NON-LINEAR STABILITY OF PLASMA OSCILLATIONS*

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The collective behavior of a fully ionized plasma in which the number of particles in a sphere of radius a, the Debye length, is very large compared to one is governed by the collisionless Boltzmann or Vlasov equation. In an infinite homogeneous plasma of this type, it is well known that in the "linearized" theory a velocity distribution $f_0(v)$ consisting of a main part that is a monotonically decreasing function of energy plus a small gentle bump on the tail of the main part (e.g. a Maxwellian plus runaway electrons) leads to unstable (growing) plasma oscillations, and that the unstable oscillations are those for which $v\,\partial f_0(v)/\partial v > 0$ for $v = \omega/k$ (ω is the frequency and k the wave number).

After a sufficient time these waves grow to such an amplitude that the non-linear terms in the Vlasov equation are important and the linearization is no longer valid. The question then arises as to the behavior of these waves in the non-linear region and it is this question which we consider.

The method is to divide the non-linear terms into two groups, one of which combined with the linear terms yields a non-linear dispersion relation, while the other provides a weak coupling between the different modes. The non-linear dispersion relation leads to the establishment of an equilibrium spectrum, which then decays slowly to zero due to the mode-coupling terms. The limiting of the wave amplitudes to the equilibrium spectrum is due to flattening of the bump in the velocity distribution by non-linear effects. The slow decay of the equilibrium spectrum leads to further changes in the velocity distribution so that asymptotically the distribution function is a monotonically decreasing function of energy and hence stable. Analytic expressions for the equilibrium spectrum and the equilibrium velocity distribution are obtained. An approximate value for the maximum energy in the equilibrium electric field is given by the geometric mean of the thermal energy and the drift energy of the particles in the bump.

1. Introduction

The collective behavior of a fully ionized plasma in which the number of particles in a sphere of radius a, the Debye length, is very large compared to one is governed by the collisionless Boltzmann or Vlasov equation. In an infinite homogeneous plasma of this type it is well known that certain velocity distributions lead to unstable (growing) oscillations. The frequencies and growth rates of these oscillations are obtained by linearizing the Vlasov equation about the unperturbed distribution function, and this leads to exponentially damped (stable) or exponentially growing (unstable) solutions. After a sufficient time the unstable solutions evidently grow to such an amplitude that the non-linear terms become important and the linearizing of the Vlasov equation is no longer valid. The question then arises as to the ultimate fate of such unstable oscillations. It is this question that we wish to consider.

It will be shown that the development in the non-linear regime for certain types of unstable modes can be followed in considerable detail for long times. This is illustrated for the case of unstable electron-plasma oscillations. The result is that these waves, which are initially unstable, grow in a short time to an equilibrium spectrum (in $\mathbf{k}$ space) and then decay slowly to zero. The limiting of these waves to an equilibrium spectrum is a result of a diffusion in the velocity distribution due to non-linear effects and cannot be obtained in the magnetohydrodynamic (MHD) approximation. The decay of this spectrum is due to a combination of non-linear changes in the distribution function and MHD effects.

The method is to divide the non-linear terms into two groups. One of these, combined with the linear terms, yields a non-linear dispersion relation, and leads to the establishment of an equilibrium spectrum. The other group of non-linear terms provides coupling between the different modes, and it is this coupling that leads to the eventual damping of the spectrum. In practical cases, this decay time may be so long that collision effects will dominate the damping process.

The instabilities to which the theory applies are those for which the growth rate depends on the velocity gradient of the distribution function, and the dispersion relation is such that the interaction between modes is non-resonant.

In Section 2 the linearized solutions are discussed, and the non-linear dispersion relations are developed in Section 3. In Section 4 the non-linear dispersion relations are applied to a one-dimensional example. The damping due to mode coupling is discussed in Section 5, and the results are discussed in Section 6.

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2. Linearized theory

In treating the dynamics of the electrons we shall treat the ions as a uniform background of positive charge. In so doing we neglect the influence of the ionic polarizability, an approximation which is valid for phenomena occurring at frequencies large compared to characteristic ionic frequencies. We study the Vlasov equation which governs the time rate of change of the single-particle distribution function, $F(\mathbf{r}, \mathbf{v}, t)$, according to

$$\frac{\partial F}{\partial t} + \mathbf{v}\cdot\nabla F - \frac{e\mathbf{E}}{m}\cdot\nabla_{\mathbf{v}} F = 0 \quad (2.1)$$

Here $\mathbf{E}$ is the electric field produced by the particles, and is determined by Poisson's equation

$$\nabla\cdot\mathbf{E} = +4\pi e(n-\varrho) = +4\pi e\left[n - \int d^3\mathbf{v}\, F(\mathbf{v})\right] \quad (2.2)$$

where $\varrho(\mathbf{r}, t)$ is the local electron density, and n is the average electron and ion density, and e and m are the electronic charge and mass respectively.

The use of Eq. (2.1) represents an approximation of the following kind [1, 2]. First, it is valid only when

$$n a^3 \gg 1 \quad (2.3)$$

where a is the Debye length,

$$a^2 = \mathrm{k}T/4\pi n e^2 \quad (2.4)$$

and T is the electron temperature. This approximation is well satisfied in all classical plasmas of physical interest. Next we remark that in general the right-hand side of (2.1) should not be zero but should contain Fokker-Planck-type collision terms, which are themselves non-linear functions of F. These collision terms give rise to relaxation phenomena in the electron gas which are characterized by a time

$$\tau = n a^3/\omega_p \quad (2.5)$$

when $\omega_p = (4\pi n e^2/m)^{1/2}$ is the plasma frequency. Our use of Eq. (2.1) is therefore only valid as long as the phenomena we consider take place in a time short compared to τ.

The solution of Eqs. (2.1) and (2.2) is obtained by splitting F into two parts

$$F(\mathbf{r}, \mathbf{v}, t) = F_0(\mathbf{v}) + f(\mathbf{r}, \mathbf{v}, t) \quad (2.6)$$

Here $F_0(\mathbf{v})$ is the unperturbed homogeneous time-independent distribution function, while $f(\mathbf{r}, \mathbf{v}, t)$ represents the correction to $F_0(\mathbf{v})$ brought about by the Coulomb interaction between the electrons. If we expand f in a Fourier series in a box of size L^3,

$$f(\mathbf{r}, \mathbf{v}, t) = \sum_k f_k(\mathbf{v}, t)\, e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$f_{\mathbf{k}}(\mathbf{v}, t) = \frac{1}{L^3}\int d^3\mathbf{r}\, f(\mathbf{r}, \mathbf{v}, t)\, e^{-i\mathbf{k}\cdot\mathbf{r}} \quad (2.7)$$

we obtain

$$\frac{\partial f_{\mathbf{k}}}{\partial t} + i\mathbf{k}\cdot\mathbf{v} f_{\mathbf{k}} = +\frac{e}{m}\mathbf{E}_{\mathbf{k}}\cdot\nabla_{\mathbf{v}} F_0 + \frac{e}{m}\sum_{\mathbf{q}} \mathbf{E}_{\mathbf{k}-\mathbf{q}}\cdot\nabla_{\mathbf{v}} f_{\mathbf{q}} \quad (2.8)$$

and

$$\mathbf{E}_{\mathbf{q}}(t) = \frac{4\pi e}{q^2} i\mathbf{q}\int d^3\mathbf{v}\, f_{\mathbf{q}}(t) \quad (2.9)$$

where we have made use of the fact that we consider only longitudinal electric fields.

In this section we consider the perturbation series solution of Eqs. (2.8) and (2.9), in which $f(\mathbf{v}, t)$ is regarded as representing a small correction to the initial velocity distribution, $F_0(\mathbf{v})$. Our motivation in doing so is, first, to bring out the nature of the difficulties that arise when $F_0(\mathbf{v})$ is such as to yield growing plasma waves; and second, to provide a framework within which to view the non-perturbation solution discussed in the following section.

The perturbation-theoretical solution consists in expanding $f_{\mathbf{k}}$ and $\mathbf{E}_{\mathbf{k}}$ as follows

$$f_{\mathbf{k}} = f_{\mathbf{k}}^{(1)} + f_{\mathbf{k}}^{(2)} + f_{\mathbf{k}}^{(3)} + \dots$$

$$\mathbf{E}_{\mathbf{k}} = \mathbf{E}_{\mathbf{k}}^{(1)} + \mathbf{E}_{\mathbf{k}}^{(2)} + \mathbf{E}_{\mathbf{k}}^{(3)} + \dots \quad (2.10)$$

while regarding $F_0(\mathbf{v})$ as a zero-order quantity. The first three equations are

$$\frac{\partial f_{\mathbf{k}}^{(1)}}{\partial t} + i\mathbf{k}\cdot\mathbf{v} f_{\mathbf{k}}^{(1)} = \left(\frac{e}{m}\right)\mathbf{E}_{\mathbf{k}}^{(1)}\cdot\nabla_{\mathbf{v}} F_0 \quad (2.11)$$

$$\frac{\partial f_{\mathbf{k}}^{(2)}}{\partial t} + i\mathbf{k}\cdot\mathbf{v} f_{\mathbf{k}}^{(2)} = \left(\frac{e}{m}\right)\left[\mathbf{E}_{\mathbf{k}}^{(2)}\cdot\nabla_{\mathbf{v}} F_0 + \sum_{\mathbf{q}} \mathbf{E}_{\mathbf{k}-\mathbf{q}}^{(1)}\cdot\nabla_{\mathbf{v}} f_{\mathbf{q}}^{(1)}\right] \quad (2.12)$$

$$\frac{\partial f_{\mathbf{k}}^{(3)}}{\partial t} + i\mathbf{k}\cdot\mathbf{v} f_{\mathbf{k}}^{(3)} = \left(\frac{e}{m}\right)\left\{\mathbf{E}_{\mathbf{k}}^{(3)}\cdot\nabla_{\mathbf{v}} F_0 + \sum_{\mathbf{q}}\left[\mathbf{E}_{\mathbf{k}-\mathbf{q}}^{(2)}\cdot\nabla_{\mathbf{v}} f_{\mathbf{q}}^{(1)} + \mathbf{E}_{\mathbf{k}-\mathbf{q}}^{(1)}\cdot\nabla_{\mathbf{v}} f_{\mathbf{q}}^{(2)}\right]\right\} \quad (2.13)$$

and

$$\mathbf{E}_{\mathbf{q}}^{(1)} = \frac{4\pi e}{q^2} i\mathbf{q}\int d^3\mathbf{v}\, f_{\mathbf{q}}^{(1)}(\mathbf{v}) \quad (2.14)$$

with corresponding equations connecting $\mathbf{E}_{\mathbf{q}}^{(2)}$ and $f_{\mathbf{q}}^{(2)}$, $\mathbf{E}_{\mathbf{q}}^{(3)}$ and $f_{\mathbf{q}}^{(3)}$. One then finds, from (2.11) and (2.14) that a non-zero solution of $\mathbf{E}_{\mathbf{q}}^{(1)}$ with a time dependence of the form $\exp - i\Omega t$ exists only for frequencies Ω which satisfy the dispersion relation

$$\varepsilon^{(1)}(\mathbf{q}, \Omega) = 1 + \frac{4\pi e^2}{mq^2}\int d^3\mathbf{v}\, \frac{\mathbf{q}\cdot\nabla_{\mathbf{v}} F_0(\mathbf{v})}{\Omega - \mathbf{q}\cdot\mathbf{v}} = 0 \quad (2.15)$$

Here $\varepsilon^{(1)}(\mathbf{q}, \Omega)$ is the first-order dielectric constant of the plasma, and $\mathrm{Im}\,\Omega > 0$. $\varepsilon^{(1)}(\mathbf{q}, \Omega)$ is thus defined only for Ω in the upper half of the complex Ω plane and is to be continued analytically into the lower half plane where necessary to obtain solutions for which $\mathrm{Im}\,\Omega \leq 0$.

In the present paper we confine our attention to distribution functions $F_0(\mathbf{v})$ which are such that one will find for certain values of $\mathbf{q}$ growing plasma oscillations with

$$\Omega(\mathbf{q}) = \omega_{\mathbf{q}}^{(1)} + i\gamma_{\mathbf{q}}^{(1)} \tag{2.16}$$

such that the growth rate $\gamma_{\mathbf{q}}^{(1)}$ is small compared to the oscillation frequency, $\omega_{\mathbf{q}}^{(1)}$, the latter being given by its long wavelength expansion

$$\omega_{\mathbf{q}}^2 \approx \omega_p^2 + 3\int d^3v\, F_0(\mathbf{v})\,(\mathbf{q}\cdot\mathbf{v})^2 + \cdots \tag{2.17}$$

In addition we require $\gamma_{\mathbf{q}}/q\Delta v \ll 1$ where Δv is the characteristic velocity increment in which $F_0(\omega_p/q)$ changes. The growth rate $\gamma_{\mathbf{q}}^{(1)}$, is then given by

$$\frac{\gamma_{\mathbf{q}}^{(1)}}{\omega_{\mathbf{q}}^{(1)}} = \frac{2\pi^2 e^2}{m q^2}\int d^3v\, \mathbf{q}\cdot\nabla_{\mathbf{v}} F_0(\mathbf{v})\,\delta(\omega_{\mathbf{q}}^{(1)} - \mathbf{q}\cdot\mathbf{v}) \ll 1 \tag{2.18}$$

In the second order one finds, on substituting the appropriate first-order solutions for $f_{\mathbf{k}}^{(1)}$ and $\mathbf{E}_{\mathbf{k}}^{(1)}$ into (2.12), that $\mathbf{E}_{\mathbf{k}}^{(2)}$ (and $f_{\mathbf{k}}^{(2)}$) oscillate at the sum and difference frequencies $\omega_{\mathbf{q}} \pm \omega_{\mathbf{k}-\mathbf{q}}$ as one might expect. These solutions, when substituted into the equation for $f_{\mathbf{q}}^{(3)}$ (2.13) will yield corrections to the dispersion relation (2.15). There is also in second order a correction to $F_0(\mathbf{v})$ arising from $f_0^{(2)}(\mathbf{v})$. The equation for the time rate of change of $f_0^{(2)}(\mathbf{v})$ reads

$$\frac{\partial f_0^{(2)}(\mathbf{v})}{\partial t} = \sum_{q} \frac{8\pi}{q^2}\left(\frac{e}{m}\right)^2 \mathscr{E}_{\mathbf{q}}(t)\,\mathbf{q}\cdot\nabla_{\mathbf{v}} \frac{\gamma_{\mathbf{q}}^{(1)}}{(\omega^{(1)} - \mathbf{q}\cdot\mathbf{v})^2 + (\gamma_{\mathbf{q}}^{(1)})^2} \times \mathbf{q}\cdot\nabla_{\mathbf{v}} F_0(\mathbf{v}) \tag{2.19}$$

where we have introduced $\mathscr{E}_{\mathbf{q}}(t)$, the electrostatic energy in the $\mathbf{q}$th plasma mode

$$\mathscr{E}_{\mathbf{q}}(t) = \frac{|E_{\mathbf{q}}(t)|^2}{8\pi} \tag{2.20}$$

According to (2.16), one further has

$$\frac{\partial \mathscr{E}_{\mathbf{q}}}{\partial t} = 2\gamma_{\mathbf{q}}^{(1)}\mathscr{E}_{\mathbf{q}} \tag{2.21}$$

Plasma waves that are damped (and which possess an initial energy level $\mathscr{E}_{\mathbf{q}}$ not much greater than the thermal level, kT) do not make a sizeable contribution to $f_0^{(2)}(\mathbf{v})$. Their contribution is such that $f_0^{(2)}(\mathbf{v})$ will be of order $1/na^3$ compared to $F_0(\mathbf{v})$ and therefore represent a small correction consistent with the assumptions underlying the use of perturbation theory. In fact, under these circumstances, the terms on the right-hand side of (2.19) are of the same order as certain of the Fokker-Planck terms we have neglected. Again, for damped plasma waves, it is necessary to consider corrections to (2.21) that are contained in the more complete Fokker-Planck equation, which correspond to spontaneous emission of plasma waves, and which we have neglected.

In third order one finds corrections to the dispersion relation (2.15) the most important of which comes from $f_0^{(2)}(\mathbf{v})$ and is of the form

$$\frac{\delta\gamma_{\mathbf{q}}}{\omega_{\mathbf{q}}} = \frac{2\pi^2 e^2}{m_{\mathbf{q}}^2}\int d^3v\, \mathbf{q}\cdot\nabla_{\mathbf{v}} f_0^{(2)}(\mathbf{v},t)\,\delta(\omega_{\mathbf{q}} - \mathbf{q}\cdot\mathbf{v}) \tag{2.22}$$

This correction term $\delta\gamma_{\mathbf{q}}$ is of order $(1/na^3)\gamma_{\mathbf{q}}$ for damped waves or waves with energies $\mathscr{E}_{\mathbf{k}}(t)$ near the thermal level kT. Similar behavior is found for the other third-order terms; so for these circumstances the use of perturbation-theory is well justified.

For growing plasma waves the situation is quite different. If one begins with a thermal level of energy kT in such a wave, after a few e-folding times $(t \gtrsim 1/\gamma_{\mathbf{q}})$, the corrections to (2.19) and (2.21) arising from the Fokker-Planck collision terms will be small. Moreover, as the energy in these growing plasma modes increases, the correction to $\gamma_{\mathbf{q}}$ arising from $f_0^{(2)}(\mathbf{v})$ will eventually become comparable to $\gamma_{\mathbf{q}}$. Further inspection shows that at this point the corrections to $\gamma_{\mathbf{q}}$ from $f_0^{(4)}$, $f_0^{(6)}$, etc., are also comparable to $\gamma_{\mathbf{q}}$ and thus the perturbation-theory solution breaks down. One is forced, then, to search for a more accurate set of equations to describe this time-dependent interrelationship between the spatially homogeneous part of the distribution function

$$g(\mathbf{v},t) = F_0(\mathbf{v}) + f_0(\mathbf{v},t) \tag{2.33}$$

and the growth rate, now also time dependent $\gamma(\mathbf{q},t)$.

3. Non-linear theory

The breakdown in the perturbation-theoretical solution of (2.8) and (2.9) for plasmas in which growing waves exist leads us to consider an alternative set of solutions in which the spatially homogeneous part of the distribution function $g(\mathbf{v}, t)$ plays a special role. From among the terms in the sum over $\mathbf{q}$ on the right hand side of Eq. (2.8) we single out the term with $\mathbf{q}=0$ so that the non-linear Vlasov equation becomes

$$\frac{\partial f_{\mathbf{K}}}{\partial t} + i\mathbf{K}\cdot\mathbf{v} f_{\mathbf{K}} = + \frac{e}{m}\mathbf{E}_{\mathbf{K}}\cdot\nabla_{\mathbf{v}} g + \frac{e}{m}\sum_{\mathbf{q}}{}' \mathbf{E}_{\mathbf{K}-\mathbf{q}}\cdot\nabla_{\mathbf{v}} f_{\mathbf{q}} \tag{3.1}$$

where $g(\mathbf{v}, t) = F_0(\mathbf{v}) + f_0(\mathbf{v}, t)$ and the prime in the summation indicates that the term with $q=0$ is to be deleted. For $\mathbf{K}=0$, $\partial f_0/\partial t = \partial g/\partial t$, we likewise have

$$\frac{\partial g}{\partial t} = + \frac{e}{m}\mathbf{E}_0\cdot\nabla_{\mathbf{v}} g + \frac{e}{m}\sum_{\mathbf{q}}{}' \mathbf{E}_{-\mathbf{q}}\cdot\nabla_{\mathbf{v}} f_{\mathbf{q}} \tag{3.2}$$

In the term $\mathbf{E}_0\cdot\nabla_{\mathbf{v}} g$ in Eq. (3.2), $\mathbf{E}_0$ is determined by the boundary conditions. If the different boundaries are held at different fixed potentials $\mathbf{E}_0$ is non-zero. However, if we apply periodic boundary conditions (to the potential), $\mathbf{E}_0=0$. We shall take $\mathbf{E}_0=0$ for the sake of simplicity.

The non-linear terms in the second term on the right-hand side of (3.1) represent an interaction between different modes whereas the non-linear part of $\mathbf{E}_{\mathbf{K}}\cdot\nabla_{\mathbf{v}} g$ combined with (3.2) leads to a slow variation of the frequency and growth rate with time.

Our procedure is first to solve Eqs. (3.1) and (3.2) neglecting the mode-coupling terms and then to treat these terms as a perturbation. The justification for this is that when we neglect mode coupling we find that the energy $\mathscr{E}_{\mathbf{q}}$ in the unstable modes does not grow indefinitely as in the linearized theory but instead comes rapidly to an equilibrium spectrum the amplitude of which is of order $\gamma_{\mathbf{q}}^{(1)}/\omega_{\mathbf{q}}$ and hence small. Since the equilibrium amplitude is small, the non-linear mode coupling can be treated by perturbation-theoretical methods and leads to a rather slow decay of the equilibrium spectrum.

The basic equations which we consider with (2.9), are therefore

$$\frac{\partial f_{\mathbf{K}}}{\partial t} + \mathrm{i}\,\mathbf{K}\cdot\mathbf{v}\,f_{\mathbf{K}} = \frac{e}{m}\mathbf{E}_{\mathbf{K}}\cdot\nabla_{\mathbf{v}}\,g \tag{3.3}$$

$$\frac{\partial g}{\partial t} = +\frac{e}{m}\sum_{\mathbf{q}}{}'\,\mathbf{E}_{\mathbf{q}}\cdot\nabla_{\mathbf{v}}\,f_{\mathbf{q}} \tag{3.4}$$

Our solution is based on the fact that the frequency and growth rate determined from these equations are slowly varying functions of time. Thus we wish to assume that the change in γ (or ω) during a time of interest (γ^{-1} or ω^{-1}) is small compared to γ (or ω) and then demonstrate that this assumption is consistent. The derivation is based on a type of WKB approximation, and is given in the appendix. The resulting equations are

$$\frac{\partial g(\mathbf{v})}{\partial t} = \sum_{\mathbf{q}}{}'\frac{8\pi e^2\mathscr{E}_{\mathbf{q}}(t)}{m^2q^2}\mathbf{q}\cdot\nabla_{\mathbf{v}}\frac{\gamma(\mathbf{q},t)}{(\omega_{\mathbf{q}}-\mathbf{q}\cdot\mathbf{v})^2+\gamma^2}\mathbf{q}\cdot\nabla_{\mathbf{v}}\,g(\mathbf{v},t) \tag{3.5}$$

$$\gamma(\mathbf{q},t) = \frac{2\pi^2e^2}{mq^2}\,\omega_{\mathbf{q}}\int \mathrm{d}^3\mathbf{v}\,\mathbf{q}\cdot\nabla_{\mathbf{v}}\,g(\mathbf{v},t)\,\delta(\omega_{\mathbf{q}}-\mathbf{q}\cdot\mathbf{v}) \tag{3.6}$$

$$\frac{\partial\mathscr{E}_{\mathbf{q}}}{\partial t} = 2\gamma(\mathbf{q},t)\,\mathscr{E}_{\mathbf{q}} \tag{3.7}$$

It is instructive to compare these equations with their perturbation-theoretical counterparts. We remark that (3.5) differs from (2.19) in that $g(\mathbf{v},t)$ now relaxes toward itself rather than toward $F_0(\mathbf{v},t)$. Equation (3.6) (there is an analogous equation for $\omega_{\mathbf{q}}(t)$ of which we shall not have need) states that the growth rate is determined by the total spatially homogeneous distribution function existing at the time t rather than the initial value $F_0(\mathbf{v})$. This result seems eminently reasonable since one would expect the plasma dispersion relation to follow $g(\mathbf{v},t)$ adiabatically provided the latter changed slowly over characteristic plasma times. Indeed one is so naturally led on intuitive grounds to write down (3.5) and (3.6) that it is hardly surprising that their derivation is possible.

We may further remark that from the point of view of a perturbation-theoretical approach, we have essentially summed a whole class of higher-order terms (those corresponding to $f_0^{(4)}$, $f_0^{(6)}$, $f_0^{(8)}$, etc.) in writing down these basic equations. We also note that these equations may be obtained by means of a quantum treatment based on the explicit introduction of collective coordinates, a plasmon distribution function and the random phase approximation [3].

We consider some general properties of (3.5) and (3.6). First we note that (3.5) resembles the diffusion term in a Fokker-Planck equation. Thus we may write

$$\frac{\partial g(\mathbf{v},t)}{\partial t} = \sum_{i,j}\frac{\partial}{\partial v_i}T_{ij}\frac{\partial}{\partial v_j}g(\mathbf{v},t) \tag{3.8}$$

where the diffusion coefficient T_{ij} is given by

$$T_{ij}(\mathbf{v}) = \sum_{\mathbf{q}}{}'\frac{8\pi e^2\mathscr{E}_{\mathbf{q}}(t)}{m^2q^2}\,\frac{\gamma(\mathbf{q},t)}{(\omega_{\mathbf{q}}-\mathbf{q}\cdot\mathbf{v})^2+\gamma_{\mathbf{q}}^2}\,q_i\,q_j \tag{3.9}$$

$T_{ij}(\mathbf{v})$ possesses a rather different character depending on whether or not $\mathbf{v}$ is such that $\mathbf{q}\cdot\mathbf{v}\approx\omega_{\mathbf{q}}$. Where $\mathbf{q}\cdot\mathbf{v}\approx\omega_{\mathbf{q}}$, since $\gamma\ll\omega_{\mathbf{q}}$, the denominator displays a characteristic resonance behavior, and we have

$$\frac{\gamma(\mathbf{q},t)}{(\omega_{\mathbf{q}}-\mathbf{q}\cdot\mathbf{v})^2+\gamma_{\mathbf{q}}^2}\approx\pi\,\delta(\omega_{\mathbf{q}}-\mathbf{q}\cdot\mathbf{v})$$

We thus find an explicit plasma-oscillation diffusion coefficient

$$T_{ij}^{p'}(\mathbf{v}) = \sum_{\mathbf{q}}{}'\frac{8\pi^2e^2}{m^2q^2}\,q_i\,q_j\,\delta(\omega_{\mathbf{q}}-\mathbf{q}\cdot\mathbf{v})\,\mathscr{E}_{\mathbf{q}} \tag{3.10}$$

These equations have a physical significance which is most easily understood in a one-dimensional case. In this case particles traveling slightly slower than the phase velocity of a particular wave are accelerated by the wave and thus take energy from the wave, whereas particles traveling slightly faster than the wave are slowed down and give up energy to the wave. Thus if there are more particles going slightly faster than the wave than there are particles going slightly slower than the wave, i.e., if $\partial f/\partial v>0$ for $v=\omega_{\mathbf{q}}/q$, then there is a net energy transfer to the wave. In three dimensions it is the projection of the particle velocity in the direction of wave propagation that determines the growth or damping, and thus if $\int\mathrm{d}^3\mathbf{v}\,\mathbf{q}\cdot\nabla_{\mathbf{v}}\,g(\mathbf{v},t)\,\delta(\omega_{\mathbf{q}}-\mathbf{q}\cdot\mathbf{v})>0$, the wave grows. This is the physical significance of Eq. (3.6). The accompanying diffusion of the particle distribution function is given by Eq. (3.8) and is such as to reduce the growth rate.

This process continues until the particle distribution function is readjusted such that $\int\mathrm{d}^3\mathbf{v}\,\mathbf{q}\cdot\nabla_{\mathbf{v}}\,g(\mathbf{v},t)\,\delta(\omega_{\mathbf{q}}-\mathbf{q}\cdot\mathbf{v})=0$ for those particles which interact with the waves. The waves then have reached an equilibrium spectrum and their amplitude no longer changes. This process is discussed in detail for a one-dimensional example in the next section.

Next on a time scale longer compared to that of the above process the mode-coupling terms act to distort and damp the equilibrium spectrum. This process is considered in detail in Section 5. Before going to that, we wish to discuss the question of energy conservation as determined by Eqs. (3.5)-(3.7).

Particle energy U is given by

$$U = \int\mathrm{d}^3\mathbf{v}\,\frac{mv^2}{2}\,g(\mathbf{v}) \tag{3.11}$$

We have, using (3.5)

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \sum_{\mathbf{q}}{}' \frac{8\pi e^2}{m^2 q^2} \mathscr{E}_{\mathbf{q}}(t) \int \mathrm{d}^3\mathbf{v} \frac{mv^2}{2} \times \left[\mathbf{q}\cdot\nabla_{\mathbf{v}} \frac{\gamma(\mathbf{q},t)}{(\omega_{\mathbf{q}} - \mathbf{q}\cdot\mathbf{v})^2 + \gamma_{\mathbf{q}}^2} \mathbf{q}\cdot\nabla_{\mathbf{v}} g(\mathbf{v},t)\right] \quad (3.12)$$

From the integral over velocities, there will be two distinct contributions to $\mathrm{d}U/\mathrm{d}t$, according to whether or not one has resonant transfer of energy to plasma waves at $\mathbf{q}\cdot\mathbf{v} \approx \omega_{\mathrm{p}}$.

We consider the resonant contribution first. We then have

$$\left(\frac{\mathrm{d}U}{\mathrm{d}t}\right)_{\mathrm{res}} = \sum_{\mathbf{q}}{}' \frac{4\pi^2 e^2}{m^2 q^2} \mathscr{E}_{\mathbf{q}}(t) \int \mathrm{d}^3\mathbf{v}\, m\, v^2 (\mathbf{q}\cdot\nabla_{\mathbf{v}}) \times [\delta(\omega_{\mathbf{q}} - \mathbf{q}\cdot\mathbf{v})\, \mathbf{q}\cdot\nabla_{\mathbf{v}} g(\mathbf{v},t)] \quad (3.13)$$

If we integrate by parts, we find

$$\left(\frac{\mathrm{d}U}{\mathrm{d}t}\right)_{\mathrm{res}} = -\sum_{\mathbf{q}}{}' \frac{8\pi^2 e^2}{m q^2} \mathscr{E}_{\mathbf{q}}(t) \times \int \mathrm{d}^3\mathbf{v}\, \mathbf{q}\cdot\mathbf{v}\, \delta(\omega_{\mathbf{q}} - \mathbf{q}\cdot\mathbf{v})\, \mathbf{q}\cdot\nabla_{\mathbf{v}} g(\mathbf{v},t) \quad (3.14)$$

The integrated part vanishes since $\mathbf{q}\cdot\nabla_{\mathbf{v}} g(\mathbf{v},t)$ vanishes for the limits at which growing waves no longer occur. We then see, on comparing (3.14) and (3.6) (and making use of the replacement of $\mathbf{q}\cdot\mathbf{v}$ in the integrand by $\omega_{\mathbf{q}}$ according to the properties of the δ function), that

$$\left(\frac{\mathrm{d}U}{\mathrm{d}t}\right)_{\mathrm{res}} = -\sum_{\mathbf{q}} 4\gamma(\mathbf{q},t)\mathscr{E}_{\mathbf{q}} = -2\sum_{\mathbf{q}} \frac{\partial \mathscr{E}_{\mathbf{q}}}{\partial t} \quad (3.15)$$

Thus where the resonances occur, the particle energy is transferred directly to the plasma waves, as we might expect.

There is also the non-resonant contribution to (3.12). The major contribution to this comes from small $\mathbf{v}$. After integrating by parts twice we obtain

$$\left(\frac{\mathrm{d}U}{\mathrm{d}t}\right)_{\text{non-res}} \approx \sum_{\mathbf{q}}{}' \frac{8\pi e^2}{m} \mathscr{E}_{\mathbf{q}}(\mathrm{t})\, \gamma(\mathbf{q},t) \times \overline{\int} \mathrm{d}^3\mathbf{v} \frac{1}{(\omega_{\mathbf{q}} - \mathbf{q}\cdot\mathbf{v})^2 + \gamma_{\mathbf{q}}^2} g(\mathbf{v},t) \quad (3.16)$$

The bar over the integral denotes that principal parts are to be taken, and we neglect a term of order $(qa)^2$. If we now make use of the dispersion relation

$$1 = \frac{4\pi e^2}{m} \overline{\int} \mathrm{d}^3\mathbf{v} \frac{1}{(\omega - \mathbf{q}\cdot\mathbf{v})^2 + \gamma_{\mathbf{q}}^2} g(\mathbf{v},t) \quad (3.17)$$

we see that

$$\left(\frac{\mathrm{d}U}{\mathrm{d}t}\right)_{\text{non-res}} \approx \sum_{\mathbf{q}}{}' 2\gamma(\mathbf{q},t)\mathscr{E}_{\mathbf{q}}(t) = \sum_{\mathbf{q}} \frac{\partial \mathscr{E}_{\mathbf{q}}}{\partial t} \quad (3.18)$$

This result possesses a simple physical interpretation. Those particles with velocities near the phase velocity of the waves give up an energy $2\Sigma_{\mathbf{q}}\mathscr{E}_{\mathbf{q}}$ to the waves. Half of this, $\Sigma_{\mathbf{q}}\mathscr{E}_{\mathbf{q}}$, goes to potential energy and the other half goes into the kinetic energy of oscillation of the bulk of the particles. The over-all energy transfer is therefore given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(U + \sum_{\mathbf{q}}{}' \frac{|E_{\mathbf{q}}|^2}{8\pi}\right) = 0 \quad (3.19)$$

4. Application to one-dimensional case

We now specialize to the one-dimensional case and examine in detail the development of the equilibrium spectrum. We take $g(v,0) = F_0(v)$ to consist of a main part which is a monatonically decreasing function of energy plus a small gentle bump on the tail of the main part (see Fig. 1).

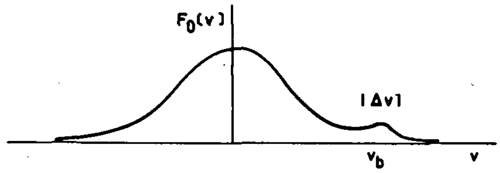

Fig. 1 Initial velocity-distribution function for one-dimensional case has bump on tail of monotonically decreasing main part.

The mean drift velocity of the bump is denoted by v_{b} and the width of the bump by Δv. The number of particles in the bump is denoted by n_{b}, and it is assumed that $n_{\mathrm{b}}/n \ll 3\bar{v}^2(\Delta v)^2/v_{\mathrm{b}}^4$, $\frac{\gamma^{(1)}}{\omega_{\mathrm{p}}}\frac{v_{\mathrm{b}}}{\Delta v} \ll 1$ and $n_{\mathrm{b}} v_{\mathrm{b}}^2 \ll n\bar{v}^2$ where $\bar{v}$ is the root-mean-square velocity of the main part, so that the approximations made in Section 3 are satisfied.

Denoting $\mathscr{E}_q(t)$ by $\mathscr{E}(v,t)$ where $v = \omega_q/q$, Eqs. (3.5) to (3.7) become, for v near v_{b},

$$\frac{\partial \mathscr{E}(v,t)}{\partial t} = \alpha(v)\, \mathscr{E}(v,t) \frac{\partial g(v,t)}{\partial v} \quad (4.1)$$

$$\frac{\partial g(v,t)}{\partial t} = \frac{\partial}{\partial v}\left[\beta(v)\, \mathscr{E}(v,t) \frac{\partial g(v,t)}{\partial v}\right] \quad (4.2)$$

where $\alpha(v) = 4\pi^2 e^2 v^2/m\omega_{\mathrm{p}}$, $\beta = 8\pi L e^2/m^2 v$, and we have neglected $(ka)^2 \approx (\bar{v}/v_{\mathrm{b}})^2 \ll 1$.

The temporal behavior of this pair is described as follows: If $\partial g/\partial v$ is positive at v, then $\mathscr{E}(v)$ increases in time. However $\beta\mathscr{E}(v)$ plays the role of a diffusion coefficient in Eq. (4.2) and hence as $\mathscr{E}(v)$ increases, $g(v)$ diffuses in such a way as to reduce $\partial g/\partial v$ at v. Thus the behavior of the pair of equations is such as to limit the amplitude of $\mathscr{E}(v)$.

To determine the resulting equilibrium spectrum we combine Eqs. (4.1) and (4.2) to obtain

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial v}\beta\mathscr{E}\frac{\partial g}{\partial v} = \frac{\partial}{\partial v}\frac{\beta}{\alpha}\frac{\partial \mathscr{E}}{\partial t}$$

and therefore

$$\frac{\partial}{\partial t}\left(g - \frac{\partial}{\partial v}\frac{\beta}{\alpha}\mathscr{E}\right) = 0 \quad (4.3)$$

We assume that $\frac{\beta}{\alpha}\frac{\partial \mathscr{E}}{\partial v}$ is negligible at $t = 0$ and thus

$$\frac{\beta}{\alpha}\frac{\partial \mathscr{E}}{\partial v} = g(v) - F_0(v) \quad (4.4)$$

We seek a solution as $t \to \infty$ for which $\partial \mathscr{E}/\partial t = \partial g/\partial t = 0$. This is given by

$$\frac{\beta}{\alpha}\mathscr{E}_\infty(v) = \int_{v_0}^{v} [g_\infty - F_0(v)]\,\mathrm{d}v, \qquad v_0 < v < v_1$$
$$= 0 \qquad v < v_0;\ v > v_1 \tag{4.5}$$

where g_∞ is a constant which together with v_0 and v_1 is determined by

$$\int_{v_0}^{v_1} g_\infty\,\mathrm{d}v = g_\infty(v_1 - v_0) = \int_{v_0}^{v_1} F_0(v)\,\mathrm{d}v \tag{4.6}$$

This result is illustrated in Fig. 2. It is worth noting that the equilibrium spectrum is independent of the initial data, provided only that the initial data are smooth enough so that the sum in Eq. (3.9) can be evaluated by replacing $\gamma/(\omega_q - qv)^2 + \gamma^2$ by $\pi\delta(\omega_q - qv)$. From Fig. 2 we also note that the energy given up to the plasma waves is of the order of $|g_\infty - F_0(v)|_{\max}$ for $v_0 < v < v_1$ times $mv_b\Delta v$ and is thus much less than the drift energy of the particles in the bump.

The development of this equilibrium spectrum in time has been calculated numerically for a typical

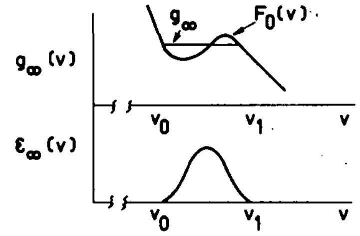

Fig. 2 Equilibrium velocity distribution and electric field spectrum from one-dimensional calculation.

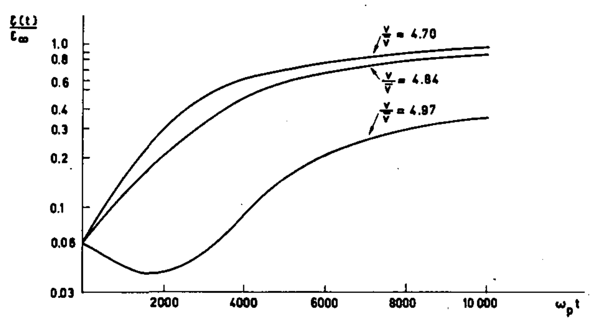

Fig. 3 Development in time of $\mathscr{E}(v)$ for several values of v.

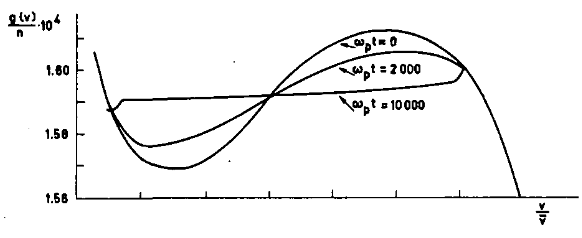

Fig. 4 Diffusion of $g(v)$.

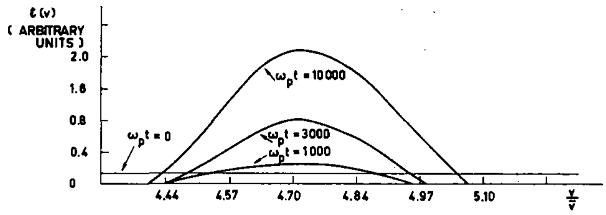

Fig. 5 Spectrum of $\mathscr{E}(v)$ at different times.

case and is illustrated in Figs. 3, 4, and 5. $F_0(v)$ was taken to be

$$\frac{n}{(2\pi)^{1/2}\bar{v}}\left\{\exp\left[-\frac{1}{2}\left(\frac{v}{\bar{v}}\right)^2\right] + 4\times 10^{-4}\exp\left[-\frac{1}{2}\left(\frac{v}{\bar{v}} - 5\right)^2\right]\right\}$$

and the initial data were taken to be constant in the region of interest and of such a magnitude that equilibrium was reached after a few e-folding times. Fig. 3 shows the development in time of $\mathscr{E}(v)$ for several values of v, and we note that the linear theory is valid for small times even for the relatively large initial data used. Figure 5 shows the spectrum of $\mathscr{E}(v)$ at different times while Fig. 4 shows the accompanying diffusion of $g(v)$.

From Eq. (4.5) we can estimate the amplitude of the equilibrium spectrum. We have

$$\mathscr{E}_\infty \approx \frac{\alpha}{\beta}(g_\infty - F_0)\Delta v \approx \frac{\alpha}{\beta}\frac{\partial F_0}{\partial v}(\Delta v)^2 \approx \frac{(\Delta v)^2}{\beta}\gamma^{(1)}$$

and the total electrostatic energy in the equilibrium spectrum is approximated by

$$\sum_q \mathscr{E}_q = \frac{2L}{2\pi}\int_0^\infty \mathrm{d}_q \mathscr{E}_q \approx \frac{L}{\pi}\mathscr{E}_q \Delta_q$$
$$\approx \frac{1}{2\pi}\left(\frac{\Delta v}{v_b}\right)\left(\frac{\Delta v}{\bar{v}}\right)^2 \frac{\gamma^{(1)}}{\omega_p} n m \bar{v}^2 \ll \frac{n m \bar{v}^2}{2}$$

Thus the amplitude of the equilibrium spectrum is in fact of order $\gamma^{(1)}/\omega_p$, which is extremely small. As discussed in the next section, the mode-coupling terms that affect the equilibrium spectrum are of order $\mathscr{E}_q$, $\mathscr{E}^2$, etc., compared to the terms of the non-linear dispersion relation. Thus we may view the non-linear dispersion relation as the lowest order result of an expansion in powers of $\gamma^{(1)}/\omega_p$, and the mode-coupling terms can be safely treated by perturbation-theoretical methods.

5. Mode coupling

We now wish to consider for the one-dimensional case the effect of the non-linear mode-coupling terms in Eq. (3.1) on the equilibrium spectrum. We first note that the equilibrium spectrum is relatively narrow and f_q and E_{k-q} are large only if $|q|$, $|k - q| \approx |k_0|$, where k_0 denotes the center of the spectrum. It follows that k must lie near 0 or $\pm 2k_0$. Thus to second order (in E) the mode coupling terms lead to waves near

$k=0$ or $\pm 2k_0$, and to no change in the amplitude near k_0. Moreover, since the natural frequency of these waves $\omega_k \approx \omega_p$ is very different from $\omega_{k-q}+\omega_q \approx 0$ or $2\omega_p$, there are no "time proportional" transitions to these waves. In third order (in E) we find that the mode-coupling terms introduce an additional time dependence of the waves near k_0 which is given by

$$\frac{\partial E_k}{\partial t} = -\frac{\mathrm{i}\,\omega_k}{2}\sum_{q,q'}{}' H(k,q,q')\,E_{k-q}(t)\,E_{q-q'}(t')\,E_{q'}(t) \qquad q \ll k_0 \qquad (5.1)$$

The prime on the sum denotes that the terms with $q=0$ are to be deleted. In what follows only Im $[H(k,q,k)+H(k,q,q-k)]$ for $q \ll k_0$ is needed and this is given, after some tedious algebra, by

$$\mathrm{Im}\,[H(k,q,k)+H(k,q,q-k)] = 9\left(\frac{k}{q}\right)^2 \frac{k^2}{\omega_p^4}\left(\frac{e}{m}\right)^2\left(1-\frac{q}{k}\right)^2 \mathrm{Im}\,\frac{1}{\varepsilon(q,\omega_{q-q'}+\omega_{q'})} \qquad q \ll k_0$$

$$= \frac{9\pi k^4}{\omega_p^2}\left(\frac{e}{m}\right)^2 \frac{\left(1-\frac{q}{k}\right)^2}{q^3|q|}\frac{1}{|\varepsilon|^2}\frac{\partial q}{\partial v}\bigg|_{v=v_0} \qquad (5.2)$$

where $v_0 = 3|k|[1-(2q/k)]\,a\bar{v} \ll \bar{v}$.

We consider first the terms in Eq. (4.1) that are in phase with E_k. These have $q'=k$ or $q-q'=k$, and it is only the imaginary parts of these that give a change in amplitude. Thus

$$\frac{\partial E_k}{\partial t} = \frac{\omega_k}{2}\,\mathrm{Im}\sum_q [H(k,q,k)+H(k,q,q-k)]\,E_k|E_{q-k}|^2 \qquad q \ll k_0 \qquad (5.3)$$

and the in-phase coupling terms yield

$$\frac{\partial |E_k|^2}{\partial t} = |E_k|^2\,\omega_k\,9\pi k^4\left(\frac{e}{m}\right)^2\frac{1}{\omega_p^2} \times \sum_q \frac{\left(1-\frac{q}{K}\right)^2}{q^5}\frac{|q|}{|\varepsilon|^2}\frac{\partial q}{\partial v}\bigg|_{v_0}|E_{k-q}|^2 \qquad (5.4)$$

$$q \ll k_0$$

This term arises from the fact that in second order in E the waves interact with the distribution function at $v=v_0$. For $k \geq k_0$ this interaction is such as to damp the waves while for $k \leq k_0$ this interaction leads to growth. However, it is easy to show that $\Sigma_k \partial|E_k|^2/\partial t = 0$ and thus there is no net transfer of energy between the particles and the waves, and the interaction simply distorts the equilibrium spectrum towards the lower k values. It should be noted that formally this "in-phase" mode coupling is of the same order in γ/ω_p as terms in the non-linear dispersion relation and might have been considered as part of the non-linear dispersion relation. However, for the problem considered it is numerically small and can thus be treated as a perturbation.

To estimate the size of this interaction we note that

$$\frac{1}{|\varepsilon|^2} \approx (qa)^4 \text{ and } \left(\frac{\partial f}{\partial v}\right)_{v_0} \propto -\frac{3ka}{\bar{v}^2}$$

Thus

$$\left|\frac{\partial \mathscr{E}}{\partial t}\right| \approx \left|\frac{27\pi(ka)^5}{v^2}\left(\frac{e}{m}\right)^2\frac{1}{\omega_p^2}\frac{L}{2\pi}\int \mathrm{d}q\,\frac{q}{|q|}\left(1-\frac{q}{k}\right)^2 \times (E_{k-q})^2\right| \mathscr{E}_q \approx 2\gamma^{(1)}\mathscr{E}_q\left[\frac{27}{2}(ka)^5\left(\frac{\Delta v}{\bar{v}}\right)^2 M\right] \qquad (M<1) \qquad (5.5)$$

Thus the in-phase mode-coupling terms have a time scale long compared to that of the initial growth. For the example considered in Section 4, it is longer by a factor of about 10^3.

The out-of-phase terms in Eq. (4.1) lead to time-proportional transitions and can be treated by the familiar methods of quantum mechanics. For these terms we obtain for the transitions to the kth mode

$$|E(k,t+\tau)|^2 - |E(k,t)|^2 = \left(\frac{\omega_K}{2}\right)^2 \sum_{\substack{q,q'\\p,p'}} H(k,q,q')\,H^*(k,p,p') \times \left(\frac{\exp[\mathrm{i}(\omega_k-\omega_{k-q}-\omega_{q-q'}-\omega_{q'})\tau]-1}{\mathrm{i}(\omega_k-\omega_{k-q}-\omega_{q-q'}-\omega_{q'})}\right) \times \left(\frac{\exp[\mathrm{i}(\omega_k-\omega_{k-p}-\omega_{p-p'}-\omega_{p'})\tau]-1}{\mathrm{i}(\omega_k-\omega_{k-p}-\omega_{p-p'}-\omega_{p'})}\right) \times \mathfrak{E}_{k-q}\,\mathfrak{E}_{q-q'}\,\mathfrak{E}_{q'}\,\mathfrak{E}_{k-p}\,\mathfrak{E}_{p-p'}\,\mathfrak{E}_{p'} \qquad (5.6)$$

where the * denotes complex conjugate and $E_{k-q} = \mathfrak{E}_{k-q}\exp -\mathrm{i}\omega_{k-q}t$, $E_{q-q'} = \mathfrak{E}_{q-q'}\exp -\mathrm{i}\omega_{q-q'}t$, etc.

We assume that the phases of the initial data are random and since no phase correlation is introduced by the non-linear dispersion relation the only terms which survive the q, q', p, p' summation are those for which the phases cancel. The result is

$$|E(k,t'+\tau)|^2 - |E(k,t)|^2 = \left(\frac{\omega_k}{2}\right)^2\sum_{q,q'}|E_{K-q}|^2|E_{q-q'}|^2|E_{q'}{}^2| \times \left|\frac{\exp[\mathrm{i}(\omega_k-\omega_{k-q}-\omega_{q-q'}-\omega_{q'})\tau]-1}{\mathrm{i}(\omega_k-\omega_{k-q}-\omega_{q-q'}-\omega_{q'})}\right|^2 \times H^*(k,q,q')\{H^*(k,q,q') + H^*(k,q,q-q') + H^*(k,k+q'-q,q) + H^*(k,k+q'-q,k-q) + H^*(k,k-q',q-q') + H^*(k,k-q',k-q)\} \qquad (5.7)$$

The term

$$\times\left|\frac{\exp[\mathrm{i}(\omega_k-\omega_{k-q}-\omega_{q-q'}-\omega_{q'})\tau]-1}{\mathrm{i}(\omega_k-\omega_{k-q}-\omega_{q-q'}-\omega_{q'})}\right|^2$$

has resonances at $q'=k$, $q-k$, for $q \ll k_0$, $q \approx 2k$ and at $q=0$ for $q' \approx \pm k_0$. Evaluating the q, q' sum at these resonances thus yields for the transition rate to the kth mode

$$\left(\frac{\partial |E_k|^2}{\partial t}\right)_{\lambda \to K} = \frac{\omega_p L}{12 a^2} |E_k|^2 \sum_{\lambda} \frac{|E_\lambda|^4}{|k-\lambda|} |\mathscr{H}_{k,\lambda}|^2 \quad (\lambda \approx k) \tag{5.8}$$

where

$$\mathscr{H}_{k,\lambda} = H(k, k-\lambda, k) + H(k, k-\lambda, -\lambda) + H(k, k+\lambda, k) + H(k, k+\lambda, \lambda) + H(k, 0, -\lambda) + H(k, 0, \lambda) \tag{5.9}$$

Similarly the transitions from the kth mode to all other modes are given by

$$\left(\frac{\partial |E_k|^2}{\partial t}\right)_{K \to \lambda} = \frac{\omega_p L}{12 a^2} |E_k|^4 \sum_{\lambda} \frac{|E_\lambda|^2 |\mathscr{H}_{\lambda k}|^2}{|k-\lambda|} \quad (\lambda \approx k) \tag{5.10}$$

and the net change of the kth mode is given by

$$\frac{\partial |E_k|^2}{\partial t} = \frac{\omega_p L}{12 a^2} |E_k|^2 \sum_{\lambda} \frac{|E_\lambda|^4 |\mathscr{H}_{k\lambda}|^2 - |E_k|^2 |E_\lambda|^2 |\mathscr{H}_{\lambda k}|^2}{|k-\lambda|} \quad \lambda \approx k \tag{5.11}$$

This term arises from the scattering of waves from each other and we note that for those k's for which $|E_k|^2$ is less than the average $\partial |E_k|^2/\partial t > 0$ while for k's near the peak of the spectrum $\partial |E_k|^2/\partial t < 0$. Thus the scattering of waves tends to flatten the spectrum.

$\mathscr{H}_{k\lambda}$ is relatively complicated but since this term turns out to be extremely small the exact form of $\mathscr{H}_{k\lambda}$ is unimportant. However, it is worth noting that the leading term of $\mathscr{H}_{k\lambda}$ is proportional to (k^2/ω_p^4) $(e/m)^2$ $(ka)^2$.

Estimating $\partial \mathscr{E}/\partial t$ we obtain

$$\left|\frac{\partial \mathscr{E}}{\partial t}\right| \approx 2 \gamma \mathscr{E} \frac{\gamma}{\omega} (k a)^6 \left(\frac{\Delta v}{V}\right)^4 M \tag{5.12}$$

Thus, the out-of-phase mode-coupling terms have a time scale longer than that of the initial growth. For the example of Section 4, it is longer by a factor of about 10^8, and we can neglect the out-of-phase-mode-coupling terms compared to in-phase terms.

The damping due to the in-phase mode-coupling terms arises from the distortion of the spectrum towards the lower values of k. This feeds energy into those modes that have phase velocities greater than v_1 and hence for which $\partial g/\partial v < 0$. However since these modes are naturally damped, their amplitude will be small and the rate of energy transfer will be correspondingly small. On the other hand the Fokker-Planck terms, which we have neglected, may have a time scale comparable to that of the in-phase mode-coupling terms and these will tend to drive $\partial f/\partial v$ negative over the entire range of interest, and the ultimate decay may actually be dominated by collisions.

6. Conclusions

The basic requirements for validity of the theory described above are two fold. First, the solution of the linearized problem must yield a growth rate γ, which depends on the velocity gradient of the distribution function and $\gamma/\omega \ll 1$. This is necessary in order that the non-linear dispersion relation lead to a diffusion of the distribution function. Second, the dispersion relation is such that neither the sum nor difference of the frequencies of two of the unstable modes is equal to the natural frequency of another mode. It is this requirement that keeps the energy of the unstable modes from being fed into harmonics.

These restrictions apply to a large class of waves in a plasma with a magnetic field and the method can be generalized in a straightforward way to these waves. The result as in the case illustrated above will be a non-linear dispersion relation that leads to the establishment of an equilibrium spectrum of waves. This result represents a special type of turbulence in which the wave spectrum is confined to a relatively narrow band of wave-lengths.

Although there are no "time-proportional" transitions to waves outside of the equilibrium spectrum, the second order interaction between waves in the equilibrium spectrum do produce a stationary spectrum of very-small-amplitude [$\mathscr{E}_q$ of order $(\gamma/\omega)^2$] waves near $q=0$. These waves will interact with the bulk of the particles and produce a diffusion of the bulk of the particles, an effect which is important for the case of a plasma with a magnetic field.

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Appendix

We wish to show that for $\gamma/\omega \ll 1$, Eqs. (3.3), (3.4) and (2.9) reduce to Eqs. (3.5), (3.6) and (3.7).

Integrating Eq. (3.3) along the unperturbet orbits yields

$$f_{\mathbf{K}} = \frac{e}{m} \int_{-\infty}^{t} \mathrm{d}t' \, G_{\mathbf{K}}(t-t') \, \mathbf{E}_{\mathbf{K}}(t') \cdot \nabla_{\mathbf{v}} g(t') \tag{A1}$$

where we have set the lower limit to $-\infty$ and neglected $f_K(v, 0)$—a good approximation for growing waves. Integrating over velocities then yields

$$E_{\mathbf{K}}(t) = -\frac{4\pi e^2}{i m K} \int \mathrm{d}^3 \mathbf{v} \int_{-\infty}^{t} \mathrm{d}t' \, G_{\mathbf{K}}(t-t') \, \mathbf{E}_{\mathbf{K}}(t') \cdot \nabla_{\mathbf{v}} g(t') \tag{A2}$$

We expect the solution of Eq. (A2) to have the same general form as the solution of the linearized problem except that the frequency and growth rate will be slowly varying functions of time. We then take $E_{\mathbf{K}}(t')$ to be of the form

$$\begin{aligned} \mathbf{E}_{\mathbf{K}}(t') &= E_{\mathbf{K}}(0)\exp\left\{\int_0^{t'} S_{\mathbf{K}}(\tau)\,\mathrm{d}\tau\right\} \\ &= E_{\mathbf{K}}(t)\exp\left\{-\int_{t'}^{t} S_{\mathbf{K}}(\tau)\,\mathrm{d}\tau\right\} \\ &= E_{\mathbf{K}}(t)\exp\{-S_{\mathbf{K}}(t)(t-t')\} \\ &\quad\times\left\{1+\frac{1}{2}\frac{\partial S_k}{\partial t}(t-t')^2+\dots\right\} \end{aligned} \tag{A3}$$

and $g(v, t') = F_0(v) + f_0(v, t) + (\partial f_0/\partial t)(t'-t) + \dots$

We shall assume and verify later that $S_{\mathbf{K}}(\tau)$ changes slowly in a period of oscillation and that the change in frequency and of growth rate are of the same order of magnitude. In particular we make use of the result, Eq. (A8), that $S_{\mathbf{K}}—S_{\mathbf{K}_0}/S_{\mathbf{K}}$ depends linearly on $f_0(v, t)$ and that $\partial f_0/\partial t \approx 2\gamma f_0$ so that

$$\frac{1}{S_{\mathbf{K}}^2}\frac{\partial S_{\mathbf{K}}}{\partial t} \approx \frac{2\gamma}{S_{\mathbf{K}}}\frac{S_{\mathbf{K}}—S_{\mathbf{K}_0}}{S_{\mathbf{K}_0}} \ll \frac{S_{\mathbf{K}}—S_{\mathbf{K}_0}}{S_{\mathbf{K}}} \tag{A4}$$

where $S_{\mathbf{K}_0}$ is the solution of the linearized problem.

Using Eqs. (A3) and (A4), Eq. (A2) yields

$$\begin{aligned} E_{\mathbf{K}}(t) = E_{\mathbf{K}}(t)\Big\{1 &- \frac{1}{2}\frac{\partial S_{\mathbf{K}}}{\partial t}\frac{\partial^2}{\partial S_{\mathbf{K}}^2}+\dots\Big\} \\ &\times\left\{\frac{4\pi e^2}{mK^2}\int \mathrm{d}^3\mathbf{v}\,\frac{\mathrm{i}\mathbf{K}\cdot\nabla_{\mathbf{v}}}{S_{\mathbf{K}}(t)+\mathrm{i}\mathbf{K}\cdot\mathbf{v}}[F_0+f_0]\right. \\ &\left.+\frac{4\pi e^2}{mK^2}\frac{\partial}{\partial S_{\mathbf{K}}}\int \mathrm{d}^3\mathbf{v}\,\frac{\mathrm{i}\mathbf{K}\cdot\nabla_{\mathbf{v}}}{(S_K+\mathrm{i}\mathbf{K}\cdot\mathbf{v})}\frac{\partial}{\partial t}f_0(t)\right\} \end{aligned} \tag{A5}$$

The term involving $F_0(v)$ has the same form as in the linearized theory and thus we have

$$\begin{aligned} E_{\mathbf{K}}(t)\Big\{1-\frac{1}{2}\frac{\partial S_{\mathbf{K}}}{\partial t}\frac{\partial^2}{\partial S_{\mathbf{K}}^2}+\dots\Big\}\varepsilon[\mathbf{K}, S_{\mathbf{K}}(t)] \\ = \Big\{1-\frac{1}{2}\frac{\partial S_{\mathbf{K}}}{\partial t}\frac{\partial^2}{\partial S_{\mathbf{K}}}\Big\}\frac{4\pi e^2}{mK^2}\int \mathrm{d}^3\mathbf{v}\,\frac{\mathrm{i}\mathbf{K}\cdot\nabla_{\mathbf{v}}f_0}{S_{\mathbf{K}}(t)+\mathrm{i}\mathbf{K}\cdot\mathbf{v}} \\ +\frac{4\pi e^2}{mK^2}\frac{\partial}{\partial S_{\mathbf{K}}}\int \mathrm{d}^3\mathbf{v}\,\frac{\mathrm{i}\mathbf{K}\cdot\nabla_{\mathbf{v}}}{S_{\mathbf{K}}(t)+\mathrm{i}\mathbf{K}\cdot\mathbf{v}}\frac{\partial f_0}{\partial t}+\dots \end{aligned} \tag{A6}$$

Now $(S_{\mathbf{K}}(t)—S_{\mathbf{K}_0})/S_{\mathbf{K}_0} \ll 1$ and thus we can expand $\varepsilon[\mathbf{K}, S_{\mathbf{K}}(t)]$ about $S_{\mathbf{K}_0}$ to obtain

$$\begin{aligned} \varepsilon[\mathbf{K}, S_{\mathbf{K}}(t)] &= \varepsilon(\mathbf{K}, S_{\mathbf{K}_0}) + \frac{\partial\varepsilon}{\partial S_{\mathbf{K}_0}}(S_{\mathbf{K}}-S_{\mathbf{K}_0})+\dots \\ &\approx \frac{\partial\varepsilon}{\partial S_{\mathbf{K}_0}}(S_{\mathbf{K}}-S_{\mathbf{K}_0}) \approx -2\frac{\omega_p^2}{S_{\mathbf{K}_0}^3}[S_{\mathbf{K}}(t)-S_{\mathbf{K}_0}] \\ &\approx 2\frac{[S_{\mathbf{K}}(t)-S_{\mathbf{K}_0}]}{S_{\mathbf{K}_0}} \end{aligned}$$

since $\varepsilon(\mathbf{K}, S_{\mathbf{K}_0}) = 0$. The correction term on the left-hand side of Eq. (A6) is, since $\varepsilon \approx 1+\omega_p^2/S_{\mathbf{K}}^2$, of the order of

$$\frac{\partial S_{\mathbf{K}}}{\partial t}\,3\frac{\omega_p^2}{S_{\mathbf{K}}^4} \approx \frac{3}{\omega_p^2}\frac{\partial S_{\mathbf{K}}}{\partial t} \ll \frac{S_{\mathbf{K}_0}-S_{\mathbf{K}}}{S_{\mathbf{K}_0}}$$

and can thus be neglected.

$f_0(v, t)$ is non zero only in the small neighborhood Δv. Defining (for simplicity we consider only the one-dimensional case)

$$I = \int\frac{\mathrm{d}v\,(\partial f_0/\partial v)}{S_K(t)+iK\cdot v} = \mathrm{i}\,P\int\frac{(\partial f_0/\partial v)\,\mathrm{d}v}{(\omega-Kv)} + \frac{\pi}{K}\frac{\partial f_0}{\partial v}\bigg|_{v=\omega_K/K} \tag{A7}$$

$\partial^2 I/\partial S^2$ is thus of the order of $I/(K\Delta v)^2$ and the first correction term on the right-hand side of Eq. (A6) is of order

$$I\frac{\partial S}{\partial t}\frac{1}{(K\Delta v)^2} = I\left(\frac{v_b}{\Delta v}\right)^2\frac{\partial S}{\partial t}\frac{1}{(Kv_b)^2} \ll \frac{S_K-S_{K_0}}{S_K}$$

where $Kv_b \approx \omega_p$. Similarly since $\partial f_0/\partial t \approx 2\gamma f_0$, the second correction term on the right-hand side of Eq. (A6) can be neglected.

The result, neglecting $(Ka)^2 \ll 1$, is

$$S_K(t) - S_{K_0} = \frac{2\pi e^2}{mK}\omega_{K_0}\left\{\frac{\pi}{K}\frac{\partial f_0}{\partial v}\bigg|_{v=\omega_K/K} + \mathrm{i}\,P\int\frac{\mathrm{d}v\,\partial f_0/\partial v}{(\omega-Kv)}\right\} \tag{A8}$$

Using $S_{\mathbf{K}}(t) = -\mathrm{i}\,\omega_{\mathbf{K}}(t)+\gamma_{\mathbf{K}}(t)$ we obtain

$$\omega_K(t) = \frac{K}{|K|}\omega_{K_0}\left[1-\frac{2\pi e^2}{mK}P\int\frac{\partial f_0/\partial v\,\mathrm{d}v}{(\omega-Kv)}\right] \tag{A9}$$

$$\frac{\gamma_K}{\omega_K} = \frac{2\pi^2 e^2}{K^2 m}\frac{\partial g(v,t)}{\partial v}\bigg|_{v=\omega_K/\omega}$$

and the three-dimensional result is given by Eq. (3.6).

The change in $\omega_{\mathbf{K}}(t)$ is of the same order as the change in $\gamma_{\mathbf{K}}$ and since $\gamma_{\mathbf{K}}/\omega_{\mathbf{K}} \ll 1$ we can neglect the correction to $\omega_{\mathbf{K}}$. The change in $\gamma_{\mathbf{K}}$ is however significant.

The energy in the Kth mode thus grows according to

$$\frac{\partial|E_K|^2}{\partial t} = 2|E_K|^2\gamma_K(t) \tag{A10}$$

In the same way Eq. (A1) can be integrated to obtain

$$f_K = -\frac{e}{m}\frac{E_K}{S_K+\mathrm{i}Kv}\frac{\partial g}{\partial v} \tag{A11}$$

and thus

$$\frac{\partial g}{\partial t} = \left(\frac{e}{m}\right)^2\frac{\partial}{\partial v}\sum_{q=-\infty}^{\infty}\frac{|E_q|^2}{(Sq+\mathrm{i}qv)}\frac{\partial g}{\partial v} \tag{A12}$$

and the three-dimensional result is given by Eq. (3.5).