Thermodynamics of Unstable Plasmas

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I. Motivation

Plasmas are seldom completely quiet and stable. Thus it is necessary in applications to be able to calculate such properties as heat and mass transport in the turbulent state. This is a difficult nonlinear problem, very complex and largely unsolved. It would be useful, therefore, to develop an intuition in the subject and an ability to make simple estimates. This article treats a thermodynamic approach having primarily these aims.

Our main objective is to calculate the energy in electromagnetic and kinetic fluctuations after instability, or turbulence, is fully developed. Generally, this is not the total energy in the plasma. We shall call that part which can transfer to fluctuations *free energy*. This free energy plays the role of a potential energy from which other quantities, such as the transport coefficients, can be estimated.

The advantage of dealing with free energy is that, while an exact calculation would still require the complete nonlinear dynamics, an upper bound on free energy is easily calculated even if only a few constraint laws are known. The conceptually simplest procedure, applicable to collisionless plasmas, was given by Gardner (1). He notes that the appropriate equation for each particle species, the nonlinear Vlasov equation neglecting collisions, is of the Liouville form and, hence, preserves the phase volume for each particle species separately. Thus, no actual motion can transfer more energy from the particles to electromagnetic fluctuations than would a transition to the state of lowest kinetic energy consistent with phase-volume conservation.

That this lowest allowed state is one in which the phase-space distribution for each species is a monotone decreasing function of energy $\mathscr{E} = \frac{1}{2}mv^2$ can be seen as follows. Certainly, the final energy is least when the maximum allowed number of particles is near the points in phase space where $\mathscr{E} = 0$. As Gardner observes, the allowed state for each species is analogous to the final configuration assumed by tiers of incompressible liquids of different densities all acted on by gravity. Regarding $\mathscr{E}$ as "height" and greater density in phase space as greater "weight," the final state of the liquids, being that of least gravitational potential, is analogous to the allowed plasma state of least kinetic energy. Evidently, the heaviest layer sinks to the bottom ($\mathscr{E} = 0$), the next heaviest lies next, and so on [a monotonic decrease of weight (phase density) with increasing height ($\mathscr{E}$)]. Of course, energy is not invariant with respect to the reference frame and this argument would hold in whatever frame $\mathscr{E}$ is defined. The proper frame is that in which the energy change is least equivalent to momentum conservation.

Thus, Gardner's theory is characterized by two qualitative conclusions. The monotone states, which can give up no energy, are nonlinearly stable. For other equilibria, if they are unstable, the fluctuation energy cannot exceed that which would be given up in a transition to a neighboring monotone state. For high-pressure plasmas in a magnetic field, the latter must be understood to include not only the kinetic energy change discussed above but also, in some circumstances, a portion of the plasma diamagnetic energy.

In Section II we present a thermodynamic theory which retains the above characteristics but makes calculations easier and takes account of collisions when necessary. In this, we apply the work of Newcomb (2) and of Kruskal and Oberman (3). In comparison with Gardner, these authors replace the preservation of phase volume by a related constraint which is the monotonic increase of entropy when collisions are important and by something similar otherwise. Again, the main physical content is that no real transition can liberate more free energy than would a transition to monotone distributions. The best estimate corresponds to the "nearest" such state, which is found by varying state parameters such as density and temperature and net momentum to obtain the least free energy. In the final section we shall comment briefly on how this approach relates to the search for better variational theories. For the most part, however, consistent with our practical aims, we shall emphasize monotone trial functions of Maxwellian form as the most interesting case.

The thermodynamic approach cannot be expected to give all the answers in detail, because the thermodynamic constraints are insensitive to certain dynamical variables. Evidently, then, thermodynamics at best represents the extreme case within the range of these hidden parameters. Nonetheless, it tells us much of what we want to know about turbulence. The thermodynamic upper bounds on free energy are frequently useful in a negative way by predicting when turbulence can likely be neglected.

Actually, there is evidence that a very simple thermodynamic bound gives many of the dominant features correctly. This is our bound restricted to Maxwellian trial distributions. Physically, one is supposing that in an *unstable* plasma the collective processes drive the system toward thermal equilibrium. This is motivated first by the fact that there is already known at least one instability mechanism corresponding essentially to each way in which a plasma differs from equilibrium. There is scattered experimental evidence that these instabilities do drive the system toward equilibrium, and this is also borne out by several calculations in the regime of weak turbulence where the quasilinear perturbation theory applies. Secondly, we may guess that nonthermodynamic constraints have limited influence once instability sets in strongly, for the following reason. Since the thermodynamic laws do constrain the free energy (an arbitrary set of constraints may not), other dynamical features, at most, constrain the energy still further by exacting work as the price for a transition to thermal equilibrium. An example treated in Section V is motion perpendicular to an increasing magnetic field if the motion preserves the magnetic moment. Such motions increase the perpendicular kinetic energy. As in the principle of virtual work, if this increase would exceed the free energy that would otherwise be liberated, nothing actually happens and the system is stable. Thus, a constraint such as magnetic-moment conservation may set a threshold for instability, for example, a threshold in field gradient. On the other hand, once the threshold is exceeded so that the thermodynamically available energy greatly exceeds the work done against nonthermodynamic constraints, these constraints have little effect on the final fluctuation energy attained. Any exceptions would be rather weak instabilities.

Even so, the thermodynamic laws generally must be supplemented in applications. The reason is that these laws are most meaningful for isolated systems, while real systems are never fully isolated. Thus in classical thermodynamics, typically one must decide independently as to how the system contacts its environment; i.e., whether it is adiabatic, or isothermal, and so on. This is partly a matter of the time scale of interest. For plasmas, the question of time scale is all the more pressing for the interesting case in which the plasma is confined in one or more dimensions by a magnetic field. Magnetic confinement is not permanent, since even collisions destroy it, and thus we shall find that the magnetic forces do not enter the most general thermodynamic constraints valid for all time. Consequently, these most general constraints alone would allow the plasma to give up all its thermal energy to fluctuations by expanding to a large spatial volume, like expansion cooling of a gas. To find when plasma expansion can actually occur on other than collisional time scales, we must add supplemental, approximate constraints such as $\mathbf{E} \cdot \mathbf{B} = 0$ in magnetohydrodynamics, or conservation of the magnetic moment as discussed above. In Section IV, we discuss a less restrictive assumption; namely, different portions of plasma are supposed to be effectively isolated from each other if transport between them would take a time longer than, say, an instability growth time. The simplicity of thermodynamics lies in the fact that the consequences of such assumptions consistent with the thermodynamic laws can then be readily calculated.

II. Fluctuation Energy

We wish to calculate the free energy and also its rate of change. The final results, as they appear in later sections, are simple and transparent. In this and the next section, we give a derivation.

Let each particle species j of mass m_j and charge q_j be described by a phase-space distribution function $f_j(\mathbf{x},\mathbf{v},t)$ which satisfies a Vlasov equation (not linearized),

$$\frac{\partial f_j}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} f_j + \frac{q_j}{m_j}\left(\mathbf{E} + \frac{1}{c}\,\mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f_j}{\partial \mathbf{v}} = \left(\frac{\partial f_j}{\partial t}\right)_{\text{coll}} \quad \text{(II.1)}$$

where the right side represents collisions, and so on. Fields $\mathbf{E}$ and $\mathbf{B}$ satisfy

$$\frac{\partial \mathbf{E}}{\partial t} = -4\pi \sum_{j=i,e} q_j \int d\mathbf{v}\, \mathbf{v} f_j + c\nabla \times \mathbf{B} \quad \text{(II.2)}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} \quad \text{(II.3)}$$

Sums are taken over particle species.

The fields are required to satisfy Maxwell's divergence equations initially with charge density $\sum q_j \int d\mathbf{v}\, f_j$; then they do so thereafter by Eqs. (II.1) and (II.3) since the collision term conserves charge. There may exist an external magnetic field $\mathbf{B}_c$, included in $\mathbf{B}$. Let $\mathbf{B}_c$ be independent of t, and, since we are only interested in the region outside the coils which generate it, we assume

$$\nabla \times \mathbf{B}_c = 0 \quad \text{(II.4)}$$

A. *Bound on Free Energy*

The main task is finding a useful free-energy function. The one most used is a generalization of the Helmholtz function,

$$A = U - TS \quad \text{(II.5)}$$

where U is the kinetic and electromagnetic plasma energy, T is temperature, and S is entropy. To the extent that the plasma is isolated, U is of course conserved. The object is to choose $(-S)$ to be likewise conserved, or decreasing, from which A is also. Then, as we shall see, states which minimize the kinetic part of A play the role of the stable monotone states in Gardner's method discussed above.

As indicated earlier, following reference 4, we shall limit ourselves to the familiar entropy $S \sim -\int f \ln f$ which is maximized by the Maxwell distribution, denoted by g. Then, written out in full, the appropriate Helmholtz function with kinetic part normalized to zero at $f_j = g_j$ is

$$A = \sum_{j=i,e} \int_V d\mathbf{x}\, d\mathbf{v}[G_j(f_j) - G_j(g_j) + (f_j - g_j)\mathscr{E}_j] + \Phi \quad \text{(II.6)}$$

where $\mathscr{E}_j = \frac{1}{2}m_j v^2$ and

$$\Phi = (8\pi)^{-1} \int_V d\mathbf{x}(\mathbf{E}^2 + \Delta\mathbf{B}^2) \tag{II.7}$$

$$G_j(x) = T_j[x \ln (C_j^{-1}x) - x] \tag{II.8}$$

$$g_j = C_j \exp(-\mathscr{E}_j/T_j) \tag{II.9}$$

The somewhat peculiar definition of entropy, through G, is chosen so that the kinetic part of A is minimal at $f = g$ for all choices of the variational parameters C_j and T_j. The parameters C_j and T_j, to be chosen later by variation, can be different for each species. The subscript V denotes integration over a finite spatial volume V. Again, sums are taken over species. Note that $\Delta\mathbf{B} = \mathbf{B} - \mathbf{B}_c$ omits the external field energy in Φ, so that Φ is essentially the field-fluctuation energy. As Gerjuoy has pointed out (5), the fact that this field energy is positive is a consequence of neglecting negative-pair correlation energy (e.g., binding) in the Vlasov fluid approximation, generally valid at densities and temperatures such that turbulence due to plasma instability is of interest.

Since g minimizes the integrand in the kinetic part of A, and the minimum is zero, the kinetic part is positive,

$$A_K \equiv A - \Phi \geq 0 \tag{II.10}$$

The other important property of A is its time dependence obtained by differentiating the kinetic and electromagnetic parts of Eq. (II.6) by t and evaluating the resulting partial derivatives from Eqs. (II.1)–(II.3), together with Eq. (II.4),

$$\frac{dA}{dt} \leq P_V \equiv \int_V d\mathbf{x} \frac{\partial}{\partial \mathbf{x}} \cdot \left\{ \sum_{j=i,e} \int d\mathbf{v}\, \mathbf{v}[\mathscr{E}_j f_j + G_j(f_j)] \times c(4\pi)^{-1}\mathbf{E} \times \Delta\mathbf{B} \right\} \tag{II.11}$$

The inequality permits the collisional increase of entropy. As would be expected, on integrating by parts, the right side yields a surface integral which is just the net rate of energy flow across the surface S bounding the integration volume V. Integrating in time gives

$$A(t) \leq A(0) + \int_0^t dt'\, P_V \tag{II.12}$$

where $A(0)$ is the initial value.

Since $A(t) \geq \Phi(t)$ by Eq. (II.10), the right side of Eq. (II.12) is an upper bound on $\Phi(t)$. However, the bound is generally an overestimate. As Chen has pointed out (6), Maxwell's equations require that charge and current fluctuate in space like $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{B}$, whereas our trial functions are smooth. Similarly, these trial states must omit $\mathbf{E} \times \mathbf{B}$ drift energy that fluctuates reversibly with $\mathbf{E}$. Because the trial functions are states of minimum kinetic energy under the thermodynamic constraints, the correct state consistent with Maxwell's equations and dynamics has greater kinetic energy. The difference is a part of A_K which is, on the average, not available to Φ but appears as a kinetic fluctuation proportional to and varying reversibly with Φ, like polarization energy in a dielectric. In other words, for some dielectric constant ϵ and permeability μ,

$$(8\pi)^{-1} \int_V d\mathbf{x}[(\epsilon - 1)\mathbf{E}^2 + (\mu - 1)\,\Delta\mathbf{B}^2] \leq A_K(t) \tag{II.13}$$

Always, $\epsilon, \mu \geq 1$, and generally for plasmas, $\mu \simeq 1$. While a rigorous nonlinear derivation of ϵ is lacking, its general order of magnitude is easily deduced for two important limiting cases for states not too far from equilibrium. For electrostatic oscillations, $(\epsilon - 1) \sim (k\lambda_{Dj})^{-2}$ where k is the wave number and λ_{Dj} is the Debye length for species j. This corresponds to the adiabatic relation between charge and potential fluctuations, $\delta\rho \sim e^2 n\, \delta\varphi/T$, so that $\frac{1}{2}\, \delta\rho\, \delta\varphi \sim (k^2\lambda_{Dj}^2)^{-1}(E^2/8\pi)$ with $E \sim k\, \delta\varphi$. In a magnetic field, if fluctuation frequencies fall below the cyclotron frequency for a given species and $k^{-1} > r_{Lj}$, the gyroradius, we take $(\epsilon - 1) \sim (r_{Lj}/\lambda_D)_j^2$ which corresponds to $\mathbf{E} \times \mathbf{B}$ drift for that species since $[\frac{1}{2}nm_j(cE/B)^2] \sim (r_{Lj}/\lambda_{Dj})^2(E^2/8\pi)$. An approximate expression correct in these two limits is

$$\epsilon - 1 = \sum_{j=i,e} (r_{Lj}^{-2}\lambda_{Dj}^2 + k^2\lambda_{Dj}^2)^{-1} \tag{II.14}$$

This will be recognized as the typical contribution of "nonresonant" particles for Maxwellian distributions.

Combining Eqs. (II.12) and (II.13) gives the bound, with $\mu = 1$,

$$(8\pi)^{-1} \int_V d\mathbf{x}(\epsilon\mathbf{E}^2 + \Delta\mathbf{B}^2) \leq A(0) + \int_0^t dt'\, P_V \tag{II.15}$$

where ϵ was estimated above, and in any case, $\epsilon = 1$ would yield a rigorous bound by Eqs. (II.10) and (II.12).

This bound is most useful when that portion of the plasma within V is effectively isolated, which is to say the P_V term is negligible compared to $A(0)$. Indeed, most of the physics in our approach, beyond the thermodynamic laws, lies either in choosing V such that this condition holds or in choosing some other assumption equivalent to isolation. We shall return to this important question in Section IV. One example is the idealized plasma uniform over infinite V so that one may take $P_V \to 0$. Then $\Phi \leq A(0)$, from which it follows that Maxwellian states g are stable since for them $A(0)$ is zero aside from the negligible energy in initial perturbations. This proof was first given by Newcomb (2) and, in magnetohydrodynamics, by Berkowitz, Grad, and Rubin (7). Kruskal and Oberman (3) introduced a generalized entropy by which the proof extends to all monotone functions of the energy in the absence of collisions, the same class of stable states found by Gardner. They show that for any monotone state f, there exists a corresponding entropy which is maximum for that state. It has the form above but for G we take the solution of $G' \equiv dG/df = -\mathscr{E}$ since this gives an extremum of A which is a minimum inasmuch as $G'' = -(df/d\mathscr{E})^{-1} > 0$ if $df/d\mathscr{E} < 0$. With their entropy function, bound (II.15) could similarly be generalized in principle, but care must be taken to restrict the perturbation amplitude for all time if G''' or higher derivatives are negative, which is not a problem with Maxwellian distributions and G given by Eq. (II.8).

In evaluating the bound, $A(0)$ should be calculated with f equal to the initial state f_0 since energy in initial perturbations about f_0 is assumed negligible in that instability has not yet developed. Note that it is not necessary that f_0 be an exact equilibrium state; for example, it could oscillate.

B. *Variational Procedure*

Parameters C_j and T_j in the trial functions for each species are as yet arbitrary. Since the bound holds for all values, the best bound is obtained by varying C_j and T_j for each species so as to minimize $A(0)$. Because energy depends on the reference frame, it is also important to be able to change frames. This is accomplished, while preserving the essential features of A, by adding to A any known momenta conserved for the most part. For example, if $\mathbf{B}_c$ is azimuthally symmetric, the total angular momentum is conserved,

$$P_\theta = \sum_{j=i,e} \int_V d\mathbf{x}\, d\mathbf{v}\, f_j p_{\theta j} + (4\pi c)^{-1} \int_V d\mathbf{x}\, \mathbf{z} \cdot \mathbf{r} \times (\mathbf{E} \times \Delta\mathbf{B}) \qquad \text{(II.16)}$$

where $p_{\theta j} = m_j r v_\theta + (q_j/c) r A_\theta$ and A_θ is the azimuthal component of the vector potential generating $\mathbf{B}_c$. To effect a transformation to a reference frame rotating at angular frequency μ, we add μP_θ to A above. The new result retains the original form,

$$A = \sum_{j=i,e} \int_V d\mathbf{x}\, d\mathbf{v} \{G_j(f_j) - G_j(g_j') + (f_j - g_j')\mathscr{E}_j'\} + \Phi \qquad \text{(II.17)}$$

$$g_j' = C_j \exp(-\mathscr{E}_j'/T_j) \qquad \text{(II.18)}$$

$$\mathscr{E}_j' = \mathscr{E}_j + \mu p_{\theta j} = \tfrac{1}{2} m_j (\mathbf{v} + \hat{\theta} r \mu)^2 + U_j \qquad \text{(II.19)}$$

$$U_j = \mu (q_j/C) r A_\theta(r,z) - \tfrac{1}{2} m_j r^2 \mu^2 \qquad \text{(II.20)}$$

However, $\mathscr{E}_j$ has been replaced by $\mathscr{E}_j'$, which is the particle energy in the new frame, and Φ' is the sum of Φ and the field term from μP_θ. It can be shown that Φ' is positive definite and is the field energy transformed to the new frame, though it is not quite relativistically correct because of our nonrelativistic treatment of the particles. In any case, if $r\mu \ll c$ within V,

$$\Phi' \simeq \Phi \qquad \text{(II.21)}$$

Other known canonical momenta $p_{\sigma j}$ may be added similarly to again yield Eqs. (II.17), (II.18), and (II.21) but with

$$\mathscr{E}_j' = \mathscr{E}_j + \sum_\sigma \mu_\sigma p_{\sigma j} \qquad \text{(II.22)}$$

Again, bound (II.15) follows with an expression for P_V different in form but having the same content. Now the constants μ_σ determining the reference frame are to be varied along with C_j and T_j. Note, however, that while C_j and T_j can be different for each species, the μ_σ's cannot. They must be the same for all species able to exchange momentum.

Generally, the optimum values for the variational parameters depend on the situation. However, the optimum C_j obtained by equating to zero the first variation of A with respect to C_j is always that which separately conserves particles of each species,

$$\int_V d\mathbf{x}\, d\mathbf{v}\, C_j \exp(-\mathscr{E}_j'/T_j) = \int_V d\mathbf{x}\, d\mathbf{v}\, f_j \qquad \text{(II.23)}$$

C. *An Example*

As an example of how to apply the bound on field fluctuations, consider counterstreaming ions and electrons with Maxwellian distributions of the form $f_e = N_e \exp(-\mathscr{E}_e/T_0)$ and $f_i = N_i \exp[-(\mathscr{E}_i + up_{zi})/T_0]$

where $p_{zi} = mv_{zi}$. There may be a uniform magnetic field **B** along z. We may consider this uniform plasma to be so large that P_V is negligible; we also neglect collisions; and initially Φ is likewise negligible. Then, evaluating the bound (II.15) calls only for calculating the kinetic part of $A(0)$ by introducing the above initial distributions into Eq. (II.17) and taking the form $\mathscr{E}'_j = \mathscr{E}_j + \mu_z p_{zj}$ for each species to permit variation of reference frames translating along $\mathbf{B}_c$, the direction of relative motion of the ions and electrons. From Eq. (II.23), for ions, the optimum function is

$$C_i = N_i(T_0/T)^{3/2} \exp\left[(m_i u^2/2T_0) - (m_i\mu_z^2/2T)\right] \tag{II.24}$$

and similarly for C_e, but with $u = 0$. With this normalization, we find

$$A(0) = nV\left\{\tfrac{1}{2}m_i(\mu_z - u)^2 + \tfrac{1}{2}m_e\mu_z^2 + \sum_{j=i,e}\frac{3}{2}\left[T_j \ln\frac{T_j}{T_0} - T_j + T_0\right]\right\} \tag{II.25}$$

where $n = \int d\mathbf{v} f_i = \int d\mathbf{v} f_e$. This is minimized by $T_j = T_0$ for each species and $\mu_z \simeq u$, from which

$$A(0) = nV(\tfrac{1}{2}m_e u^2) \tag{II.26}$$

This simple result could have been estimated easily by asking how the system could come to thermal equilibrium. The Maxwellian ions and electrons can do so merely by dissipating their relative motion into fluctuations. To conserve momentum, it is mainly the lighter electrons which decelerate, and from this the bound on free energy is just their kinetic energy of translation relative to the ions, $\tfrac{1}{2}m_e u^2$ per electron. This is readily generalized to several species of different densities, which just yield other terms like those of Eq. (II.25) with n different for each. With the optimum choice $T = T_0$, $A(0)$ is just the sum of translation energies of each species in the reference frame moving at speed μ_z and is minimized by choosing μ_z to eliminate terms with the largest product nm. For example, for an electron beam of density n_b penetrating a denser plasma, we find $A(0) = n_b V(\tfrac{1}{2}m_e u^2)$, which agrees in order of magnitude with the efficiency of utilizing beam energy in several beam-plasma experiments. Also an approximate quasi-linear calculation by Shapiro (8) of two-stream instability for a very weak beam gave a result only a factor of six less than this thermodynamic bound, with $\epsilon \sim 1$ since $k\lambda_D \sim 1$ for plasma oscillations.

The above example illustrates some of the virtues and weaknesses of the thermodynamic method. The bound is insensitive to dynamically important parameters such as mu^2/T, T_e/T_i, nT/B_c^2 (in fact, $\mathbf{B}_c$ could be zero) and whether $\nabla \times \mathbf{E} \neq 0$ (transverse mode). As was indicated in the introduction, we expect these hidden parameters mainly to affect thresholds for various modes and not their nonlinear development except perhaps for very weak modes. We have already seen that the thermodynamic bound is about the right nonlinear limit for the extreme case of the two-stream instability which requires $u > v_e = (2T_e/m_e)^{1/2}$. At a lower streaming speed $u < v_e$, there persists in a beam-plasma system an ion acoustic mode if $T_e > T_i$, and a transverse magnetoacoustic mode if $\beta = (8\pi nT/B_c^2)$ is large (9). These strong instabilities are then probably those which fulfill the thermodynamic bound. At low β and $T_e \lesssim T_i$ when the strong modes go away, there exists an electrostatic ion cyclotron mode, but it saturates at an energy a factor $(u/v_e)^3$ below the thermodynamic limit according to a quasi-linear calculation by Drummond and Rosenbluth (10). Thus, as anticipated, what evidence there is indicates that the thermodynamic bound on free energy represents the strong instabilities fairly well but overestimates the very weak ones.

III. Growth Constant

We can also give an energetic bound on the growth rate of perturbations about any state f_{0j}, $\mathbf{E}_0$, $\mathbf{B}_0$ which is a solution of the Vlasov-Maxwell Eqs. (II.1)–(II.3). In principle, the state may be time dependent; for example, it might incorporate a spectrum of fluctuations already developed. To obtain a thermodynamic bound on the growth constant γ, we imagine that we add enough artificial damping to the system to make f_0 stable. To show this thermodynamically, we would seek a free energy function $H = H_K + \Phi$ such that H_K has a zero minimum at $f = f_0$, whence $H \geq \Phi$ as in the previous section. Then f_0 is stable if H is constant or damping in time, and the artificial damping which must be added to accomplish this, just equal to the actual fastest rate of change of H, is a bound on the growth constant of the real system;

$$\gamma < \max \frac{1}{2}\frac{1}{H}\frac{dH}{dt} \tag{III.1}$$

where "max" means that the maximum value for any allowed perturbation is taken. More precisely, the maximum growth constant $\gamma_{\max}$ is defined as the greatest lower bound on ν such that

$$\int_0^\infty dt\, \Phi \exp(-2\nu t) < \infty$$

for all perturbations. Then, if $(d/dt \ln H) < 2\gamma_0 \equiv (d/dt \ln H)_{\max}$, $\Phi \leq H < C \exp 2\gamma_0 t$ and Eq. (III.1) follows.

To illustrate the method, we shall treat just the following simple function useful for states not too far from thermal equilibrium,

$$H = \tfrac{1}{2} \sum_{j=i,e} \int_V d\mathbf{x}\, d\mathbf{v}\, f_{1j}^2 \frac{T_j}{f_{0j}} + (8\pi)^{-1} \int_V d\mathbf{x}\left(\mathbf{E}_1^2 + \mathbf{B}_1^2 + \frac{2}{c}\mathbf{a}\cdot\mathbf{E}_1 \times \mathbf{B}_1\right) \tag{III.2}$$

Details and other forms of H are given in reference 11. Here $f_{1j} = f_j - f_{0j}$, $\mathbf{E}_1 = \mathbf{E} - \mathbf{E}_0$ and $\mathbf{B}_1 = \mathbf{B} - \mathbf{B}_0$ are the perturbations, and T_j is a variational parameter. Also, we have added the field-momentum term to permit easy variation of the reference frame by varying $\mathbf{a}$, which can depend on $\mathbf{x}$. Since we want the instantaneous rate of change away from whatever solution f_0 exists at that moment, it is sufficient to consider only small perturbations and linearize Eq. (II.1) in computing dH/dt,

$$\frac{\partial f_{1j}}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} f_{1j} + \frac{q_j}{m_j}\left(\mathbf{E}_0 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_0\right)\cdot\frac{\partial f_{1j}}{\partial \mathbf{v}} + \frac{q_j}{m_j}\left(\mathbf{E}_1 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_1\right)\cdot\frac{\partial f_{0j}}{\partial \mathbf{v}} = 0 \tag{III.3}$$

Similarly, in Eqs. (II.2)–(II.3), which are already linear, replace f_j by f_{1j} and so on. Note that we have dropped the collision term here.

When $f_{0j} = g_j$ (Maxwellian) and $\mathbf{E}_0 = 0$ and $\mathbf{B}_0 = \mathbf{B}_c$, H in Eq. (III.2) becomes A to second order in $(f - g)_j$ and $dH/dt = 0$ indicating stability. Otherwise, dH/dt computed from Eq. (III.3) is

$$\frac{dH}{dt} = -\sum_{j=i,e} \int d\mathbf{x}\, d\mathbf{v} (f_{1j}/f_{0j}) \mathbf{J}_j \cdot \mathbf{E}_1 \tag{III.4}$$

with

$$\mathbf{J}_j = q_j\left[(\mathbf{v} + \mathbf{a}) f_{0j} + \frac{T_j}{m_j}\frac{\partial f_{0j}}{\partial \mathbf{v}}\right] \tag{III.5}$$

We have used the fact that f_{0j}, $\mathbf{E}_0$, $\mathbf{B}_0$ is a solution of Eq. (II.1). Also, we have taken the integration volume V to be large and have therefore dropped surface integrals analogous to P_V in Eq. (II.11); we have also dropped the magnetic force $\mathbf{v} \times \mathbf{B}_1$, whose contribution to the bound is smaller by v/c as shown in reference 11.

To complete the calculation, we apply Schwarz' inequality twice to Eq. (III.4) and also the relation $\int d\mathbf{x}\, E_{1i}^2 \leq 8\pi\Phi$, with the notation $\mathbf{E}_1 \cdot \mathbf{J}_j = \sum_i E_{1i} J_{ji}$ and Φ_1 denotes the (positive definite) field term in H. The resulting chain of inequalities is

$$\begin{aligned}
\int d\mathbf{x}\, d\mathbf{v} (f_{1j}/f_{0j}) \mathbf{J}_j \cdot \mathbf{E}_1 \\
&\leq \int d\mathbf{v} \sum_i \left[\int d\mathbf{x}\, 8\pi f_{1j}^2 J_{ji}^2 / f_{0j}^2\right]^{1/2} \left[\int d\mathbf{x}\, E_{1i}^2/8\pi\right]^{1/2} \\
&\leq \nu \int d\mathbf{v}\left[\left(d\mathbf{x}\, f_{1j}^2 \frac{T_j}{f_{0j}}\right)\left(\frac{8\pi \mathbf{J}_j^2}{T_j f_{0j}}\right)_{\max} \Phi_1\right]^{1/2} \\
&\leq 2\sqrt{2}\nu\omega_{pj}(H_V\Phi_1)^{1/2}(A_{vj}/nVT_j)^{1/2}
\end{aligned} \tag{III.6}$$

In the second line of Eq. (III.6), we have removed (J_j^2/f_{0j}) from the $\mathbf{x}$ integration by taking its maximum value at any $\mathbf{x}$ in V, denoted by "max." The quantity $\nu \leq 3$ is the number of nonzero components of the vector $\mathbf{J}_j$, and

$$\begin{aligned}
A_{vj} &= nV \int d\mathbf{v} \frac{2\pi}{\omega_{pj}^2}\left(\frac{\mathbf{J}_j^2}{f_{0j}}\right)_{\max} \\
&= V \int d\mathbf{v}\left[f_{0j}\frac{m_j}{2}\left(\mathbf{v} + \mathbf{a} + \frac{T_j}{m_j f_{0j}}\frac{\partial f_{0j}}{\partial \mathbf{v}}\right)^2\right]_{\max}
\end{aligned} \tag{III.7}$$

Again, H_K and Φ are the kinetic and electromagnetic parts of H.

The combination of Eqs. (III.1), (III.4), and (III.6) gives the bound,

$$\gamma = \sum_{j=i,e} \sqrt{2}\nu\omega_{pj}\left(\frac{A_{vj}}{\epsilon_1 nVT_j}\right)^{1/2} \tag{III.8}$$

where the sum is taken over species and for each $\omega_{pj}^2 = 4\pi n q_j^2/m_j$ with $n = \int d\mathbf{v}\, f_{0j}$. The quantity ϵ_1 is

$$\begin{aligned}
\epsilon_1 = (H_K + \Phi_1)^2/H_K\Phi_1 &\geq 4 && \text{all } H_K, \Phi_1 \\
&\to (H_K/\Phi_1) && \text{if } H_K \gg \Phi_1
\end{aligned} \tag{III.9}$$

Thus ϵ_1, interpreted as the ratio of kinetic and field-fluctuation energies, acts as a dielectric constant quite analogous to our discussion of Eq. (II.13).

The quantity A_{vj} for each species is closely related to the contribution of that species to the free energy. Indeed $\sum A_{vj}$ is the free energy in our earlier example of counterstreaming Maxwell distributions of the form $f_{0j} \propto \exp[-(\mathscr{E} + up_z)/T_0]_j$ with $u_e = 0$ for electrons and $u_i = u$ for ions. Then, choosing $\mathbf{a} = u\hat{\mathbf{z}}$ to minimize the bound, Eq. (III.8), we find from Eq. (III.7) that $A_{vi} = 0$ and A_{ve} is the free energy for this case found previously, given by Eq. (II.26). Then, the bound on γ can be seen to have the form $\gamma \lesssim e\langle E\rangle/m_e v_e$ where v_e is the electron thermal speed and $\langle E\rangle$ is the maximum field allowed thermodynamically with the relation $\epsilon_1\langle E\rangle^2/8\pi = A/V$. An example illustrating the accuracy of the bound on γ for a state near thermal equilibrium is given in the next section.

IV. Free Energy of Confined Plasma

In our examples so far we treated plasmas with uniform density and pressure, the only source of free energy being the deceleration of current streams through the plasma. We now consider the case of a plasma confined in one or more dimensions.

In a confined plasma, some of the random energy, or thermal energy, can contribute to free energy if the plasma can expand in volume. On expanding, the plasma might cool and the heat lost may do work in driving fluctuations. Expansion may not be possible. For example, a plasma confined by a gravitational potential could be in thermal equilibrium and hence be stable with no free energy. The distributions for each species could be monotonic decreasing functions of the particle energy $\mathscr{E}_j = \frac{1}{2}m_j v^2 + V_{gj}$, which now includes the gravitational potential V_{gj}; for example, $f_{0j} \propto \exp(-\mathscr{E}_j/T_j)$. Here the gravitational force prevents expansion.

By contrast, a plasma confined by a magnetic field can never be in thermal equilibrium because diamagnetic current must flow if the field exerts a force against the plasma pressure. Now the plasma can expand if the diamagnetic current is somehow dissipated. Also, the kinetic energy of streaming would be given up, as well as the energy in the diamagnetic field itself.

Thus, in a magnetically confined plasma there are always three possible sources of free energy associated with dissipating the diamagnetic current: kinetic streaming, the diamagnetic field, and expansion. These "fundamental" sources of free energy are those associated with a plasma as near to thermal equilibrium as is possible while still remaining consistent with magnetic confinement. For example, they are generally the principal sources if field lines close within the plasma or if the plasma is magnetically confined only in directions perpendicular to the lines. Then ions and electrons may be Maxwellian but they are in different frames of reference so that their relative motion produces the diamagnetic current. Departures from Maxwellians, such as a loss cone in velocity space or pressure anisotropy, are additional sources sometimes present.

The largest of the fundamental free-energy sources is generally expansion, unless the plasma pressure approaches the external field energy density $B_c^2/8\pi$. Indeed, the most general thermodynamic constraints employed in Section II would permit unlimited expansion and hence the total thermal energy would be free. As proposed in the introduction, here we shall also consider the role of expansion in more localized processes. That is, we subdivide the plasma and consider the different subregions to be isolated from each other in the sense that the free energy in a given region comes from plasma in that region only. The regions are isolated in this sense if the power flow between them, P_V, is negligible for times of interest.

As an example calculation, we consider a long plasma column confined radially by a magnetic field $\mathbf{B}_c$ which is azimuthally symmetric. We take the distributions to be Maxwellian, of the form (for each species)

$$f_{0j} = n(m_j/2\pi T_{0j})^{3/2} \exp[-(\tfrac{1}{2}m_j(\mathbf{v} + \hat{\theta} r\alpha_j)^2 + V_j)/T_{0j}] \qquad \text{(IV.1)}$$

$$V_j = \alpha_j(q_j/c)r(A_0)_\theta - \tfrac{1}{2}m_j r^2\alpha_j^2 + q_j\varphi_0 \qquad \text{(IV.2)}$$

The exponent in Eq. (IV.1) is the sum of particle energy $\frac{1}{2}m_j v^2 + q_j\varphi_0$ and α_j times the canonical angular momentum in the equilibrium field $\mathbf{B}_0$ [not $\mathbf{B}_c$ as in the argument of g_j, Eq. (II.19)]. Here $\mathbf{E}_0 = -\nabla\varphi_0$ and $\mathbf{B}_0 = \nabla \times \mathbf{A}_0(r,z)$ are also assumed symmetric and must be made consistent with Maxwell's equations. One charge species (but not both) can be confined by φ_0, therefore, for that species $\alpha_j \simeq 0$; or both can be confined magnetically, from which for charge quasi-neutrality $(\alpha_i/T_{0i}) \simeq -(\alpha_e/T_{0e})$. In either case, the ions and electrons are in relative rotation as rigid bodies and an azimuthal current flows of the order $j_\theta \simeq enr(\alpha_e - \alpha_i)$.

A bound on the growth constant for this case, of use presently, can be found by evaluating Eq. (III.8). The quantity $\nu = 1$, since $\mathbf{J}_j$ has one component, and $T_j = T_{0j}$ and $\mathbf{a} = 0$ are found to be about optimal choices if $\alpha_i/T_i = -\alpha_e/T_e$ (for neutrality). Taking the field uniform for simplicity (which is not necessary), we find

$$\alpha_i = (v_i r_{Li}/R^2) = -\alpha_e(T_i/T_e) \tag{IV.3}$$

where $\frac{1}{2}m_i v_i^2 = T_i$, r_{Li} is the gyroradius, and R is the plasma radius defined by $V_i/T_i = r^2/R^2$. Here we have assumed $r_{Le} \lesssim r_{Li} \ll R$ and dropped $m_j r^2 \alpha_j^2$ in V_j. The maximum of the integrand occurs around $r = R$, where $r^2 \exp(-V_i/T_i) = 0.37R^2$. Introducing these quantities into Eq. (III.7) with $\mathbf{a} = 0$ gives for each species

$$A_{vj} = nVT_j(0.37r_{Lj}^2/R^2) \tag{IV.4}$$

and hence, by Eq. (III.8),

$$\gamma \leq 0.6\sqrt{2}(\omega_{pi}/\sqrt{\epsilon_1})[(\sqrt{T_i} + \sqrt{T_e})/\sqrt{T_i}]r_{Li}/R \tag{IV.5}$$

For long-wavelength oscillations ($kr_{Li} \leq 1$), we estimate $\epsilon_1 \simeq (r_{Li}^2/\lambda_{Di}^2)$ by Eq. (II.14). Then $\gamma \lesssim v_i/R$, which is about the correct upper limit on the growth constant for the various kinds of universal instabilities which can occur in the plasma column, the maximum occurring at $T_e \sim T_i$. For shorter wavelengths, with $\epsilon_1 = 4$ by Eq. (III.9), the bound (Eq. IV.5) gives about the correct upper limit on the growth constant for the drift-cyclotron instability (12) aside from a factor $(kr_{Li})^{-1/2} \leq 1$ which is the fraction of the ions which are resonant. Thus, the bound on γ compares well with known results for this example in which the plasma is near equilibrium.

To continue the calculation of free energy, let us now divide the plasma into cylindrical shells concentric to the axis of symmetry. We sample instability in different regions of the plasma by calculating the free energy in each of these shell regions separately according to the bound Eq. (II.15) with the integration volume V covering the shell in question. Thus we take V to be the volume within $r_1 < r < r_1 + \lambda$, $-L < z < L$ with $L \to \infty$, and we vary r_1 to change shells.

We first neglect P_V and later determine the error. For generality, we evaluate $A(0)$ by Eq. (II.17) written in a reference frame rotating at angular velocity μ. The optimum variational parameters other than μ are $T_j = T_{0j}$ and C_j given by Eq. (II.23). Then, introducing into Eq.

(II.17) the initial state f_{0j}, $\mathbf{E}_0$, $\mathbf{B}_0$ with f_{0j} given by Eq. (IV.1), we find

$$A(0) = \sum_{j=i,e} nV[\tfrac{1}{2}m_j R^2(\alpha_j - \mu)^2 + \tfrac{1}{4}\beta_j T_{0j}] + A_{\text{exp}} \tag{IV.6}$$

This is the maximum value of $A(0)$ for any of the shell regions, occurring around $r_1 = R$.

As expected, the three terms in $A(0)$ are the three fundamental sources of free energy in a magnetically confined plasma. The first term is kinetic energy in diamagnetic streaming as seen in the reference frame rotating at angular velocity μ. The second term is $\int_V d\mathbf{x}(\mathbf{B}_0 - \mathbf{B}_c)^2/8\pi$, the magnetic term of $\Phi'(0)$ taken to be $\simeq \Phi(0)$ by Eq. (II.21). This is the diamagnetic field energy with $|\mathbf{B}_c - \mathbf{B}_0| \simeq 2\sum_j \beta_j|\mathbf{B}_c|$ and $\beta_j = (8\pi n T_{0j}/B_j^2)$ assumed $\ll 1$ (if $\beta_j \to 1$, merely drop the factor $\frac{1}{4}$). We have neglected energy in $\int d\mathbf{x}\, \mathbf{E}_0^2/8\pi$, of the order λ_D^2/R^2. The third term is the expansion energy,

$$A_{\text{exp}} = \sum_{j=i,e} nT_{0j} \int_V d\mathbf{x} \exp\left[-\frac{V_j}{T_{0j}}\right]$$
$$\left\{\ln\left(\int_V d\mathbf{x}\exp\left[-\frac{U_j}{T_{0j}}\right] \Big/ \int_V d\mathbf{x}\exp\left[-\frac{V_j}{T_{0j}}\right]\right) + \frac{U_j - V_j}{T_{0j}}\right\} \tag{IV.7}$$

The expansion energy can be greatly simplified for not too large λ and field curvature sufficiently gentle so that $(V/T_0)_j \simeq r^2/R^2 \simeq (\alpha_j/\mu)(U/T_0)_j$ for each species. With the restriction that $\lambda/R \ll 1$ and $\lambda/R < \frac{1}{2}|\alpha_j/\mu|$, we may then expand Eq. (IV.7) in powers of λ/R to obtain

$$A_{\text{exp}} = \sum_{j=i,e} nVT_{0j}\frac{1}{6}\frac{\lambda^2}{R^2}\left(\frac{\mu}{\alpha_j} - 1\right)^2 \tag{IV.8}$$

Again we took $r_1 \simeq R$ to sample the worst region.

The quantity $A(0)$ is the desired bound on the fluctuation energy, by Eq. (II.15), if P_V is negligible. To obtain the best bound, the parameter μ should be chosen to minimize $A(0)$. The optimum value depends on λ/R and T_{0e}/T_{0i}. For example, if λ is large enough so that expansion energy dominates, the optimum choice is generally $\mu = 0$; but if $\lambda \to 0$ so that the kinetic drift is greater, the optimum is $\mu \simeq \alpha_i$, analogous to our previous discussion of streaming in connection with Eq. (II.26).

There remains the question of when P_V can be neglected. It is useful to rewrite this quantity, given in the laboratory frame in Eq. (II.11), in the following form, appropriate also in other reference frames,

$$P_V = \int_V d\mathbf{x}\, d\mathbf{v}\, \nabla \cdot \mathbf{v}\mathscr{A}_K \tag{IV.9}$$

where $\mathscr{A}_K$ is the integrand of $A_K = A - \Phi'$ in the chosen frame having the form given by Eq. (II.17). The terms in $(\mathbf{v}\mathscr{A}_K)$ which contain g_j [not present in Eq. (II.11)] integrate to zero so long as the current $\int d\mathbf{v}\, \mathbf{v}g_j$ has no component perpendicular to the surface bounding V, true in our above example with azimuthal symmetry. In Eq. (IV.9) we have neglected the electromagnetic radiation term that is usually small.

While we cannot give a rigorous calculation of P_V for all cases, we can see that the essence of the matter is that P_V is a transient, a consequence of the fluctuations themselves, and it can certainly be neglected in calculating free energy if instability growth saturates within about one growth time γ^{-1}, where γ is the growth constant. Three facts point toward these conclusions. The first is that initially $P_V = 0$ if only we choose V such that the initial currents $\int d\mathbf{v}\, \mathbf{v}f_0$ do not represent a net flow of energy into or out of V, true in our example above. Second, P_V is also zero after "saturation," that is, after a transition to thermal equilibrium, $f_j \to g_j$. Thus, it is consistent with our extreme estimate of free energy as that given up in a transition to thermal equilibrium if we assume P_V to be a transient caused by the fluctuations. Third, we can show, at least for our example and in certain other cases, that an estimate of the peak value of the transient P_V is

$$P_V \leq \gamma A(0) \tag{IV.10}$$

Then, if τ is the duration of the transient P_V, it follows that P_V can be neglected in calculating the bound on free energy if $\tau < \gamma^{-1}$, since the integrated power then satisfies $\int dt'\, P_V \simeq \tau P_V < A(0)$.

Equation (IV.10) does hold for our above example of a plasma column confined radially. Since we found currents across the boundary of V to be caused by the fluctuations, we make what should be the pessimistic replacement $\mathbf{v} \to [A(0)/nmV]^{1/2}$, which supposes that $A_K \simeq A(0)$, whereupon we estimate $P_V \lesssim k(k\lambda)^{-1}[A(0)/nmV]^{1/2}A(0)$, k^{-1} being the scale length of radial variation of $\int d\mathbf{v}\, \mathbf{v}\mathscr{A}_K$. The most interesting case is that when expansion energy dominates, $A(0) \to nVT(\lambda^2/R^2)$, with $T_e = T_i = T$ for simplicity. What mass to take in the above calculations is somewhat ambiguous, but probably it is m_i, by analogy with ambipolar flow. If so, the above estimate of P_V yields Eq. (IV.10) with $\gamma \simeq v_i/R$, which we previously found to be the maximum growth constant in the expansion regime.

V. Applications

In this section, we present in brief some sample applications of the foregoing results, which have been discussed more fully in the literature. As outlined in the introduction, these are not intended as rigorous calculations but rather as simple estimates relating intuitional assumptions to experimental data with the help of our estimate of free energy.

A. *Diffusion*

As a first example, we estimate diffusion in velocity and in space as a consequence of microinstability, following reference 4. Velocity diffusion involves slowing down of particles by the fluctuation fields and reacceleration in a different direction. Then, if D_{vj} is the velocity diffusion coefficient for species j, the quantity $\sum nm_jD_{vj}$ is essentially the rate of energy exchange with the fluctuations and hence must equal the rate of change of the fluctuation energy*

$$\sum_{j=i,e} nm_jD_{vj} \simeq \frac{d}{dt}\frac{\epsilon E^2}{8\pi} = \frac{\epsilon E^2}{8\pi}\frac{d}{dt}\ln E^2 \tag{V.1}$$

On the average $(\epsilon E^2/8\pi) \lesssim [A(0)/V]$, by Eq. (II.15) and neglecting P_V, and for fluctuations not too large we assume $\frac{1}{2}d(\ln E^2)/dt < \gamma_{\max}$, the maximum growth constant bounded by Eq. (III.8). Then, for each species,

$$D_{vj} \leq v_j^2\nu_j \tag{V.2}$$

$$\nu_j \lesssim [A(0)/nVT_j]\gamma_{\max} \tag{V.3}$$

where again $v_j^2 = (2T_j/m_j)$.

Various turbulent transport coefficients can now be calculated by analogy with collisional transport if we regard ν_j as an effective collision frequency. Thus the coefficient of spatial diffusion perpendicular to $\mathbf{B}_c$ would be $D_{\perp j} = r_{Lj}^2\nu_j$ at low $\beta_j \ll 1$; resistivity would be $\eta = (4\pi\nu_e/\omega_{pe}^2)$

* In reference 4, the factor ϵ is omitted, but probably the present estimate, taking the time derivative of the total fluctuation energy, is better.

with $\omega_{pe}^2 = (4\pi ne^2/m_e)$; and so on. The turbulent transport is important only if $\nu_j \gg \nu_c$, the classical binary collision frequency. The latter can be written in a form parallel to Eq. (V.3), $\nu_c \simeq (n\lambda_D^3)^{-1}\omega_{pe}$ where the first factor is the ratio of thermal fluctuation energy to the total thermal energy. On comparing this with Eq. (V.3), we find that the energy and frequency factors compete oppositely. Though thermal fluctuations are generally much smaller than turbulent fluctuations, typically $\gamma_{max} \ll \omega_{pe}$ also. In most cases, it turns out that ν_j does not much exceed ν_c unless the plasma is far from thermal equilibrium. From this result one draws the important conclusion, made quantitative in reference 4, that microturbulence poses only a limited threat to long confinement of plasmas if free-energy sources, in addition to the fundamental sources discussed in Section IV, are minimized.

B. *Magnetic Well Stabilization*

As a second application, following reference 14 we consider stabilization of instabilities driven by the expansion free-energy by means of a sufficiently large positive magnetic gradient, or magnetic well. In expanding in a magnetic well, the plasma moves on the average to regions of higher field and thereby gains energy if fluctuations involved have frequencies below the ion-cyclotron frequency so that the magnetic moment is approximately conserved. The energy gain tends to compensate the expansion free energy. This compensating effect has not been included in our earlier estimate of expansion energy, which is applicable no matter how rapidly energy is transferred (any frequency). Thus no expansion actually occurs and the plasma is stable if the energy gain exceeds our estimate of the expansion energy. The net gain on expanding a distance λ would be

$$W_{mag} = \sum_{j=i,e} \frac{1}{3}\left(\lambda \frac{dn}{dr}\right)\left(\mu_j \lambda \frac{dB}{dr}\right) \tag{V.4}$$

where $\mu_j = T_j/B$ is the magnetic moment and the factor $(\frac{1}{3})$ arises from averaging $(dr)^2$ over displacements in the range $0 < dr < \lambda$. Then stability follows if $W_{mag} > A_{exp}$ given by Eq. (IV.7). This criterion is applied to universal instabilities in reference 14. Since these modes are known to be driven primarily by electrons (the dispersion relation is relatively insensitive to T_i), we take $A_{exp} = nVT_e(\lambda^2/6R^2)$, corresponding to $\mu = 0$ and $T_i = 0$ in Eq. (IV.8). But since ions must move across the field if electrons do to preserve quasi-neutrality, both species contribute to stabilization, see Eq. (V.4). Thus for universal modes, the stability criterion becomes

$$\Delta B/B > \tfrac{1}{2}[T_e/(T_e + T_i)] \tag{V.5}$$

with $R = n(dn/dr)^{-1}$ and $\Delta B = R(dB/dr)$. This result has also been derived directly from dispersion relations (14).

Stability of certain special distributions against fluctuations preserving μ_j can be shown more directly. There are two essential points as emphasized by Taylor (15) and recently criticized and extended by Grad (16). First, μ_j conservation can be used to reduce the number of phase-space dimensions. Then μ_j acts merely as a species label. Applying thermodynamic arguments to the reduced space, either in Gardner's form discussed in the introduction or by defining a free energy analogous to Eq. (II.6) as Taylor does, one concludes stability if f_{0j} depends on $\mathscr{E}_j$ and μ_j only and f_{0j} is a monotonic decreasing function of $\mathscr{E}_j$ for each value of μ_j, $(\partial f_{0j}/\partial \mathscr{E})_j\mu_j < 0$. The second point is that functions of μ_j and $\mathscr{E}_j$ satisfying this stability criterion can actually represent real plasmas confined in a magnetic well.

The above theoretical results have had a considerable influence in furthering the interest in magnetic wells in fusion research.

C. *Fluctuation Spectra*

R. Chen (6) has obtained a thermodynamic bound on the power spectrum of plasma oscillations. He considers a one-species, one-dimensional plasma unstable to electrostatic oscillations with total free energy $\leq A(0)$ computed by methods above. As we discussed earlier in Section II-A, Chen observes that, because $A(0)$ is obtained from the transition to a state of thermal equilibrium and uniform density, it is an overestimate in that it neglects the fact that electrostatic field fluctuations imply density fluctuations Δn. He estimates the correction, essentially $(\epsilon - 1)(E^2/8\pi)$ in Eq. (II.13), as

$$\mathscr{E}_n \simeq 0.6\langle(\Delta n/n)^2\rangle nVT \tag{V.6}$$

Chen's objective is to bound a partial sum of the field power spectrum E_k^2, which he relates to $\mathscr{E}_n$ through the inequality

$$V\int_k^\infty dk'(E_{k'}^2/8\pi) = 2\pi V\int_k^\infty dk' \frac{q^2(\Delta n_{k'})^2}{k'^2} \leq (1.2k^2\lambda_D^2)^{-1}\mathscr{E}_n \tag{V.7}$$

Here E_k is the Fourier component of $E(x)$ with wave number k and in the second step we use $kE_k = 4\pi q\,\Delta n_k$ from Poisson's equation and λ_D is the Debye length. Now

$$\mathscr{E}_n + V\int_k^\infty dk'\, E_{k'}^2/8\pi \leq \mathscr{E}_n + \Phi \leq A(0) \tag{V.8}$$

which is essentially Eq. (II.15). Combining Eqs. (V.7) and (V.8) gives the bound

$$V\int_k^\infty dk'\, E_{k'}^2/8\pi \leq \frac{1}{1 + 1.2k^2\lambda_D^2} A(0) \tag{V.9}$$

When the bound is achieved, differentiating by k gives the spectral shape $E_k^2 \propto k^{-3}$ for $k\lambda_D \gg 1$.

As a final example, we calculate a bound on the power spectrum of fluctuations driven by expansion energy, following reference 17. To do so, we neglect mode coupling from short- to long-wavelength fluctuations. This is perhaps reasonable for expansion modes. A possible picture of mode coupling in this case is that expansion on a scale k^{-1} creates a local, steeper pressure gradient which in turn drives expansion on a smaller scale (bigger k). In any case, neglecting mode coupling to long wavelengths leads immediately to a relation between the power spectrum $\mathscr{E}(k)$ and A_{exp}. We then assume that energy released by expansion on a scale k^{-1} exists in fluctuations with wavelengths less than this scale size, from which

$$V\int_k^\infty dk'\mathscr{E}(k') \leq A_{\text{exp}}(k) \tag{V.10}$$

where A_{exp} given by Eq. (IV.8) with the replacement $k \equiv \lambda^{-1}$. Again assuming low frequencies and long wavelengths, we take $\epsilon = r_{Li}^2/\lambda_{Di}^2$ by Eq. (II.14) and

$$\mathscr{E}(k) = \epsilon\langle \mathbf{E}_k^2\rangle/8\pi \tag{V.11}$$

$$\int_0^\infty dk\,\langle \mathbf{E}_k^2\rangle = \frac{1}{V}\int_V dx\, \mathbf{E}^2(\mathbf{x},t) \tag{V.12}$$

When the bound (V.10) is achieved, we may differentiate this expression to obtain $\langle E_k^2\rangle \propto k^{-3}$, since $A_{\text{exp}} \propto k^{-2}$, a result also obtained by F. Chen (18). A prescription for inverting the above procedure in order to obtain A_{exp} from experimental spectral data is discussed in reference 17.

VI. Further Developments

Continued progress along thermodynamic lines hinges on finding better free-energy functions. The search has gone on for some time, mostly within linearized theory with the hope that the results would later lead to a nonlinear theory.

We can prove the existence of free-energy functions within linear theory in the following sense. Through an extension of the stability theory of Liapunov (19), we can show that if the state f_{0j} is stable there exists a quantity H which has a zero minimum at $f_j = f_{0j}$ and $d/dt \ln H$ is as small as we please (H is nearly constant). One such H can be constructed as follows. Let ψ denote a column vector whose components are the perturbations f_{1j} and the components of $\mathbf{E}_1$ and $\mathbf{B}_1$, and define a scalar product by

$$(\psi_a,\psi_b) = C_1 \sum_{j=i,e} \int d\mathbf{x}\, d\mathbf{v}(f_{1j}^{(a)})^* f_{1j}^{(b)} + C_2\int d\mathbf{x}[(\mathbf{E}_1^{(a)})^*\cdot \mathbf{E}_1^{(b)} + (\mathbf{B}_1^{(a)})^*\cdot \mathbf{B}_1^{(b)}] \tag{VI.1}$$

where C_1 and C_2 are positive constants and (*) denotes the complex conjugate. Write $\psi(t) = T(t)\psi(0)$, $T(t)$ being the solution operator of the system of linearized Vlasov and Maxwell's equations with the property $T(t + \Delta) = T(t)T(\Delta)$. Now suppose f_{0j} is stable so that $[\psi(t),\psi(t)]\exp(-\mu t)$ is integrable on t for all ψ and any $\mu > 0$. Then a quantity having the required properties of H above is

$$\begin{aligned} H &= \int_t^\infty dt'[\psi(t'),\psi(t')]\exp[-\mu(t'-t)] \\ &= \left(\psi(t),\left[\int_0^\infty ds\, T^\dagger(s)T(s)e^{-\mu s}\right]\psi(t)\right) \end{aligned} \tag{VI.2}$$

since $H \geq 0$ and $H = 0$ if $f_{1j} = \mathbf{E}_1 = \mathbf{B}_1 = 0$, and $dH/dt = -(\psi,\psi) + \mu H \leq \mu H$ so that $(d/dt)\ln H < \mu$, any $\mu > 0$. Here (†) denotes the Hermitian conjugate, and $s = t' - t$. An example is H given by Eq. (III.2) if $f_{0j} \propto \exp(-\mathscr{E}_j/T_j)$. Similarly, one can show that there exists a positive quadratic form H for which a bound on the growth constant analogous to Eq. (III.1) becomes arbitrarily accurate. A more detailed proof is given in reference 19.

While constants of motion having the properties of H must exist, as yet there is no really practical method for finding them. Several approaches have been explored. One utilizes the fact that finding the free-energy function relative to a stable state is equivalent to finding a nonunitary similarity transformation bringing the Vlasov equation linearized about this state into Hermitian form (19,20). Sturrock (21), Low (22), and Buneman (23) have pursued a Lagrangean formulation. However, with the possible exception of new constants found by Kruskal and Oberman (24), none of these efforts has yet disclosed a free energy materially different from the Helmholtz function. Along related lines, several authors have identified variational energies with marginal mode analysis (25–28), and Minardi (29) has advocated abandoning the search for constants and looking instead for an entropy displaying Landau damping. What will come of these diverse approaches remains to be seen.

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