Part 4

Theory of Weakly Turbulent Plasma

Weak Plasma Turbulence Approximation

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Thus far, only laminar plasma motions that can be described by the behavior of local hydrodynamic plasma and field parameters or by the particle distribution function have been considered. The effectiveness of such descriptions in many cases is explained by the fact that plasma motions as a rule have a collective character, and therefore a simple relation between plasma motion at different points exists. However, plasma as a many-body system has a large number of degrees of freedom and thus has a wide variety of possible collective motions. When the amplitudes of these collective motions are infinitely small, the superposition principle is valid and this is the basis for linear plasma theory. According to this principle, an arbitrary perturbation is represented by the superposition of the collective motions, where each motion is considered independent. But when the amplitudes become finite as a result of the appropriate instability development, then nonlinear effects result in the interaction of these motions with each other, similar to the interaction of different-scale motions in hydrodynamic turbulence. The plasma could be considered turbulent in this case.

The main difficulty in the mathematical description of hydrodynamic turbulence is that the interaction between the different turbulence scales is generally so strong that these different motions cannot be considered even approximately independent. Plasma turbulence is often much simpler in this respect. The reason for this is that the wide class of kinetic plasma instabilities results in the excitation of the plasma eigenmodes with small amplitudes. The nonlinear interaction of such modes is weak, and this permits them to be considered independent in a first approximation. As a result, an arbitrary disturbance can be represented in the form of an eigenmode expansion.

In the next approximation the coefficients of such an expansion slowly change in time because of the interaction of the eigenmodes, and finally strongly deviate from

Handbook of Plasma Physics, Eds. M.N. Rosenbluth and R.Z. Sagdeev
Volume 1: Basic Plasma Physics I, edited by A.A. Galeev and R.N. Sudan

the initial values predicted by linear theory. This approach is widely known as weak turbulence theory (Sagdeev and Galeev, 1969, 1973; Kadomtsev, 1964, 1966; Tsytovich, 1967; Davidson, 1971, Ichimaru, 1973). The equations of this theory can be derived from first principles by the expansion of the basic plasma equations on the small parameter—the ratio of the collective oscillation energy to the total plasma energy. Moreover, these equations can be essentially simplified in the approximation of the random phases of different oscillations when a statistical description is used instead of a dynamic one. The precise criteria of the transition to a statistical description of a many-body system are well known and can be easily found for any specific case.

It is convenient to lay down the theory of weakly turbulent plasma in terms of the three basic types of interaction: *quasilinear wave–particle interaction*, *nonlinear wave–wave interaction* and, finally, *wave–particle–wave interaction* (known also as nonlinear wave–particle interaction).

The first type, quasilinear wave–particle interaction, is particularly strong near the Cherenkov resonance, $\omega = \boldsymbol{k} \cdot \boldsymbol{v}$, since a particle with velocity satisfying this relation conserves a constant phase relative to the wave and thus is accelerated (or decelerated) by the wave field. The analogous resonance in the magnetic field takes place under the condition

$$\omega - l\omega_c = k_{\parallel} v_{\parallel}, \qquad l = 0, \pm 1, \ldots,$$

where ω_c is the particle Larmor frequency. Since this type of interaction is described by a group of resonance particles, it is necessary to use a kinetic description. The time variation of the wave amplitude due to this interaction is Landau damping (cyclotron damping in the case of resonance in a magnetic field) (which was considered in Chapters 2.2 and 3.3 of this volume). The corresponding time variation of the resonant particles' velocity distribution has the form of diffusion in the velocity space (so-called quasilinear diffusion) and will be considered below.

The second type of interaction, nonlinear wave–wave interaction, is often referred to as wave scattering by waves. The condition for such resonance can be written as follows:

$$\sum_i \omega_i = 0, \qquad \sum_i \boldsymbol{k}_i = 0,$$

where ω_i and $\boldsymbol{k}_i$ are, respectively, the frequencies and the wavevectors of the waves taking part in the interaction. Since this interaction does not involve the resonant particles, it can be described by a hydrodynamic approximation, assuming it to be valid for noninteracting waves.

The third type of interaction, wave–particle–wave interaction, is considered to be either nonlinear Landau damping or induced wave scattering in a plasma. The resonance condition for this interaction is $\omega_1 - \omega_2 = (\boldsymbol{k}_1 - \boldsymbol{k}_2) \cdot \boldsymbol{v}$, and the basic mechanism is similar to that for linear wave–particle interaction. In this case, the particle keeps its phase constant relative to the beating of the two waves. This interaction also involves resonant particles and should be considered in the framework of the kinetic approach.

It should be noted that each of the three basic types of interaction has a corresponding quantum analog. Also, the resonance conditions given above are the consequences of the energy and momentum conservation laws for elementary interaction processes. For example, while the particle emits a radiation quantum with frequency ω and wavevector $\boldsymbol{k}$, it gains a recoil momentum, $\Delta \boldsymbol{p} = -\hbar \boldsymbol{k}$ ($\hbar$ is the Planck constant), and its energy decreases by the value $\Delta \mathcal{E} = \Delta \boldsymbol{p} \cdot \boldsymbol{v}$, which is equal to the emitted quantum energy, $\hbar\omega$, assuming the resonance condition is satisfied. To consider the absorption process instead of the emission process, simply change the sign of the frequency and the wavevector. It is obvious that these processes conserve the sum of the particle and wave energies.

As a rule, the quantum approach to the derivation of equations for weakly turbulent plasma equations are much more complicated than the consistent application of perturbation theory to classical field equations and kinetic equations for the particle distribution function. Nevertheless, the consideration of elementary processes for a given type of interaction allows one to draw general conclusions concerning the conservation laws and to check them in specific computations. For example, the resonance condition for the wave–particle–wave interaction, written in the form $\omega_1 - \omega_2 = (\boldsymbol{k}_1 - \boldsymbol{k}_2) \cdot \boldsymbol{v}$, where the frequencies ω_1 and ω_2 have the same sign, corresponds to the elementary process of wave absorption by a particle followed by the emission of another wave by the same particle (in other words, the scattering process). Obviously, this type of interaction should conserve both the energy and the number of the radiation quanta. In the classical limit the number of quanta can be defined as the ratio of the wave energy, w_k, to the frequency, i.e. w_k/ω_k is the action of the $(\omega, \boldsymbol{k})$ wave. This kind of argument will be used to discuss all three types of interaction.

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Chapter 4.1

Wave – Particle Interaction

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Contents

4.1.1. Statistical description criteria

To illustrate the transition from a dynamical description of monochromatic wave–particle resonant interaction to a statistical description of a many-wave interaction, consider one-dimensional perturbations in a plasma without a magnetic field. The complete system of equations to solve this problem consists of the electron kinetic equation with self-consistent electric field and the Poisson equation:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial v} + \frac{e}{m_e}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = 0 \qquad (1)$$

$$\partial^2\phi/\partial x^2 = 4\pi e n\left(1 - \int f \, \mathrm{d}v\right), \qquad (2)$$

where $f(x, v, t)$ is the electron distribution function and ϕ is the electric field

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potential. In the case of a monochromatic wave the potential can be represented by

$$\phi(x,t)=\phi\cos(kx-\omega t).$$

The nonlinear term in the kinetic equation (1) is responsible for the resonant wave–particle interaction. In the linear approximation, the distribution function entering this term is considered unperturbed and the wave amplitude varies slowly with time relative to its oscillation period. Neglecting the variation of the distribution function is justified in this approximation if the characteristic time of this variation is much larger than the time of wave damping with a linear Landau damping rate γ_k^{L}. It is natural to choose the bounce period of the resonant electrons trapped by the wave electric field, $\tau_{\mathrm{b}}=(2m_{\mathrm{e}}/ek^2\phi)^{1/2}$, as the characteristic time scale of the distribution function variation in the resonant velocity region. Interest here is in the opposite limiting case, $\gamma_k^{\mathrm{L}}\tau_{\mathrm{b}}\ll 1$, corresponding to not very small wave amplitudes, i.e.,

$$\phi\gg m_{\mathrm{e}}\left(\gamma_k^{\mathrm{L}}\right)^2/ek^2. \tag{3}$$

In this limit, in contrast, the wave amplitude is practically constant, i.e. $\phi(x,t)\cong\phi_0\cos(kx-\omega t)$, and the resonant particles' distribution function is essentially varying. To study the evolution of the distribution function, consider the electron trajectories in the phase plane (Fig. 4.1.1). In a coordinate system moving with the wave, these trajectories can be found from the conservation law of total energy, $\mathcal{E}=m_{\mathrm{e}}v^2/2-\phi_0\cos kx$. The electrons with $\mathcal{E}<e\phi_0$ are trapped by a wave, and the electrons with $\mathcal{E}>e\phi_0$ are untrapped. It is convenient to consider the distribution function in terms of the energy and angle $(\mathcal{E},\vartheta)$ variables, where $\mathcal{E}$ defines the trajectory and ϑ defines a point on the latter. In the problem under consideration, the function f depends initially on both $\mathcal{E}$ and ϑ, but later strong mixing over the phase of particle motion along the trajectory takes place. To prove this, consider the behavior of trapped particles. Two particles at neighboring trajectories, i.e., two particles with somewhat different energies $\mathcal{E}_1$ and $\mathcal{E}_2$, have, generally speaking, somewhat different frequencies of rotation in phase space (see Fig. 4.1.1), $\omega(\mathcal{E}_1)-\omega(\mathcal{E}_2)\simeq(\mathrm{d}\omega/\mathrm{d}\mathcal{E})(\mathcal{E}_1-\mathcal{E}_2)$. If these particles start moving with the same phase ϑ, then after a time period $\Delta t\sim 1/(\omega_1-\omega_2)$, their phases will deviate from each other by $\Delta\vartheta\sim 1$. This is how phase mixing takes place, and as a result, the function becomes constant along the trajectory if one considers the distribution averaged even over a small $\mathcal{E}$ interval. Similar arguments can be applied to untrapped particles if

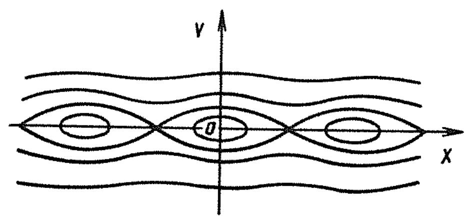

Fig. 4.1.1. Phase space trajectories of particles moving in a monochromatic wave field.

the function f is periodic in the space. The energy $\mathcal{E}$ dependence of the particle distribution function can be explicitly obtained in terms of elliptic integrals (see Chapter 2.2 by Oraevsky).

The case when two monochromatic waves with amplitudes of the same order are excited in a plasma has no simple analytical solution. However, if the phase velocities of these waves are so wide apart that the region of the electrons trapped in the first of these waves does not overlap with the region of the electrons trapped in the second wave, then their mutual interference can be approximately neglected and the evolution of the distribution functions in these phase space regions can be considered independently, as in the case of one wave. Qualitatively new, effects appear in the opposite limit, i.e. for

$$2(e\phi_0/m_e)^{1/2} > \left|\frac{\omega_1}{k_1} - \frac{\omega_2}{k_2}\right|. \tag{4}$$

Then the electron trajectories nearest the separatrix of the regions of trapped and untrapped electrons are essentially perturbed and cannot stay forever inside one region. Under the action of mutual disturbance they are transferred from the trapping region of one wave into that of the other wave, i.e. their "collectivization" takes place. The dynamical description of the electron motion becomes so complicated that the use of computers is necessary. The problem is much simplified in the case when a large number of modes are excited in a plasma and the trapping regions of these modes overlap. Then, the statistical description can replace the dynamical one.

Let a large set of the plasma oscillations with phase velocities in the range $(\omega/k)_{\max} > (\omega/k) > (\omega/k)_{\min}$ be excited in a plasma. Then the electric field can be represented in the form of an electric field superposition of the different waves:

$$E(x,t) = \sum_k E_k \exp[-\mathrm{i}(\omega_k t - kx)]. \tag{5}$$

It is convenient here to perform the summation over both the positive and negative k. In order for the electric field to be real, it is necessary to fulfill the following conditions:

$$E_{-k} = E_k^*, \qquad \omega_{-k} = \omega_k^*. \tag{6}$$

Assume also that the resonances of the neighboring modes overlap, i.e. [compare with (4)]

$$\left(e^2|E_k|^2\delta k/m_e^2k^2\right)^{1/4} \gg \delta(\omega/k), \tag{7}$$

where $\delta(\omega/k)$ and δk are the distance between the neighboring harmonics in the phase velocity and in the k-space, respectively. The plasma noise energy in the range $(k, k+\delta k)$ is represented here by the product of the electric field spectral energy density, $|E_k|^2/8\pi$, and the width of the range, δk. The electric field potential of the separate harmonic is estimated, therefore, as $(|E_k|^2\delta k)^{1/2}/k$. When condition (7) is satisfied the resonant particles can be transferred from the trapping region of one wave into that of a neighboring wave. If, in addition, the phases of the different harmonics are random, then such a transfer is also random. As a result, the resonant particles perform Brownian motion in velocity space. In phase space, this Brownian motion is superimposed on the free particle motion. In time, the Brownian trajectories cover the whole range of particle resonant velocities in the phase plane:

$$(\omega/k)_{\max} > v > (\omega/k)_{\min}. \tag{8}$$

As a result, the distribution function asymptotically approaches a constant value in the strip of phase space between these limiting velocities, although it still remains very complicated and rough. The roughness can be smoothed either by using Coulomb collisions (see Chapter 2.2 by Oraevsky) or by averaging. The true (rough) distribution function obviously conserves entropy and the averaged one does not. The time evolution of the smoothed (averaged) distribution function is governed by the so-called quasilinear diffusion equation (Vedenov et al., 1961; Drummond and Pines, 1962; Romanov and Filippov, 1961).

In conclusion, note that condition (7) imposes only a lower limit to the plasma noise level. However, it is obvious that at high noise level the quasilinear approximation fails, and not only because the nonlinear interaction between the different modes (discussed later) was neglected. Note also that the spacing between the wavenumbers of the neighboring discrete modes is usually so small in the case of large sizes, L, of the plasma volume that condition (7) is more than satisfied. This means that our arguments regarding electron trapping by the wave field are applicable not only to separate harmonics but also to wave packets of width δk_*, defined by the inequality (7):

$$\delta k_*{}^{3/4} \lesssim \left(\frac{e^2|E_k|^2}{m_e^2 k^2}\right)^{1/4} \Big/ \frac{\partial}{\partial k}\left(\frac{\omega}{k}\right). \tag{9}$$

Therefore the random step in velocity space because of Brownian motion is equal to the width of the electron trapping region in such a wave packet:

$$w = \left(e^2|E_k|^2\delta k_*/m_e^2k^2\right)^{1/4}, \tag{10}$$

and the time between these steps is equal to the bounce time of the trapped electron:

$$\tau \simeq 1/kw. \tag{11}$$

The diffusion approximation describing this motion is valid if the width of the wave packet with nearby velocities in the sense of (7) is smaller than the width of the whole spectrum of excited waves. It is convenient to write the latter condition in a form similar to that of (7):

$$\left(\frac{e^2}{m_e^2 k^2}\sum_k |E_k|^2\right)^{1/4} \ll \left(\frac{\omega}{k}\right)_{\max} - \left(\frac{\omega}{k}\right)_{\min} \equiv \Delta\left(\frac{\omega}{k}\right). \tag{12}$$

When this condition is satisfied, the diffusion coefficient can be estimated from the well-known relation

$$D = \frac{w^2}{\tau} = \frac{e^2|E_k|^2}{m_e^2 k^2} \bigg/ \left[\frac{\partial}{\partial k}\left(\frac{\omega}{k}\right)\right]. \tag{13}$$

4.1.2. Quasilinear diffusion in the one-dimensional case

The most rigorous derivation of the quasilinear equation is based on the solution of the initial-value problem, taking into account the overlapping of the resonant regions of neighboring monochromatic waves [see, for example, Altshul and Karpman (1965); Rogister and Oberman (1968)]. However, in practice, as a rule, one uses a simpler derivation procedure in terms of the Fourier components of the electric field and distribution function. As in the case of Landau damping, the solution of this problem, with the help of Fourier transform, gives the same results as the solution of the initial value problem, if the same rules are used to choose the integration path around the pole as used in the Landau damping problem. This simple derivation of the quasilinear equations, resembling the known Van der Pol equations, is based on the separation of the fast and slow processes in the problem. A fast process here is the phase variation of the resonant particles in the wave field. The variation of the separate Fourier harmonics of the electric field and the quasilinear relaxation of the particle distribution function are slow processes. The criterion to separate the electron motion into fast and slowly varying components, $\Delta(\omega_k - kv) \gg \gamma_k, \tau_R^{-1}$ can be written in a form of limitation on the width of the excited plasma wave packet and on their growth rate. To do this, express the time of the quasilinear relaxation of the resonant electron distribution in the velocity range (12) through the velocity space diffusion coefficient (13):

$$\tau_R = [\Delta(\omega/k)]^2/D \simeq [\Delta(\omega/k)]^3 \Big/ \left(e^2 \sum_k |E_k|^2/m_e^2 k^2\right). \tag{14}$$

As a result, the inequality $\Delta(\omega_k - kv) \gg \tau_R^{-1}$ is reduced to the condition that the width of the trapping region is much smaller than the phase velocity spread [see (12)]. The limitation on the instability growth rate is reduced, then, to an inequality similar to that represented by (3) with the effective potential $\phi_{k,\text{eff}} \simeq (|E_k|^2 \delta k_*)^{1/2}/k$.

The fulfilment of the above stated conditions permits one to represent the particle distribution function in the form of a sum of the slowly and rapidly varying parts:

$$f = f_0(v,t) + \delta f(x,v,t). \tag{15}$$

Averaging over a time interval large in comparison with the fast time of the problem and smaller than the quasilinear relaxation time, and also averaging over a space interval large in comparison with the wavelength gives, in accordance with our definition,

$$\langle f \rangle = f_0(v,t). \tag{16}$$

For the rapidly varying part, one can use the linearized equation (1):

$$\frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial x} - \frac{e}{m_e}E(x,t)\frac{\partial}{\partial v}f_0 = 0, \tag{17}$$

where $E(x,t) = -\partial\phi(x,t)/\partial x$ is the plasma wave electric field. Here the rapidly oscillating nonlinear term $(eE/m_e)(\partial\delta f/\partial v)$ responsible for the nonlinear interaction of the different modes is neglected and will be discussed later. Since the function f_0 changes very little during one period of oscillations, to find δf one can use the WKB approximation in time, and then perform the Fourier expansion of the perturbation electric field and of the rapidly varying part of the distribution function:

$$E(x,t) = \sum_k E_k(t)\exp[-\mathrm{i}(\omega_k t - kx)]; \tag{18}$$

$$\delta f(x,v,t) = \sum_k f_k(v,t)\exp[-\mathrm{i}(\omega_k t - kx)], \tag{19}$$

where $\mathrm{Im}\,\omega_k = 0$. In the case considered here of a slowly changing background the expansion coefficients also vary slowly in time and obey the relations (6).

The equation for the slowly-varying part of the distribution function is obtained by simple averaging of the kinetic equation (1), performed as described above:

$$\partial f_0/\partial t = (e/m_e)\langle E(\partial/\partial v)\delta f\rangle. \tag{20}$$

Here the relation (16) is used, and it is assumed that a steady electric field is absent in the plasma. The term on the right-hand side describes the distribution variation under the action of the mean square effect of the fast electric field oscillations. With the help of the relations (18) and (19), the averaging is performed explicitly:

$$\frac{\partial f_0}{\partial t} = \frac{e}{m_e}\sum_k E_k^*(t)\frac{\partial}{\partial v}f_k(v,t). \tag{21}$$

The terms with $k_1 \neq k_2$ disappear as a result of averaging. In agreement with the linear theory of plasma waves, the Fourier coefficient, $f_k(v,t)$, is represented in the WKB approximation by (see Chapter 2.2):

$$f_k = \left(\frac{e}{m_e}\right)E_k(t)\frac{\partial f_0}{\partial v}\left(\mathrm{P}\frac{\mathrm{i}}{\omega_k - kv + \mathrm{i}\gamma_k} + \pi\delta(\omega_k - kv)\right), \tag{22}$$

where the symbol P denotes that the velocity range given by $|\omega_k - kv| < \gamma_k$ is excluded. Introducing this epxression into (21) gives the quasilinear equation for the slowly-varying part of the distribution function in the form of the velocity diffusion equation:

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}D\frac{\partial f_0}{\partial v}, \tag{23}$$

where

$$D(v,t) = \frac{e^2}{m_e^2}\sum_k |E_k|^2\left(\mathrm{P}\frac{\gamma_k}{(kv-\omega_k)^2 + \gamma_k^2} + \pi\delta(\omega_k - kv)\right). \tag{24}$$

The evolution of the wave amplitude and hence the dependence on time of the growth (damping) rate γ_k is defined here by linear theory relations (see Chapter 2.2), where instead of the unperturbed distribution function there enters the distribution function, $f(x, v, t)$, averaged along the complicated particle trajectories:

$$(\partial/\partial t)|E_k(t)|^2 = 2\gamma_k(t)|E_k|^2, \tag{25}$$

where

$$\gamma_k(t) = \left(\frac{\pi}{2}\right)\omega_k\left(\frac{\omega_{pe}}{k}\right)^2 \frac{\partial f_0(v,t)}{\partial v}\bigg|_{v=\omega_k/k} \tag{26}$$

The equations (23) and (25) form a complete set of equations describing the plasma behavior in the quasilinear approximation. Two terms in the diffusion coefficient (specifically, the δ-function term and the principal-value part) have different physical meaning. The δ-function term is positive-definite and responsible for smoothing the distribution function in the resonant region. This is an irreversible process. On the other hand, the principal-value term describes a reversible process since $2\gamma_k|E_k|^2 = \partial|E_k|^2/\partial t$ changes sign when the time is reversed. This apparent (or "adiabatic") diffusion reflects the nonresonant particle response on the variation of the wave amplitude. To prove this, rewrite the part of the quasilinear equation (25) responsible for the nonresonant diffusion in the form:

$$\frac{\partial f_0}{\partial t} = \frac{e^2}{m_e^2}\frac{\partial}{\partial v}\sum_k \frac{\gamma_k|E_k|^2}{\omega_{pe}^2}\frac{\partial f_0}{\partial v}, \tag{27}$$

where, in the case of Langmuir oscillations considered here, the denominator $[(kv - \omega_k)^2 + \gamma_k^2]$ is approximated by ω_{pe}^2. Multiplying both sides of this equation by $m_e v^2/2$ and integrating the result over the velocities, with the help of (25) for the wave growth gives:

$$\frac{d}{dt}\frac{m_e}{2}\int_{-\infty}^{+\infty} dv\, v^2 f_0(v,t) = \frac{d}{dt}\sum_k \frac{|E_k|^2}{8\pi}. \tag{28}$$

In other words, the electron kinetic energy within the main part of the distribution function increases together with the wave electrostatic energy. This is, apparently, the consequence of the well-known result that the total plasma wave energy consists of two equal parts: the electrostatic energy and the electron kinetic energy. Similarly, it can be shown that the main part of the distribution also carries the momentum attributed to waves. However, to do this one should keep the velocity dependence of the denominator, $[(\omega_k - kv)^2 + \gamma_k^2]$, in the diffusion coefficient expression (24). This dependence provides a shift of the distribution function maximum in the direction of wave propagation which corresponds to taking the momentum into account.

The character of the resonant quasilinear diffusion can be most easily illustrated in the example of electron beam relaxation in a plasma [Fig. 4.1.2(a) and (b)]. The presence of the δ-function in the diffusion coefficient (24) assumes the transition from the discrete spectrum to the continuous one. Mathematically this transition can be performed considering the possible wave spectra in a system of finite size L with

$L \to \infty$. The distance between the separate Fourier harmonics in the wavevector space tends to zero since $\delta k = 2\pi/L \to 0$. Therefore, instead of the harmonics summation, one can perform the integration over the wavevectors using the simple relation:

$$\sum_k = \frac{L}{2\pi}\int \mathrm{d}k, \tag{29}$$

taking into account that the number of states in the unit $\mathrm{d}k$ of wavevector space is equal to the length of this element divided by the elementary interval δk corresponding to a single oscillation.

As a result, the diffusion coefficient takes the form:

$$D = (e^2/m_e^2)|E_k|^2(k = \omega_k/v)/|v - \mathrm{d}\omega_k/\mathrm{d}k|, \tag{30}$$

where contributions from both positive and negative k are included in the sum over the harmonics. Assume that the initial wave spectrum is described by some smooth function of (ω/k) [see Fig. 4.1.2(a)]. The waves with phase velocities such that $(\partial f_0/\partial v)(v = \omega_k/k) > 0$ should grow and, after some time, be large enough to cause the velocity diffusion of the resonant electrons. This diffusion results in the flux of resonant particles in the direction opposite to that of the distribution function gradient, i.e. towards the lower velocities. Though the distribution function gradient decreases it continues to be positive, and this leads to wave growth in the low phase velocity region as well. In the limit as $t \to \infty$, the distribution function should relax to the state in which there will be no parts of the distribution function with a positive derivative. Obviously, this condition is satisfied by the distribution with a "plateau" extending from the velocities of the beam particles to the Maxwellian tail of the thermal particles. The height of such an asymptotic distribution in the resonance region is defined uniquely by the condition that the number of particles is conserved in the process of quasilinear diffusion [see Fig. 4.1.2(b)]:

$$\int_{v_1}^{v_2} f_0(v, t=0)\,\mathrm{d}v = f_0(t \to \infty)(v_2 - v_1);$$

$$f_0(v_1, t=0) = f_0(v_2, t=0) = f_0(t \to \infty). \tag{31}$$

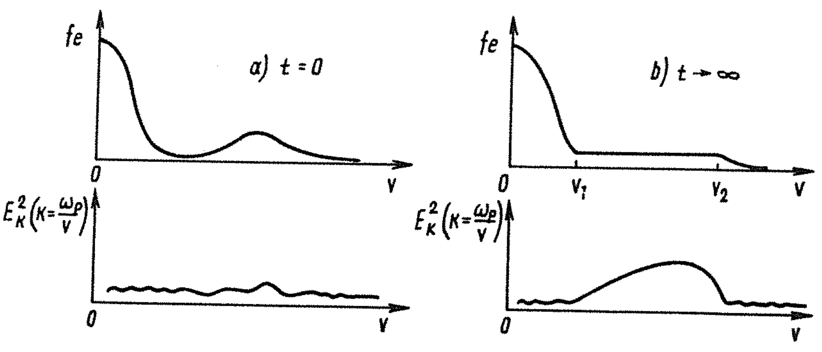

Fig. 4.1.2. Initial (a) and time-asymptotic (b) particle distribution and wave spectrum in the case of a spread beam interaction with a plasma.

To find the asymptotic shape of the wave spectrum, rewrite the quasilinear equation (23) with the help of the wave equation (25) and the resonant diffusion coefficient expression (30) in the form

$$\frac{\partial}{\partial t}\left[f_0(v,t)-\frac{\partial}{\partial v}\frac{e^2}{m_e^2}\frac{2k^2}{\pi\omega_k\omega_{pe}^2}|E_k|^2\left(k=\frac{\omega_k}{v}\right)\Big/\left|v-\frac{d\omega_k}{dk}\right|\right]=0. \tag{32}$$

Assuming that the initial plasma noise energy in the resonant phase velocity region is small in comparison with the spread of electron beam energy, by integrating this equation the spectral energy density of the plasma waves is obtained:

$$|E_k|^2=2\pi^2 m_e\left(\frac{\omega_p^2}{k^3}\right)n_0\int_{v_1}^{\omega_p/k}[f_0(v,t\to\infty)-f_0(v,0)]\,dv, \tag{33}$$

using the approximation $\omega_k \simeq \omega_{pe}=$ constant, which applies to long-wavelength plasma waves. The plasma wave intensity grows rapidly toward the large phase velocity region until it drops sharply at the plateau boundary by virtue of the condition (31) [see Fig. 4.1.2(b)].

It should be noted that the applicability of the quasilinear theory of beam relaxation to a plasma is not great since, due to specific properties of Langmuir waves, their nonlinear interaction is already important for very small wave energy density (see Chapters by Shapiro and Shevchenko; and by Rostoker and Sudan in Volume II).

4.1.3. Relaxation in the case of two- and three-dimensional wave spectra

The quasilinear diffusion equation derived above for the simplest case of a one-dimensional wave spectrum is easily generalized to the more general case of two- or three-dimensional spectra:

$$\frac{\partial f_0}{\partial t}=\sum_{\alpha,\beta=1}^{3}\frac{\partial}{\partial v_\alpha}D_{\alpha\beta}\frac{\partial}{\partial v_\beta}f_0;$$

$$D_{\alpha\beta}=\frac{e^2}{m_e^2}\int\frac{dk}{(2\pi)^3}|E_k|^2\frac{k_\alpha k_\beta}{k^2}\left(\mathrm{P}\frac{\gamma_k}{(\omega_k-kv)^2+\gamma_k^2}+\pi\delta(\omega_k-kv)\right). \tag{34}$$

Although it looks very much like the one-dimensional case, its solutions are qualitatively different from the solutions for a one-dimensional spectrum. The reason for this is an essential increase of the resonant velocity region, even for a wave packet localized in k-space. Since the qualitative differences between two- and three-dimensional relaxations are the same, only the two-dimensional case, which is easier to observe, is considered in detail here.

The relaxation can be visualized in the form of a time variation of the level (equal-value) curves of the distribution function, $f(v_x, v_y, t)$ (Fig. 4.1.3). Let these lines be circles at the initial moment (isotropic distribution), and consider first what happens when a sufficiently narrow wave packet propagates in a plasma in the x-axis direction. In this case the quasilinear diffusion over the v_x velocities results only in a plateau formation in the narrow resonant velocity strip. The state of the plateau corresponds to level lines parallel to the v_x-axis and smoothly matching the circles outside the resonant velocity strip. Such reconstruction of the distribution function requires a finite wave energy in the wave packet. When, in addition to this packet, the other packets propagate in a plasma at different angles to the x-axis, then its own level-line system should correspond to each of those packets. It is obvious that in the intersection region of different resonant strips the distribution function should have the same constant value. Therefore, in the case when the resonant strips of different wave packets mutually overlap and densely cover some range of angles, the distribution function has to be constant in this whole region up to infinitely large velocities. It is clear that the relaxation to that state would require an infinite energy supply from the waves. With finite wave energy, either the waves damp to zero or only one or several nonoverlapping narrow wave packets are left out of the whole wave spectrum, before relaxation can be completed.

As an illustration, consider the simplest case of a two-dimensional wave packet consisting of waves with the same phase velocity (ω/k) and possessing cylindrical symmetry. The resonant region in the (v_x, v_y) plane is outside the circle with the radius (ω/k) since every part of this region belongs to at least two resonant strips. Because of the symmetry of the problem, the solution for f_0 has to be isotropic: $f_0 = f_0(v_x^2 + v_y^2)$. Taking into account this isotropy, one can perform the azimuthal angle integration in the quasilinear equation (34) and represent it as follows:

$$\frac{\partial f_0}{\partial t} = \frac{e^2\omega^2}{m_e^2 k_0^4}|E_0|^2 \frac{1}{v}\frac{\partial}{\partial v}\left(v^2 - \omega^2/k_0^2\right)^{-1/2}\frac{1}{v}\frac{\partial}{\partial v}f_0. \tag{35}$$

Here, the integration was performed explicitly, assuming that the spectral energy density of a sufficiently narrow wave packet can be approximately represented by $|E_k|^2 = 2\pi|E_0|^2 k^{-1}\delta(k - k_0)$. The quasilinear diffusion approximation is still valid here since the overlap of the different resonant regions is provided by the angular

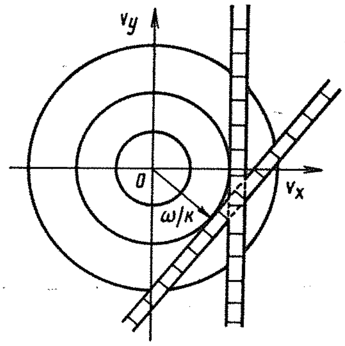

Fig. 4.1.3. Initial and final level curves of the distribution function in the case of two wave packets.

spread inside the wave packet. Particles inside a circle with radius (ω/k) are not resonant with waves and therefore the distribution function stays unchanged in this region.

The diffusion equation should be complemented by the wave equation (25), where the expression for γ_k in the general case of an arbitrary wave propagation angle is:

$$\gamma_k(t) = \left(\frac{\pi}{2}\right)\omega_k\left(\frac{\omega_{pe}}{k}\right)^2 \sum_\alpha \int \mathrm{d}\boldsymbol{v}\, k_\alpha \frac{\partial f_0(v,t)}{\partial v_\alpha}\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}). \tag{36}$$

Again using the velocity distribution isotropy,

$$\gamma_k(t) = \pi\omega_k^2\left(\frac{\omega_{pe}^2}{k^3}\right)\int_{\omega_k/k}^{\infty} \mathrm{d}v \frac{\partial f_0(v,t)}{\partial v}\left(v^2 - \frac{\omega_k^2}{k^2}\right)^{-1/2}. \tag{37}$$

An exact solution of (35) and (37) can be found by introducing an additional simplification. When the initial wave energy is sufficiently large, then as a result of diffusion toward higher velocities, the inequality $v \gg (\omega/k)$ holds for most of the velocity space occupied by the particles. An example of a situation for which one can neglect (ω/k) in comparison with v in (35) and (37) is electron interaction with ion sound waves, since $(\omega/k) = (T_e/m_i)^{1/2} \ll (T_e/m_e)^{1/2} \sim v$ (see details in the Chapter by Sagdeev in Volume II). In the case of Langmuir waves, this applies only after a sufficiently long time for the distribution function to spread to large values of v as a result of diffusion.

In the limit when $v \gg (\omega/k)$, one can neglect the terms of the order $(\omega/kv)^2 \ll 1$ in (35) and (37) and introduce a new variable:

$$\tau = \left(25e^2/m_e^2\right)\int_0^t \left(\omega^2/k_0^3\right)|E_0|^2 \mathrm{d}t \equiv \int_0^t D(t')\,\mathrm{d}t'. \tag{38}$$

Then (35) can be rewritten in the form

$$\frac{\partial f_0}{\partial \tau} = \frac{4}{25}\frac{\partial}{\partial v^2}\frac{1}{v}\frac{\partial}{\partial v^2} f_0. \tag{39}$$

Independent of the initial distribution function, in the limit $\tau \to \infty$, this function asymptotically approaches the self-similar solution (Sagdeev and Galeev, 1969):

$$2\pi f_0(v, t\to\infty) = A\left(\int_0^t D(t')\,\mathrm{d}t'\right)^{-2/5} \exp\left(-v^5\Big/\int_0^t D(t')\,\mathrm{d}t'\right), \tag{40}$$

where

$$A = \left[5/\Gamma\left(\tfrac{2}{5}\right)\right]\int_{\omega/k}^{\infty} f_0(v, t=0)\, v\,\mathrm{d}v, \qquad v > \omega/k.$$

Substitution of this solution into (37) gives the asymptotic behavior of the growth rate:

$$\gamma_k = -A\omega_k^2\omega_p^2\Gamma\left(\tfrac{4}{5}\right)/2k^3\left(\int_0^t D(t')\,\mathrm{d}t'\right)^{3/5}. \tag{41}$$

Thus, the particle distribution function and instability growth rate evolution is described in an explicit form through the integral of the spectral wave energy density. The equation (35) for the spectral energy density can be reduced to a second-order differential equation whose solution is cumbersome. Nevertheless, the expected qualitative behavior of this solution is sufficiently evident. The wave energy damps and, finally, in the limit as $t \to \infty$ it damps to zero. The damping rate that is initially equal to the linear Landau damping rate decreases due to the variation of the distribution function slope. In the limit as $t \to \infty$, the growth rate (γ_k) approaches a constant value as the energy needed to reconstruct the distribution function is exhausted. This is essentially different from the one-dimensional case when a "plateau" is formed.

It should be noted that in the previous consideration the wave spectrum was assumed to be isotropic. However, in the presence of a magnetic field the same equations [(35) and (37)] are still valid, even in the cases when the excitation mechanism (current-driven instability, drift-cone instability, etc.) results in wave spectrum anisotropy. In these cases the distribution function isotropy in the (v_x, v_y) plane is a result of particle cyclotron rotation in the magnetic field directed along the z-axis (see the chapters by Galeev and Sagdeev, and Trakhtengerts in Volume II).

4.1.4. Plasma quasilinear relaxation in a magnetic field (two-dimensional quasiplateau)

In the general case of electromagnetic waves in a plasma immersed in a steady and uniform magnetic field $\boldsymbol{B}_0$ the equation for the distribution function f_j of each particle species j is:

$$\frac{\partial f_j}{\partial t} + \boldsymbol{v}\cdot\nabla f_j + \frac{e_j}{m_j c}[\boldsymbol{v}\times \boldsymbol{B}_0]\cdot\frac{\partial f_j}{\partial \boldsymbol{v}} + \frac{e_j}{m_j}\left(\boldsymbol{E}_1 + \frac{1}{c}[\boldsymbol{v}\times\boldsymbol{B}_1]\right)\cdot\frac{\partial f_j}{\partial \boldsymbol{v}} = 0, \tag{42}$$

and the Maxwell equations determine the electric ($\boldsymbol{E}_1$) and magnetic ($\boldsymbol{B}_1$) fields. The distribution function is again represented as the sum of the slowly (f_{0j}) and rapidly (δf_j) varying parts. The expression for δf_j is found by linearizing (42) and integrating the resulting equation along the unperturbed particle trajectories (see Chapters 2.2 and 3.3):

$$\delta f_j = -\frac{e_j}{m_j}\sum_k \int_{-\infty}^{t} \mathrm{d}t'\left(\boldsymbol{E}_k(t') + \frac{1}{c}[\boldsymbol{v}\times\boldsymbol{B}_k(t')]\right)\cdot\frac{\partial f_{0j}}{\partial \boldsymbol{v}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}(t') - \omega_k t')], \tag{43}$$

where the Fourier expansion (5) for the electric and magnetic fields has been used. An appropriate quasilinear equation is obtained by averaging (42), and retaining the term quadratic in the wave amplitude:

$$\frac{\partial f_{0j}}{\partial t}=\frac{e_j^2}{m_j^2}\sum_k\left(\boldsymbol{E}_k^*(t)+\frac{1}{c}[\boldsymbol{v}\times\boldsymbol{B}_k^*(t)]\right)\cdot\frac{\partial}{\partial \boldsymbol{v}}$$
$$\times\int_{-\infty}^{t}\mathrm{d}t'\left(\boldsymbol{E}_k(t')+\frac{1}{c}[\boldsymbol{v}\times\boldsymbol{B}_k(t')]\right)\cdot\frac{\partial f_{0j}}{\partial \boldsymbol{v}}$$
$$\times\exp\{\mathrm{i}\omega(t-t')-\mathrm{i}\boldsymbol{k}\cdot[\boldsymbol{r}-\boldsymbol{r}(t')]\}. \tag{44}$$

Performing the integration along the particle trajectories explicitly, the resulting velocity diffusion equation is:

$$\frac{\partial f_{0j}}{\partial t}=\sum_k\hat{Q}_{jk}^+|\boldsymbol{E}_k|^2\pi\delta(\omega_k-k_\parallel v_\parallel+l\omega_{cj})\hat{Q}_{jk}f_{0j}, \tag{45}$$

where the operator $\hat{Q}_{jk}$ and the conjugate operator $\hat{Q}_{jk}^+$ are linear combinations of the derivatives on different velocity components, and the resonance conditions are given by the δ-function (the value of the integer l depends on the type and polarization of the wave). The explicit expressions for these operators will be obtained in Volume II, Part 6, in a discussion of different applications of quasilinear theory. Here, the discussion is limited to the relaxation character described by such equations.

First, it should be noted that the wave–particle resonance condition in a magnetic field imposes a limitation only on the particle velocity along the magnetic field. Therefore, in the case of a narrow wave packet, quasilinear relaxation takes place in the narrow strip of the resonant velocities. Apparently, the quasilinear diffusion velocity within this strip leads to a state where the diffusion stops. The asymptotic particle distribution can be found from the equation

$$\hat{Q}_{jk}f_{0j}(t\to\infty)\big|_{v_\parallel=(\omega+l\omega_{cj})/k_\parallel}=0. \tag{46}$$

The level lines of the steady velocity distribution function satisfying (46) are the characteristics of this partial differential equation.

In the case of a uniform plasma in a magnetic field, when the particle perpendicular velocity distribution is isotropic the operator can contain only the derivatives $\partial/\partial v_\perp$ and $\partial/\partial v_\parallel$. In agreement with that, the characteristics of (46) are some lines in the $(v_\perp, v_\parallel)$ plane governed by the equation in a general form:

$$w_k(v_\perp^2, v_\parallel)=\text{constant}. \tag{47}$$

The subscript $\boldsymbol{k}$ here takes into account the fact that the level curves generally depend on the wavevector. When the packet is narrow, then the characteristics for different $v_\perp$ are far enough from each other and, as a result of relaxation, a "plateau" is rapidly formed along these lines although the height of the "plateau" is different for different $v_\perp$. The plasma state with such a plateau is stable, as in the case of one-dimensional relaxation in a plasma without a magnetic field. This can be proved by explicit calculation of the damping (growth) rate, with the help of linear theory.

For example, in the case of an anisotropic plasma instability resulting in excitation of Whistler waves propagating along the magnetic field, the growth rate is zero (see Chapter 3.3):

$$\int \mathrm{d}v_\perp\, v_\perp^2 \left[\left(1-\frac{k_\| v_\|}{\omega_k}\right)\frac{\partial f_{0e}}{\partial v_\perp}+\frac{k_\| v_\perp}{\omega_k}\frac{\partial f_{0e}}{\partial v_\|}\right]\Bigg|_{v_\|=(\omega+\omega_{ce})/k_\|}=0. \tag{48}$$

The corresponding equation for the steady electron distribution function is obviously:

$$\left[\left(1-\frac{k_\| v_\|}{\omega_k}\right)\frac{\partial f_{0e}}{\partial v_\perp}+\frac{k_\| v_\perp}{\omega_k}\frac{\partial f_{0e}}{\partial v_\|}\right]\Bigg|_{v_\|=(\omega+\omega_{ce})/k_\|}=0. \tag{49}$$

Note that in this particular case the equation for characteristics playing the role of f_{0j} level curves for $t\to\infty$ has the form of a circle equation with the center at a distance $\omega/k_\|$ from the origin (see Fig. 4.1.4):

$$w_k\left(v_\perp^2, v_\|\right)=v_\perp^2/2+v_\|^2/2-\omega_k v_\|/k_\|. \tag{50}$$

Anisotropic plasma relaxation will be discussed in detail in the chapter by Trakhtengerts, Volume II. Here, the discussion of general properties of (45) is continued for broad wave packets. In this case, quasilinear diffusion caused by interaction with one wave out of the wave packet tends to establish a plateau along the characteristic corresponding to the wavevector of that wave. Since the different k characteristics are intersecting, a stationary state can be reached only by forcing f_{0j} to be constant in the whole resonant velocity region. In this sense, the situation is similar to the case of the two- (or three-) dimensional wave packet in a plasma without a magnetic field. However, in those cases the angles at which the characteristics intersect are small, a "quasiplateau" occurs. This is due to the fact that the plateau along the characteristics is formed much faster than the equalization of the distribution function across them.

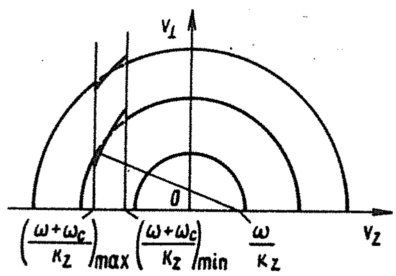

Fig. 4.1.4. Initial (———) and final (–) level curves of the resonant electron distribution function in the case of the Whistler packet.

Finally, note that in the case of nonuniform plasma the operator Q_k is a linear combination of the spatial and velocity derivatives and therefore one can refer to quasiplateau formation in the mixed space of space and velocity coordinates (see the chapter by Horton in Volume II).

4.1.5. Influence of collisions on the wave–particle interaction

It has been shown that in the case of one-dimensional or quasi-one-dimensional (in a magnetic field) quasilinear diffusion in a velocity space, the distribution function is strongly distorted near the resonant velocity region. For narrow wave packets this results in such a sharp increase of the distribution function derivatives that one should take into account long-range Coulomb collisions sensitive to the fine structure of the distribution function. The latter statement is mathematically expressed through the fact that the Landau collisional integral contains a term with the second derivative in velocity.

As an illustration, consider the effect of collisions on one-dimensional plateau formation. In this case, quasistationary distribution occurs under the influence of two effects: quasilinear diffusion leading to a plateau, and collisions tending to restore the Maxwellian distribution function. A balance of the corresponding terms in the kinetic equation has the form:

$$0=\frac{\mathrm{d}}{\mathrm{d}v}D(v)\frac{\mathrm{d}f}{\mathrm{d}v}+St\{f\}. \tag{51}$$

The following approximate expressions describe a qualitative picture of the relaxation for the quasilinear diffusion coefficient and the collisional term,

$$D(v)=(\pi e^2/m_e^2)\sum_k|E_k|^2\delta(\omega_k-kv)\simeq e^2\langle E^2\rangle/m_e^2\omega;$$

$$St\{f\}=\nu(\omega/k)^2\frac{\mathrm{d}^2}{\mathrm{d}v^2}(f_M-f),$$

where f_M is the Maxwellian distribution function. Integrating (51) gives

$$\frac{\mathrm{d}f}{\mathrm{d}v}=\frac{\mathrm{d}f_M}{\mathrm{d}v}\Big/\left(1+\frac{e^2\langle E^2\rangle}{m_e^2\omega\nu(\omega_k/k)^2}\right). \tag{52}$$

Then, the slope of the distribution function found above is introduced into the damping rate expression, $\gamma=(\pi/2)(\omega^3/k^2)(\mathrm{d}f_M/\mathrm{d}v)(v=\omega/k)$.

The result is:

$$\gamma=\gamma^L/\left[1+e^2\langle E^2\rangle/m_e^2\omega\nu(\omega_k/k)^2\right]. \tag{53}$$

The effect of collisions is now clear. When the wave amplitudes in a packet are sufficiently small then the damping rate approaches the linear Landau damping rate, γ^L. This is due to the fact that collisions have enough time to restore the slope of the distribution function in a resonant region corresponding to the Maxwellian velocity distribution. In the case of large amplitudes, the slope of the distribution function is proportional to the frequency of Coulomb collisions and inversely proportional to the level of plasma waves forcing the distribution to a state approaching a plateau. In agreement with this, the damping rate drops. The expression (53) can be generalized to the arbitrary case of relaxation to a "quasiplateau":

$$\gamma = \gamma^{L}/(1 + \tau_1/\tau_2), \qquad (54)$$

where τ_1 is the characteristic time to reach a local Maxwellian distribution under the influence of collisions (or, more generally, an unstable distribution formation under the influence of external forces), and τ_2 is the characteristic time of "quasiplateau" formation under the influence of the wave packet.

These arguments are particularly important for drift instabilities since, in a rarefied plasma, rare Coulomb collisions are not able to stop quasiplateau formation in the $(x, v_{\parallel})$ space or instability saturation. As a result of this, anomalous diffusion also stops (see the chapter by Horton in Volume II).

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Chapter 4.2

Wave – Wave Interaction

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Contents

4.2.1. Dynamic equations for plasma wave resonant interaction

Thus far, in the framework of linear and quasilinear descriptions of a plasma containing collective excited oscillations, the interaction of different oscillation modes has been neglected by considering them as a simple superposition of the separate modes. This could be justified by the fact that the mode coupling effects are caused by the nonlinearity of the medium and therefore they are weak when mode amplitudes are small. But even a weak interaction can show its effect after a sufficiently long time. Naturally this time is longer when the oscillation level is lower. In the lowest expansion order, the time after which the interaction effects start to be important is inversely proportional to the wave energy, W. In addition, as shown previously, the quasilinear relaxation of an initially unstable particle distribution under the influence of developing oscillations can stabilize their growth at best

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Volume 1: Basic Plasma Physics I, edited by A.A. Galeev and R.N. Sudan

during a time inversely proportional to the wave energy. This means that neglect of nonlinear wave interaction cannot be justified, even for small amplitudes; to estimate its importance one should compare the quasilinear relaxation time with the time of energy redistribution for different modes.

As mentioned previously, mode coupling effects are caused by plasma nonlinearity. Precisely because of the nonlinearity, forced beat waves at mixed frequencies appear in a plasma and can augment the resonance with plasma eigenmodes. Whether or not resonance conditions are satisfied depends, in turn, on the dispersion of the appropriate waves in a plasma. For example, in the simplest case of quadratic nonlinearity, the beats at the frequency $(\omega_{k_1} \pm \omega_{k_2})$ with wavevector $(\boldsymbol{k}_1 + \boldsymbol{k}_2)$ can reasonate with some other eigenmode only if the resonance condition in the form $\omega_{k_1} \pm \omega_{k_2} = \omega_{k_1 \pm k_2}$ is satisfied. Possible dependencies of the frequency on the wavevector in an isotropic plasma are shown in Fig. 4.2.1. Using simple geometrical arguments (Vedenov et al., 1961), it is easy to verify that in the case when all three interacting waves belong to the same oscillation branch, the resonance condition can be satisfied only for branches 2 and 4. For branches 1 and 3, the interaction between oscillations of the same type is possible only in the next expansion order (of the wave energy) when the cubic nonlinearity terms are retained. This does not mean, of course, that the lowest-order resonant interaction of types 1 and 3 waves can be neglected, since the interaction of two waves of the same type can take place in this case with the involvement of a third wave of another type. [Note that the linear dispersion law for sound waves in a hydrodynamic medium allows the interaction in all orders of the perturbation theory and corresponds to the strong mode coupling case, which is outside the scope of weak turbulence theory.]

As an example to illustrate the technique for deriving dynamical equations for interacting wave amplitudes, consider a mixed interaction between Langmuir and ion sound waves in nonisothermal plasma, having dispersion of types 1 and 3, respectively (Oraevskii and Sagdeev, 1962). Since the resonant interaction of waves does not involve particles, it is convenient to perform all computations in the framework of the simpler hydrodynamic description. In the next Chapter (4.3), a

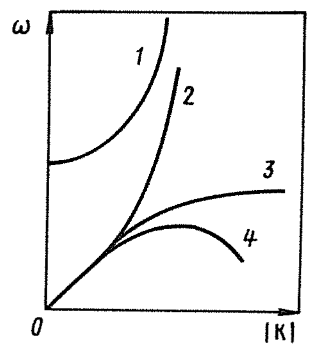

Fig. 4.2.1. Dispersion curves in an isotropic case. A resonance between three waves of the same type is possible for the dispersion of the types 2 or 4 and impossible for the dispersion of the types 1 or 3.

more rigorous kinetic approach will be shown to give the same results for this case. The equation for interacting waves in terms of the polarizability of the medium by expansion in the wave amplitudes will also be included.

A plasma in which Langmuir and ion sound waves are excited is described by a system of equations consisting of the momentum and continuity equations for the electron and ion components separately, and a Poisson equation for the wave electric field:

$$n_j m_j\left[\partial \boldsymbol{v}_j/\partial t+(\boldsymbol{v}_j\cdot\nabla)\boldsymbol{v}_j\right]=-T_j\nabla n_j+e_j n_j \boldsymbol{E}; \tag{1}$$

$$\partial n_j/\partial t+\operatorname{div}(n_j\boldsymbol{v}_j)=0; \tag{2}$$

$$\operatorname{div}\boldsymbol{E}=-4\pi\sum_j e_j n_j, \tag{3}$$

where n_j, $\boldsymbol{v}_j$ and T_j are the density, velocity and temperature, respectively, of the species; and E is the electric field strength. By restricting attention to the simplest case of one-dimensional wave propagation along the x-axis, then labeling by the indices 1 and 3 the hydrodynamical parameters of the Langmuir waves, and by the index 2 the corresponding parameters of the ion-sound wave, one can represent the electric field, the density and the velocity of the electron and ion components in the form:

$$\begin{aligned}
&\boldsymbol{E}=\sum_\alpha \boldsymbol{E}_\alpha(x,t);\\
&n_e=n_0+n_1(x,t)+n_3(x,t)+N_2(x,t);\\
&n_i=n_0+N_2(x,t);\\
&\boldsymbol{v}_e=v_1(x,t)+v_3(x,t)+V_2(x,t);\\
&\boldsymbol{v}_i=V_2(x,t).
\end{aligned} \tag{4}$$

Considering all the wave amplitudes to be small, and using perturbation theory to solve equations (1)–(3), in a linear approximation they are reduced to the wave equations for Langmuir and ion sound waves. In the next approximation, quadratic terms are retained for wave amplitudes in (1)–(3). As a result:

$$\frac{\partial v_1}{\partial t}+\frac{e}{m_e}E_1+\frac{T_e}{m_e n_0}\frac{\partial n_1}{\partial x}=-\frac{\partial}{\partial x}(v_3V_2)+\frac{T_e}{m_e n_0^2}\frac{\partial}{\partial x}(n_3N_2); \tag{5}$$

$$\frac{\partial n_1}{\partial t}+n_0\frac{\partial v_1}{\partial x}=-\frac{\partial}{\partial x}(N_2v_3+n_3V_2); \tag{6}$$

$$n_1+\frac{1}{4\pi e}\frac{\partial E_1}{\partial x}=0; \tag{7}$$

$$m_i\frac{\partial V_2}{\partial t}-eE_2=0; \tag{8}$$

$$en_0E_2+T_e\frac{\partial N_2}{\partial x}=-\frac{\partial}{\partial x}(n_0m_ev_1v_3)-T_e\frac{\partial}{\partial x}\frac{n_1n_3}{n_0}; \tag{9}$$

$$\frac{\partial N_2}{\partial t}+n_0\frac{\partial V_2}{\partial x}=0. \tag{10}$$

The left-hand sides of (5)–(7) describe the Langmuir wave and the right-hand sides couple it to the ion sound wave and a second Langmuir wave. Similarly, the left-hand sides of (8)–(10) describe the ion sound wave and the right-hand sides couple it to two Langmuir waves. The equation for wave 3 is not included here because it is easily obtained by the interchange of the indices 1 and 3 in (5)–(7). Since the hydrodynamic description of Langmuir waves is valid only for waves with wavelength exceeding the Debye length, $\lambda_D = v_{Te}/\omega_{pe}$, then a small contribution of electron thermal motion to the wave dispersion should be taken into account only in the linear approximation. This permits one to neglect the last nonlinear terms on the right-hand sides of (5) and (9). The relative importance of the remaining nonlinear terms in (5) and (6) for the Langmuir waves is estimated by comparison with the corresponding linear terms:

$$R_1 = \frac{\partial}{\partial x}(v_3 V_2) \Big/ \frac{\partial v_1}{\partial t} \sim \frac{k_1 V_2 v_3}{\omega_1 v_1};$$

$$R_2 = \frac{\partial}{\partial x}(v_3 N_2) \Big/ \frac{\partial}{\partial x}(n_0 v_1) \sim \frac{N_2 v_3}{n_0 v_1},$$

where k_α and ω_α are the wavevector and the frequency of the α component. Use of the linear relation between the parameters V_2 and N_2 in the sound wave gives $R_2 \sim (\omega_1/\omega_2) R_1 \gg R_1$. Therefore it is sufficient to retain in (5)–(7) only the nonlinearity terms corresponding to R_2. By simple transformation, (5)–(10) can be represented by inhomogeneous equations for E_1 and N_2:

$$\frac{\partial^2 E_1}{\partial t^2} + \omega_{pe}^2 E_1 - \frac{T_e}{m_e}\frac{\partial^2 E_1}{\partial x^2} = \frac{\partial}{\partial t}\frac{N_2}{n_0}\frac{\partial E_3}{\partial t}; \tag{11}$$

$$\frac{\partial^2 N_2}{\partial t^2} - c_s^2 \frac{\partial^2 N_2}{\partial x^2} = \frac{1}{4\pi\omega_{pe}^2 m_i}\frac{\partial^2}{\partial x^2}\left(\frac{\partial E_1}{\partial t}\frac{\partial E_3}{\partial t}\right), \tag{12}$$

where $c_s = (T_e/m_i)^{1/2}$ is the ion sound speed in a nonisothermal plasma.

As in quasilinear theory, the weak nonlinear wave interaction results only in a slow variation of the amplitude in time. In accordance with this, the oscillations of E, n, and v values of a wave can be represented in the form of harmonic oscillations with slowly varying amplitude. For example:

$$E_\alpha(x,t) = E_{k_\alpha}(t)\exp\left[-i(\omega_{k_\alpha} t - k_\alpha x)\right]. \tag{13}$$

Neglecting small nonlinear terms in (11)–(12) responsible for wave coupling shows that the wave frequencies ω_{k_α} are related to the wavevectors k_α by known linear dispersion relations. A slow wave amplitude variation in the next approximation can take place only when the coordinate dependence of the left- and right-hand sides is the same, i.e. the wavevector resonance condition, $k_1 = k_2 + k_3$, is satisfied. This variation is slow in comparison with the oscillation period (i.e. $\partial/\partial t \ll \omega$). The equations describing it are obtained by the substitution of (13) into (11) and (12):

$$\mathrm{i}\frac{\partial E_{k_1}}{\partial t}=\frac{(\omega_{k_3}+\omega_{k_2})\omega_{k_3}N_{k_2}}{2\omega_{k_1}n_0}E_{k_3}\exp[\mathrm{i}(\omega_{k_1}-\omega_{k_2}-\omega_{k_3})t]; \tag{14}$$

$$\mathrm{i}\frac{\partial N_{k_2}}{\partial t}=\frac{k_2^2\omega_{k_1}\omega_{k_3}}{m_i\omega_{k_2}\omega_{\mathrm{pe}}^2}\frac{E_{k_1}E_{k_3}^*}{8\pi}\exp[-\mathrm{i}(\omega_{k_1}-\omega_{k_2}-\omega_{k_3})t]. \tag{15}$$

Finally, these equations should be rewritten in terms of the probability amplitudes, so that the equations for interacting waves take a symmetrical form characteristic of a Hamiltonian system (Galeev and Karpman, 1963). The relation between the probability amplitude and the amplitude of the oscillations of the physical parameters in a wave is found using the expression for the probability amplitude squared, i.e. the number of wave quanta, in terms of the wave energy:

$$|C_k|^2\equiv n_k=W_k/\omega_k. \tag{16}$$

The Langmuir wave energy is approximately the sum of two equal parts: the electric field oscillation energy and the kinetic energy of electrons participating in oscillations, i.e. $W_{k_1}=|E_{k_1}|^2/2\pi$. (Here, it was taken into account that $\langle E_1^2\rangle=2|E_{k_1}|^2$.) Similarly, the ion sound energy consists of the plasma compression energy and the ion kinetic energy: $W_{k_2}=2|N_{k_2}|^2T_e/n_0$. In agreement with that, the probability amplitudes are defined as

$$C_{k_{1,3}}(t)=\frac{E_{k_{1,3}}(t)}{(2\pi|\omega_{k_{1,3}}|)^{1/2}},\qquad C_{k_2}=\frac{N_{k_2}}{(n_0|\omega_{k_2}|/2T_e)^{1/2}}. \tag{17}$$

As a result, (14) and (15) take the form of the Schrödinger-type equation in the representation ($\omega_k-\omega_{k_1}-\omega_{k_2}\ll\omega_c$):

$$\mathrm{i}\,\partial C_{k_1}/\partial t=V_{k_1,k_2,k_3}C_{k_2}C_{k_3}\exp[-\mathrm{i}(\omega_{k_1}-\omega_{k_2}-\omega_{k_3})t], \tag{18}$$

$$\mathrm{i}\,\partial C_{k_2}/\partial t=V_{k_2,-k_3,k_1}C_{k_3}^*C_{k_1}\exp[\mathrm{i}(\omega_{k_2}+\omega_{k_3}-\omega_{k_1})t], \tag{19}$$

where:

$$V_{k_1,k_2,k_3}=V_{k_2,-k_3,k_1}\mathrm{sign}(\omega_{k_1},\omega_{k_2})=-(|\omega_{k_1}\omega_{k_2}\omega_{k_3}|/8\pi n_0T_e)^{1/2}\mathrm{sign}\,\omega_{k_1}.$$

Such a relation between the matrix elements of the interaction operator in (18) and (19) is a consequence of the Hamiltonian form of the two-fluid hydrodynamic equations. Strictly speaking, this relation has been proved only with accuracy of the terms of the order $\sim(\omega_{k_2}/\omega_{k_1})\ll 1$. When this inequality is satisfied then the so-called adiabatic approximation for wave coupling is applicable and this allows simplification of the derivation of the dynamical equations. The plasma density variation in an ion sound wave is actually adiabatic with respect to a high-frequency Langmuir wave packet. Therefore, the equation for Langmuir waves can be obtained simply from the linear wave equation, taking into account a slow variation in plasma density:

$$\frac{\partial^2 E}{\partial t^2} + \frac{4\pi n_0 e^2}{m_e} E = -\frac{4\pi e^2}{m_e} NE. \tag{20}$$

The influence of the Langmuir waves on the ion sound waves is described by high-frequency pressure added to the usual kinetic pressure:

$$n_0 m_e \langle v^2 \rangle = \langle E^2 \rangle / 4\pi.$$

As a result, the ion sound equation takes the form:

$$\frac{\partial^2}{\partial t^2} N = \frac{1}{m_i} \frac{\partial^2}{\partial x^2} \left(T_e N_2 + \frac{EE^*}{4\pi} \right). \tag{21}$$

It is evident that these equations coincide with (11) and (12) in the limit $k^2 \lambda_D^2 \ll 1$ and $(\omega_2/\omega_{1,3}) \ll 1$.

4.2.2. Criteria of the transition from dynamical to statistical description

A detailed investigation of resonant interaction of finite amplitude waves is the subject of a later section devoted to parametric instabilities. Here, the particular case is considered of the decay of a wave with frequency ω_{k_3} and wavevector k_3 into two waves, (ω_{k_1}, k_1) and (ω_{k_2}, k_2), with infinitely small amplitudes (Oraevsky and Sagdeev, 1962). In this case, the amplitude of the initial wave (pumping wave) can be considered constant. As a result, the system of equations (18) and (19) becomes linear and its solution has an exponential form:

$$C_{k_1}, C_{k_2} \sim e^{\nu t}. \tag{22}$$

Using the symmetry relation for matrix elements in these equations, the growth rate expression is obtained (Galeev and Karpman, 1963):

$$\nu = \left[-|V_{k_1,k_2,k_3}|^2 |C_3^{(0)}|^2 \operatorname{sign}(\omega_{k_1}\omega_{k_2}) - \tfrac{1}{4}(\Delta\omega)^2 \right]^{1/2}, \tag{23}$$

where $\Delta\omega = \omega_{k_1} - \omega_{k_2} - \omega_{k_3}$ is the frequency mismatch of three interacting waves. Perturbations grow only when the signs of the frequencies ω_{k_1} and ω_{k_2} are different, i.e. when the initial wave frequency, $|\omega_{k_3}|$, is larger than the frequencies $|\omega_{k_1}|$ and $|\omega_{k_2}|$ of the pumped waves. In other words, the initial quantum energy according to the energy conservation law is approximately equal to the sum of the resulting quanta energies, and therefore it is larger than the energy of either of these quanta.

Here, it should be noted that, in agreement with the quantum mechanical uncertainty principle, for the finite mode growth time interval it is not necessary to satisfy exactly the frequency resonance condition. As a consequence, even in discrete systems, two waves with given (ω_{k_3}, k_3) and (ω_{k_2}, k_2) can interact resonantly through the wave packet (ω_{k_1}, k_1).

A more detailed discussion of the conditions for the transition from three-wave interaction to many-wave interaction follows. For one-dimensional propagation of the waves considered above, the resonance condition has the form (here, the more correct Langmuir wave dispersion relation derived from kinetic theory is used):

$$k_1 = k_2 + k_3;$$
$$\omega_{pe}\left(1+\tfrac{3}{2}k_1^2\lambda_D^2\right) = -k_2c_s + \omega_{pe}\left(1+\tfrac{3}{2}k_3^2\lambda_D^2\right). \tag{24}$$

Obviously, due to the smallness of the ion sound frequency, these conditions can be satisfied only for $k_3 \simeq -k_1$ and $k_2 \simeq 2k_1$. In a system of size L, the wavevector spectrum is discrete, with the spacing between the neighboring harmonics $\delta k = 2\pi/L$. For Langmuir waves this corresponds to a frequency spacing of the order of $\delta\omega = (\partial\omega/\partial k)\delta k = 3k\,\delta k\,\lambda_D^2\omega_{pe}$. Therefore, the transition from the three-wave interaction, when conditions (24) are exactly satisfied, to the many-wave interaction takes place for an initial (pumping) wave amplitude higher than critical:

$$|V_{k_1,k_2,k_3}|^2|C_3|^2 > \tfrac{1}{4}(\partial\omega/\partial k_1)^2\delta k^2. \tag{25}$$

As the amplitudes of the waves (ω_{k_1}, k_1) and (ω_{k_2}, k_2) grow, they start interacting between themselves without participation of the initial wave (ω_{k_3}, k_3), i.e. via growth of nearby harmonics. It is precisely this overlapping process of the different possible resonances with width of order ν that results in the stochastization of the interacting wave phases.

The phase stochastization time is estimated for the stage when all the spectrum harmonics are of the same order [a more rigorous consideration is given by Zaslavskii and Sagdeev (1967); and Kaufman (1971)]. At this stage, instead of $|C_3|^2$, the number of coherent waves in a packet of width δk_* enters the expression (23). The coherence condition obviously has the form

$$\tfrac{1}{4}(\partial\omega/\partial k)^2\delta k_*^2 \lesssim \mathcal{K}^2 n_k \delta k_*, \tag{26}$$

where $\mathcal{K}$ is the matrix element for the interacting wave packets; i.e. $\mathcal{K} = |V_{k_1,k_2,k_3}| =$ constant. This condition is used to obtain the resonance overlapping condition and the phase stochastization time:

$$\delta k_* = 4\mathcal{K}^2 n_k/(\partial\omega/\partial k)^2 > \pi/L; \tag{27}$$
$$\tau^{-1} = \tfrac{1}{2}(\partial\omega/\partial k)\delta k_* = 2\mathcal{K}^2 n_k/(\partial\omega/\partial k). \tag{28}$$

Note that this time exceeds the decay time of a given monochromatic wave. Therefore, one can speak about the phase stochastization of waves born as a result of the decay instability only in the case of a steady pumping wave. A weak turbulence approximation is still valid if the uncertainty of the frequency resonance condition does not exceed the ion sound wave frequency spread:

$$\Delta k c_s \gg (\partial\omega/\partial k)\delta k_* \simeq 4\mathcal{K}^2 n_k/(\partial\omega/\partial k). \tag{29}$$

Here, the ion sound spectrum width coincides with that of Langmuir waves by virtue of fulfilment of the wavevector resonance condition. Using the matrix element expression found in the course of the derivation of (18) and (19), this condition is rewritten in terms of the Langmuir wave energy:

$$\sum_k |E_k|^2/n_0 T_e \ll 6k\,\Delta k \lambda_D^2.$$

It will be shown in the chapter by Shapiro and Shevchenko in Volume II that when this inequality is violated, instead of the stochastization of Langmuir wave phases they are fragmentated into coherent entities collapsing to very small space scale.

4.2.3. *Wave kinetic equation in the random-phase approximation*

Phase mixing as a result of wave resonant interaction allows the random-phase approximation, in which evolution of the wave field can be described in terms of the number of wave quanta (occupation number) varying for a given $\boldsymbol{k}$. In other words, the wave amplitudes are followed only while averaging over the wave phases. To obtain the equation for the number of waves, a classical analog of quantum mechanical perturbation theory is used (Peierls, 1965), as was first done by Galeev and Karpman (1963); Camae et al. (1962).

As a starting point, the dynamical equation for the probability amplitudes is used, which is a simple generalization of (18) and (19) to the many-wave case:

$$\mathrm{i}\frac{\mathrm{d}C_k}{\mathrm{d}t} = \sum_{k'} V_{k,k',k''} C_{k'}(t) C_{k-k'}(t) \exp[\mathrm{i}(\omega_k - \omega_{k'} - \omega_{k''})t]. \tag{30}$$

The summation over $\boldsymbol{k}'$ here takes into account the fact that a given wave $(\omega_k, \boldsymbol{k})$ can interact with pairs of waves, $(\omega_{k'}, \boldsymbol{k}')$ and $(\omega_{k''}, \boldsymbol{k}'')$, with wavevectors satisfying the space resonance condition $(\boldsymbol{k} = \boldsymbol{k}' + \boldsymbol{k}'')$. Wave amplitudes are normalized again in such a way that the square of the probability amplitudes is equal to the number of waves with given $\boldsymbol{k}$:

$$|C_k|^2 = n_k. \tag{31}$$

With this choice of normalization, the matrix elements satisfy the symmetry properties, generalizing earlier results to the case of arbitrary plasma modes:

$$V_{k,k',k-k'} = V_{k-k',-k',k} \operatorname{sign}(\omega_k \omega_{k-k'});$$

$$V_{k,k',k-k'} = V_{k,k-k',k'} \simeq -V_{-k,-k',k'-k}. \tag{32}$$

These properties apply to any Hamiltonian system. Expanding C_k in a series over the interaction operator $\hat{V}$:

$$C_k(t) = C_k^{(0)} + C_k^{(1)} + C_k^{(2)} + \cdots$$

and substituting the result into (30),

$$C_k^{(1)} = -\mathrm{i}\sum_{k',k''} C_{k'}^{(0)}C_{k''}^{(0)}\int_0^t V_{k,k',k''}(t')\,\mathrm{d}t';$$

$$C_k^{(2)} = -\sum_{k',k'',q',q''}\Big[C_{k'}^{(0)}C_{q'}^{(0)}C_{q''}^{(0)}\int_0^t \mathrm{d}t'\int_0^{t'}\mathrm{d}t''V_{k,k',k''}(t')V_{k'',q',q''}(t'')$$

$$+ C_{k''}^{(0)}C_{q'}^{(0)}C_{q''}^{(0)}\int_0^t \mathrm{d}t'\int_0^{t'}\mathrm{d}t''V_{k,k',k''}(t')V_{k',q',q''}(t'')\Big], \tag{33}$$

where

$$V_{k,k',k''}(t) = V_{k,k',k''}\delta_{k,k'+k''}\exp[\mathrm{i}(\omega_k - \omega_{k'} - \omega_{k''})t];$$

$$\delta_{k,q} = \begin{cases}1, & k=q\\ 0, & k\neq q\end{cases}. \tag{34}$$

The values $C_k^{(0)}$ do not depend upon time and correspond to the solution in the absence of mode interaction. They can be represented in the form of a product of the positive amplitude and the phase factor, $\exp(\mathrm{i}\phi_k)$. Although the phases ϕ_k are defined by initial conditions in any given experiment, nevertheless, it is reasonable to assume them random, when the conditions described in the previous paragraph are satisfied. In the case of random phases, the following relation holds:

$$\langle C_k^{(0)}C_{k'}^{(0)}\rangle = |C_k^{(0)}|^2\delta_{k,-k'}. \tag{35}$$

It is used to average the variation of the number of waves (i.e. the value $|C_k|^2 - |C_k^{(0)}|^2$), giving, in the lowest order,

$$|C_k|^2 = |C_k^{(0)}|^2 + \langle|C_k^{(1)}|^2\rangle + \langle C_k^{(0)}C_k^{(2)*} + C_k^{(0)*}C_k^{(2)}\rangle. \tag{36}$$

Using the values $C_k^{(1)}$ and $C_k^{(2)}$ given by (33), this is rewritten:

$$|C_k(t)|^2 - |C_k(0)|^2$$

$$= \sum_{k',k'',q',q''}\Big(C_{k'}^{(0)}C_{k''}^{(0)}C_{q'}^{(0)*}C_{q''}^{(0)*}\int_0^t \mathrm{d}t'V_{k,k',k''}(t')\int_0^t \mathrm{d}t'V_{k,q',q''}(t')$$

$$-\mathrm{Re}\,2C_k^{(0)*}C_{k'}^{(0)}C_{q'}^{(0)}C_{q''}^{(0)}\int_0^t \mathrm{d}t'V_{k,k',k''}(t')\int_0^{t'}\mathrm{d}t''V_{k'',q',q''}(t'')$$

$$-\mathrm{Re}\,2C_k^{(0)*}C_{k''}^{(0)}C_{q'}^{(0)}C_{q''}^{(0)}\int_0^t \mathrm{d}t'V_{k,k',k''}(t')\int_0^{t'}\mathrm{d}t''V_{k',q',q''}(t'')\Big). \tag{37}$$

As a result of averaging this equation over random phases, the product of four amplitudes, $C_k^{(0)}$, is reduced to the product of two occupation numbers. Two possible options of amplitude pairing are shown in (37) by dashed and solid brackets. In the first term the amplitudes $C_k^{(0)}$ are combined into the product

$$|C_{k'}^{(0)}|^2|C_{k''}^{(0)}|^2 = n_{k'}^{(0)}n_{k''}^{(0)},$$

and in the other two terms they are combined into

$$|C_k^{(0)}|^2|C_{k'}^{(0)}|^2 = n_k^{(0)}n_{k'}^{(0)}$$

and

$$|C_k^{(0)}|^2|C_{k''}^{(0)}|^2 = n_k^{(0)}n_{k''}^{(0)},$$

respectively. Symmetry properties allow one to write the product of the two matrix elements entering (37) in the form of its square modulus,

$$\left|\int_0^t \mathrm{d}t' V_{k,k',k''}(t')\right|^2,$$

with the sign depending on the signs of the frequencies ω_k, $\omega_{k'}$ and $\omega_{k''}$. For time intervals much larger than the oscillation period of any wave, the time integration can be carried out approximately:

$$\left|\int_0^t \mathrm{d}t' V_{k,k',k''}(t')\right|^2 = \frac{4\sin^2[(\omega_k - \omega_{k'} - \omega_{k''})t/2]}{(\omega_k - \omega_{k'} - \omega_{k''})}|V_{k,k',k''}|^2\delta_{k,k'+k''}$$

$$= 2\pi\delta(\omega_k - \omega_{k'} - \omega_{k''})\delta_{k,k'+k''}|V_{k,k',k''}|^2 t.$$

As a result, the time variation of the number of waves can be written in the form:

$$\Delta n_k = 4\pi\Delta t \sum_{k',k''} |V_{k,k',k''}|^2 \left[n_{k'}^{(0)}n_{k''}^{(0)} - \mathrm{sign}(\omega_k\omega_{k''})n_k^{(0)}n_{k'}^{(0)}\right.$$

$$\left. - \mathrm{sign}(\omega_k\omega_{k'})n_k^{(0)}n_{k''}^{(0)}\right]\delta(\omega_k - \omega_{k'} - \omega_{k''})\delta_{k,k'+k''}. \tag{38}$$

This can be represented in the form of a differential equation if the above averaging procedure is applied at any moment, t, thus defining the variation of the number of waves at the next moment, $t + \mathrm{d}t$. In other words,

$$\Delta n_k/\Delta t = \mathrm{d}n_k/\mathrm{d}t; \qquad n_k^{(0)} = n_k(t).$$

Thus, the wave kinetic equation is obtained from (38):

$$\frac{\mathrm{d}n_k}{\mathrm{d}t} = 4\pi \sum_{k',k''} |V_{k,k',k''}|^2\{n_{k'}n_{k''} - \mathrm{sign}(\omega_k\omega_{k''})n_k n_{k'}$$

$$- \mathrm{sign}(\omega_k\omega_{k'})n_k n_{k''}\}\delta(\omega_k - \omega_{k'} - \omega_{k''})\delta_{k,k'+k''}. \tag{39}$$

A plasma wave equation in such a form was written first by Camac et al. (1962) using an analogy with the quantum mechanical equation for phonons in the solid state and later was derived rigorously (Galeev and Karpman, 1963; Kadomtsev and Petviashvily, 1962). A quantum mechanical analog of this equation is usually written for positive frequencies since the frequency of the quantum is always associated with its energy expression, $\hbar\omega_k$. It can be easily obtained from the same dynamic equations (3) using quantum mechanical perturbation theory and the "Golden Rule."

Consider for example the interaction of a wave with frequency ω_k with two other waves with lower frequencies, $\omega_{k'}$ and $\omega_{k''}$ ($\omega_k > \omega_{k'}$, $\omega_{k''} > 0$). The interaction processes in this case consists of the ω_k-wave decay processes and the reversed merging processes with the participation of two waves with frequencies $\omega_{k'}$ and $\omega_{k''}$. The variation of the occupation number in the course of these processes can be written (in units where $\hbar = 1$) (Peierls, 1965):

$$\frac{\mathrm{d}n_k}{\mathrm{d}t} = -4\pi \sum_{k',k''} |V_{k,k',k''}|^2 \left[n_k(n_{k'}+1)(n_{k''}+1) - (n_k+1)n_{k'}n_{k''} \right] \times \delta(\omega_k - \omega_{k'} - \omega_{k''}) \delta_{k,k'+k''}. \tag{40}$$

In the classical limit ($n_k \gg 1$) this reduces to (39). Similarly, the "collisional term" for the four-wave interaction is derived. The latter is proportional to the third power of the occupation number. However, it is very seldom used in plasma turbulence studies since, for nondecay spectrum types, there is a possibility of nonlinear wave interaction via resonance with particles (this is the *wave – particle – wave interaction* discussed below). The latter is, as a rule, more important than the four-wave interaction since it is of lower order in the wave energy. Usually in hydrodynamics, where wave–particle resonances are absent, the four-wave interaction can be decisive, as for gravitational surface waves.

The kinetic equation (40) has been used for a long time in solid state theory to describe the interaction of phonons with lattice irregularities (Peierls, 1965). However, there is a principal difference between the application of this equation to phonons and to plasma turbulence. Regarding the solid state, one usually deals with a state close to thermodynamic equilibrium. In this case, the problem is reduced to the calculation of small corrections to the equilibrium occupation numbers, i.e. to the Rayleigh–Jeans distribution of phonons. In a plasma, in contrast, a strongly nonequilibrium situation is usually encountered when a strong wave source is present in one region of wavevector space, and wave dissipation is taking place in a completely different region. This situation is more likely than the usual hydrodynamic turbulence and, therefore, the turbulence spectra corresponding to the constant wave energy flux in wavevector space (Kolmogorov–Obuchov hypothesis) are more characteristic for a plasma (for details, see the chapter by Zacharov in Volume II).

4.2.4. Interaction of waves with different energy signs

The qualitative picture of wave resonant interaction changes when negative energy waves take part in this interaction. Attention to the existence of such waves in a plasma was drawn by Kadomtsev et al. (1964). The negative energy term means that the total (kinetic and potential) plasma energy decreases while the wave amplitude increases. Negative energy waves can arise only in a nonequilibrium medium. This can be directly verified by considering the known expression for electromagnetic field energy in a dispersive medium:

$$w=\frac{1}{8\pi}\left(\frac{\mathrm{d}}{\mathrm{d}\omega}(\omega\varepsilon)\langle E^2\rangle+\frac{\mathrm{d}}{\mathrm{d}\omega}(\omega\mu)\langle H^2\rangle\right), \tag{41}$$

where ε and μ are the dielectric and magnetic permeability of the medium, respectively. If consideration is limited to electrostatic waves, then the sign of the energy depends only on the sign of $(\mathrm{d}\varepsilon/\mathrm{d}\omega)$, which can be negative in a thermodynamically nonequilibrium medium and is excluded in the equilibrium medium by the Kramers–Kronig relationships. Some specific examples of negative energy waves have been discussed in Chapter 3.3 (kinetic plasma instabilities).

To find out what qualitative differences arise in a wave interaction in which negative energy waves participate the dynamic and kinetic equations are generalized to this case. For simplicity, consideration is limited to electrostatic modes. The probability amplitude satisfying the definition (31) can be expressed through the wave electric field potential:

$$C_k(t)=\left[(k^2/8\pi)|\partial\varepsilon(\omega_k,k)/\partial\omega_k|\right]^{1/2}\phi_k. \tag{42}$$

The symmetry properties of the matrix element can be obtained from (32) with the help of the substitution:

$$\operatorname{sign}\omega_k\to\operatorname{sign}\left(\omega_k^{-1}\frac{\partial}{\partial\omega_k}\left[\omega_k\varepsilon(\omega_k,k)\right]\right)\equiv\operatorname{sign}\frac{\partial\varepsilon}{\partial\omega_k}.$$

The result is that

$$V_{k-k',-k',k}=V_{k,k',k-k'}\operatorname{sign}\left(\frac{\partial\varepsilon}{\partial\omega_k}\frac{\partial\varepsilon}{\partial\omega_{k-k'}}\right). \tag{43}$$

A rigorous proof of this relation will be obtained in the next section, where the equation for interacting waves in terms of the dielectric permeability expansion (more correctly, medium polarizability) on the wave amplitudes will be derived. The change of symmetry properties results in a corresponding change in the kinetic equation for positive occupation numbers:

$$\begin{aligned}\operatorname{sign}\left(\frac{\partial\varepsilon}{\partial\omega_k}\right)\frac{\partial n_k}{\partial t}=\sum_{k',k''}|V_{k,k',k''}|^2\Big[&n_{k'}n_{k''}\\&-\operatorname{sign}\left(\frac{\partial\varepsilon}{\partial\omega_k}\frac{\partial\varepsilon}{\partial\omega_{k'}}\right)n_kn_{k''}-\operatorname{sign}\left(\frac{\partial\varepsilon}{\partial\omega_k}\frac{\partial\varepsilon}{\partial\omega_{k''}}\right)n_kn_{k'}\Big]\\&\times\delta(\omega_k-\omega_{k'}-\omega_{k''})\delta_{k,k'+k''}.\end{aligned} \tag{44}$$

Note that a specific nonlinear instability is possible in a system of waves with different energy signs. The reason is that when the negative energy wave gives energy to the positive energy wave, then the amplitudes of both waves grow. The simplest example of such an instability is negative energy wave decay into two waves of each type, which occurs explosively (Dikasov et al., 1965; Coppi et al., 1969).

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Chapter 4.3

Wave – Particle – Wave Interaction

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Contents

4.3.1. Kinetic derivation of the wave equation

In the discussion of resonant wave interaction in the previous chapter, the hydrodynamic description of plasma was used intentionally in order to separate it from the wave interaction with participation of resonant particles. The fact is that the beats with frequencies $(\omega_k \pm \omega_{k'})$ and wavevectors $(\boldsymbol{k} \pm \boldsymbol{k}')$ resulting from plasma nonlinearity can come into resonance both with the third wave and the particles moving with a velocity satisfying the Cerenkov resonance condition, $\omega_k \pm \omega_{k'} = (\boldsymbol{k} \pm \boldsymbol{k}')\cdot \boldsymbol{v}$ (or cyclotron resonance in a magnetic field). At first glance, it might seem sufficient to take into account the wave–particle resonance only in the quasilinear approximation, that is in the lowest-order approximation in the wave energy. However, the existence of weakly damped waves also requires the quasilinear

Handbook of Plasma Physics, Eds. M.N. Rosenbluth and R.Z. Sagdeev
Volume 1: Basic Plasma Physics I, edited by A.A. Galeev and R.N. Sudan

interaction to be weak. The latter, as a rule, takes place for waves with phase velocities much higher than the particle thermal velocities. Therefore, for example, the ion sound oscillations exist only in a nonisothermal plasma when $\omega/k = (T_e/m_i)^{1/2} \gg v_{Ti} = (T_i/m_i)^{1/2}$, and Langmuir turbulence consists of long-wavelength plasmons, the phase velocity of which is $\omega/k \gg v_{Te}$. For such oscillations the number of particles involved in linear resonance is small. On the other hand, the number of particles resonantly interacting with beats can be large, and this necessitates the consideration of such processes.

From the above discussion it is clear that the equation for weakly interacting waves obtained in the framework of the kinetic description of plasma should already contain both effects: the resonant three-wave interaction and the two-wave interaction with participation of resonant particles.

As in the two preceding chapters, classical perturbation theory is used, considering terms up to third order in the expansion in wave amplitude. The simplest case of plasma electrostatic waves is considered first. The wave electric field potential is expanded both in time* and space in Fourier series. The time expansion assumes good behavior of the potential for $t \to \infty$. Although this condition is definitely violated in the linear approximation when wave growth or damping takes place, the nonlinear effects limiting perturbation growth can justify such an assumption. In agreement with classical perturbation theory, a distribution function is sought in the form of an expansion in wave amplitudes. The nonlinear term of the kinetic equation is transferred to the right-hand side:

$$\frac{\partial f_j}{\partial t} + \boldsymbol{v}\cdot\nabla f_j + \frac{e_j}{m_j c}[\boldsymbol{v}\times\boldsymbol{B}_0]\cdot\frac{\partial f_j}{\partial \boldsymbol{v}} = \frac{\mathrm{i}e_j}{m_j}\sum_{\boldsymbol{k},\omega}\phi(\boldsymbol{k},\omega)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\boldsymbol{k}\cdot\frac{\partial f_j}{\partial \boldsymbol{v}}. \tag{1}$$

The left-hand side here is a complete time derivative of the particle distribution function along the particle trajectory defined by the equations of motion:

$$\mathrm{d}\boldsymbol{r}/\mathrm{d}t = \boldsymbol{v}; \qquad \mathrm{d}\boldsymbol{v}/\mathrm{d}t = (e/c)(\boldsymbol{v}\times\boldsymbol{B}_0). \tag{2}$$

This allows (1) to be rewritten in the form of an integral along the particle trajectory:

$$f_j = \frac{\mathrm{i}e_j}{m_j}\sum_{\boldsymbol{k},\omega}\int_{-\infty}^{t}\mathrm{d}t'\phi(\boldsymbol{k},\omega)\exp\{\mathrm{i}[\boldsymbol{k}\cdot\boldsymbol{r}(t')-\omega t']\}\boldsymbol{k}\cdot\frac{\partial f_j}{\partial \boldsymbol{v}}, \tag{3}$$

where $\phi(\boldsymbol{k},\omega)$ is the Fourier transform of the potential. Solving this equation by

*It is, actually, more convenient to use the integral transformation in time instead of the Fourier series in time. Therefore, to simplify notation, the summation sign over frequencies means the integration over frequencies.

iteration, the Fourier transform of the distribution function is obtained:

$$f_j(\boldsymbol{k},\omega,\boldsymbol{v})=\sum_n^{\infty} f_j^{(n)}(\boldsymbol{k},\omega,\boldsymbol{v}); \tag{4}$$

$$\begin{aligned} f_j^{(n)}(\boldsymbol{k},\omega,\boldsymbol{v})\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)] &= \frac{\mathrm{i}e_j}{m_j}\sum_{\substack{\boldsymbol{k}'+\boldsymbol{k}''=\boldsymbol{k}\\ \omega'+\omega''=\omega}}\int_{-\infty}^{t}\mathrm{d}t'\phi(\boldsymbol{k},\omega)\exp[\mathrm{i}(\boldsymbol{k}'\cdot\boldsymbol{r}(t')-\omega t')] \\ &\times \boldsymbol{k}'\cdot\frac{\partial f_j^{(n-1)}(\boldsymbol{k}'',\omega'',\boldsymbol{v})}{\partial\boldsymbol{v}}\exp[\mathrm{i}(\boldsymbol{k}''\cdot\boldsymbol{r}(t')-\omega''t')]. \end{aligned} \tag{5}$$

Substitution of this expression into the Poisson equation gives the wave dynamical equation in the form of a plasma polarizability expansion in the Fourier harmonic amplitudes of the potential:

$$\begin{aligned} &\varepsilon_{\boldsymbol{k}}^{(1)}(\omega)\phi(\boldsymbol{k},\omega)+\sum_{\substack{\boldsymbol{k}'+\boldsymbol{k}''=\boldsymbol{k}\\ \omega'+\omega''=\omega}}\varepsilon_{\boldsymbol{k}',\boldsymbol{k}''}^{(2)}(\omega',\omega'')\phi(\boldsymbol{k}',\omega)\phi(\boldsymbol{k}'',\omega) \\ &+\sum_{\substack{\boldsymbol{k}'+\boldsymbol{k}''+\boldsymbol{k}'''=\boldsymbol{k}\\ \omega'+\omega''+\omega'''=\omega}}\varepsilon_{\boldsymbol{k}',\boldsymbol{k}'',\boldsymbol{k}'''}^{(3)}(\omega',\omega'',\omega''')\phi(\boldsymbol{k}',\omega')\phi(\boldsymbol{k}'',\omega'')\phi(\boldsymbol{k}''',\omega''') \\ &+\cdots=0. \end{aligned} \tag{6}$$

Here, $\varepsilon_{\boldsymbol{k}}^{(1)}(\omega)$ is the linear dielectric permeability of a plasma, and the expressions for $\varepsilon^{(2)}$ and $\varepsilon^{(3)}$ can be found from (5), where the corresponding term (4) of the distribution function expansion in wave amplitudes should be substituted. Next, these coefficients are calculated for the simplest case of Langmuir waves in a plasma without a magnetic field.

To solve the dynamical equation, consider the Fourier expansion coefficients, $\phi(\boldsymbol{k},\omega)$, as a small parameter. Obviously, $\phi(\boldsymbol{k},\omega)$ has a narrow peak near the eigenfrequency; i.e. it can be approximated as:

$$\phi(\boldsymbol{k},\omega)\simeq\phi_{\boldsymbol{k}}^{(1)}\delta(\omega-\omega(\boldsymbol{k})), \tag{7}$$

where $\omega(\boldsymbol{k})$ is the solution of the equation $\mathrm{Re}\,\varepsilon_{\boldsymbol{k}}^{(1)}(\omega)=0$. The width of this peak at the quasilinear stage is of the order of $\gamma_{\boldsymbol{k}}$. Thus it is small for $\gamma_{\boldsymbol{k}}\ll\omega_{\boldsymbol{k}}$. One can expect some broadening of the peak at the nonlinear stage that is proportional to the level of oscillations. However, in a weakly unstable plasma ($\gamma_{\boldsymbol{k}}\ll\omega_{\boldsymbol{k}}$), the level of oscillations is also small ($|\phi_{\boldsymbol{k}}^{(1)}|^2\sim\gamma_{\boldsymbol{k}}/\omega_{\boldsymbol{k}})n_0T$), and one can still use the approximation (7).

In the next approximation, (6) gives

$$\phi^{(2)}(\boldsymbol{k},\omega)=-\sum_{\boldsymbol{k}'+\boldsymbol{k}''=\boldsymbol{k}}\frac{\varepsilon_{\boldsymbol{k}',\boldsymbol{k}''}^{(2)}(\omega_{\boldsymbol{k}'},\omega_{\boldsymbol{k}''})}{\varepsilon_{\boldsymbol{k}}^{(1)}(\omega)}\phi_{\boldsymbol{k}'}^{(1)}\phi_{\boldsymbol{k}''}^{(1)}\delta(\omega_{\boldsymbol{k}'}+\omega_{\boldsymbol{k}''}-\omega). \tag{8}$$

To derive the wave kinetic equation, multiply (6) by $\phi^*(\boldsymbol{k},\omega)\exp[\mathrm{i}(\tilde{\omega}-\omega)t]$ and integrate the result over $\mathrm{d}\omega\,\mathrm{d}\tilde{\omega}$. The first term of the equation thus obtained is:

$$\int \mathrm{d}\omega \int \mathrm{d}\omega' \varepsilon_k^{(1)}(\omega)\phi^*(\boldsymbol{k},\omega)\phi(\boldsymbol{k},\omega)\exp[\mathrm{i}(\tilde{\omega}-\omega)t]. \tag{9}$$

Since $\phi(\boldsymbol{k},\omega)$ has a peak near ω_k, the dielectric permeability here can be represented as:

$$\varepsilon_k^{(1)}(\omega) \simeq \frac{\partial \varepsilon_k^{(1)\prime}(\omega_k)}{\partial \omega_k}(\omega-\omega_k)+\mathrm{i}\varepsilon_k^{(1)\prime\prime}(\omega_k),$$

where $\varepsilon_k^{(1)\prime}$ and $\varepsilon_k^{(1)\prime\prime}$ are the real and imaginary parts of the dielectric permeability, respectively. Symmetrizing the integrand (9) over ω and $\tilde{\omega}$ and performing the integration, the first term may be rewritten in the form:

$$\frac{\mathrm{i}}{2}\frac{\partial \varepsilon_k^{(1)\prime}(\omega_k)}{\partial \omega_k}\frac{\partial}{\partial t}|\phi_k(t)|^2+\mathrm{i}\varepsilon_k^{(1)\prime\prime}(\omega_k)|\phi_k(t)|^2, \tag{10}$$

where the time dependence of the eigenoscillation amplitude is defined as:

$$\phi_k(t)=\int \mathrm{d}\omega\phi(\boldsymbol{k},\omega)\exp[-\mathrm{i}(\omega-\omega_k)t]. \tag{11}$$

Substituting the expressions (7) and (8) into the remaining terms of (6) and averaging the result over the phases (i.e. $\langle \phi_k^{(1)}\phi_{k'}^{(1)}\rangle = |\phi_k^{(1)}|^2\delta_{k,-k'}$) the wave equation is obtained:

$$\begin{aligned}
&\frac{1}{2}\frac{\partial \varepsilon_k^{(1)}}{\partial \omega_k}\frac{\partial}{\partial t}|\phi_k|^2 \\
&\quad= -\operatorname{Im}\varepsilon_k^{(1)}(\omega_k)|\phi_k|^2+\operatorname{Im}\sum_{k'+k''=k}\frac{2|\varepsilon_{k',k''}^{(2)}(\omega_{k'},\omega_{k''})|^2|\phi_{k'}|^2|\phi_{k''}|^2}{\varepsilon_{k'+k''}^{(1)*}(\omega_{k'}+\omega_{k''})} \\
&\qquad+\operatorname{Im}\sum_{k'}\left(\frac{4\varepsilon_{k',k-k'}^{(2)}(\omega_{k'},\omega_k-\omega_{k'})\varepsilon_{k,-k'}^{(2)}(\omega_k,-\omega_{k'})}{\varepsilon_{k-k'}^{(1)}(\omega_k-\omega_{k'})}\right. \\
&\qquad\qquad\left.-3\varepsilon_{k',-k',k}^{(3)}(\omega_{k'},-\omega_{k'},\omega_k)\right)|\phi_k|^2|\phi_{k'}|^2.
\end{aligned} \tag{12}$$

Here, wave energy terms are limited to the second order and the superscript (1) of the wave amplitude, $\phi_k^{(1)}$, is dropped. This equation was derived first by Drummond and Pines (1962) for the particular case of the one-dimensional Langmuir wave packet and later was generalized by a number of authors (Kadomtsev and Petviashvili, 1962; Galeev et al., 1964; Silin, 1964) to more general cases. The first term on the right-hand side describes linear wave damping (growth). The contribution to the second term comes from the poles arising when the beating frequency coincides with one of the eigenfrequencies. Note that, as in the case of a Landau pole, here one should use the specific path of integration to calculate the contribution from the pole. This rule can be formulated as follows: independently of the sign of the imaginary part of the dielectric permeability, the pole contribution is calculated assuming that this sign is the same as in the equilibrium medium. According to the Kramers–Kronig relation the sign of the imaginary part of the dielectric permeability in an equilibrium medium coincides with the frequency sign. Therefore,

$$\operatorname{Im}\frac{1}{\varepsilon_k^{(1)}(\omega)} = -\pi \operatorname{sign}\omega_k\, \delta\left[\varepsilon_k^{(1)}(\omega)\right] = \frac{\pi \operatorname{sign}\omega_k\, \delta(\omega-\omega_k)}{|\partial\varepsilon_k(\omega)/\partial\omega|}. \tag{13}$$

Thus, in the case of positive energy waves, the second term describes the merging of the waves $\phi_{k'}$ and $\phi_{k''}$ with the production of the wave ϕ_k. In the case of a negative energy wave, ϕ_k, the sign of its amplitude variation due to the merging of the positive energy waves $\phi_{k'}$ and $\phi_{k''}$ is correctly taken into account by rule (13) and the relation (10). The third term gives the contributions both to the decay processes $(\omega_k - \omega_{k'} = \omega_{k''})$ and to induced wave scattering $(\omega_k - \omega_{k'} = (\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v})$. The fourth term is obviously responsible for induced wave scattering by free particles. (The role of the last two terms in the induced scattering of Langmuir and electromagnetic waves is discussed in detail in Sections 4.3.2 and 4.3.3.)

Finally it has been shown by Rosenbluth et al. (1969) that the nonlinear wave–particle–wave interaction can also lead to an instability when both positive and negative energy waves are excited in an active medium. For electrostatic waves in the absence of a static magnetic field $\boldsymbol{B}_0$ the required condition is

$$S_k(\boldsymbol{k}+\boldsymbol{k}')\cdot\int \mathrm{d}^3v\,(\partial f/\partial \boldsymbol{v}) > 0$$

where

$$S_k = \left(\partial\varepsilon^{(1)\prime}/\partial\omega_k\right)/|\partial\varepsilon^{(1)\prime}/\partial\omega_k|.$$

In a magnetic field for waves with $\boldsymbol{k}\cdot\boldsymbol{B}_0 = 0$, the resonant condition is $\omega_k + \omega'_k = l\omega_c$, $l\omega_c$ being a multiple of the cyclotron frequency. A nonlinear instability occurs if $S_k = S_{k'}$ and

$$(\omega_k + \omega_{k'})S_k\int \mathrm{d}E_\perp\,|v|^2\,\partial f/\partial E_\perp > 0$$

where $E_\perp = \frac{1}{2}mv_\perp^2$ is the perpendicular particle kinetic energy.

4.3.2. *The weak Langmuir wave turbulence equations*

As a first example, consider the interaction of Langmuir waves with random phases in a plasma without a magnetic field. The calculation of coefficients for plasma polarizability expansion on wave amplitudes is particularly simple in this case since the integration in (4) is performed along the straight particle trajectories

$r(t)=vt$. As a result, the first two expansion coefficients are:

$$\varepsilon_k^{(1)}(\omega)=1+\sum_j \frac{\omega_{pj}^2}{k^2}\int d^3v \frac{k\cdot\partial f_0/\partial v}{\omega-k\cdot v+i0}; \tag{14}$$

$$\begin{aligned}\varepsilon^{(2)}_{k',k''}(\omega',\omega'')=&-\sum_j \frac{\omega_{pj}^2}{2(k'+k'')^2}\left(\frac{e_j}{m_j}\right)\\ &\times\int d^3v \frac{1}{\omega'+\omega''-(k'+k'')\cdot v+i0}\\ &\times\left(k'\cdot\frac{\partial}{\partial v}\frac{1}{\omega''-k''\cdot v+i0}k''\cdot\frac{\partial}{\partial v}+k''\cdot\frac{\partial}{\partial v}\frac{1}{\omega'-k'\cdot v+i0}k'\cdot\frac{\partial}{\partial v}\right)f_{0j};\end{aligned} \tag{15}$$

$$\begin{aligned}\varepsilon^{(3)}_{k',k'',k'''}(\omega',\omega'',\omega''')=&\sum_j \frac{\omega_{pj}^2e_j^2/m_j^2}{6(k'+k''+k''')^2}\\ &\times\int d^3v \frac{1}{\omega'+\omega''+\omega'''-(k'+k''+k''')\cdot v+i0}\\ &\times k'''\cdot\frac{\partial}{\partial v}\frac{1}{\omega'+\omega''-(k'+k'')\cdot v+i0}\\ &\times\left(k'\cdot\frac{\partial}{\partial v}\frac{1}{\omega''-k''\cdot v+i0}k''\cdot\frac{\partial}{\partial v}+k''\cdot\frac{\partial}{\partial v}\frac{1}{\omega'-k'\cdot v+i0}k'\cdot\frac{\partial}{\partial v}\right)f_{0j}\\ &+\text{two other terms with the interchange of } k''' \text{ and } k' \text{ or } k''' \text{ and } k''.\end{aligned} \tag{16}$$

The infinitely small positive value (+0) defines here an integration path around the pole to integrate over velocities. It does not arise naturally here, as was the case in linear theory when the Laplace transform was used to solve an initial problem. It has been introduced simply to maintain the causality principle (to provide an adiabatic "turn-off" for both the eigenoscillation and the forced beats as $t\to\infty$).

It was shown first by Drummond and Pines (1962) that the contribution of $\varepsilon^{(3)}$ term to induced wave scattering by electrons is compensated by the second-term contribution in the large brackets of (12). The physical mechanism for the weakening of scattering is the electron charge screening by the ion cloud (so-called polarization effect). Then the dominant process is induced plasmon scattering by ions described by the third term of (12) (and, of course, decay processes with participation of phonons in the case of a non-isothermal plasma). It is easy to see that the dielectric permeability nonlinearity, $\sim\varepsilon^{(2)}_{k,-k'}$, is caused by the nonlinearity of the electron motion equations, and the ions contribute to the imaginary part of $\varepsilon^{(1)}_{k-k'}(\omega_k-\omega_{k'})$. To evaluate $\varepsilon^{(2)}_{k,-k'}$ the integral in (14) is expanded in terms of the small parameter

$(\omega - \omega')/|k - k'|v_{Te} \ll 1$:

$$\varepsilon^{(2)}_{k,-k'}(\omega_k, -\omega_{k'}) = \frac{\omega_{pe}^2}{2(k-k')^2}\frac{e}{m_e}\int d^3v[\omega_k - \omega_{k'} - (k-k')\cdot v + i0]^{-1}$$

$$\times\left[\frac{k\cdot k'}{(\omega_k - k\cdot v)^2}k\cdot\frac{\partial f_{0e}}{\partial v} - \frac{k\cdot k'}{(\omega_{k'} - k'\cdot v)^2}k'\cdot\frac{\partial f_{0e}}{\partial v}\right.$$

$$\left. + \frac{\omega_k - \omega_{k'} - (k-k')\cdot v}{(\omega_k - k\cdot v)(\omega_{k'} - k'\cdot v)}\left(k\cdot\frac{\partial}{\partial v}\right)\left(k'\cdot\frac{\partial}{\partial v}\right)f_{0e}\right]$$

$$\simeq \frac{e}{2m_e}\frac{(k\cdot k')}{\omega_k\omega_{k'}}\frac{\omega_{pe}^2}{(k-k')^2}\int d^3v \frac{(k-k')\cdot\partial f_{0e}/\partial v}{\omega_k - \omega_{k'} - (k-k')\cdot v + i0}$$

$$\simeq \frac{e}{2m_e}\frac{(k\cdot k')}{\omega_k\omega_{k'}}\varepsilon^{(1)e}_{k-k'}(\omega_k - \omega_{k'}). \tag{17}$$

In a similar way

$$k^2\varepsilon^{(2)}_{k',k-k'}(\omega_k, \omega_k - \omega_{k'}) \simeq \frac{e}{2m_e}\frac{(k\cdot k')}{\omega_k\omega_{k'}}(k-k')^2\varepsilon^{(1)e}_{k-k'}(\omega_k - \omega_{k'}) \tag{18}$$

and

$$k^2\varepsilon^{(3)}_{k',k,-k'}(\omega_{k'}, \omega_k, -\omega_{k'}) \simeq \frac{e^2}{3m_e^2}\frac{(k\cdot k')}{\omega_k^2\omega_{k'}^2}(k-k')^2\varepsilon^{(1)e}_{k-k'}(\omega_k - \omega_{k'}). \tag{19}$$

It is evident that in the limit when $(\omega - \omega')/|k - k'|v_{Te} \ll 1$, the contribution of the last two terms in (12) to the induced wave scattering by electrons cancel each other. Moreover, in the case of a sufficiently long (or narrow) wave packet, the induced wave scattering by electrons is neglibile compared with that by ions. In this case one can obtain a particular solutions of the induced wave scattering equation. Because of that the electron contribution to scattering is neglected and the expressions (17) and (18) for $\varepsilon^{(2)}$ are used to reduce (12) to the form (Galeev et al., 1964):

$$\frac{1}{2}\left[\frac{\partial}{\partial t} - \gamma_k\right]\left(\frac{\partial\varepsilon^{(1)}_k}{\partial\omega_k}k^2|\phi_k|^2\right) = \sum_{k'}\frac{e^2|\phi_k|^2|\phi_{k'}|^2(k\cdot k')}{m_e^2\omega_k^2\omega_{k'}^2}$$

$$\times|k-k'|^2\left|\frac{\varepsilon^{(1)e}_{k-k'}(\omega_k - \omega_{k'})}{\varepsilon^{(1)}_{k-k'}(\omega_k - \omega_{k'})}\right|^2 \operatorname{Im}\varepsilon^{(1)i}_{k-k'}(\omega_k - \omega_{k'}), \tag{20}$$

where $\gamma_k = -\varepsilon_k^{(1)''}/(\partial\varepsilon_k^{(1)'}/\partial\omega_k)$ is the linear growth (damping) rate. The nonlinear term in this equation differs from zero when resonant ions with velocities close to the beating phase velocity $(\omega_k \pm \omega_{k'})/(k \pm k')$ are present in the plasma. Since most of the thermal ions are moving with velocities much lower than the wave phase velocity, resonance can take place only with beats at the difference frequency. The resonance condition $\omega_k - \omega_{k'} = (k - k')\cdot v$, rewritten in the form of the conservation law $[\hbar(\omega_k - \omega_{k'}) = \partial\mathcal{E}/\partial p \cdot \Delta p$ ($\mathcal{E} = mv^2/2$ is the particle energy, $\Delta p = \hbar(k - k')$ is the momentum transferred to it from waves], shows clearly that absorption of the quantum (ω_k, k) by particles is occurring, followed by radiation of the quantum $(\omega_{k'}, k')$, i.e. with induced wave scattering. Naturally, the number of waves is conserved in this process. To check this, rewrite (20) in terms of the number of waves. Using the relation (17) gives:

$$\frac{\partial n_k}{\partial t} = -\frac{4\pi e^2}{m_e^2\omega_{pe}^2}\int \frac{d^3k}{(2\pi)^3} n_k n_{k'} \frac{(k\cdot k')^2}{k^2k'^2}(k-k')^2$$

$$\times \operatorname{Im}\frac{|\varepsilon^{(1)e}_{k-k'}(0)|^2}{1+\varepsilon^{(1)e}_{k-k'}(0)+\varepsilon^{i}_{k-k'}(\omega_k-\omega_{k'})}. \tag{21}$$

Here, integration over the phase space has been used instead of the summation over wave vectors [see equation (29) of Chapter 4.1).

It is clear that such a nonlinear non-one-dimensional integral equation is very difficult to solve in a general case. Therefore, this equation is simplified for some specific cases. Two limiting cases are considered below: an isotropic Langmuir wave packet and a streamer-type wave spectrum in k-space when some specifics of induced scattering can be clarified due to great simplification of (20).

4.3.3. Approximation of the differential transfer over the spectrum

Consider the evolution of an isotropic Langmuir wave packet in k-space in the limit when the phase velocity spread is much larger than the ion thermal velocity:

$$\Delta k/k \gg (m_e/m_i)^{1/2}. \tag{22}$$

When this condition is fulfilled the number of waves depends on the wave vector modulus only, and the wave packet width in frequencies $\Delta\omega$ is much larger than the width of the narrow kernel of the integral operator on the right-hand side of (21) at $\omega_{k'} \to \omega_k$. In other words, within an entire wave spectrum, only the waves with similar frequencies interact intensely, so that the integrand in (21) can be expanded over the frequency difference of the interacting waves. Because of this oddity of the kernel as a function of $(\omega_k - \omega_{k'})$ the zero-order expansion term disappears after integration. The contribution of the first-order expansion term [linear in $(\omega_k - \omega_{k'})$] is calculated with the help of the following formula from dispersion relation theory (Galeev and Sunyaev, 1972):

$$\operatorname{Im}\int_{-\infty}^{+\infty}\frac{|\varepsilon^{(1)e}_{k-k'}(0)|^2(\omega_k-\omega_{k'})\,d(\omega_k-\omega_{k'})}{1+\varepsilon^{(1)e}_{k-k'}(0)+\varepsilon^{(1)i}_{k-k'}(\omega_k-\omega_{k'})} = \frac{\pi\omega_{pe}^2(m_e/m_i)}{[1+1/\varepsilon^{(1)e}_{k-k'}(0)]^2}. \tag{23}$$

As a result, (21) is reduced to a differential equation in **k**-space (Galeev et al., 1964):

$$\partial N_\kappa/\partial\tau - N_\kappa \partial N_\kappa/\partial\kappa = 0, \tag{24}$$

where

$$N_\kappa = \frac{k^2\Delta k}{6\pi^2}\frac{n_k\omega_{\rm pe}}{4\pi n_0 T_{\rm e}};$$

$$\tau = \frac{4\pi^2\omega_{\rm pe}^2 m_{\rm e} t}{9m_{\rm i}}\frac{1}{(\Delta k)^2\lambda_{\rm D}^2}; \qquad \kappa = k/\Delta k.$$

This equation is well known from hydrodynamics. It describes the flow of plasmons toward the lower frequencies (i.e. smaller **k**) with the transfer velocity proportional to the number of plasmons. For the wave packet under consideration, the N_k profile has the form of a broad spectral line that steepens in time on the front side (i.e. in a region of smaller **k**) until overlapping takes place (Fig. 4.3.1). Such behavior is caused by the probability of induced scattering proportional to the wave intensity at that point of **k**-space to which the waves are scattered. Therefore, scattering to the foot of the N_k profile takes place more slowly than to its top, resulting in overlapping. However, the conclusion about the final overlapping made on the basis of the simplified equation (24) is not correct because a smooth spectrum assumption was used to derive it. To correct the equation, one should take into account the next-order terms in the expansion of the integrand over the frequency difference on the right-hand side of (21). Then, (24) is replaced by:

$$\frac{\partial N_\kappa}{\partial\tau} - N_\kappa\frac{\partial N_\kappa}{\partial\kappa} - \beta N_\kappa\frac{\partial^3 N_\kappa}{\partial\kappa^3} = 0, \tag{25}$$

where the coefficient β can be estimated to an order of magnitude as $\beta \simeq (m_{\rm e}T_{\rm i}/9m_{\rm i}T_{\rm e})\,(\Delta k\lambda_{\rm D})^{-2}$ and is small when the condition (22) is satisfied. This equation as well as the representation of the integral operator in the form of a

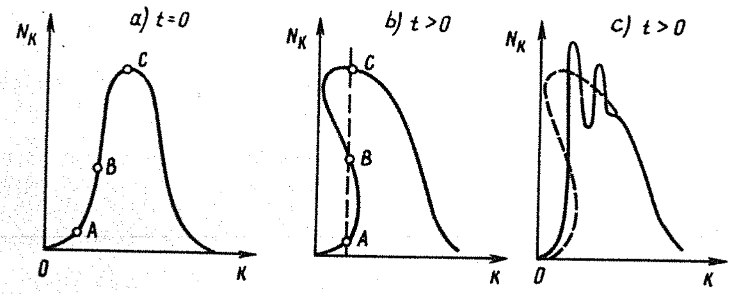

Fig. 4.3.1. Langmuir wave spectrum evolution in the course of an induced scattering by ions: (a) initial spectrum; (b) formation of a three-valued region for the solution of the simplified equation (23); (c) single-valued spectrum for the same moment of time as in the case (b), but with the dispersion taken into account (Zeldovich, 1975).

differential operator series is valid in general unless the gradients in k-space become large (i.e. for $\beta \partial^2 N_\kappa / \partial \kappa^2 \ll N_\kappa$). This last condition is definitely satisfied when a low and sufficiently broad spectral line is moving on the background of a nearly uniform spectral distribution of waves. In this case, the problem of the line-profile evolution is similar to the problem of the evolution of the initial disturbance in a nonlinear dispersive medium. The *velocity dispersion* (see the chapter by Oraevsky in Volume II) of the motion of the line in k-space stops the steepening of the profile's leading edge. Simultaneously, solitons are detached from the front and carried back since shorter harmonics generated in the course of steepening have lower phase velocities in k-space, according to (25). When a line height is low enough, the width of solitons satisfies the condition (22), which is necessary for (25) to be valid.

When relaxation of an isolated spectral line in $\boldsymbol{k}$-space takes place, then the resulting gradients in k-space are not small and the parameter β is no longer small. Solution of an exact integral equation is necessary to describe the line evolution in this case. However, the qualitative picture of relaxation remains the same: the leading front of the line steepens and solitons are detached from it (see Fig. 4.3.1) (Zeldovich, 1975; Zeldovich and Sunyaev, 1972).

4.3.4. Streamer-type Langmuir wave turbulence spectrum

The idealized case of isotropic Langmuir wave turbulence, when the time evolution of a Langmuir wave packet is described by a partial differential equation, was considered above. However, in real conditions Langmuir waves are excited as a result of the development of plasma instabilities and this excitation is usually anisotropic in k-space. Following the work of Breizman et al. (1973) it will be shown here that even a small angular asymmetry of an instability growth rate is hypertrophied in the course of induced wave scattering by particles, and ultimately the stationary Langmuir wave turbulence spectra become even more anisotropic, and the Langmuir waves are concentrated on lines or surfaces in $\boldsymbol{k}$-space. For the sake of simplicity attention is restricted to the axially symmetric case that can be realized when a preferred direction in $\boldsymbol{k}$-space exists along which preferred excitation takes place (for example by particle beams or by the electric field of an external polarized radiation). The Langmuir wave spectrum evolution is then described by (21), where one can use the differential form of the interaction operator (assuming the wave spectrum to be broad in $\boldsymbol{k}$ but not in angles):

$$\frac{\partial N(k,x)}{\partial t} = N(k,x)\left(\gamma(k,x) + \frac{\partial}{\partial k}\int_{-1}^{+1} T(x,y) N(k,y)\,\mathrm{d}y\right), \tag{26}$$

where $N(k,x) = k^2 n(k,x)/(2\pi)^3$;

$$T(x,y) = \frac{\pi^2}{9}\frac{m_\mathrm{e}}{m_\mathrm{i}}\frac{\omega_\mathrm{pe}^2}{n_0 T_\mathrm{e} \lambda_\mathrm{D}^2}\left(1 - x^2 - y^2 + 3x^2y^2 - 3xy + 3xy^2 + 3x^3y - 5x^3y^3\right);$$

$x = \cos\theta$; $y = \cos\theta$; and (k, θ, φ) are spherical coordinates in $\boldsymbol{k}$-space with the polar axis along the preferred direction. In contrast to (24), the number of waves, $n(k, x)$, depends on the angle θ.

Consider now what stationary solutions of (26) exist. On the face of it, it seems fairly arbitrary: $N(k, x)$ can be set equal to zero at any predetermined region, (k, x). However, such predetermined solutions tend to be unstable as a rule (Galeev et al., 1965). Stability conditions coupled with stationarity conditions for the solution of (26) lead to the relations:

$$\begin{aligned} &\gamma(k, x) = \gamma^N(k, x) \qquad \text{for } N(k, x) \neq 0; \\ &\gamma(k, x) < \gamma^N(k, x) \qquad \text{for } N(k, x) = 0, \end{aligned} \tag{27}$$

where

$$\gamma^N(k, x) = -\frac{\partial}{\partial k}\int_{-1}^{+1} T(x, y) N(k, y)\,\mathrm{d}y.$$

In other words, the Langmuir waves are concentrated on the conical surfaces, $x = x(k)$, where the first of the relations (27) is satisfied. In a more general case when there is no axial symmetry, the contact of the functions $\gamma(k)$ and $\gamma^N(k)$ could take place at lines only. Such lines (or surfaces) at which the spectral distribution is concentrated were named (Breizman et al., 1973) *streamers* (one-dimensional or two-dimensional, respectively). When the number and the form of the streamers $x = x_i(k)$ $(i = 1, \ldots, r)$ are known the solution can be represented in the form:

$$N(k, x) = \sum_i N_i(k)\,\delta[x - x_i(k)], \tag{28}$$

where $N_i(k)$ is found from the equation

$$\begin{aligned} &\gamma[k, x_i(k)] + \sum_j T[x_i(k), x_j(k)]\frac{\mathrm{d}N_j}{\mathrm{d}k} \\ &\qquad -\sum_j \frac{\partial}{\partial x_j} T[x_i(k), x_j(k)] N_j \frac{\mathrm{d}x_j}{\mathrm{d}k} = 0, \end{aligned} \tag{29}$$

and the streamer form is defined by the contact condition (27):

$$\frac{\mathrm{d}}{\mathrm{d}x}[\gamma(k, x) - \gamma^N(k, x)]|_{x = x_i(k)} = 0. \tag{30}$$

The energy pumped into Langmuir waves due to an instability is transferred to the long-wavelength region along the streamer.

As an example of streamer form determination consider the parametric instability of the pumping wave $\boldsymbol{E} = \boldsymbol{E}_0 \cos\Omega t$ resulting in an excitation of Langmuir waves due to pumping wave scattering by ions (Valeo et al., 1972; DuBois and Goldman, 1972). The preferred direction is defined in this case by the $\boldsymbol{E}_0$ vector and streamer formation at $x = \pm 1$ can be expected since the parametric instability growth rate is maximum in this direction [see (20)]:

$$\gamma(k,x) = \frac{\omega_{pe}|E_0|^2}{8\pi n_0 T_e} x^2 \operatorname{Im} \frac{k^2\lambda_D^2 |\varepsilon_k^{(1)e}(\Omega - \omega_k)|^2}{|\varepsilon_k^{(1)}(\Omega - \omega_k)|^2} \varepsilon_k^{(1)i}(\omega_k - \Omega) - \nu_{ei}, \tag{31}$$

where ν_{ei} is the frequency of rare electron–ion collisions, allowing for the collisional dissipation of Langmuir waves. Because of the growth rate symmetry [$\gamma(k,x) = \gamma(k,-x)$], the streamers are also symmetric:

$$N(k,x) = 2N(k)\,\delta(x^2 - 1);$$
$$\mathrm{d}N(k)/\mathrm{d}k = \gamma(k,1)/T(1,-1). \tag{32}$$

Then the stability condition (27) takes a simple form:

$$\gamma(k,x) < x^2\gamma(k,1) \qquad \text{for } |x| < 1. \tag{33}$$

This condition is satisfied due to the small collisional dissipation in (36). In a similar way the streamer forms can be determined in the cases where the growth rate maximum is achieved at angles corresponding to the two-dimensional streamers (Breizman et al., 1973).

4.3.5. The lack of renormalization in a quasilinear theory of Langmuir waves

The simple derivation of the Langmuir wave quasilinear equations in Section 4.1.2 was based on the assumption that the wave packet is sufficiently broad that rapid phase mixing takes place. The condition for such mixing is formulated in the following way: if Δv is the wave packet width on phase velocities then the phase mixing time in a wave packet is $t_1 \sim 1/k\Delta v$. This time must be considerably smaller than the quasilinear diffusion time:

$$t_1 \ll \tau_R = (\Delta v)^2/D_k, \tag{34}$$

where

$$D_k = \frac{\pi e^2}{m_e^2} \sum_k |E_k|^2 \delta(\omega_k - \boldsymbol{k}\boldsymbol{v})$$

is the quasilinear diffusion coefficient. This inequality can be rewritten in terms of the wandering time of the resonant electrons [the bounce period of trapped electrons, see (11) and (13) of Section 4.1.1]:

$$k\Delta v \gg \tau_b^{-1} = (k^2 D_k)^{1/3}. \tag{35}$$

It is necessary, of course, in addition to this inequality, to satisfy the condition for the overlapping of neighboring mode resonances [see (7) of Section 4.1.1.]:

$$w = \left(k^2 D_k\right)^{1/3}/k > \delta v, \tag{36}$$

where w is the width of an individual wave–particle resonance, and δv is the phase velocity difference for two neighboring harmonics in a spectrum.

Statements have been published in the literature claiming that it is insufficient to satisfy the conditions (35) and (36), and the applicability of the quasilinear equation is restricted by the limit of very small field amplitudes: $(k^2 D_k)^{1/3} \ll \gamma_k$ (Bakai and Sigov, 1977; Sigov, 1977; Adam et al., 1978, 1979). These authors refer to computer simulations showing supposedly that the phase correlation of individual harmonics takes place during a time of the order of $(k^2 D_k)^{-1/3}$. Therefore, they believe that in the opposite limit,

$$\left(k^2 D_k\right)^{1/3} \gg \gamma_k, \tag{37}$$

it is impossible to use the rapid phase-mixing assumption. In addition, Adam et al. (1979) have carried out a partial summation of higher-order expansion terms that have not been taken into account in quasilinear theory. They conclude that renormalization is necessary when the condition (37) holds. The analysis of the computer simulation mentioned above will not be dealt with here. However, note that the strict conditions (34) and (35) of quasilinear theory applicability were not satisfied in these studies. The contribution of higher-order terms to the resonant wave–particle interaction (i.e. to the growth rate) will be analyzed here in the framework of the exact perturbation theory developed in this Section.

Following the work of Galeev et al. (1980), the nonlinear equation for interacting waves is used. Here, the contribution to the wave–particle interaction is related to the terms that diverge formally at the resonance point $\boldsymbol{k} \cdot \boldsymbol{v} = \omega_k$. Although the divergences are removed by the nonlinear resonance broadening effect, nevertheless, in the one-dimensional case, their contribution to the growth rate is comparable with the linear growth rate when a very rough estimate is used. (In the two- and three-dimensional cases, the contribution of these terms could be neglected on the grounds of such a rough estimate alone.) This requires a more exact consideration of these terms in the one-dimensional case.

Consider first the second-order terms in the expression for wave energy that are contained in (12). It is evident that the main nonlinear contribution to the wave–particle resonant interaction comes from the last two terms. In contrast to the above case of induced wave scattering, there is no mutual cancellation of these terms' contributions at the resonance point $kv = \omega_k$. This is due to the fact that the first of these two terms takes into account the electric field of the resonance particles' beats, which contains an additional small parameter proportional to the number of these particles. Thus only the last term should be analyzed. Using integration by parts in the expression (16) for $\varepsilon^{(3)}$ its contribution to (12) in the one-dimensional case is represented by:

$$\frac{1}{2}\left(\frac{\partial \varepsilon_k^{(1)}}{\partial \omega_k}\right)\left(\frac{\partial |\phi_k|^2}{\partial t}\right)_{\text{scat}} = -\operatorname{Im}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_1}{2\pi}\varepsilon^{(3)}_{-k_1,k,k_1}|\phi_{k_1}|^2|\phi_k|^2$$

$$\equiv \frac{4\pi e^4}{m_e^3}|\phi_k|^2\int_{-\infty}^{+\infty}\mathrm{d}v\frac{\partial f_0}{\partial v}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_1}{2\pi}k_1^2|\phi_{k_1}|^2$$

$$\times\left(\frac{k}{(\omega_k - kv + \mathrm{i}\delta)^4(\omega_{k_1} - k_1 v + \mathrm{i}\delta)} + \frac{(k+k_1)}{2(\omega_k - kv + \mathrm{i}\delta)^3(\omega_{k_1} - k_1 v + \mathrm{i}\delta)\left[\omega_k + \omega_{k_1} - (k+k_1)v + \mathrm{i}\delta\right]}\right). \tag{38}$$

Here, the addend $\mathrm{i}\delta \sim \mathrm{i}(k^2 D_k)^{1/3}$ in the denominators formally takes into account the resonance broadening effect caused by resonant electrons wandering in a wave field with characteristic time τ_b. To make a rough estimate of the resonant velocity $(v = \omega/k \approx \omega_1/k_1)$ region contribution to this integral in the resonant denominators only the $\mathrm{i}\delta$ term is retained

$$\gamma_L \frac{e^2|E_k|^2 k^2}{m_e^2 v\, \delta_k^3}, \tag{39}$$

where

$$\gamma_L = \frac{2\pi^2 e^2}{m_e k^2}\omega_{pe}(\partial f_0/\partial v)$$

is the linear growth rate. Using the expression for δ_k in terms of the diffusion coefficient, the nonlinear contribution to the growth rate is found to be of the order of the linear contribution. If this were true, then the use of such terms could lead to the renormalization of the wave–particle resonant interaction strength without changing the form of the quasilinear equation. However, an exact calculation shows that the integral on the right-hand side of (38) becomes zero to the highest (third) order in the parameter $(k\Delta v/\delta_k)$, and therefore such normalization is not necessary. The proof of this is based on the fact that all singularities of the integrand (38) are represented in explicit form in the resonant denominators, whereas the numerators are smooth functions of variables v and k_1. This allows one to compute all integrals with the help of the residue theorem and to represent the result in the form of the nth-order derivatives of those smooth functions. When the condition (35) is satisfied the value of the integral is $(k\Delta v/\delta_k)^n$ times smaller than that obtained by the rough estimate.

Consider the first addend on the right-hand side of (38). It has the form:

$$\int_{-\infty}^{+\infty}\mathrm{d}v\varphi_1(v)(\omega_k - kv + \mathrm{i}\delta_k)^{-4}, \tag{40}$$

where

$$\varphi_1(v) \sim \frac{\partial f_0}{\partial v}\int_{-\infty}^{+\infty} \mathrm{d}k_1 \frac{|E_{k_1}|^2}{\omega_{k_1} - k_1 v + \mathrm{i}\delta} \equiv -\frac{\mathrm{i}\pi}{v}|E_{k_1}|^2\left(k_1 = \frac{\omega_{k_1}}{v}\right)\frac{\partial f_0}{\partial v}$$

is a smooth function of v. This can be transformed to:

$$-\frac{1}{k^3}\int_{-\infty}^{+\infty} \mathrm{d}v \frac{1}{\omega_k - kv + \mathrm{i}\delta}\frac{\partial^3 \varphi_1}{\partial v^3} = \frac{\mathrm{i}\pi}{k^4}\frac{\partial^3\varphi_1}{\partial v^3}(v = \omega_k/k). \tag{41}$$

It is $(k\Delta v/\delta_k)^3$ times smaller than its rough estimate.

The second addend is proportional to the integral

$$\int_{-\infty}^{+\infty} \mathrm{d}v \frac{\partial f_0/\partial v}{(\omega_k - kv + \mathrm{i}\delta_k)^3}\int_{-\infty}^{+\infty}\frac{(k+k_1)|E_{k_1}|^2 \mathrm{d}k_1}{(\omega_{k_1} - k_1 v + \mathrm{i}\delta)\left[\omega_k + \omega_{k_1} - (k+k_1)v + \mathrm{i}\delta\right]}. \tag{42}$$

Since both poles in the integral over k_1 are in one half plane (above the real k_1-axis for $v > 0$), then at the resonance point $kv = \omega_k$ it is reduced to the form:

$$\varphi_2(v) = \int_{-\infty}^{+\infty}\frac{(k+k_1)|E_{k_1}|^2 \mathrm{d}k_1}{(\omega_{k_1} - k_1 v + \mathrm{i}\delta)^2} = \frac{\mathrm{i}\pi}{v^2}\frac{\partial}{\partial k_1}(k+k_1)|E_{k_1}|^2\left(k_1 = \frac{\omega_{k_1}}{v}\right).$$

The remaining integral over v can also be taken easily:

$$\int_{-\infty}^{+\infty}\frac{(\partial f_0/\partial v)\varphi_2(v)}{(\omega - kv + \mathrm{i}\delta)^3} = -\frac{\mathrm{i}\pi}{k^2}\frac{\partial^2}{\partial v^2}\left(\frac{\partial f_0}{\partial v}\varphi_2(v)\right). \tag{43}$$

In a similar way one can easily show that the terms for induced scattering do not contribute to the resonant wave–particle interaction in any order of perturbation theory.

In a slightly different way, one can calculate the contribution to the resonant wave–particle interaction from the perturbation theory terms describing the decay-type interaction. Such terms appear first in the third order of the wave energy. Because of the symmetry properties of the interaction operator matrix elements it is sufficient to consider only one of these terms [compare with the term describing the three-wave interaction in (12)]:

$$\frac{\partial \varepsilon_k^{(1)}}{\partial \omega_k}\left(\frac{\partial|\phi_k|^2}{\partial t}\right)_{\mathrm{d}} = \mathrm{Im}\iint\frac{\mathrm{d}k_1\mathrm{d}k_2}{(2\pi)^2}\frac{|\varepsilon^{(3)}_{k+k_1-k_2,k_1,-k_2}|^2}{\varepsilon_k^{(1)}(\omega_{k+k_1-k_2} + \omega_{k_2} - \omega_{k_1})}$$
$$\times |\phi_{k+k_1-k_2}|^2|\phi_{k_1}|^2|\phi_k|^2. \tag{44}$$

When the dispersion of the interacting waves in the resonance region ($k_i v - \omega_{k_i} \sim \delta_k \ll \omega_{k_i}$) can be neglected, i.e. for

$$\omega_{k+k_1-k_2} + \omega_{k_2} - \omega_k - \omega_{k_1} \simeq k^2\frac{\partial^2\omega_k}{\partial k^2}\left(\frac{\delta_k}{\omega_k}\right)^2 \ll \gamma_k, \tag{45}$$

(44) takes the form:

$$\frac{\partial |E_k|^2}{\partial t} = \left(\frac{4\pi e^4}{k m_e^3 \partial \varepsilon_k / \partial \omega_k} \right)^2 \gamma_k^{-1} \mathrm{Im} \iint \mathrm{d}v \mathrm{d}v' \frac{1}{\omega_k - kv + \mathrm{i}\delta} \left(\frac{1}{\omega_k - kv' + \mathrm{i}\delta} \right)^*$$
$$\times \frac{\partial}{\partial v} \frac{\partial}{\partial v'} \int \frac{\mathrm{d}k_1}{2\pi} \frac{|E_{k_1}|^2}{\omega_k + \omega_{k_1} - (k + k_1)v + \mathrm{i}\delta}$$
$$\times \left(\frac{1}{\omega_k + \omega_{k_1} - (k + k_1)v' + \mathrm{i}\delta} \right)^*$$
$$\times \frac{\partial}{\partial v} \frac{\partial}{\partial v'} \int \frac{\mathrm{d}k_2}{2\pi} \frac{|E_{k_2}|^2 |E_{k+k_1-k_2}|^2}{\omega_{k_2} - k_2 v + \mathrm{i}\delta}$$
$$\times \left(\frac{1}{\omega_{k_2} - k_2 v' + \mathrm{i}\delta} \right)^* . \tag{46}$$

To find out whether renormalization is necessary it is sufficient to calculate the highest (sixth) order contribution in $k\Delta v/\delta_k$ which could be comparable with the linear contribution. Consequently, the dependence of the numerator on k_1, k_2 and $v_1 v'$ is neglected and its value at the resonance point $v = v' = \omega_k / k$; $k_1 \approx k_1 \approx k$ is taken out of the integral sign. The integrals over k_1 and k_2 do not become zero here in spite the high order of the poles. This is because the poles of the expression under the integral lie on different sides of the real axes k_1 and k_2. Performing these integrals gives:

$$|E_k|^{-2} \left(\frac{\partial |E_k|^2}{\partial t} \right)_\mathrm{d} = \left(\frac{4\pi^2 e^4 |E_k|^2}{m_e^3 \omega_k \partial \varepsilon_k^{(1)} / \partial \omega_k} \right)^2 \gamma_k^{-1}$$
$$\times \iint \mathrm{d}v \mathrm{d}v' \frac{1}{(\omega_k - kv + \mathrm{i}\delta)(\omega_k - kv' - \mathrm{i}\delta)}$$
$$\times \frac{\partial}{\partial v} \frac{\partial}{\partial v'} \frac{1}{k(v - v') - 2\mathrm{i}\delta} \frac{\partial}{\partial v} \frac{\partial}{\partial v'} \frac{1}{k(v - v') - 2\mathrm{i}\delta} \frac{\partial f_0}{\partial v} \frac{\partial f_0}{\partial v'} . \tag{47}$$

Here the poles in the integrals over v and v' lie on one side of the real axes v and v' (above the v axis and below the v' axis for $\omega_k / k > 0$). Because of the high order of the poles this double integral becomes zero, which proves the lack of higher-order contribution, as expected. In a similar way one can show that the higher-order decays do not give contributions comparable with the linear contribution.

Thus, it has been shown that when the conditions (35) and (36) of quasilinear theory applicability are satisfied, the nonlinear corrections to the quasilinear equations are indeed small.

4.3.6. Light-induced scattering in a nonrelativistic plasma

It is more convenient to derive the equation for interacting electromagnetic waves in terms of the plasma current expansion in powers of the wave amplitude rather than in terms of the plasma polarization expansion. To do this, start with the same iteration equation for the particle distribution function:

$$f_j = f^{j(0)}(v,t) + \int \frac{d^3k}{(2\pi)^3} f_k^{j(1)} + \iint \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3} f_{k,k'}^{j(2)} + \cdots$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right) f^{j(n+1)}_{k,k',\ldots,k^{(n)}} =$$

$$= -\frac{1}{(n+1)!}\sum \frac{e_j}{m_j}\left(\boldsymbol{E}_k^{(1)} + \frac{1}{c}\left(\boldsymbol{v}\times\boldsymbol{B}_k^{(1)}\right)\right)\cdot\frac{d}{\partial \boldsymbol{v}} f^{j(n)}_{k',\ldots,k^{(n)}}. \tag{48}$$

Here, in contrast to (4), the wavevectors of all the waves contributing to the nth-order term of the distribution function expansion in powers of the wave amplitude are shown explicitly by indices of $f^{j(n)}$. The sum on the right-hand side of (48) is taken over all possible transpositions of indices. Induced scattering of light waves with random phases is a second-order effect in the wave energy. Therefore, to describe it one ought to calculate the scattering current with an accuracy to the third order in the wave amplitudes. The latter assumes that the third-order correction $f^{(3)}$ of the distribution function should be found.

Since a wave magnetic field in a non-relativistic plasma has a weak influence on the particle motion it can be neglected while computing $f^{(n)}$. Then the expression for $f^{(n)}$ in this case coincides with that found for electrostatic Langmuir waves. Moreover, the thermal corrections of the order of $kv/\omega \ll 1$ can still be neglected. As a result, for $f^{(2)}$ [compare with (15)]:

$$f^{j(2)}_{k,-k'} = \frac{1}{2}\left(\frac{ie_j}{m_j}\right)^2 \frac{\Delta \boldsymbol{k}\cdot\partial f_j^{(0)}/\partial \boldsymbol{v}}{\Delta\omega - \Delta\boldsymbol{k}\cdot\boldsymbol{v} + i0}\,\frac{(\boldsymbol{E}_k\cdot\boldsymbol{E}^*_{k'})}{\omega_k\omega_{k'}}, \tag{49}$$

where $\Delta\omega = \omega_k - \omega_{k'}$, $\Delta \boldsymbol{k} = \boldsymbol{k} - \boldsymbol{k}'$. Substituting this expression into (48) and keeping only the term that contributes to the scattering current of frequency ω_k, gives in the next approximation

$$f^{j(3)}_{k',k,-k'} = \frac{ie_j}{m_j}\,\frac{1}{\omega_k - \boldsymbol{k}\cdot\boldsymbol{v} + i0}\left(\boldsymbol{E}_{k'} + \frac{1}{c}\left(\boldsymbol{v}\times\boldsymbol{B}_{k'}\right)\right)\cdot\frac{\partial f^{j(2)}_{k,k'}}{\partial \boldsymbol{v}}. \tag{50}$$

The intensity of the scattered wave radiation is equal to the work done by particles in a scattered wave field per unit time, i.e. $-\boldsymbol{j}^{(3)}_{k',k,-k'}\cdot\boldsymbol{E}^*_k$. However, the scattering current calculated with the help of (50) takes into account only the scattering by free particles. Particles in a plasma, however, are surrounded by a screening cloud of ions and electrons. It is clear that the screening cloud oscillations which are opposite in phase compensate for the scattering by free particles.

The electric field of the screening cloud can be considered to be potential. The potential, $\Phi^{(2)}_{\Delta k,\Delta\omega}$, is determined by the Poisson equation

$$\Delta k^2 \varepsilon(\Delta\omega,\Delta \boldsymbol{k})\Phi^{(2)}_{\Delta k,\Delta\omega} = -4\pi e\int f^{e(2)}_{k,-k'}\,\mathrm{d}^3\boldsymbol{v}.$$

From this equation the same result is obtained as for Langmuir waves [see (8) and (16)]:

$$\Phi^{(2)}_{\Delta k,\Delta\omega} = \frac{e}{2m_e}\frac{(\boldsymbol{E}_k\cdot\boldsymbol{E}_{k'})}{\omega_k\omega_{k'}}\frac{\varepsilon^e(\Delta\omega,\Delta\boldsymbol{k})}{\varepsilon(\Delta\omega,\Delta\boldsymbol{k})}, \tag{51}$$

where $\varepsilon(\Delta\omega,\Delta\boldsymbol{k})$ is the linear dielectric permeability designated earlier as $\varepsilon_k(\omega)$, for brevity of notation. The electric field of these beats perturbs in its turn the particle distribution function:

$$f^{j(1)}_{\Delta k}(\Delta\omega) = \frac{e_j}{m_j}\Phi^{(2)}_{\Delta k,\Delta\omega}\frac{\Delta\boldsymbol{k}\cdot\partial f^{j(0)}/\partial\boldsymbol{v}}{\Delta\omega-\Delta\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0}. \tag{52}$$

In this connection a new iteration of (48) is necessary:

$$f^{j(2)}_{k',\Delta k} = \frac{\mathrm{i}e}{m_e}\frac{1}{\omega_k-\Delta\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0}\left(\boldsymbol{E}_{k'}+\frac{1}{c}(\boldsymbol{v}\times\boldsymbol{B}_{k'})\right)\cdot\frac{\partial f^{j(1)}_{\Delta k}}{\partial\boldsymbol{v}}. \tag{53}$$

The indices (1) and (2) here indicate that these corrections to the distribution function were calculated by one or two iterations of (48). The order of these corrections is, in fact, higher since the field $\Phi^{(2)}_{\Delta k,\Delta\omega}$ used for a second iteration is already of second order in the wave amplitude. Therefore, to compute the scattering current, one ought to take into account both expressions (50) and (53). As a result, the scattered radiation intensity is calculated as the work of this nonlinear current of particles:

$$\frac{\partial}{\partial t}\frac{1}{8\pi}\left(\frac{\partial(\omega\varepsilon)}{\partial\omega}|\boldsymbol{E}_k|^2+|\boldsymbol{B}_k|^2\right) = \int \mathrm{d}^3\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{E}^*_k)\left[f^{j(3)}_{k',k,-k'}(\boldsymbol{v})+f^{j(2)}_{k',\Delta k}(\boldsymbol{v})\right]. \tag{54}$$

Integrating over $\mathrm{d}^3\boldsymbol{v}$ gives the equation for wave amplitudes (cf. Litvak and Trachtenhertz, 1971; Galeev and Sunyaev, 1972):

$$\frac{\partial}{\partial t}\left[\frac{1}{\omega}\frac{\partial(\varepsilon\omega^2)}{\partial\omega}\frac{|\boldsymbol{E}_k|^2}{8\pi}\right] = \frac{e^2}{m_e^2}\int\frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3}\frac{|\boldsymbol{E}_k\cdot\boldsymbol{E}^*_{k'}|^2}{\omega_k\omega_{k'}}\frac{\Delta k^2}{4\pi|\varepsilon(\Delta\omega,\Delta\boldsymbol{k})|^2}$$
$$\times\,\mathrm{Im}\left[|1+\varepsilon^{\mathrm{i}}(\Delta\omega,\Delta\boldsymbol{k})|^2\varepsilon^{\mathrm{e}}(\Delta\omega,\Delta\boldsymbol{k})+|\varepsilon^{\mathrm{e}}(\Delta\omega,\Delta\boldsymbol{k})|^2\varepsilon^{\mathrm{i}}(\Delta\omega,\Delta\boldsymbol{k})\right]. \tag{55}$$

Note that this equation has a rather restricted region of applicability. First, the fields in this case were assumed to be so weak that the potential energy of the particles in a beating field is smaller than the thermal energy ($e\Phi^{(2)}_{\Delta k,\Delta\omega}\ll T$). On the other hand, the Doppler corrections to the wave frequency which can prove to be important for

$v_{Te}^2/c^2 \gtrsim m_e/m_i$ because of the almost complete cancellation of free electron scattering (Compton scattering) and screening ion cloud scattering have been neglected.

As in the case of Langmuir oscillations, a transition to a differential equation is possible here in the limit of a sufficiently broad spectrum. In this connection it is interesting to note that such an equation for the case of induced scattering by free plasma electrons was written long ago by Kompaneets assuming radiation isotropy and the absence of its polarization. It can be derived from (55) with the help of the same procedure as used in Section 4.3.3.

$$\frac{\partial n(\nu,t)}{\partial t} = \frac{\sigma_T n_{0e} h}{m_e c} \frac{1}{\nu^2} \frac{\partial}{\partial \nu} \nu^4 \left(n^2 + n + \frac{T_e}{h} \frac{\partial n}{\partial \nu} \right), \tag{56}$$

where $\sigma_T(8\pi/3)(e^2/m_e c^2)^2$ is the Thomson scattering cross section, and $\nu = \omega/2\pi$ is the light oscillation frequency. The last two terms on the right-hand side of (56) were absent in (55). They are responsible for the light scattering by electron density fluctuations (see Chapter 2.3).

Nonlinear effects in the interaction of radiation with a plasma, including energy and momentum transfer to a plasma in the course of scattering, have been studied, especially in connection with astrophysical applications (Zeldovich, 1975). Because of the tremendous radiation intensities and large volumes these effects play an important role.

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