}\,\xi_j\right\} \\ = \frac{1}{2}\int \mathrm{d}^3 x \left\{\xi_i\, \xi_j \frac{\partial^2}{\partial x_i\, \partial x_j}\left(P+\frac{1}{2}B^2\right) + 2\,\xi_{jj}\,\xi_i \frac{\partial}{\partial x_i}\left(P+\frac{1}{2}B^2\right) \right. \\ \left. + \left(P+\frac{1}{2}B^2\right)\left[(\xi_{ii})^2 - \xi_{ij}\,\xi_{ji}\right]\right\}.\end{aligned} \tag{E8}$$

Finally, by adding this term to Eq. (6.9), we obtain the complete expression for the energy integral:

$$\begin{aligned}W = \frac{1}{2}\int \mathrm{d}^3 x \left\{\xi_i\, \xi_j \left[\varrho \frac{\partial^2 \varphi}{\partial x_i\, \partial x_j} + \frac{\partial^2}{\partial x_i\, \partial x_j}\left(P+\frac{1}{2}B^2\right)\right] \right. \\ + 2\,\xi_{jj}\,\xi_i \frac{\partial}{\partial x_i}\left(P+\frac{1}{2}B^2\right) + (\gamma P + B^2)(\xi_{ii})^2 \\ \left. + B_j B_k (\xi_{ij}\,\xi_{ik} - 2\,\xi_{ii}\,\xi_{jk})\right\}.\end{aligned} \tag{E9}$$

The stability criterion is that this integral be positive-definite relative to the boundary condition (E3).

With a tensor pressure the contribution of $\boldsymbol{\eta}$ is again given by the surface integral (E7), but with $P_\perp$ instead of P. The complete energy integral is then

$$\begin{aligned}W = \frac{1}{2}\int \mathrm{d}^3 x \left\{\xi_i\, \xi_j \left[\varrho \frac{\partial^2 \varphi}{\partial x_i\, \partial x_j} + \frac{\partial^2}{\partial x_i\, \partial x_j}\left(P_\perp+\frac{1}{2}B^2\right)\right] \right. \\ + 2\xi_{jj}\,\xi_i \frac{\partial}{\partial x_i}\left(P_\perp+\frac{1}{2}B^2\right) \\ + 2\left(P_\perp+\frac{1}{2}B^2\right)(\xi_{ii})^2 + (B^2 + P_\perp - P_\parallel)\,\tau_j\,\tau_k\,\xi_{ij}\,\xi_{ik} \\ \left. - 2\,(P_\perp + B^2)\,\tau_j\,\tau_k\,\xi_{jk}\,\xi_{ii} + (4P_\parallel - P_\perp)\,(\tau_i\,\tau_j\,\xi_{ij})^2\right\}.\end{aligned} \tag{E10}$$

In the case of steady azimuthal flow there is no additional contribution when the wall is a circular cylinder. But if it is a general surface of revolution, as it may well be in Sections 8 and 9, we obtain a surface integral

$$\pi \int r\left(P+\frac{1}{2}B^2\right)\left[\left(\xi \frac{\partial \zeta}{\partial z} - \zeta \frac{\partial \xi}{\partial z}\right)\mathrm{d}z + \left(\xi \frac{\partial \zeta}{\partial r} - \zeta \frac{\partial \xi}{\partial r}\right)\mathrm{d}r\right], \tag{E11}$$

which must be added to Eqs. (8.7), (9.8), and (9.15). This term, when reduced to a volume integral as in Eq. (E8), leads directly to Eqs. (8.9) and (10.2), and for this reason does not affect our final results in any way.

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