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General Stability Theory in Plasma Physics.

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1. – Introduction.

In this course we shall consider three energy principles for the stability of *static* magnetohydrodynamic equilibria. These three energy principles correspond to three different sets of basic equations describing the plasma, each of which applies in various limiting situations. We shall designate these three different approaches by the *fluid* theory, the *adiabatic* theory and the *double adiabatic* theory or Chew-Goldberger-Low theory.

The *fluid* theory corresponds to the strong-collision limit where collisions are so strong that the pressure always remains a scalar, but however, still so weak that the conductivity may be taken as infinite. The *adiabatic* theory corresponds to the limit of no collisions and to the limit where the gyration radius of each type of particle and the Debye length are infinitely small compared with macroscopic quantities. The *double adiabatic* theory is an intermediate theory in which collisions are not strong enough to keep the pressure isotropic but are sufficiently strong to make the heat flow negligible. This latter theory is the theory given in the paper of CHEW, GOLDBERGER and LOW [1].

In each of the three theories first the basic equations are derived, second the equilibrium is discussed, third the linearized equations for small motions about equilibrium are derived, and finally an energy principle is derived which is shown to give a necessary and sufficient condition for the stability of *all* these small motions.

In each case the basic equations are a closed system. In the *fluid* theory the equations are the usual fluid ones with p given by $p = A\varrho^{\gamma}$. In the *adiabatic* theory the pressure is found from the Boltzmann distribution function f but otherwise the equations are essentially the same as in the fluid theory. To find f it is necessary to solve the collisionless Boltzmann or Vlasov equation

but the difficulty of solving this equation is reduced by the assumption of small gyration radius. The equation reduces to a one-dimensional Boltzmann equation where the one dimension comprises the position and velocity of a particle along a magnetic line of force. The theory is designated as *adiabatic* because the motion of each particle is governed by the magnetic moment being an adiabatic invariant. In the *double adiabatic* theory a separate equation of state is derived for each of the two independent components of the stress tensor, with each equation similar to the simple equation in the *fluid* theory. Thus, since these equations arise from an adiabatic assumption (adiabatic here means no heat flow) this theory is often called the *double adiabatic* theory. The two uses of the term adiabatic in *adiabatic* theory and *double adiabatic* theory are essentially different.

The theories are developed on generally parallel lines. The arguments for the derivation of an energy principle in each case are formally the same. When the energy principles are derived they will be compared for similar equilibria. It will be shown that if one proves stability for an isotropic equilibrium on the *fluid* theory, the equilibrium will be stable in the other two theories. The comparison of the *double-adiabatic* and *adiabatic* energy principle is not completed as yet but it has been shown that the energy principle of KRUSKAL and OBERMAN [2] which is slightly more pessimistic (more unstable) than the energy principle of the adiabatic theory is definitely more pessimistic than that of the *double adiabatic* theory.

The name *double adiabatic* theory is used rather then the Chew-Goldberger-Low theory since these authors also layed the foundation for the *adiabatic* theory and we shall follow their approach. This latter work is unpublished [3].

The energy principle for the *fluid* theory is the first one to appear in plasma physics. It is the energy principle given in the paper of BERNSTEIN, FRIEMAN, KRUSKAL and KULSRUD [4]. This paper also includes the energy principle of the *double adiabatic* theory. The energy principle of the *adiabatic* theory is a slight generalization of the energy principle developed independently and simultaneously by KRUSKAL and OBERMAN [2] and ROSENBLUTH and ROSTOKER [5].

2. – Fluid theory.

2·1. *Basic equations.* – The basic equations underlying all three theories will be the Fokker-Planck equation [6] for the Boltzmann distribution function of each particle and Maxwell's equations for the electromagnetic field. In each of the three cases we shall make different assumptions of simplicity on these equations and derive three different sets of simple equations which we can handle. It should be emphasized at this point that although the exact structure of the Fokker-Planck equation is still under discussion this will not

affect our simpler equations since the uncertainties in the Fokker Planck equation are covered up by our simplifying assumptions.

The fluid equations are

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -\varrho \nabla \cdot \boldsymbol{V}\,, \tag{1}$$

$$\varrho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -\nabla \cdot p + \boldsymbol{j} \times \boldsymbol{B} + q\boldsymbol{E}\,, \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(p/\varrho^{\gamma}) = 0\,. \tag{3}$$

The Maxwell equations are

$$\nabla \cdot \boldsymbol{B} = 0\,, \tag{4}$$

$$\nabla \times \boldsymbol{B} = 4\pi \boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}\,, \tag{5}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\nabla \times \boldsymbol{E}\,, \tag{6}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi q\,. \tag{7}$$

These equations are coupled by an Ohm's law

$$\boldsymbol{E} + \frac{1}{c}\boldsymbol{V} \times \boldsymbol{B} = \frac{1}{\sigma}\boldsymbol{j}\,. \tag{8}$$

Here ϱ, $\boldsymbol{V}$, and p are the density, velocity and pressure of the fluid (assumed scalar), $\boldsymbol{j}$, $\boldsymbol{B}$, $\boldsymbol{E}$ and q are the current, magnetic field, electric field and charge density. The units used are Gaussian except $\boldsymbol{j}$ is given in e.m.u.; $\mathrm{d}/\mathrm{d}t$ indicates $\partial/\partial t + \boldsymbol{V}\cdot\nabla$ as usual. γ, c and σ are the ratio of specific heats, the velocity of light and the conductivity in abamp/abvolt.

Although these equations apply to a number of situations (for instance with $\gamma = \infty$ incompressibility) we have in mind a gas consisting of ions and electrons. ϱ is the total density, $\boldsymbol{V}$ the mass velocity of the combined fluid, and p the total pressure. The assumption is made that collisions are very strong so that the gas is everywhere nearly Maxwellian. The first two fluid equations are derived by taking the first two moments of the Fokker-Planck equations. The next moment involves the heat flow which may be computed by the method of Chapman and Enskog. Similarly a more correct Ohm's law may be derived which gives $\boldsymbol{j}$ with an anisotropic conductivity and heat flow terms. It should also be mentioned that the pressure is a tensor in eq. (2) and the off-diagonal terms give rise to a viscosity term in (2). The correct

full equations have been derived by MARSHALL [7] for the ion-electron case and are given in the review article of BERNSTEIN and TREHAN [8].

The condition for the validity of these equations is $\tau \ll T$ where τ is the mean collision time and T is a characteristic macroscopic time. However, if τ is very small all the transport terms (viscosity, heat flow) may be neglected and we get essentially eqs. (1)–(8) (σ may still be anisotropic).

We now wish to consider the limit of infinite conductivity $\sigma \to \infty$, so that (8) is replaced by

$$\boldsymbol{E} + \frac{1}{c}\boldsymbol{V}\times\boldsymbol{B} = 0\,. \tag{8}$$

However, in a certain sense $\sigma \to \infty$ corresponds to very large τ which seems to contradict our previous assumption. However, in this case, σ large means $\tau \gg T(L^2\omega_p^2/c^2)^{-1}$ so (1)–(7), and (8a) should be valid if

$$\frac{c^2}{\omega_p^2 L^2}\, T \ll \tau \ll T\,,$$

$\omega_p^2 = 4\pi n e^2/m$ where n, e and m are electron density, charge and mass, and L is a macroscopic length. The first half of this inequality is usually well satisfied so our *fluid* theory is generally valid if the second half is satisfied. This latter is more stringent.

(Of course, there are other conditions that must be satisfied in the fluid theory such as small gyration radius, small Debye length, etc. We do not stress these in the fluid theory.)

Substituting (8′) in (5) and (7) and these in (2) we have

$$\varrho\,\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -\nabla p + \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \frac{1}{4\pi c^2}\frac{\partial}{\partial t}(\boldsymbol{V}\times\boldsymbol{B})\times\boldsymbol{B} - \frac{1}{4\pi}\frac{\nabla\cdot(\boldsymbol{V}\times\boldsymbol{B})\boldsymbol{V}\times\boldsymbol{B}}{c^2}\,.$$

The last two terms are negligible if $\varrho c^2 \gg B^2/4\pi$ so we may neglect them. This corresponds to the neglect of displacement current and the electric force term in the momentum eq. (2). Also we wish at this point to eliminate all the factors 4π which may arise so we set

$$\boldsymbol{B} = \sqrt{4\pi}\,\boldsymbol{B}'\,,$$

$$\boldsymbol{J} = \frac{\boldsymbol{J}'}{\sqrt{4\pi}}\,,$$

which corresponds to a choice of units about 3 G for B', 3 A for J'.

To summarize our equations are now

Continuity:

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -\varrho\nabla\cdot\boldsymbol{V}. \tag{9}$$

Momentum:

$$\varrho\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -\nabla p + \boldsymbol{J}\times\boldsymbol{B}, \tag{10}$$

Energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\varrho^{\gamma}}\right) = 0. \tag{11}$$

Maxwell-Ohm:

$$\nabla\cdot\boldsymbol{B} = 0, \tag{12}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{V}\times\boldsymbol{B}), \tag{13}$$

$$\nabla\times\boldsymbol{B} = \boldsymbol{J}. \tag{14}$$

The other two equations give q and $\boldsymbol{E}$ which we do not need. Equations (9)–(14) are closed equations for ϱ, $\boldsymbol{V}$, p, and $\boldsymbol{B}$. These are given respectively by (9), (10), (11) and (13), $\boldsymbol{J}$ is given as a definition by (14) and could be eliminated with no trouble. (12) is a side condition; that is always satisfied if it is satisfied at any time. We shall take as *fluid* theory the study of the consequences of eqs. (9)–(14).

Equations (9)–(14) are essentially those given in the paper by BERNSTEIN, FRIEMAN, KRUSKAL and KULSRUD [4]. We parallel the development of the consequences of these equations given in this paper. The energy principle for the *fluid* theory derived in these notes was first given in this paper.

Boundary conditions: There are two types of situations which we might consider: 1) a plasma confined by a rigid infinitely conducting wall on which $\boldsymbol{B}\cdot\boldsymbol{e} = 0$, where $\boldsymbol{e}$ is the normal into the wall (Fig. 1); 2) a plasma adjacent

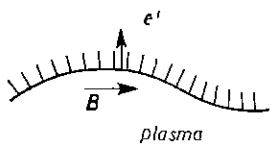

Fig. 1.

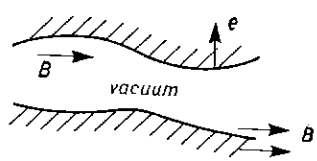

Fig. 2.

to a vacuum adjacent to a rigid wall (Fig. 2); in (2) $\boldsymbol{B}\cdot\boldsymbol{e} = 0$ on the wall and also at the plasma vacuum interface.

For simplicity consider only assumption (1). It is comparatively easy to extend the theory to case 2). We have in the wall

$$\boldsymbol{E} = 0 .$$

Also we have the usual boundary conditions (with $\langle A \rangle = A_1 - A_2$)

$$\boldsymbol{V}\cdot\boldsymbol{e} = 0 ,$$

$$\boldsymbol{e}\cdot\langle \boldsymbol{B}\rangle = 0 ,$$

$$\boldsymbol{e}\times\langle \boldsymbol{E}\rangle = 0 ,$$

$$\boldsymbol{e}\times\langle \boldsymbol{B}\rangle = \boldsymbol{J}^* ,$$

where $\boldsymbol{J}^*$ is the current flowing in the wall. Thus in the plasma

$$\boldsymbol{e}\times\boldsymbol{E} = 0 .$$

Suppose $\boldsymbol{B} = 0$ in the wall at some time; then $\partial\boldsymbol{B}/\partial t = 0$ and $\boldsymbol{B} = 0$ in the wall for all time. Hence since $\langle \boldsymbol{B}\cdot\boldsymbol{e}\rangle = 0$

$$\boldsymbol{B}\cdot\boldsymbol{e} = 0 \quad \text{in the plasma:}$$

so we have case 1) if $\boldsymbol{B} = 0$ in the wall at any time.

It is sufficient in light of (8*a*)–(14) to assume only the boundary conditions

$$\boldsymbol{V}\cdot\boldsymbol{e} = 0 , \qquad \boldsymbol{B}\cdot\boldsymbol{e} = 0 .$$

Further $\boldsymbol{B}\cdot\boldsymbol{e} = 0$ is a side condition. For,

$$-\boldsymbol{e}\times\boldsymbol{E} = \boldsymbol{e}\times(\boldsymbol{V}\times\boldsymbol{B}) = \boldsymbol{B}\cdot\boldsymbol{e}\boldsymbol{V} - \boldsymbol{V}\cdot\boldsymbol{e}\boldsymbol{B} = 0$$

and therefore

$$\frac{\partial}{\partial t}(\boldsymbol{B}\cdot\boldsymbol{e}) = -c\boldsymbol{e}\cdot\nabla\times\boldsymbol{E} = 0 .$$

The last equation follows easily on consideration of the line integral over any closed curve C in the surface

$$0 = \oint \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = \int \boldsymbol{e}\cdot\nabla\times\boldsymbol{E}\,\mathrm{d}S .$$

2·2. *Static equilibrium.* – For static equilibrium, $\boldsymbol{V} = 0$, the equations be-

come

$$\nabla p = \boldsymbol{J}\times\boldsymbol{B}\,,$$
$$\nabla\cdot\boldsymbol{B} = 0\,,$$
$$\nabla\times\boldsymbol{B} = \boldsymbol{J}\,.$$

The side condition $\nabla\cdot\boldsymbol{B}=0$ is now a full condition. Note that

$$\boldsymbol{J}\cdot\nabla p = 0\,,$$
$$\boldsymbol{B}\cdot\nabla p = 0\,,$$

so that $\boldsymbol{B}$ and $\boldsymbol{J}$ lie on surfaces of constant p. Thus $\boldsymbol{B}$ has magnetic surfaces *i.e.* surfaces everywhere tangent to $\boldsymbol{B}$. Also p is constant along lines of force. At the rigid wall

$$\boldsymbol{B}\cdot\boldsymbol{e} = 0\,.$$

2·3. *Linearized equations.* – We wish to consider motions of the fluid in the neighborhood of the static equilibrium given above. We write at a fixed point $\boldsymbol{r}$

$$p = p^0 + p'\,,$$
$$\boldsymbol{V} = \boldsymbol{V}'\,,$$
$$\boldsymbol{B} = \boldsymbol{B}^0 + \boldsymbol{B}'\,,$$
$$\varrho = \varrho^0 + \varrho'\,,$$
$$\boldsymbol{J} = \boldsymbol{J}^0 + \boldsymbol{J}'\,.$$

From (14)

$$\boldsymbol{J}' = \nabla\times\boldsymbol{B}'\,.$$

p', $\boldsymbol{V}'$, $\boldsymbol{B}'$ and ϱ' can be given independently initially and then (9)–(14) give their time development.

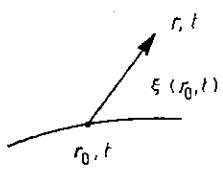

Fig. 3.

Introduce a displacement vector $\boldsymbol{\xi}$, the Lagrangian displacement, (Fig. 3) by

$$\frac{\partial\boldsymbol{\xi}}{\partial t} = \boldsymbol{V}\,.$$

However, it is not necessary to take the Lagrangian point of view. To lowest order we may regard $\boldsymbol{\xi}$ as given at a fixed point and leave it undefined to higher order by $\partial\boldsymbol{\xi}(\boldsymbol{r},t)/\partial t = \boldsymbol{V}'(\boldsymbol{r},t)$; $\boldsymbol{\xi}$ will than be an Eulerian variable. We do not consider all displacements p', $\boldsymbol{V}'$, $\boldsymbol{B}'$, ϱ' but only those obtainable in the

following manner: assume the system in equilibrium. Then introduce extraneous forces which displace the system away from equilibrium and give it a velocity. During this initial displacement all the equations except (10) hold. Then allow the system to evolve under all equations including (10). Thus initially only ξ and V can be given independently. For instance, this displacement preserves the total mass of the system.

The equations are easily integrated to first order for ϱ', p', B' in terms of ξ. Let

$$p^* = p' + \xi \cdot \nabla p \,,$$
$$\varrho^* = \varrho' + \xi \cdot \nabla \varrho \,,$$

(p^* is the perturbed pressure following the displacement). Then

$$\varrho^* = -\varrho \nabla \cdot \xi \,,$$
$$p^* = -\gamma p \nabla \cdot \xi \,,$$
$$B' = \nabla \times (\xi \times B) = B \cdot \nabla \xi - \xi \cdot \nabla B - B \nabla \cdot \xi \,,$$
$$B^* = B \cdot \nabla \xi - B \nabla \cdot \xi \,.$$

After the initial displacement ξ is given by

$$\varrho \frac{\partial^2 \xi}{\partial t^2} = \varrho \frac{\partial V}{\partial t} = -\nabla p' + J' \times B + J \times B' \,,$$

$$\varrho \frac{\partial^2 \xi}{\partial t^2} = [\nabla \times (\nabla \times (\xi \times B))] \times B + J \times [\nabla \times (\xi \times B)] + \nabla (\xi \cdot \nabla p) + \nabla (\gamma p \cdot \xi) \,. \tag{15}$$

Let

$$Q = \nabla \times (\xi \times B) \,.$$

Then we have

$$\varrho \frac{\partial^2 \xi}{\partial t^2} = F(\xi) = (\nabla \times Q) \times B + (j \times Q) + \nabla (\xi \cdot \nabla p) + \nabla (\gamma p \nabla \cdot \xi) \,, \tag{16}$$

where F is a linear (differential) operator. The boundary condition on ξ is

$$e \cdot \xi = 0 \,. \tag{17}$$

Equation (16) is self-contained giving $\ddot{\xi}$ in terms of ξ. (A dot denotes time differentiation). From condition (11), $B' \cdot e = 0$ follows.

(*Note*: one might consider independently perturbations not restricted as above. For instance take $\xi = \dot{\xi} = p' = B' = 0$, ϱ' arbitrary. This produces no

time variation. If p' is taken arbitrary and the others zero this corresponds to sound waves or magnetosonic waves. Similarly if $\boldsymbol{B}'$ is independent. Also it is possible to vary the equilibrium state by taking B' and p' together. As will be shown later, none of this extra freedom leads to more generality in the determination of stability.)

2·4. *Stability.* – We now consider the behaviour in time of the perturbations and in particular whether any of them can grow indefinitely. Of course, they must always remain small enough for the linear theory to apply but we may regard ξ as finite and later multiply it by a constant so small as to make the theory valid.

The standard method of studying small oscillations is the normal mode method. One looks for solutions of the form

$$\xi = \xi_n \exp[i\omega_n t],$$

so

$$-\varrho\omega_n^2\xi_n = \boldsymbol{F}(\xi_n), \qquad \xi_n \cdot \boldsymbol{e} = 0. \tag{18}$$

Assuming the ξ_n's form a complete set we expand any initial perturbation

$$\xi_0 = \sum a_n \xi_n,$$

and then

$$\xi = \sum a_n \xi_n \exp[i\omega_n t]$$

is a solution of (16) which equals ξ_0 at $t=0$. (Actually we expand $\dot{\xi}_0 = \sum b_n i\omega_n \xi_n$ also and use the fact that ξ_n^* is a solution to get a more complete solution in an obvious way.) If all ω_n are real ξ is bounded. If any ω is complex ω_m or $-\omega_m$ has a negative imaginary part and ξ is generally unbounded if $a_m \neq 0$, $\xi = \xi_m$ gives an unstable perturbation. Thus the question of stability reduces to examining the ω_n' to see if any is complex.

From this observation we see that no generality is lost in tying ϱ', p', and $\boldsymbol{B}'$ to ξ as we have done above, since if there is an instability, there is one with all quantities growing like $\exp[i\omega_n t]$ (with ω_n complex) and this instability must vanish if continued back to $t = -\infty$. Obviously, for this instability the equations before (15) are satisfied.

The search for a complex ω_n is made easier by the fact that E is self-adjoint *i.e.* for any ξ and η satisfying

$$\xi \cdot \boldsymbol{e} = \eta \cdot \boldsymbol{e} = 0,$$

on the the boundary, we have

$$\int \xi \cdot F(\eta)\,\mathrm{d}\tau = \int \eta \cdot F(\xi)\,\mathrm{d}\tau\,. \tag{19}$$

We show the self-adjointness of F later.
From this we find

a) ω_n^2 is always real.

Proof $\omega_n^{*2}\varrho\xi^* = F(\xi^*)$. This with (18) and (19) gives

$$\omega_n^2\int \varrho\xi^*\cdot\xi\,\mathrm{d}\tau = -\int \xi^*\cdot F(\xi)\mathrm{d}\tau = -\int \xi\cdot F(\xi^*)\,\mathrm{d}\tau = \omega_n^{*2}\int \varrho\xi^*\cdot\xi\,\mathrm{d}\tau\,,$$

from which *a*) follows.

Because of *a*) there is no « overstability », that is no growing oscillation can occur. Also if a mode is stable and the equilibrium is varied slightly this mode remains stable (see Fig. 4).

b) The modes are orthogonal

$$\int \varrho\xi_n\cdot\xi_m\,\mathrm{d}\tau = 0 \quad \text{if} \quad \omega_n^2 \neq \omega_m^2\,.$$

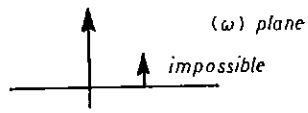

Fig. 4.

Proof

$$(\omega_n^2 - \omega_m^2)\int \varrho\xi_n\cdot\xi_m\,\mathrm{d}\tau = \int \left(\xi_n\cdot F(\xi_m) - \xi_m\cdot F(\xi_n)\right)\mathrm{d}\tau = 0\,.$$

We assume that the ξ_n are chosen orthogonal for equal ω_n's also and normalized so that

$$\int \varrho\xi_n\cdot\xi_m\,\mathrm{d}\tau = \delta_{nm}\,. \tag{20}$$

This property usually goes along with completeness for a set of functions.

c) A variational principle exists for computing ω_n's. Consider

$$\lambda = \frac{-\frac{1}{2}\int \xi\cdot F(\xi)\,\mathrm{d}\tau}{\frac{1}{2}\int \varrho\xi^2\,\mathrm{d}\tau}\,,$$

$$\delta\lambda = \frac{\int \delta\xi\cdot\left(F(\xi) + \lambda\varrho\xi\right)\mathrm{d}\tau}{\int \varrho\xi^2\,\mathrm{d}\tau}\,,$$

$\delta\lambda = 0$ for all $\delta\xi$ is equivalent to $\lambda = \omega_n^2$, $\xi = \xi_n$ for some n.

A minimum of λ gives the « most unstable » ξ_n. If λ is always positive ω_n^2

is always positive (ω_n is real). Also if λ is ever negative the smallest ω_n^2 (ω_1^2 say) is also negative and the system is unstable. Since the denominator is always positive this means we need only examine the sign of the numerator

$$\delta W = -\tfrac{1}{2}\int \boldsymbol{\xi}\cdot\boldsymbol{F}(\boldsymbol{\xi})\,\mathrm{d}\tau\,, \tag{20}$$

for all $\boldsymbol{\xi}$ with $\boldsymbol{\xi}\cdot\boldsymbol{e}=0$. We have proved $\delta W>0$ for all $\boldsymbol{\xi}$ is a necessary and sufficient condition for stability. Thus

d) There exists an energy principle for stability *i.e.* an expression $\delta W(\boldsymbol{\xi},\boldsymbol{\xi})$ quadratic in $\boldsymbol{\xi}$ such that stability can be reduced to examining the sign of $\delta W(\boldsymbol{\xi},\boldsymbol{\xi})$. δW will turn out to be the variation in potential energy of the system.

For a simple example of an energy principle consider the one-dimensional motion of a particle in a potential V:

$$\frac{\partial V}{\partial x}=0\,,$$

is the condition for equilibrium. A small motion about equilibrium is given by

$$(\delta x)''=-\frac{\partial^2 V}{\partial x^2}(\delta x)\,.$$

If $\partial^2 V/\partial x^2$ is negative the equilibrium is unstable. The sign of $\partial^2 V/\partial x^2$ may be determined by examining the second variation in ξ

$$V(x+\xi)=\frac{\xi^2}{2}\frac{\partial^2 V}{\partial x^2}=\delta W\,.$$

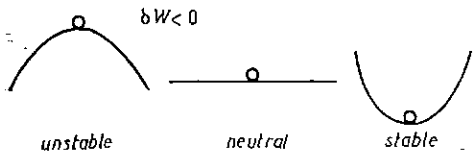

Fig. 5.

If $\delta W<0$ for some ξ then $V_{xx}<0$ and the equilibrium is unstable. Otherwise, it is stable (Fig. 5). The displacement ξ is sometimes called a virtual displacement since it is imagined to be made to test for stability of a real motion.

Similarly for n dimensions

$$\delta W=\sum\frac{\partial^2 V}{\partial x_i\,\partial x_j}\xi_i\xi_j\,,$$

and one has again an energy principle. The i component of the linearized force is $\sum_j(\partial^2 V/\partial x_i\,\partial x_j)\xi_j$.

The matrix $\partial^2 V/\partial x_i \partial x_j$ corresponds to our operator F and is obviously self-adjoint, since it is symmetric. Our theory corresponds to a continuum of dimensions but is otherwise algebraically the same.

The proof of *d*) follows the remarks in *c*) with δW given by (20). We make some more remarks on *d*) and give another proof.

The sufficiency of $\delta W > 0$ for stability does not really involve self-adjointness of F if δW is taken *a priori* to be the potential energy rather than defined by (20). For, for an unstable normal mode ω_n is imaginary

$$\delta W \sim \exp[2i\omega_n t];$$

let the kinetic energy be

$$K = \tfrac{1}{2}\int \varrho \dot{\xi}^2 \, \mathrm{d}\tau \, .$$

This is also proportional to $\exp[2i\omega_n t]$. If $\delta W > 0$ we have

$$u = K + \delta W \sim \exp[2i\omega_n t]$$

with a positive coefficient. If ω_n is unstable u must grow indefinitely which is impossible since it is a constant. Therefore ω_n is stable. This argument is essentially due to LIAPUNOFF and is used very skillfully by KRUSKAL and OBERMAN in their paper [2]. It is useful for obtaining weaker energy principles which are only sufficient for stability.

We may identify δW given by (20) with potential energy by considering

$$\frac{\partial}{\partial t}(K + \delta W) = \frac{\partial}{\partial t}\left[\int \varrho \frac{\dot{\xi}^2}{2} \mathrm{d}\tau - \frac{1}{2}\int \xi \cdot F(\xi) \, \mathrm{d}\tau\right] = \int [\varrho \ddot{\xi} \cdot \dot{\xi} - \dot{\xi} \cdot F(\xi)] \, \mathrm{d}\tau = 0 \, .$$

Here we use self-adjointness. Since K is kinetic energy δW must be the variation in potential energy.

We give another proof of the energy principle *d*)

Let $\xi = \sum a_n \xi_n$

$$\delta W = -\tfrac{1}{2}\int \xi \cdot F(\xi) \, \mathrm{d}\tau = -\tfrac{1}{2}\sum_{n,m} a_n a_m \int \xi_n \cdot F(\xi_m) = \tfrac{1}{2}\sum_{n,m} a_n a_m \omega_n^2 \int \xi_n \varrho \xi_m \, \mathrm{d}\tau \, ,$$

$$\delta W = \tfrac{1}{2}\sum_n a_n^2 \omega_n^2 \, ,$$

by (20). If all $\omega_n^2 > 0$, δW is always positive. Hence $\delta W < 0$ for some ξ implies instability. If $\omega_n^2 < 0$, $\delta W(\xi_n, \xi_n) < 0$ and thus instability $\Rightarrow \delta W < 0$ for

some a_n's (or ξ). Hence $\delta W > 0$ (for all ξ) $\Leftrightarrow$ stability. We have assumed that the ξ_n's are complete.

We wish an explicit expression for δW. From (16)

$$\delta W = -\tfrac{1}{2}\int \xi \cdot F(\xi)\, \mathrm{d}\tau =$$

$$= -\tfrac{1}{2}\int \xi \cdot [(\nabla \times \boldsymbol{Q} \times \boldsymbol{B})] + \xi \cdot \boldsymbol{J} \times \boldsymbol{Q} + \xi \cdot \nabla(\xi \cdot \nabla p) + \xi \cdot \nabla(\gamma p \nabla \cdot \xi)\,.$$

But

$$\xi \cdot (\nabla \times \boldsymbol{Q}) \times \boldsymbol{B} = -\xi \times \boldsymbol{B} \cdot \nabla \times \boldsymbol{Q} = \nabla \cdot [(\xi \times \boldsymbol{B}) \times \boldsymbol{Q}] - \nabla \times (\xi \times \boldsymbol{B}) \cdot \boldsymbol{Q}\,,$$

$$\xi \cdot \nabla(\xi \cdot \nabla p) = \nabla \cdot (\xi\, \xi \cdot \nabla p) - \nabla \cdot \xi\, \xi \cdot \nabla p\,,$$

$$\xi \cdot \nabla(\gamma p\, \nabla \cdot \xi) = \nabla \cdot (\xi \gamma p\, \nabla \cdot \xi) - \gamma p (\nabla \cdot \xi)^2\,,$$

using Gauss' theorem

$$\delta W = \tfrac{1}{2}\int [\boldsymbol{Q}^2 + \boldsymbol{J} \cdot \xi \times \boldsymbol{Q} + \gamma p (\nabla \cdot \xi)^2 + \xi \cdot \nabla p\, \nabla \cdot \xi]\, \mathrm{d}\tau -$$

$$- \int [(\xi \times \boldsymbol{B}) \times \boldsymbol{Q} + \xi(\xi \cdot \nabla p) + \xi \cdot \gamma p \nabla \cdot \xi] \cdot \mathrm{d}\boldsymbol{S}\,.$$

The last two terms vanish since $\xi \cdot \boldsymbol{e} = 0$. $(\xi \times \boldsymbol{B}) \times \boldsymbol{Q} = \boldsymbol{Q} \cdot \xi \boldsymbol{B} - \boldsymbol{B} \cdot \boldsymbol{Q} \xi$ vanishes also, since $\boldsymbol{B} \cdot \boldsymbol{e} = 0$. $\boldsymbol{B} \cdot \boldsymbol{e} = 0$ follows from (17) by the same argument as used in the nonlinearized case (p. 6).

Thus

$$\delta W = \tfrac{1}{2}\int [\boldsymbol{Q}^2 + \boldsymbol{J} \cdot \xi \times \boldsymbol{Q} + \gamma p (\nabla \cdot \xi)^2 + \xi \cdot \nabla p \nabla \cdot \xi]\, \mathrm{d}\tau\,, \tag{21}$$

and we must examine this quadratic functional of ξ for stability. This completes the derivation of the energy principle except for the proof that $\boldsymbol{F}$ is self-adjoint.

We pause to discuss the advantages of the energy principle approach to stability over the normal mode approach.

The most obvious way to check whether $\delta W > 0$ always, is to minimize it over ξ. Since it is homogeneous, it is necessary to normalize ξ. If we normalize ξ by

$$\int \varrho \xi^2\, \mathrm{d}\tau = 1\,,$$

we get back the normal mode equations by comment *c*). However, it is not necessary to choose this particular normalization but only one which makes δW bounded below. There is often a much more convenient normalization.

Sometimes δW is obviously negative by a « trivial » perturbation. In this case the energy principle has the advantage. This is facilitated by the fact that the energy has a physically intuitive significance. The first two terms represent the variation of the magnetic energy and the last two of the plasma energy.

Several important examples of this remarks are

1) Suydam's instabilities [9].

2) Gravitational instability [10].

3) Mercier's generalization of Suydam's criteria [11].

4) Sharp separation of plasma and magnetic field [4].

In each case it is sufficient to exhibit an unstable ξ. In cases 1) and 3) the ξ is localized. In case 2) the perturbation is obvious. In case 4) a complete solution is obtained in this way (see Fig. 6). (However, a more general energy principle including gravitation is used.)

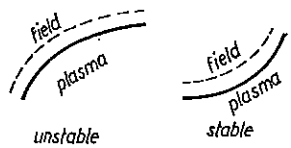

Fig. 6.

It should be remarked that whenever the energy principle is useful a corresponding trick in the normal mode technique can be found. The energy principle simply makes the trick more obvious.

In the energy principle the theorem « $\delta W > 0 \Rightarrow$ stability » is more natural. The theorem « $\delta W < 0 \Rightarrow$ instability » relies on both completeness and self-adjointness. If $\delta W < 0$ for ξ this need not imply instability in a general case since ξ need not be a normal mode. Although temporarily the system gains kinetic energy at the expense of potential energy it need not remain proportional to ξ. ξ may change direction and move into a region of δW and keep ξ finite.

Simple example. – PRENDERGAST [12] has given an example to illustrate a stable situation with $\delta W < 0$.

Consider a 2-dimensional harmonic oscillator with negative spring constant and moving in field $B \perp$ to its plane (see Fig. 7). The potential energy is $-k\xi^2$. Let the oscillator have a charge e. If $\sqrt{k/m} < \frac{1}{2}(eB/mc)$ the system is stable although $\delta W < 0$ always.

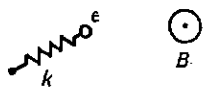

Fig. 7.

Thus it is essential to prove self-adjointness for the energy principle to work both ways.

2·5. *Proof of self-adjointness of F.* – We give two proofs of self-adjointness of F one direct and explicit and the other indirect.

Direct proof. – Let $F = F_1 + F_2$

$$F_1(\xi) = [\nabla \times (\nabla \times (\xi \times B))] \times B + \nabla(\gamma p \nabla \cdot \xi) ,$$
$$F_2(\xi) = J \times [\nabla \times (\xi \times B)] + \nabla(\xi \cdot \nabla p) ,$$

F_1 and F_2 will be separately shown to be self-adjoint.

Let η be another vector field with $\eta \cdot e = 0$

$$\int \eta \cdot F_1(\xi) \, d\tau = \int \{ - (\eta \times B) \cdot [\nabla \times (\nabla \times (\xi \times B))] + \eta \cdot \nabla(\gamma p \nabla \cdot \xi) \} d\tau =$$
$$= \int \{ \nabla \cdot [(\eta \times B) \times (\nabla \times (\xi \times B))] - [\nabla \times (\eta \times B)] \cdot [\nabla \times (\xi \times B)] +$$
$$+ \nabla \cdot (\gamma p \nabla \cdot \xi) - \nabla \cdot \eta \gamma p \nabla \cdot \xi \} .$$

But

$$(\eta \times B) \times Q = Q \cdot \eta B - Q \cdot B \eta \quad \text{and} \quad \eta \cdot e = 0$$

so when the divergence terms are integrated by Gauss' theorem they vanish. The remaining terms are completely symmetric in ξ and η and so

$$\int \eta \cdot F_1(\xi) \, d\tau = \int \xi \cdot F_1(\eta) \, d\tau ,$$

F_1 is self-adjoint. Now consider F_2

$$\int [\eta \cdot F_2(\xi) - \xi \cdot F_2(\eta)] \, d\tau = \int \{ \eta \cdot J \times (B \cdot \nabla \xi - \xi \cdot \nabla B - B \nabla \cdot \xi) +$$
$$+ \eta \cdot \nabla(\xi \cdot \nabla p) - \xi \cdot J \times [B \cdot \nabla \eta - \eta \cdot \nabla B - B \nabla \cdot \eta] - \xi \cdot \nabla \eta \cdot \nabla p \} \, d\tau .$$

But

$$- \eta \cdot (J \times B(\nabla \cdot \xi)) - \xi \cdot \nabla(\eta \cdot \nabla p) = - \eta \cdot \nabla p \, \nabla \cdot \xi - \xi \cdot \nabla(\eta \cdot \nabla p) = - \nabla \cdot (\xi \eta \cdot \nabla p)$$

which integrates to zero. Similarly

$$+ \xi \cdot J \times B(\nabla \cdot \xi) + \eta \cdot \nabla(\xi \cdot \nabla p)$$

integrates to zero.

Also

$$\eta \cdot j \times (B \cdot \nabla) \xi = \nabla \cdot [B \eta \cdot j \times \xi] - \nabla \cdot B \eta \cdot j \times \xi - (B \cdot \nabla) \eta \times j \cdot \xi - \eta \times (B \cdot \nabla) j \cdot \xi$$

and the first term integrates out while the second is zero. The third term will cancel the first term in the second bracket in the above expression. Finally

$$0 = \nabla\times(\nabla p) = \nabla(\boldsymbol{j}\times\boldsymbol{B}) = \boldsymbol{B}\cdot\nabla\boldsymbol{j} - \boldsymbol{j}\cdot\nabla\boldsymbol{B}$$

so the last term is

$$\eta\cdot(\boldsymbol{j}\cdot\nabla)\boldsymbol{B}\times\xi\,.$$

With these remarks we have

$$\int\eta\cdot\boldsymbol{F}_2(\xi) - \xi\cdot\boldsymbol{F}_2(\eta)\,\mathrm{d}\tau = \int\eta\cdot[-(\boldsymbol{j}\cdot\nabla)\boldsymbol{B}\times\xi + (\xi\cdot\nabla)\boldsymbol{B}\times\boldsymbol{j} + (\nabla\boldsymbol{B})\cdot\xi\times\boldsymbol{j}]\,\mathrm{d}\tau\,.$$

To show the bracket vanishes consider

$$[(\boldsymbol{j}\times\xi)\times\nabla]\times\boldsymbol{B} = [\xi\boldsymbol{j}\cdot\nabla - \boldsymbol{j}\xi\cdot\nabla]\times\boldsymbol{B} = \xi\times(\boldsymbol{j}\cdot\nabla)\boldsymbol{B} - \boldsymbol{j}\times(\xi\cdot\nabla)\boldsymbol{B}\,,$$

on expanding the triple product in the bracket.

Expanding the triple product with $(\boldsymbol{j}\times\xi)$ as a single unit we get

$$(\nabla\boldsymbol{B})\cdot\boldsymbol{j}\times\xi\,.$$

Equating these two we find the bracket vanishes. Thus $\boldsymbol{F}_2$ and hence $\boldsymbol{F}$ is self-adjoint.

Indirect proof. – We construct the energy of the system. The energy per unit volume of an adiabatic fluid is $p/\gamma-1$ since if we let

$$p = A\varrho^{\gamma}\,,$$

$$-\frac{1}{\tau}\int_{\infty}^{\tau} p\,\mathrm{d}\tau = \varrho\int_{0}^{\varrho} A\varrho'^{\gamma}\frac{\mathrm{d}\varrho'}{\varrho'^{2}} = \varrho\,\frac{A\varrho^{\gamma-1}}{\gamma-1}\Big|_{0}^{\varrho} = \frac{A\varrho^{\gamma}}{\gamma-1} = \frac{p}{\gamma-1}\,.$$

Therefore the energy U is

$$U = K + W = \int\left(\frac{\varrho V^2}{2} + \frac{\boldsymbol{B}^2}{2} + \frac{p}{\gamma-1}\right)\mathrm{d}\tau\,,$$

$\boldsymbol{B}^2/2$ is the energy per unit volume in the magnetic field. All we will need is

that U is constant for all motions given by (9)–(14). Explicitly using (9)–(14)

$$\dot{U} = \int \Big[- \nabla\cdot(\varrho \boldsymbol{V})\boldsymbol{V} + \boldsymbol{V}\cdot(-\nabla p + \boldsymbol{j}\times\boldsymbol{B} - \varrho \boldsymbol{V}\cdot\nabla\boldsymbol{V}) + \boldsymbol{B}\cdot\nabla\times(\boldsymbol{V}\times\boldsymbol{B}) - \\ - \frac{\gamma p}{\gamma - 1}\nabla\cdot\boldsymbol{V} - \frac{\boldsymbol{V}\cdot\nabla p}{\gamma - 1}\Big] \mathrm{d}\tau = \\ = \int \Big\{ - \nabla\cdot(\varrho \boldsymbol{V}\boldsymbol{V}) - \frac{\gamma}{\gamma - 1}\nabla\cdot(p\boldsymbol{V}) - \nabla\cdot[\boldsymbol{B}\times(\boldsymbol{V}\times\boldsymbol{B})]\Big\} \mathrm{d}\tau = 0 \,.$$

by Gauss' theorem and (15). $(\boldsymbol{B}\times(\boldsymbol{V}\times\boldsymbol{B}) = B^2\boldsymbol{V} - \boldsymbol{B}\cdot\boldsymbol{V}\boldsymbol{B}.)$

Now we wish to expand U about the equilibrium to second order in ξ. To do this we need to define ξ to next order.

We take ξ to be the absolute displacement of a fluid particle, *i. e.* a particle in equilibrium at $\boldsymbol{r}_0$ is at

$$\boldsymbol{r} = \boldsymbol{r}_0 + \xi(r_0, t) \quad \text{at time } t \,.$$

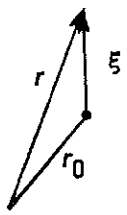

Fig. 8.

(See Fig. 8). To second order

$$\frac{\partial \xi}{\partial t}(\boldsymbol{r}_0, t) = \boldsymbol{V}(\boldsymbol{r}_0 + \xi, t) = \boldsymbol{V}(\boldsymbol{r}_0, t) + \xi\cdot\nabla\boldsymbol{V}(\boldsymbol{r}_0, t) \,.$$

In terms of ξ, it is easy to solve for $\boldsymbol{B}$ and p, and ϱ to second order in ξ. We have exactly

$$\varrho(\boldsymbol{r}_0 + \xi, t) = \varrho(\boldsymbol{r}_0, 0)\,|\boldsymbol{I} + \nabla_0\xi|^{-1} \,,$$

where $|\boldsymbol{I} + \nabla_0\xi| = \mathrm{Det}\,(\boldsymbol{I} + \nabla_0\xi)$

$$\frac{p(\boldsymbol{r}_0 + \xi, t)}{p(\boldsymbol{r}_0, 0)} = \left[\frac{\varrho(\boldsymbol{r}_0 + \xi, t)}{\varrho(\boldsymbol{r}_0, 0)}\right]^{\gamma} ,$$

$$\frac{\boldsymbol{B}(\boldsymbol{r}_0 + \xi, t)}{\varrho(\boldsymbol{r}_0 + \xi, t)} = \frac{\boldsymbol{B}_0}{\varrho_0} + \frac{\boldsymbol{B}_0}{\varrho_0}\cdot\nabla_0\xi \,,$$

$$\boldsymbol{V} = \frac{\partial \xi}{\partial t}\,\mathrm{d}\tau = \mathrm{d}\tau_0\,|\boldsymbol{I} + \nabla_0\xi| \,.$$

We do not make use of these explicit expressions except to show the possibility of expanding U.

Write

$$U = A(\dot{\xi}) + B(\xi) + K(\dot{\xi}, \dot{\xi}) + M(\dot{\xi}, \dot{\xi}) + \delta W(\xi, \xi) \,,$$

where A and B are linear functionals, K, M, and δW are bilinear functionals and we may take K and δW as symmetric.

What are A and K? To find them we take

$$\xi = 0\,; \quad \text{so} \quad V = \dot{\xi}\,, \quad \delta \boldsymbol{B} = \delta p = \delta \varrho = 0\,, \quad \mathrm{d}\tau = \mathrm{d}\tau_0$$

and

$$U = \int \frac{\varrho' \dot{\xi}^2}{2}\,\mathrm{d}\tau\,.$$

Thus

$$K = \tfrac{1}{2}\int \varrho \dot{\xi}^2 \qquad \text{and} \qquad A = 0\,.$$

Differentiate (23).

$$\ddot{U} = 0 = B(\dot{\xi}) + O(\xi^2)$$

so $B = 0$, since $\dot{\xi}$ is arbitrary ($\dot{U} = 0$ for all ξ and $\dot{\xi}$ subject to $\xi \cdot \boldsymbol{e} = \dot{\xi} \cdot \boldsymbol{e} = 0$).

Also

$$\dot{U} = 0 = 2K[\ddot{\xi}, \dot{\xi}] + M(\dot{\xi}, \dot{\xi}) + M(\xi, \ddot{\xi}) + 2\,\delta W(\dot{\xi}, \xi)\,.$$

But by (16)

$$\ddot{\xi} = \boldsymbol{F}(\xi)/\varrho$$

so

$$2K\left[\frac{\boldsymbol{F}(\xi)}{\varrho}, \dot{\xi}\right] + M(\dot{\xi}, \dot{\xi}) + M\left[\xi, \frac{\boldsymbol{F}(\xi)}{\varrho}\right] + 2\,\delta W[\dot{\xi}, \xi] = 0\,,$$

and this must hold for all ξ and $\dot{\xi}$. Setting here $\xi = 0$ we have

$$M(\dot{\xi}, \dot{\xi}) = 0\,,$$

the rest of the bilinear functionals vanishing since a bilinear functional vanishes if either argument vanishes. Also from $\dot{\xi} = 0$

$$M\left[\xi, \frac{\boldsymbol{F}(\xi)}{\varrho}\right] = 0\,.$$

Thus

$$K\left[\frac{\boldsymbol{F}(\xi)}{\varrho}, \dot{\xi}\right] = 2\,\delta W(\dot{\xi}, \xi)\,.$$

But δW is symmetrical in ξ and $\dot{\xi}$ so

$$K\left[\frac{\boldsymbol{F}(\xi)}{\varrho}, \dot{\xi}\right] = K\left[\frac{\boldsymbol{F}(\dot{\xi})}{\varrho}, \xi\right],$$

and setting $\dot{\xi} = \eta$ and using

$$K = \tfrac{1}{2}\int \varrho \ddot{\xi}^2 ,$$

we have again the self-adjointness of F obtained above by a direct method. Let $\dot{\xi} = \xi$

$$\delta W(\xi, \xi) = -K\left[\frac{\boldsymbol{F}(\xi)}{\varrho}, \xi\right] = -\frac{1}{2}\int \xi \cdot \boldsymbol{F}(\xi)\, \mathrm{d}\tau ,$$

and we have shown δW given by (20) is actually the potential energy.

Of the two proofs of self-adjointness the last generalizes the easiest to more complicated cases. *I.e.* (vacuum, adiabatic theory, double adiabatic theory) since it involves the least amount of information on $\boldsymbol{F}$. Notice it is fundamental that ξ and $\dot{\xi}$ be independent as well as that $A = 0$. Also in the establishment of an energy principle we need K to be positive definite.

3. – Adiabatic theory.

3·1. *The Boltzmann equation.* – The *adiabatic* theory corresponds to the limit of no collisions and small gyration radius.

Therefore we start with the Vlasov or collisionless Boltzmann equation for each kind of particle, ions and electrons

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f + \frac{e}{m}\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\nabla_v f = 0 . \tag{1}$$

We pass to the limit of small gyration radius formally by taking very large e (see ref. [5]). We will develop an asymptotic series for all physical quantities in $1/e$. The expansion of the Boltzmann equation follows that of CHEW, GOLDBERGER and LOW [3].

Write

$$f = f^0 + f^1 + \cdots ,$$

where f^n is of order $(1/e)^n$. Then to lowest order

$$\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\nabla_v f = 0 . \tag{2}$$

When necessary f will be distinguished by a subscript i or e. Also the charge and mass. $f(t, \boldsymbol{r}, \boldsymbol{v})$ is the density in velocity space and position space of electrons or ions, ($f\,\mathrm{d}^3v\,\mathrm{d}^3x$ is the number in $\mathrm{d}^3v\,\mathrm{d}^3x$).

Write

$$\boldsymbol{E} = -\frac{\boldsymbol{\alpha}\times\boldsymbol{B}}{c} + E_{\parallel}\boldsymbol{n}\,, \tag{3}$$

where

$$\boldsymbol{n} = \boldsymbol{B}/|B|\,, \qquad \boldsymbol{\alpha}\cdot\boldsymbol{n} = 0\,,$$

$$\boldsymbol{v} = \boldsymbol{\alpha} + \boldsymbol{s} + q\boldsymbol{n}\,, \qquad \boldsymbol{s}\cdot\boldsymbol{n} = 0\,. \tag{4}$$

(2) becomes

$$\frac{\boldsymbol{s}\times\boldsymbol{B}}{c}\cdot\nabla_v f + E_{\parallel}\boldsymbol{n}\cdot\nabla_v f = 0\,.$$

From this equation f is constant along a helix in velocity space which extends to infinity if $E_{\parallel}\neq 0$. Thus f may not approach zero at $q=\infty$ if $E_{\parallel} = O(1)$.

Hence we take

$$E_{\parallel} = O(1/c) \tag{5}$$

and regard $\boldsymbol{E}$ and $\boldsymbol{B}$ as known series in $1/c$. Later on we will expand Maxwell's equations in $1/c$ to determine $\boldsymbol{E}$ and $\boldsymbol{B}$.

From

$$\boldsymbol{s}\times\boldsymbol{B}\cdot\nabla_v f = 0 \tag{6}$$

we get

$$f^0 = F^0(t, \boldsymbol{r}, w, q)\,, \tag{7}$$

where

$$w = s^2/2\,. \tag{8}$$

(7) is the general solution of (6) where F^0 is arbitrary. From (7) F^0 may have any behavior in time. To determine it we write (1) to next order

$$\frac{\partial f^0}{\partial t} + \boldsymbol{v}\cdot\nabla f^0 + \frac{e}{mc}\boldsymbol{s}\times\boldsymbol{B}\cdot\nabla_v f' + \frac{\varepsilon}{m}\boldsymbol{n}\cdot\nabla_v f^0 = 0\,, \tag{9}$$

where $E_{\parallel} = \varepsilon e$. Let φ be the angle between $\boldsymbol{s}$ and some arbitrary vector $\perp$ to $\boldsymbol{B}$, say $\boldsymbol{n}\cdot\nabla\boldsymbol{n}$. Then

$$\boldsymbol{s}\times\boldsymbol{B}\cdot\nabla_v f' = |B|\frac{\partial f'}{\partial\varphi}\,,$$

and (9) may be solved for f'. However f' must be periodic in φ and this condition is obtained by integrating (9) over φ (for constant t, $\boldsymbol{r}$, w, q)

$$\int_0^{2\pi} \left[\frac{\partial f^0}{\partial t} + \boldsymbol{v}\cdot\nabla f^0 + \frac{\varepsilon}{m}\cdot\nabla_v f^0\right] \mathrm{d}\varphi = 0\,. \tag{10}$$

Equation (10) gives $\partial F^0/\partial t$. However, in (10) the time derivative $\partial f^0/\partial t$ is at fixed $\boldsymbol{r}$ and $\boldsymbol{v}$.

Let us formally change variables

$$t' = t\,,$$

$$\boldsymbol{r}' = \boldsymbol{r}\,,$$

$$\boldsymbol{s}' = \boldsymbol{v} - \boldsymbol{\alpha} - \boldsymbol{v}\cdot\boldsymbol{nn}\,,$$

$$q' = \boldsymbol{v}\cdot\boldsymbol{n}\,,$$

$$w' = \frac{s'^2}{2}\,,$$

$$\frac{\partial f_0}{\partial t} = \frac{\partial F_0}{\partial t'} + \frac{\partial F_0}{\partial \boldsymbol{r}'}\frac{\partial \boldsymbol{r}'}{\partial t} + \frac{\partial F_0}{\partial w'}\frac{\partial w'}{\partial t} + \frac{\partial F_0}{\partial q'}\frac{\partial q'}{\partial t}\,,$$

$$\frac{\partial t'}{\partial t} = 1 \qquad \frac{\partial \boldsymbol{r}'}{\partial t} = 0$$

$$\frac{\partial w'}{\partial t} = \boldsymbol{s}'\frac{\partial \boldsymbol{s}'}{\partial t} = \boldsymbol{s}'\cdot\frac{\partial}{\partial t}(-\boldsymbol{\alpha} - \boldsymbol{v}\cdot\boldsymbol{nn}) = -\boldsymbol{s}'\frac{\partial \boldsymbol{\alpha}}{\partial t} - q'\boldsymbol{s}'\cdot\frac{\partial \boldsymbol{n}}{\partial t}\,,$$

$$\frac{\partial q'}{\partial t} = \boldsymbol{v}\cdot\frac{\partial \boldsymbol{n}}{\partial t} = (\boldsymbol{\alpha} + q\boldsymbol{n} + \boldsymbol{s}')\cdot\frac{\partial \boldsymbol{n}}{\partial t} = (\boldsymbol{\alpha} + \boldsymbol{s}')\cdot\frac{\partial \boldsymbol{n}}{\partial t}\,.$$

Also $\partial\boldsymbol{\alpha}/\partial t$ and $\partial\boldsymbol{n}/\partial t$ are independent of φ while $\oint \boldsymbol{s}'\,\mathrm{d}\varphi = 0$. Thus

$$\frac{1}{2\pi}\int \frac{\mathrm{d}f_0}{\partial t}\,\mathrm{d}\varphi = \frac{\partial F_0}{\partial t'} + \boldsymbol{\alpha}\cdot\frac{\partial \boldsymbol{n}}{\partial t}\frac{\partial F^0}{\partial \boldsymbol{q}'}\,.$$

In the same way we find

$$\frac{1}{2\pi}\int \boldsymbol{v}\cdot\nabla f_0\,\mathrm{d}\varphi = (\boldsymbol{\alpha} + q'\boldsymbol{n})\cdot\nabla' F^0 - w'(\nabla\cdot\boldsymbol{\alpha} + q'\nabla\cdot\boldsymbol{n} - \boldsymbol{nn}:\nabla\cdot\boldsymbol{\alpha})\frac{\partial F}{\partial w'} +$$

$$+ (\boldsymbol{\alpha\alpha}:\nabla\boldsymbol{n} + q'\boldsymbol{\alpha n}:\nabla\boldsymbol{n} + w'\nabla\cdot\boldsymbol{n})\frac{\partial F}{\partial \boldsymbol{q}'}\,,$$

$$\frac{1}{2\pi}\int \frac{\varepsilon}{m}\boldsymbol{n}\cdot\nabla_v f_0\,\mathrm{d}\varphi' = \frac{\varepsilon}{m}\frac{\partial F_0}{\partial q'}\,,$$

where we use

$$\int \boldsymbol{s}\boldsymbol{s}\,\mathrm{d}\varphi = w(I - \boldsymbol{n}\boldsymbol{n}) \qquad I \text{ the unit dyadic},$$

in the first equation, and also the notation

$$\boldsymbol{a}\boldsymbol{b}:\nabla\boldsymbol{c} = \boldsymbol{a}\cdot(\boldsymbol{b}\cdot\nabla\boldsymbol{c})\,.$$

Combining these results with (10) we obtain

$$(11)\qquad \frac{\partial F^0}{\partial t} + (\boldsymbol{\alpha} + q\boldsymbol{n})\cdot\nabla F^0 - w(\nabla\cdot\boldsymbol{\alpha} + q\nabla\cdot\boldsymbol{n} - \boldsymbol{n}\boldsymbol{n}:\nabla\boldsymbol{\alpha})\frac{\partial F^0}{\partial q} + \\ + \left(\boldsymbol{\alpha}\boldsymbol{\alpha}:\nabla\boldsymbol{n} + q\boldsymbol{\alpha}\boldsymbol{n}:\nabla\boldsymbol{n} + w\nabla\cdot\boldsymbol{n} + \boldsymbol{\alpha}\frac{\partial\boldsymbol{n}}{\partial t} + \frac{\varepsilon}{m}\right)\frac{\partial F^0}{\partial q} = 0\,.$$

We have now dropped the primes.

Equation (11) is the condition on $f^0 = F^0(t, \boldsymbol{r}, w, q)$ that we can solve for f'. Also it gives the time behavior of F^0. F^0 can no longer be taken arbitrarily as a function of t, $\boldsymbol{r}$, w, q as one would conclude from the zeroth order equation, but must be taken to solve (11).

Incidentally we know that

$$(12)\qquad \frac{\partial F^0}{\partial t} + \dot{\boldsymbol{r}}\cdot\nabla F^0 + \dot{w}\frac{\partial F^0}{\partial w} + \dot{q}\,\frac{\partial F^0}{\partial q} = 0\,,$$

so we get by comparison

$$\dot{\boldsymbol{r}} = \boldsymbol{\alpha} + q\boldsymbol{n}\,,$$

$$\dot{w} = -\nabla\cdot\boldsymbol{\alpha} - q\nabla\cdot\boldsymbol{n} + \boldsymbol{n}\boldsymbol{n}:\nabla\boldsymbol{\alpha}\,,$$

$$(13)\qquad \dot{q} = \boldsymbol{\alpha}\boldsymbol{\alpha}:\nabla\boldsymbol{n} + q\boldsymbol{\alpha}\boldsymbol{n}:\nabla\boldsymbol{n} + w\nabla\cdot\boldsymbol{n} + \boldsymbol{\alpha}\frac{\partial\boldsymbol{n}}{\partial t} + \frac{\varepsilon}{m}\,,$$

to lowest order in $1/e$. The right-hand side of these equations represent the time derivatives averaged over a cyclotron period. These equations suggest an alternative way to obtain the Boltzmann equation to lowest order. Derive (13) directly from the equations of motion and use (12) to derive (11).

3·2. *Maxwell's equations to zero order.* – Equation (11) gives F_i^0 in terms of $\boldsymbol{\alpha}$, $\boldsymbol{n}$, ε as functions of $\boldsymbol{r}$ and t, Also the definitions of q and w depend on $\boldsymbol{\alpha}$ and $\boldsymbol{n}$.

To find the behavior of $\boldsymbol{\alpha} = (\boldsymbol{E}\times\boldsymbol{B})/B^2$, $\boldsymbol{n} = \boldsymbol{B}/|B|$ and $\varepsilon = (\boldsymbol{E}\cdot\boldsymbol{n})\boldsymbol{n}$ we must

express Maxwell's equation to lowest order. Our development is the same as that sketched by KRUSKAL in the Les Houches notes [13].

Maxwell's equations are

$$\nabla \cdot \boldsymbol{B} = 0 \,, \tag{14}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\nabla \times \boldsymbol{E} \,, \tag{15}$$

$$\nabla \times \boldsymbol{B} = \frac{4\pi \boldsymbol{J}}{c} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \,, \tag{16}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi\sigma \,, \tag{17}$$

where σ is the charge density. $\boldsymbol{J}$ and σ are given by

$$\boldsymbol{J} = \sum e \int f \boldsymbol{v} \, \mathrm{d}^3 v \,, \tag{18}$$

$$\sigma = \sum e \int f \, \mathrm{d}^3 v \,, \tag{19}$$

where the sum is over ions and electrons. [We must remember $\boldsymbol{E}$ and $\boldsymbol{B}$ are also expansions in $1/e$ starting with $O(1)$.]

(14) to lowest order is the same, as is (15). In (16) and (17) $\boldsymbol{J}$ and σ are minus first order in $1/e$ by (18) and (19), so (16) and (17) with (18) and (19) become to lowest order

$$\boldsymbol{J}^{-1} = \sum e \int f^0 \boldsymbol{v} \, \mathrm{d}^3 v = 0 \,, \tag{20}$$

$$\sigma^{-1} = \sum e \int f^0 \, \mathrm{d}^3 v = 0 \,. \tag{21}$$

But $f_0 = F^0(t, \boldsymbol{r}, w, q)$. Introduce w and q as integration variables. We get

$$\int f^0 \boldsymbol{v} \, \mathrm{d}^3 v = \int F^0(\boldsymbol{\alpha} + q\boldsymbol{n} + \boldsymbol{s}) 2\pi \, \mathrm{d}w \, \mathrm{d}q \,.$$

The s term vanishes and α is constant so

$$\int f^0 \boldsymbol{v} \, \mathrm{d}^3 v = N^0 \boldsymbol{\alpha} + 2\pi \int F^0 q \, \mathrm{d}q \, \mathrm{d}w \,,$$

where

$$N^0 = 2\pi \int F^0 \, \mathrm{d}q \, \mathrm{d}w \,,$$

is the density. Hence by (21) and (20)

$$\boldsymbol{J}^{-1} = \sum n^0 U_{\parallel}^0 e\boldsymbol{n} = 0$$

or

$$U_{\parallel}^i = U_{\parallel}^e ,$$

where

$$N^0 U_{\parallel} = 2\pi \int q F^0 \, \mathrm{d}q \, \mathrm{d}w .$$

Thus Maxwell's equations to minus first order give

$$\sum e \int F^0 \, \mathrm{d}w \, \mathrm{d}q = 0 , \tag{22}$$

$$\sum e \int F^0 q \, \mathrm{d}w \, \mathrm{d}q = 0 . \tag{23}$$

It is easily shown from (11) that the time derivative of (22) is zero if (23) is satisfied. (This is just $(\partial\sigma^{-1}/\partial t) + \nabla \cdot \boldsymbol{J}^{-1} = 0$.)

Similarly the time derivative of (23) gives

$$\sum \frac{Ne^2}{m} E_{\parallel} = \sum \frac{e}{m} \boldsymbol{n} \cdot (\nabla \cdot \mathbf{P}^0) , \tag{24}$$

where

$$\mathbf{P}^0 = m \int (\boldsymbol{v} - \boldsymbol{\alpha} - v_{\parallel} \boldsymbol{n})(\boldsymbol{v} - \alpha - v_{\parallel} \boldsymbol{n}) f^0 \, \mathrm{d}^3 v ,$$

is the zero order pressure. This may be obtained more simply by taking a moment of the unexpanded Boltzmann equation for ions and electrons dotting with $\boldsymbol{n}$ and subtracting (see ref. [13]). It is easily shown that

$$\mathbf{P}^0 = p_{\perp}(I - \boldsymbol{nn}) + p_{\parallel} \boldsymbol{nn} , \tag{25}$$

where

$$p_{\perp} = m \int w F^0 2\pi \, \mathrm{d}w \, \mathrm{d}q , \tag{26}$$

$$p_{\parallel} = m \int q^2 F^0 2\pi \, \mathrm{d}w \, \mathrm{d}q . \tag{27}$$

Thus, Maxwell's equations to lowest order (together with (11)) give $E_{\parallel}$ and $\partial \boldsymbol{B}/\partial t$ but not $\boldsymbol{E}_{\perp}$. (22) and (23) are side conditions which if once satisfied are always satisfied. (14) is also a side condition.

$\partial \boldsymbol{E}/\partial t$ is given by (16) to zeroth order. Hence we must proceed to zeroth order in (16) to find $\partial \boldsymbol{E}/\partial t$ (strictly speaking (16) is three equations). The component parallel to $\boldsymbol{n}$ is minus first order, which we used to find $E_{\parallel}$. The part perpendicular to $\boldsymbol{n}$ is zero order which we use to find $E_{\perp}$).

To proceed to zero order in (16) we need $\boldsymbol{J}$ to zero order which by (18) involves f'. Because of (11) eq. (9) may be solved for f' but not uniquely; only up to a function of t, $\boldsymbol{r}$, w and q as in the case of the solution for f^0 (at first). However such a function gives no component of $\boldsymbol{J}$ perpendicular to $\boldsymbol{B}$ except for the convected current $\sigma^0\boldsymbol{\alpha}$, so $J_{\perp}$ is uniquely determined in terms of F^0 (and E^0). We find $\boldsymbol{j}$ by multiplying (19) by $m\boldsymbol{v}$ and integrating. Let $\boldsymbol{U}=\boldsymbol{\alpha}+U_{\parallel}\boldsymbol{n}$. The result is

$$\varrho^0 \frac{\mathrm{d}\boldsymbol{U}^0}{\mathrm{d}t} = \varrho^0\left(\frac{\partial \boldsymbol{U}^0}{\partial t} + \boldsymbol{U}^0\cdot\nabla\boldsymbol{U}^0\right) = \boldsymbol{J}^0\times\boldsymbol{B}^0 - \nabla\cdot\mathbf{P}^0 + \sigma^0\boldsymbol{E}^0 . \tag{28}$$

Thus the perpendicular part of (28) gives $\boldsymbol{J}_{\perp}$. The parallel part of (28) is already satisfied in virtue of (11). Remember $\boldsymbol{E}^0\cdot\boldsymbol{n}=0$, $\varrho=\sum mN$. $\mathbf{P}$ is here the total pressure. (28) is obviously the equation of motion. σ^0 is found from (17) to zero order. Solving (28) for $\boldsymbol{J}_{\perp}$ we may substitute in (16) to find $\partial \boldsymbol{E}_{\perp}/\partial t$.

This would complete our system of equations for zeroth order quantities. This system is (11), (15), $(16)_{\perp}$ and (24) with side conditions (14), (22) and (23). [$(16)_{\perp}$ means the perpendicular part of (16), the parallel part is actually (24)] (11) is an equation for F^0, (15) for $\boldsymbol{B}$, $(16)_{\perp}$ for $\boldsymbol{E}_{\perp}$ and (24) for $\boldsymbol{E}_{\parallel}$. In (11), $\boldsymbol{\alpha}$, $\boldsymbol{n}$, and ε occur. These are defined by (3), $\boldsymbol{B}/|\boldsymbol{B}|$ and $e\boldsymbol{E}_{\parallel}$ respectively. In $(16)_{\perp}$, $\boldsymbol{j}_{\perp}$ occurs which is defined by $(28)_{\perp}$. In (24) $\mathbf{P}$, and N occur which are given by (25)–(27) and $2\pi\int F^0\,\mathrm{d}q\,\mathrm{d}w$ respectively. σ^0 in eq. (28) is given by (17). These equations which form the system include all of the Boltzmann and Maxwell equations each to their lowest order. $(16)_{\perp}$ and $(16)_{\parallel}$ are of different orders of course. (17) is taken to minus first and zeroth order but in zeroth order it determines σ^0 which gives a condition on f' which we do not use. It is clear that this procedure may be iterated to obtain a set of equations valid to first order [*i.e.* order $1/e$]. All those conditions on zeroth-order quantities which are necessary conditions for the solutions of the first order equations have been obtained. Thus, we have attained our goal of determining a system of equations for the zeroth-order quantities.

In these notes we shall work only with lowest order quantities. In general, these equations are valid if 1) the gyration radius and Debye length of both ions and electrons are small compared to macroscopic lengths and 2) both gyration frequencies and plasma frequencies are large compared to macroscopic frequencies. It is not always the case that these conditions are sufficient for finding stability from the lowest order equations. For instance, if ω/Ω is of

order $(a/R)^2$ where ω and R are a characteristic macroscopic frequency and length, and Ω and a are the gyration frequency and radius, then it is necessary to proceed to next order in the expansion to determine stability. (See the paper of ROSENBLUTH, KRALL and ROSTOKER [14].)

3·3. *Summary of equations for the adiabatic theory.* – For convenience we collect in one place the zero order system of equations. However, we note first that instead of solving (28) for $\boldsymbol{J}$, and substituting in (17) we may solve (17) for $\boldsymbol{J}_\perp$ and substitute in (28). The result is

$$\varrho \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = -\nabla\cdot\mathbf{P} + \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} - \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\times\frac{\boldsymbol{B}}{4\pi} + \frac{\nabla\cdot\boldsymbol{EE}}{4\pi}\,. \tag{29}$$

Since $\boldsymbol{E}^0 = -\boldsymbol{\alpha}\times\boldsymbol{B}/c$, the last two terms are small by the factor $B^2/\varrho c^2$ when compared with the left-hand side. We may thus write (29) as

$$\varrho \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = -\nabla\cdot\mathbf{P} + \boldsymbol{J}\times\boldsymbol{B}\,, \tag{30}$$

where now $4\pi\boldsymbol{J} = \nabla\times\boldsymbol{B}$.

Of course, only the perpendicular part of (30) is independent of (11). To make our scheme closer to the fluid theory we derive the continuity equation from (11). Our equations may now be written:

Continuity:

$$\frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\boldsymbol{U}) = 0\,. \tag{31}$$

Momentum:

$$\varrho \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = \boldsymbol{J}\times\boldsymbol{B} - \nabla\cdot\mathbf{P}\,. \tag{32}$$

Energy:

$$\mathbf{P} = p_\parallel \boldsymbol{nn} + p_\perp(I - \boldsymbol{nn})\,, \tag{33}$$

$$p_\parallel = \sum m\int F^0(q - U_\parallel)^2 2\pi\,\mathrm{d}q\,\mathrm{d}w\,, \tag{34}$$

$$p_\perp = \sum m\int F^0 w 2\pi\,\mathrm{d}q\,\mathrm{d}w\,, \tag{35}$$

$$\frac{\partial F^0}{\partial t} + (\boldsymbol{\alpha} + q\boldsymbol{n})\cdot\nabla f_0 + w(\boldsymbol{nn}:\nabla\boldsymbol{\alpha} - \nabla\cdot\boldsymbol{\alpha} - q\nabla\cdot\boldsymbol{n})\frac{\partial F_0}{\partial w_i} + \left(\boldsymbol{\alpha}\cdot\frac{\partial\boldsymbol{n}}{\partial t} + \boldsymbol{\alpha\alpha}:\nabla\boldsymbol{n} + q\boldsymbol{\alpha n}:\nabla\boldsymbol{n} + \frac{w}{r_L}\nabla\cdot\boldsymbol{n} + \frac{eE_\parallel}{m}\right)\frac{\partial F_0}{\partial q} = 0\,, \tag{36}$$

$$\boldsymbol{\alpha} = \boldsymbol{U} - \boldsymbol{U}\cdot\boldsymbol{nn} \tag{37}$$

Maxwell's equations:

$$\nabla \cdot \boldsymbol{B} = 0 \,, \tag{38}$$

$$\nabla \times \boldsymbol{B} = \boldsymbol{J} \,, \tag{39}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{\alpha} \times \boldsymbol{B}) \,, \tag{40}$$

$$\sum e \int F^0 \, \mathrm{d}q \, \mathrm{d}w = 0 \,, \tag{41}$$

$$\sum e \int F^0 q \, \mathrm{d}q \, \mathrm{d}w = 0 \,. \tag{42}$$

Ohm's law:

$$E_{\parallel} = \frac{\sum (e/m) \boldsymbol{n} \cdot (\nabla \cdot \mathbf{P})}{\sum N e^2 / M} \,. \tag{43}$$

Definitions (besides $\mathbf{P}$ and $\boldsymbol{\alpha}$):

$$\boldsymbol{n} = \frac{\boldsymbol{B}}{|B|} \,, \tag{44}$$

$$\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} \,, \tag{45}$$

$$\sum \text{ means sum over } i \text{ and } e \,, \tag{46}$$

$$N = \int F \, \mathrm{d}^3 v = \int F 2\pi \, \mathrm{d}q \, \mathrm{d}w \,. \tag{47}$$

The names are in general heuristic. The equations are complete and independent. New dependent variables have been taken to replace $\boldsymbol{E}$, $\boldsymbol{B}$ and F^0. For instance,

$$\varrho = \sum m \int f \mathrm{d}^3 v \,, \tag{48}$$

originally, but this definition can be replaced by a differential equation (31). (48) is now a side condition. Similarly we have the side condition

$$\varrho \, \boldsymbol{U} \cdot \boldsymbol{n} = \sum m \int F^0 q 2\pi \, \mathrm{d}w \, \mathrm{d}q \,. \tag{49}$$

$\boldsymbol{E}_{\perp}$ has been eliminated. The Boltzmann equation now plays only the role of giving the equation of state. (38), (41) and (42) are side conditions. Rationalized units are used for $\boldsymbol{J}$ and $\boldsymbol{B}$ as in the fluid theory.

The system is (31)–(49). The differential equations are (31), (32), (36), (40), for ϱ, $\boldsymbol{u}$, F^0, $\boldsymbol{B}$. The definitions are (33)–(35), (37), (39), (43)–(47). The side conditions (initial conditions) are (38), (41), (42), (48), (49).

This system of equations is for all quantities to zero order. It is in principle easy to develop the system of equations to next order or to any order in $1/e$ by iteration.

It is possible to simplify the Boltzmann equation somewhat by introducing the magnetic moment $\nu = w/\beta$ for a particle in place of w, where $\beta = |B|$. Also it is possible to write

$$E_{\parallel} = -\boldsymbol{n}\cdot\nabla\psi\,, \tag{50}$$

where this equation of course only expresses part of E' as a gradient. Then let

$$\varepsilon = w + \frac{q^2}{2} + \frac{e}{m}\psi\,, \tag{51}$$

so

$$q = \sqrt{2\left(\varepsilon - \nu\beta - \frac{e}{m}\psi\right)}\,. \tag{52}$$

$$w = \nu\beta\,. \tag{53}$$

We may transform the Boltzmann equation to these variables, t, $\boldsymbol{v}$, ν and ε just as we did from trv to $trwq$. The notation is the same as that of Kruskal and Oberman [2]. The introduction of ψ is due to NEWCOMB [15].

The equation becomes after some algebra

$$\frac{\partial f}{\partial t} + (\boldsymbol{\alpha} + q\boldsymbol{n})\cdot\nabla f + \Big[\nu\beta\boldsymbol{nn}\cdot\nabla\boldsymbol{\alpha} - \nu\beta\nabla\cdot\boldsymbol{\alpha} - q^2\boldsymbol{nn}:\nabla\boldsymbol{\alpha} + q\boldsymbol{n}\cdot\nabla\frac{\alpha^2}{2} + \frac{e}{m}\frac{D\psi}{Dt}\Big]\frac{\partial f}{\partial q} = 0\,, \tag{54}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{\alpha}\cdot\nabla\,.$$

In deriving (54) we had to use (40) and (41) to show

$$\boldsymbol{\alpha}\cdot\frac{D\boldsymbol{n}}{Dt} = \boldsymbol{\alpha n}:\nabla\boldsymbol{\alpha} = \frac{1}{2}\boldsymbol{n}\cdot\nabla\frac{\alpha^2}{2}\,.$$

Also we have set

$$f(t, \boldsymbol{r}, \nu, \varepsilon) = 2\pi F^0(t, \boldsymbol{r}, w, q) \tag{55}$$

and in (54) q is given by (52) in terms of t, $\boldsymbol{r}$, ν, ε. f is a different function than $f(t, \boldsymbol{r}, \boldsymbol{v})$.

We note that there is no coefficient of $\partial f/\partial \nu$ which corresponds to the fact that the zero-order ν is a constant of the motion. Also the coefficient of $\partial f/\partial t$ vanishes if α and $\partial\psi/\partial t$ vanish which corresponds to the fact that for $\alpha = 0$ $E^0 = 0$) and $E_\parallel$ constant, ε is a constant of the motion. These facts may be derived directly from eq. (13).

In computing moments we also need to know the Jacobian of the transformation to $\nu\varepsilon$ which if β/q $(2\pi\,(\beta/q)\,\mathrm{d}\nu\,\mathrm{d}\varepsilon = \mathrm{d}^3V)$. Hence

$$
(56)\qquad \begin{cases} N &= \sum_\pm \int \frac{\beta}{q}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon f\,, \\ U_\parallel &= \sum_\pm \int \frac{\beta}{q}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon f q\,, \\ p_\perp &= \sum_\pm m \int \frac{\beta}{q}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon f \nu\beta\,, \\ p_\parallel &= \sum_\pm m \int \frac{\beta}{q}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon\, f q^2\,, \end{cases}
$$

the factor 2π having been absorbed into f. Since two values of q correspond to one of f we must sum the integrals over the + and — values. Thus the $\sum_\pm$.

Note also for toroidal situations ψ is in general multivalued which is permissible but awkward. We shall, henceforth, restrict ourselves to situations where ψ may be taken singlevalued. Such a situation is a mirror machine where all the particles are « trapped », turned around by the magnetic field as in a mirror machine.

Boundary conditions. – We will assume the simplest boundary conditions namely $\boldsymbol{\alpha}\cdot\boldsymbol{n} = 0$ on all boundaries. The magnetic lines of force of the mirror machine are assumed to enter a rigid infinitely conducting wall (see Fig. 9).

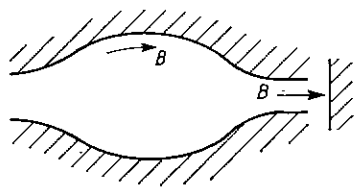

Fig. 9.

For particles which are turning around we have

$$
(57)\qquad f^+(t, \boldsymbol{r}, \nu, \varepsilon) = f^-(t, \boldsymbol{r}, \nu, \varepsilon) \qquad \text{when } q = 0.
$$

Where the plus and minus refer to the different signs of q as one approaches the point where $q = 0$.

We now consider the consequences of the equations given in the summary with (36) replaced by (54) and where appropriate integrals over w and q are

replaced by integrals over ε and ν (and summed over $\pm$) with the the corresponding factor of β/q.

3·4. *Static equilibrium.* – We first consider the case of a static equilibrium satisfying the equations of the adiabatic theory given in Section 3·3. The side conditions now become full equations. Denoting the equilibrium f by g we find from the Boltzmann equation with $\alpha = 0$ $(\boldsymbol{u} = 0)$

$$q\,\boldsymbol{n}\cdot\nabla g = 0 \tag{58}$$

or g is constant along each line of force L

$$g = g(L, \nu, \varepsilon)\,. \tag{59}$$

Also from (42) and $u_{\parallel} = 0$

$$0 = \int \frac{\beta}{|q|}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon f q g\,. \tag{60}$$

The other non-trivial equations are

$$\nabla\cdot\mathbf{P} = \boldsymbol{J}\times\boldsymbol{B}\,, \tag{61}$$

$$\boldsymbol{J} = \nabla\times\boldsymbol{B}\,, \tag{62}$$

$$\nabla\cdot\boldsymbol{B} = 0\,, \tag{63}$$

$$\textstyle\sum_{\pm i,e} e \int \frac{\beta}{|q|}\, g\,\mathrm{d}\nu\,\mathrm{d}\varepsilon = 0\,. \tag{64}$$

The other equations including (43) become redundant. The simplicity of (58) explains the choice of ε as an independent variable.

3·5. *Linearized theory.* – As in the fluid theory we do not consider any arbitrary perturbation of the equations of Section 3·3 from the state given in Section 3·4 but restrict ourselves to those satisfying constraints which do not involve loss in generality with regard to stability.

Let

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \boldsymbol{\alpha}\,. \tag{65}$$

Then since $\boldsymbol{B}\cdot\boldsymbol{\alpha} = 0$

$$\frac{\partial}{\partial t}(\boldsymbol{\xi}\cdot\boldsymbol{B}) = \boldsymbol{B}\cdot\boldsymbol{\alpha} + \boldsymbol{\xi}\cdot\frac{\partial \boldsymbol{B}}{\partial t} = O(\xi^2)\,. \tag{66}$$

Thus we may assume

$$\boldsymbol{\xi}\cdot\boldsymbol{B}=0\,. \tag{71}$$

We take

$$\boldsymbol{B}'=\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B})\,, \tag{67}$$

which is a solution of the linearized equations. (Remember an arbitrary constant in time can be added to (67) in general, so (67) involves a constraint on the perturbation.)

The linearized momentum equation is

$$\varrho\,\frac{\mathrm{d}^2\boldsymbol{\xi}}{\partial t^2}+\nabla\cdot\mathbf{P}'=(\nabla\times\boldsymbol{B}')\times\boldsymbol{B}+\boldsymbol{J}\times\boldsymbol{B}'\,. \tag{68}$$

(We remind the reader $\boldsymbol{B}'$ is the perturbed $\boldsymbol{B}$ at a fixed point; $\boldsymbol{B}^*=\boldsymbol{B}'+\boldsymbol{\xi}\cdot\nabla\boldsymbol{B}$ is the perturbed quantity following $\boldsymbol{\xi}$.)

Now

$$\mathbf{P}'=\mathbf{P}^*-\boldsymbol{\xi}\cdot\nabla\mathbf{P}^0\,,$$

and

$$\mathbf{P}^*=p_\perp^*(I-\boldsymbol{nn})+(p_\parallel-p_\perp)(nn)^*+p_\parallel^*\boldsymbol{nn}\,.$$

$p_\perp^*$ and $p_\parallel^*$ are found from (56)

In evaluating $p_\perp^*$ care must be taken to vary the factor β/q. (q is given by (52)). To do this note that $(\partial q/\partial\varepsilon)_{\mu,r,\nu}=1/q$ so, for instance,

$$\left[\int\frac{\beta}{q}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon\nu\beta f\right]^*=-\left[\int\beta q\,\mathrm{d}\nu\,\mathrm{d}\varepsilon\nu\beta f_\varepsilon\right]^*=-\int(\beta^2 q)^*\,\mathrm{d}\nu\,\mathrm{d}\varepsilon\nu g_\varepsilon+\int\frac{\beta}{q}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon\nu\beta f^*\,.$$

In this way the expression for $\mathbf{P}^*$ is found to be

$$\begin{aligned}\mathbf{P}^*=(c+2\beta_\perp)(nn:\nabla\xi-\nabla\cdot\xi)(I-nn)+\\+(p_\parallel-p_\perp)[nn\cdot\nabla\xi+n\cdot\nabla\xi n-nn(nn\cdot\nabla\xi+\nabla\cdot\xi)]+\\+\sum_{i,e}{}_{\pm}\,m\int\frac{\beta}{|q|}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon[\nu\beta I+q^2-\nu\beta nn]\left(f^*+\frac{e}{m}\,g_\varepsilon\psi^*\right),\end{aligned} \tag{69}$$

where

$$c=\sum_{i,e}{}_{\pm}\,m\int\frac{\beta}{|q|}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon\nu^2\beta g_\varepsilon\,. \tag{70}$$

To complete the linearized equations we need equations for f^* and one for ψ^*.

The equation for f^* is from (54)

$$\frac{\mathrm{d}f^*}{\mathrm{d}t} + q\boldsymbol{n}\cdot\nabla f^* - \frac{\mathrm{d}\zeta}{\mathrm{d}t}\, g_\varepsilon = 0\,, \tag{71}$$

where

$$\zeta = q^2 nn : \nabla\xi + \nu\beta(\nabla\cdot\xi - nn : \nabla\xi) - \frac{e}{m}\,\psi^*\,, \tag{72}$$

and the equation for ψ^* is (from (41))

$$0 = \sum_{i,e}{}_{\pm}\, e\int \frac{\beta}{|q|}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon(f^* - \zeta g_\varepsilon)\,. \tag{73}$$

Equation (42) is easily shown to follow automatically.

From boundary conditions (57) and (71) we can show that

$$\sum\nolimits_{\pm}\int \frac{\beta}{|q|}\,\mathrm{d}l(f_t - \zeta_t g_\varepsilon) = 0\,, \tag{74}$$

where the line integral is between the points where $q = 0$. We restrict ourselves to perturbations satisfying

$$\sum\nolimits_{\pm}\int \frac{\beta}{|q|}\,\mathrm{d}l(f - \zeta g_\varepsilon) = 0\,, \tag{75}$$

which once satisfied is always satisfied by eq. (71). Hence, no loss in generality is involved by restriction (75).

3·6. *Stability theory.* – The stability theory follows closely the *fluid theory* of stability. The details are given in the author's paper « On the Necessity of the Energy Principle of Kruskal and Oberman », *Physics of Fluids*, 2, 192 (1962) [16].

Instead of a single equation for ξ as in the fluid theory we have another equation for f which is first order. To arrive at two second-order equations we make use of a suggestion of NEWCOMB [15] and set

$$\bar{f}(t, r, \nu, \varepsilon) = f_+^*(t, r, \nu, \varepsilon) + f_-^*(t, r, \nu, \varepsilon)\,,$$
$$h(t, r, \nu, \varepsilon) = f_+^*(t, r, \nu, \varepsilon) - f_-^*(t, r, \nu, \varepsilon)\,,$$

$\bar{f}$ is the total number of particles with given ε regardless of the sign of q, and h represents the asymmetry of the particles between the two signs of q. Writing

(71) for both f_+^* and f_-^* and adding and subtracting we obtain for $\bar{f}$ and h

$$\begin{cases} \dfrac{\partial \bar{f}}{\partial t} + |q|\boldsymbol{n}\cdot\nabla h - 2\dot{\zeta}g_s = 0 \,, \\ \dfrac{\partial h}{\partial t} + |q|\boldsymbol{n}\cdot\nabla \bar{f} = 0 \,. \end{cases} \tag{76}$$

Eliminating h from these equations we have

$$\frac{\partial^2 \bar{f}}{\partial t^2} - q\boldsymbol{n}\cdot\nabla(q\boldsymbol{n}\cdot\nabla\bar{f}) - 2\ddot{\zeta}g_s = 0 \,, \tag{77}$$

a second order equation for $\bar{f}$. Since only $\bar{f}$ enters into $\mathbf{P}^*$ and (73) which gives ψ^* we may regard (68), (73) and (77) as a closed system of equations. Regarding (73) as a definition for ψ^* we may write the two second-order equations as

$$\begin{cases} \ddot{\boldsymbol{\xi}} = \boldsymbol{F}(\xi, \bar{f}) \,, \\ \dot{\bar{f}} = J(\xi, \bar{f}) \,, \end{cases} \tag{78}$$

and derive the energy principle from these equations. For this purpose it is necessary that $\boldsymbol{F}$ and J together satisfy a self-adjointness. This self-adjointness was first discovered by NEWCOMB [15] who gave a direct proof of it. In these notes we follow the indirect method of the fluid theory in finding and demonstrating this self-adjointness.

We consider an arbitrary perturbation away from the equilibrium described in Section 3·4. Ordinarily one could specify $\boldsymbol{\xi}_t$, and f^* while f_t^* is determined by (71). These are subject to the constraints $\boldsymbol{\xi}\cdot\boldsymbol{n} = \boldsymbol{\xi}_t\cdot\boldsymbol{n} = 0$ and (75). However, it is possible to assign both $\bar{f}$ and $\bar{f}_t$ arbitrarily since $\bar{f}$ satisfies a second-order differential equation. Of course, $\bar{f}$ must satisfy restriction (75), and one must be able to determine h from eq. (76). It is easily seen that the latter requires that (75) be satisfied by $\bar{f}_t$. In summary the stability problem is reduced to examining all solutions of (78) subject to the constraints (on the initial values of the perturbations) $\boldsymbol{\xi}\cdot\boldsymbol{n} = \boldsymbol{\xi}_t\cdot\boldsymbol{n} = 0$ and $\bar{f}$ and $\bar{f}_t$ satisfy (75).

In order to define our perturbations to higher order (as in the fluid theory) we define

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \alpha(\boldsymbol{r} + \boldsymbol{\xi}, t) \,, \tag{79}$$

so $\boldsymbol{\xi}$ is the true Lagrangian variable. We generalize (75) by noting that it is a linearized version of a nonlinear constraint given in the paper of KRUSKAL

and OBERMAN [2]. Namely, introduce a labeling (*i.e.*, two parameters) of the lines of force, L, which gives the original position of a line of force passing through $\boldsymbol{r}$ at time t. Then for any function G

$$\mu_G = \int G(v, f, L)\, \mathrm{d}^3v\, \mathrm{d}\tau\,, \tag{80}$$

is a constant. We consider only those perturbation which have the same value of μ as the equilibrium for any choice of G. It is easy to show that to first order in the displacement this is just constraint (75). Because (80) is nonlinear f^{**}, the second-order perturbation of f, is not completely independent but may be related to f^{*2} and therefore it is possible to express the energy to second order in terms of ξ and $\bar{f}$ alone.

After these remarks we now follow the argument given in the fluid theory to establish the self-adjointness of the operators in (78). For convenience, we write (78) as one equation

$$\ddot{\Lambda} = H(\Lambda)\,, \tag{81}$$

where Λ represents ξ and $\bar{f}$ together. The energy U is

$$U = \sum m \int \frac{\beta}{q}\, \mathrm{d}v\, \mathrm{d}\varepsilon f \varepsilon\, \mathrm{d}\tau + \int \frac{B^2}{2}\, \mathrm{d}\tau\,,$$

which may be shown to be constant by means of the equations of Section 3·3 Expanding it to second order in Λ, we get

$$U = A(\Lambda) + B(\dot{\Lambda}) + K(\ddot{\Lambda}, \ddot{\Lambda}) + M(\Lambda, \dot{\Lambda}) + W(\Lambda, \Lambda)$$

which must be constant for all Λ and $\dot{\Lambda}$ subject to the constraints. We find A and K by setting $\Lambda = \bar{f}$, $\xi = 0$

$$U = \frac{1}{2}\int \varrho \dot{\xi}^2 + \sum m \int \frac{\beta}{q}\, \mathrm{d}v\, \mathrm{d}\varepsilon \left(f^{**} + \frac{e}{m}\psi^{**} \right) \varepsilon\,.$$

(We may ignore the change in β/q produced by ξ, since $\xi = 0$ and the change produced by ψ^{**} is simply $(e/m)\beta/q\psi^{**}$.) On the other hand the nonlinear constraint (80) to second order gives

$$\int \frac{\beta}{q} \left[G_f \left(f^{**} + \frac{e}{m}\psi^{**} \right) + G_{ff} \frac{f^{*2}}{2} \right] \mathrm{d}v\, \mathrm{d}\varepsilon = 0\,.$$

Taking $G_f = \varepsilon$ (for $f = g$) we have $G_{ff} = -1/g_\varepsilon$ and

$$U = \frac{1}{2}\int \varrho \dot{\xi}^2 - \sum \frac{m}{8}\int \frac{\beta}{q}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon\,\frac{1}{g_\varepsilon}\,h^2 = K(\dot{\Lambda},\dot{\Lambda})\,, \tag{82}$$

and $\Lambda = 0$. We must assume $g_\varepsilon < 0$ since it must vanish nowhere, and this makes K positive definite. h is to be expressed in terms of $\bar{f}_t$ by means of (76). Now from the same formal argument as in the fluid theory one obtains

$$K[H(\Lambda), \Lambda'] = K[H(\Lambda'), \Lambda]\,, \tag{83}$$

where Λ and Λ' are any two Λ's satisfying the constraints. Also

$$W(\Lambda, \Lambda) = -K[H(\Lambda), \Lambda] \tag{84}$$

and $W(\Lambda, \Lambda)$ being positive for all Λ gives a necessary and sufficient condition for stability as can be shown by the last proof of condition *d*) in Section 2·4. The analogous equation here is

$$W(\Lambda, \Lambda) = \sum a_n^2\, K(\Lambda_n, \Lambda_n)\omega_n^2$$

and K is positive definite.

To obtain an explicit energy principle it is necessary to find W by means of eq. (84). Let $h(\Lambda) = h(f, \xi)$ represent an abbreviation for the solution of

$$q\,\boldsymbol{n}\cdot\nabla h = f + 2\zeta g_\varepsilon \tag{85}$$

then

$$W(\Lambda, \Lambda) = -\frac{1}{2}\int \varrho\xi\cdot\boldsymbol{F}(\xi, f)\,\mathrm{d}\tau + \sum \frac{m}{4}\int \frac{\beta}{q}\frac{1}{g_\varepsilon}\,h(\Lambda)h[H(\Lambda)]\,\mathrm{d}\tau\,\mathrm{d}\varepsilon\,\mathrm{d}\nu\,. \tag{86}$$

But is is easily seen from (71) that

$$h[H(\Lambda)] = -|q|\,\boldsymbol{n}\cdot\nabla\bar{f}\,. \tag{87}$$

Using (87) in (86), integrating the last term by parts and using (85) we have

$$W(\Lambda, \Lambda) = -\frac{1}{2}\int \varrho\xi\cdot\boldsymbol{F}(\xi, f) - \sum m\int \frac{\beta}{q}\frac{1}{g_\varepsilon}\,(\bar{f} - \zeta g_\varepsilon)\bar{f}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon\,\mathrm{d}\tau\,.$$

The remaining integrations by parts (on $\boldsymbol{F}$) are standard and one obtains after

some algebra

$$\delta W_A = \frac{1}{2}\int\{Q^2 + \boldsymbol{J}\cdot\boldsymbol{\xi}\times\boldsymbol{Q} + \boldsymbol{\xi}\cdot\nabla p_\perp \nabla\cdot\boldsymbol{\xi} + \tag{88}$$

$$+ 2p_\perp(\boldsymbol{nn}:\nabla\boldsymbol{\xi} - \nabla\cdot\boldsymbol{\xi})^2 + (p_\parallel - p_\perp)[-\boldsymbol{nn}:(\boldsymbol{\xi}\cdot\nabla\nabla\boldsymbol{\xi}) +$$

$$+ (\boldsymbol{nn}:\nabla\boldsymbol{\xi})^2 - (\boldsymbol{n}\cdot\nabla\boldsymbol{\xi})^2 - (\boldsymbol{n}\cdot\nabla\boldsymbol{\xi})(\nabla\boldsymbol{\xi}\cdot\boldsymbol{n})] +$$

$$+ e(\boldsymbol{nn}:\nabla\boldsymbol{\xi} - \nabla\cdot\boldsymbol{\xi})^2\}\mathrm{d}\tau - \frac{1}{4}\sum_{i,e} m\int \frac{1}{g_\varepsilon} f^{A^2}\frac{\beta}{q}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon\,\mathrm{d}\tau =$$

$$= W_1 - \frac{1}{4}\sum m\int \frac{f^{A^2}}{g_\varepsilon}\frac{\beta}{q}\,\mathrm{d}\nu\,\mathrm{d}\varepsilon\,\mathrm{d}\tau\,,$$

where

$$f^A = \bar{f} + \frac{2e}{m}\psi^* g_\varepsilon\,,$$

f^A is the value of f^* given by KRUSKAL and OBERMAN and differs from ours due to a difference in definition of ε. Their ε is

$$\varepsilon_{\mathrm{KO}} = \nu\beta + q^2/2$$

with no ψ.

Expression (88) is identical with that given by KRUSKAL and OBERMAN after one sets $f_+ = f_-$ in their expression (a trivial minimization). Thus their δW gives a necessary and sufficient condition for stability.

To be useful this expression should be minimized over f^A (or $\bar{f}$) subject to constraints (73) and (75). This has not yet been done but a good sufficient condition (for stability) is obtained by minimization subject only to (75). The resulting expression is

$$\delta W_{\mathrm{KO}} = W_1 - \sum m\int \frac{\beta}{q}\lambda^2 g_\varepsilon\,\mathrm{d}\nu\,\mathrm{d}\varepsilon\,\mathrm{d}\tau\,, \tag{89}$$

where

$$\lambda = -\frac{\int[(q^2 - \nu\beta)\boldsymbol{nn}:\nabla\boldsymbol{\xi} + \nu\beta\nabla\cdot\boldsymbol{\xi}]\mathrm{d}l/|q|}{\int \mathrm{d}l/|q|}\,. \tag{90}$$

3·7. *Comparison theorems.* – To compare δW_A given by (88) with δW_F of fluid theory we must consider the case of an isotropic g. We know that $\delta W_A > > \delta W_{\mathrm{KO}}$ of (89) and δW_{KO}, simplifies considerably for isotropic g,

$$\delta W_{\mathrm{KO}} = \frac{1}{2}\int[Q^2 + j\cdot\boldsymbol{\xi}\times Q + \boldsymbol{\xi}\cdot\nabla p\,\nabla\cdot\boldsymbol{\xi}] - \sum m\int\frac{\beta}{q} g_\varepsilon\lambda^2\,\mathrm{d}\nu\,\mathrm{d}\varepsilon\,\mathrm{d}\tau\,. \tag{91}$$

Write

$$\lambda = \frac{\varepsilon\int[2-3y\beta)\boldsymbol{nn}:\nabla\boldsymbol{\xi}+y\beta\nabla\boldsymbol{\xi}]\,\mathrm{d}l/\sqrt{1-y\beta}}{\int \mathrm{d}l/\sqrt{1-y\beta}} = \frac{J}{K};\quad K=\int\frac{\mathrm{d}l}{\sqrt{1-y\beta}};\quad y=\frac{v}{\varepsilon}. \tag{92}$$

Then the last term of (91) is

$$W_2 = \frac{15}{4}\int \mathrm{d}\psi\int \mathrm{d}y\int \mathrm{d}l\,\frac{p}{(1-y\beta)^{\frac{1}{2}}}\,\frac{J^2}{K^2}, \tag{93}$$

where $\mathrm{d}\psi$ is a flux element and we have used

$$p = \frac{8\sqrt{2}}{15}\sum m\int \varepsilon^{5/2} g_\varepsilon\,\mathrm{d}\varepsilon\,.$$

But by Schwarz's inequality

$$\int\frac{J^2}{K}\,\mathrm{d}y = \int \mathrm{d}y\left(\frac{J}{K}\right)^2 K \geqslant \frac{[\int \mathrm{d}yJ]^2}{\int \mathrm{d}yK}, \tag{1}$$

so

$$W_2 \geqslant \frac{15}{4}\int \mathrm{d}\psi p\,\frac{[\int \mathrm{d}yJ]^2}{\int \mathrm{d}yK} \tag{2}$$

Now

$$\int \mathrm{d}yK = \frac{2}{\beta}, \tag{3}$$

$$\int \mathrm{d}yJ = \frac{2}{3}\int \mathrm{d}l\,\frac{\nabla\cdot\boldsymbol{\xi}}{\beta}, \tag{4}$$

so

$$W_2 \geqslant \frac{1}{2}\frac{5}{3}\int \mathrm{d}\tau p\langle\nabla\cdot\boldsymbol{\xi}\rangle^2, \tag{5}$$

where

$$\langle\nabla\cdot\boldsymbol{\xi}\rangle = \frac{\int(\nabla\cdot\boldsymbol{\xi})\,\mathrm{d}l/\beta}{\int \mathrm{d}l/\beta}\,. \tag{6}$$

However we must remember that in the adiabatic theory $\boldsymbol{\xi}\cdot\boldsymbol{n}=0$ while in the fluid theory $\boldsymbol{\xi}\cdot\boldsymbol{n}\neq 0$. Minimizing δW_F over $\boldsymbol{\xi}\cdot\boldsymbol{n}$ is equivalent to making $\boldsymbol{n}\cdot\nabla(\nabla\cdot\boldsymbol{\xi})=0$ or $\nabla\cdot\boldsymbol{\xi}=\langle\nabla\cdot\boldsymbol{\xi}\rangle$. Call δW_F minimized over $\boldsymbol{\xi}\cdot\boldsymbol{n}$, $\delta W_F'$. Hence we obtain the important result

$$\delta W_A > \delta W_{\mathrm{KO}} > \delta W_F'\,. \tag{7}$$

But $\delta W_F'$ gives a necessary and sufficient result for stability on the fluid theory. Hence if the fluid theory gives stability so must the adiabatic theory. Thus, we see the fluid theory gives reliable results even in cases where one would suppose it was unreliable. The comparison theorem and its proof is given in both references [2] and [5].

4. – Double adiabatic theory.

4·1. *Basic equations.* – We now consider the situations intermediate between those of weak collisions, adiabatic theory, and strong collisions, fluid theory. That is the situations in which collisions are not sufficiently strong to keep the pressure a tensor but sufficiently strong to prevent heat flow and other transport processes. The basic equations are

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -\varrho \nabla \cdot \boldsymbol{V}\,, \tag{1}$$

$$\varrho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -\nabla \cdot \mathbf{P} + \boldsymbol{J}\times\boldsymbol{B}\,, \tag{2}$$

$$\mathbf{P} = p_\perp (I - \boldsymbol{nn}) + p_\parallel \boldsymbol{nn}\,, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_\perp}{\varrho\beta}\right) = 0 \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_\parallel \beta^2}{\varrho^3}\right) = 0\,, \tag{4}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{V}\times\boldsymbol{B})\,, \tag{5}$$

$$\nabla \cdot \boldsymbol{B} = 0\,, \tag{6}$$

$$\boldsymbol{J} = \nabla \times \boldsymbol{B}\,. \tag{7}$$

The only difference between these equations and the fluid eqs. (9)–(14) of Section 2 is that in (10) the pressure is replaced by a tensor whose two independent components are determined by the two adiabatic equations of state (4) instead of the single equation of state (11) of the fluid theory.

One derives (3) and (4) as follows: Starting with the Boltzmann equation (without collisions)

$$\frac{\partial f}{\partial t} + \boldsymbol{V}\cdot\nabla f + e\left(\boldsymbol{E} + \frac{\boldsymbol{V}\times\boldsymbol{B}}{c}\right)\cdot \nabla_v f = 0\,, \tag{8}$$

and introducing

$$N = \int f \,\mathrm{d}^3 v\,, \tag{9}$$

$$N\boldsymbol{U} = \int \boldsymbol{V} f \, \mathrm{d}^3 v \,, \tag{10}$$

$$\mathbf{P} = m \int (\boldsymbol{v} - \boldsymbol{U})(\boldsymbol{v} - \boldsymbol{U}) f \, \mathrm{d}^3 v \,, \tag{11}$$

$$Q = m \int (\boldsymbol{v} - \boldsymbol{U})(\boldsymbol{v} - \boldsymbol{U})(\boldsymbol{v} - \boldsymbol{U}) f \, \mathrm{d}^3 v \,, \tag{12}$$

one finds after multiplying (8) by $(\boldsymbol{v} - \boldsymbol{U})(\boldsymbol{v} - \boldsymbol{U})$ and integrating by parts

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} + \nabla \cdot Q + \mathbf{P}(\nabla \cdot \boldsymbol{U}) + \mathbf{P} \cdot \nabla \boldsymbol{U} + (\mathbf{P} \cdot \nabla \boldsymbol{U})^{\mathrm{tr}} + \frac{e}{mc}(\boldsymbol{B} \times \mathbf{P} - \mathbf{P} \times \boldsymbol{B}) = 0 \,, \tag{13}$$

where $(\mathbf{P} \cdot \nabla \boldsymbol{U})^{\mathrm{tr}}$ is the transposed dyadic of $\mathbf{P} \cdot \nabla \boldsymbol{U}$.

Let e (or B) be large and expand $\mathbf{P}$

$$\mathbf{P} = \mathbf{P}^0 + \mathbf{P}' + \ldots \,,$$

then from (13)

$$\boldsymbol{B} \times \mathbf{P}^0 = \mathbf{P}^0 \times \boldsymbol{B}$$

or

$$\mathbf{P} = p_\perp (I - \boldsymbol{nn}) + p_\parallel \boldsymbol{nn} \,. \tag{14}$$

Equation (13) to first order involves $\mathbf{P}'$ but first taking the trace and second double-dotting with $\boldsymbol{nn}$ eliminates $\mathbf{P}$, and gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(2p_\perp + p_\parallel) + (2p_\perp + p_\parallel)\nabla \cdot \boldsymbol{U} + 2p_\perp(\nabla \cdot \boldsymbol{U} - \boldsymbol{nn} : \nabla \boldsymbol{U}) + \\ + 2p_\parallel(\boldsymbol{nn} : \nabla \boldsymbol{U} - \nabla \cdot \boldsymbol{U}) + 2p_\parallel \nabla \cdot \boldsymbol{U} = 0 \,, \tag{15}$$

and

$$\frac{\mathrm{d}p_\parallel}{\mathrm{d}t} + p_\parallel(\nabla \cdot \boldsymbol{U}_0) + 2p_\parallel \boldsymbol{nn} : \nabla \boldsymbol{U} = 0 \,, \tag{16}$$

respectively. Now from (1) and (5)

$$\nabla \cdot U = -\frac{1}{\varrho}\frac{\mathrm{d}\varrho}{\mathrm{d}t} \,, \tag{17}$$

$$\boldsymbol{nn} : \nabla \boldsymbol{U} - \nabla \cdot \boldsymbol{U} = \frac{1}{\beta}\frac{\mathrm{d}\beta}{\mathrm{d}t} \,, \tag{18}$$

and combining these with (16) gives the second half of (4). Combining these with (15) and the second half of (4) now gives the first half of (4). The derivation is essentially given in reference [8].

4·2. *Static equilibrium.* – The equations for static equilibrium are

$$\boldsymbol{J}\times\boldsymbol{B} = \nabla\cdot\mathbf{P}\,, \tag{19}$$

$$\boldsymbol{J} = \nabla\times\boldsymbol{B}\,, \tag{20}$$

$$\nabla\cdot\boldsymbol{B} = 0\,. \tag{21}$$

These equations are the same as those of the adiabatic theory except $p_\perp$ and $p_\parallel$ are free and not determined by f.

4·3. *Linearized equations.* – Introduce ξ as before

$$\frac{\partial\xi}{\partial t} = \boldsymbol{U}\,.$$

Then the basic linearized equation is

$$\varrho\,\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = -\nabla\cdot\mathbf{P}^* + \nabla\cdot(\xi\cdot\nabla\mathbf{P}) + \boldsymbol{J}'\times\boldsymbol{Q} + (\nabla\times\boldsymbol{Q})\times\boldsymbol{B}\,, \tag{22}$$

where:

From (1), (3), (4) (and (5) for n^*),

$$\mathbf{P}^* = p_\perp(\boldsymbol{nn}:\nabla\xi - \nabla\cdot\xi)I - p_\perp(3\,\boldsymbol{nn}:\nabla\xi - \nabla\cdot\xi)\boldsymbol{nn} + \\ + (p_\parallel - p_\perp)[\boldsymbol{n}\cdot\nabla\xi\,\boldsymbol{n} + \boldsymbol{nn}\cdot\nabla\xi - \boldsymbol{nn}(4\,\boldsymbol{nn}:\nabla\xi + \nabla\cdot\xi)]\,. \tag{23}$$

From (22) and (23) we have

$$\varrho\,\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = \boldsymbol{F}_{\mathrm{DA}}(\xi)\,, \tag{24}$$

where $\boldsymbol{F}_{\mathrm{DA}}(\xi)$ is a linear operator in ξ corresponding to the $\boldsymbol{F}$ of the fluid theory.

4·4. *Stability.* – The treatment of stability follows the indirect proof of section 2·4. It is easy to show from (1)–(7) of this section that the energy

$$U = \int\left(\frac{\varrho}{2}V^2 + \frac{B^2}{2} + p_\perp + \frac{p_\parallel}{2}\right)\mathrm{d}\tau\,, \tag{25}$$

is constant. One expands U in ξ to second order

$$U = A(\xi) + B(\dot{\xi}) + K(\dot{\xi},\dot{\xi}) + M(\xi,\dot{\xi}) + W(\xi,\xi) \tag{26}$$

and A and B are easily shown to be zero and again

$$K(\dot{\xi}, \dot{\xi}) = \frac{1}{2}\int \varrho \dot{\xi}^2 \, d\tau \,. \tag{27}$$

As before from $\dot{U} = 0$ for all ξ and $\dot{\xi}$ one has

$$K\left(\frac{\boldsymbol{F}_{DA}(\xi)}{\varrho}, \dot{\xi}\right) = K\left[\frac{\boldsymbol{F}_{DA}(\xi')}{\varrho}, \xi\right], \tag{28}$$

or

$$\int \boldsymbol{F}_{DA}(\xi), \xi' \, d\tau = \int \boldsymbol{F}_{DA}(\xi') \cdot \xi \, d\tau \,. \tag{29}$$

From this ω^2 is real for normal modes and an energy principle exists, *i.e.*

$$\delta W_{DA} = -\int \xi \cdot \boldsymbol{F}_{DA}(\xi) \, d\tau \,, \tag{30}$$

gives a necessary and sufficient condition for the stability of the solutions of the linearized double adiabatic solutions of Section 3·3. By some integrations by parts one finds

$$\begin{aligned} \delta W_{DA} = \frac{1}{2}\int & Q^2 + \boldsymbol{J} \cdot \xi \times \boldsymbol{Q} + \xi \cdot \nabla p_\perp (\nabla \cdot \xi) + \frac{5}{3}(\nabla \cdot \xi)^2 p_\perp + \\ & + \frac{1}{3} p_\perp (\nabla \cdot \xi - \boldsymbol{nn} \nabla \xi)^2 + (p_\parallel - p_\perp)[-(\boldsymbol{n} \cdot \nabla \xi)^2 - \\ & - (\boldsymbol{n} \cdot \nabla \xi) \cdot (\nabla \xi \cdot \boldsymbol{n}) + 4(\boldsymbol{nn} : \nabla \xi)^2 + \boldsymbol{nn} \cdot \nabla \xi \nabla \cdot \xi - \boldsymbol{nn} : \nabla \cdot (\xi \cdot \nabla \xi)] \,. \end{aligned} \tag{31}$$

From (31) and eq. (21) of Section 2 one sees that if $\mathbf{P}$ is isotropic ($p_\perp = p_\perp$) and we set $\gamma = \frac{5}{3}$,

$$\delta W_{DA} > \delta W_F \,. \tag{32}$$

Thus, if an equilibrium is found to be stable on the fluid theory it will be stable on the double adiabatic theory. (It will also be an equilibrium).

To find the comparison between the adiabatic and double adiabatic theories we find from (31) of this section, and eq. (88) and (89) of Section 3, that

$$\begin{aligned} \delta W_{KO} = \delta W_{DA} + I - \frac{1}{2}\int & d\tau \boldsymbol{nn} : \nabla \xi [p_\perp (2\nabla \cdot \xi + \boldsymbol{nn} : \nabla \xi) + \\ & + 3(p_\parallel - p_\perp)(\boldsymbol{nn} : \nabla \xi)^2] \,, \end{aligned} \tag{33}$$

where

$$I = -\sum \frac{m}{2}\int \frac{\beta}{q}\,\mathrm{d}v\,\mathrm{d}\varepsilon\,\mathrm{d}\tau g_s[\lambda^2 - v^2\beta^2(\boldsymbol{nn}:\nabla\boldsymbol{\xi} - \nabla\cdot\boldsymbol{\xi})^2]\,, \tag{34}$$

and λ is given by (90) of Section 3.

But by the Schwarz inequality

$$\lambda^2 < \frac{\int(\mathrm{d}l/q)\,[q^2\boldsymbol{nn}:\nabla\boldsymbol{\xi} + v\beta(\nabla\cdot\boldsymbol{\xi} - \boldsymbol{nn}\cdot\nabla\boldsymbol{\xi})]^2}{\int \mathrm{d}l/q}\,, \tag{35}$$

(Hint: write

$$\lambda^2 = \left(\int ab\,\mathrm{d}l\right)^2 \Big/ \int\frac{\mathrm{d}l}{q} < \int a^2\,\mathrm{d}l \int b^2\,\mathrm{d}l \Big/ \left(\int\frac{\mathrm{d}l}{q}\right)^2,$$

where

$$a = \frac{1}{\sqrt{q}},\ b = \frac{1}{q}[\ ]\Big).$$

Use of (34) and (35) gives

$$I < \frac{1}{2}\int \mathrm{d}\tau\,[p_\perp(2\nabla\cdot\boldsymbol{\xi} - \boldsymbol{nn}:\nabla\boldsymbol{\xi}) + 3p_\parallel \boldsymbol{nn}:\nabla\boldsymbol{\xi}]\boldsymbol{nn}:\nabla\boldsymbol{\xi}\,. \tag{36}$$

Therefore from (33) and (36)

$$\delta W_{\mathrm{KO}} < \delta W_{\mathrm{DA}}\,. \tag{37}$$

We thus have the inequalities

$$\begin{cases} \delta W_F' < \delta W_{\mathrm{KO}} < \delta W_{\mathrm{DA}}\,, \\ \delta W_{\mathrm{KO}} < \delta W_A\,. \end{cases} \tag{38}$$

The relationship between δW_{DA} and δW_A has not yet been worked out.

From inequalities (38) we get the theorems

Stability on F.T. ⇒ Stability on K.O.T.

» » F.T. ⇒ » » A.T.

» » F.T. ⇒ » » D.A.T.

» » K.O. ⇒ » » A.T.

» » K.O. ⇒ » » D.A.

where we use the abbreviations:

F.T. ≡ Fluid theory,

A.T. ≡ Adiabatic theory,

D.A.T. ≡ Double adiabatic theory,

K.O.T. ≡ Kruskal Oberman theory,

⇒ ≡ implies.

Again the derivation of the comparison theorems follows that given in both references [2] and [5].

* * *

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