Problem Set 1

1

Getting Started!

1.1 Required: Due Fri Week 4, 11 pm

Preliminary Note:

The definition of the Kauffman Bracket invariant differs slightly depending on which reference you read. For the following problems please use this definition:

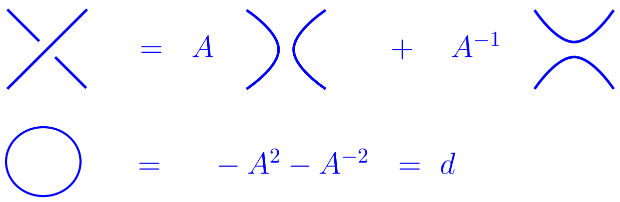


Fig. 1.1 Rules for evaluating the Kauffman bracket invariant. Note: This definition agrees with the 2022 version of the Book. The 2021 version exchanges A with A^{-1}.

Exercise 2.1 from Book
Trefoil Knot and the Kauffman Bracket

Using the Kauffman rules, calculate the Kauffman bracket invariant of the right- and left-handed trefoil knots shown in Fig. 1.2. Conclude these two knots are topologically inequivalent. While this statement appears obvious on sight, it was not proved mathematically until 1914 (by Max Dehn). It is easy using this technique!

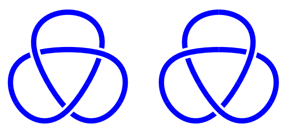


Fig. 1.2 Left- and Right-Handed Trefoil Knots (on the left and right respectively)

Exercise 2.2 from Book
Abelian Kauffman Anyons

Particles where the quantum amplitudes of their trajectories are given by the Kauffman bracket invariant with certain special values of the constant A are abelian anyons — meaning an exchange introduces only a simple phase as shown in Fig. 1.3. Here we mean that the vertical direction now means time and the knot or link describes the motion of particles in space-time.

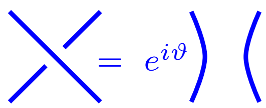


Fig. 1.3 For abelian anyons, exchange gives a phase $e^{i\vartheta}$.

(a) For $A = \pm e^{i\pi/3}$, show that the anyons are bosons or fermions respectively (i.e., $e^{i\vartheta} = \pm 1$). Further show that for these values of A any diagram calculated with A gives exactly the same result if you use the complex conjugate of A instead.

(b) For $A = \pm e^{i\pi/6}$ show the anyons are semions (i.e., $e^{i\vartheta} = \pm i$). Further show that calculating a diagram using $A = \pm e^{i\pi/6}$ gives exactly the same value as calculating the diagram using $A = \mp e^{-i\pi/6}$.

HINT: For both (a) and (b) show first the identity shown in Fig. 1.4. If you can't figure it out, try evaluating the Kauffman bracket invariant for a

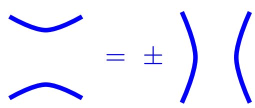


Fig. 1.4 For bosons or fermions the sign in this figure is $+$, for semions the sign is $-$.

few knots with these values of A and see how the result arises.

Case (b) corresponds to the anyons that arise for the $\nu = 1/2$ fractional quantum Hall effect of bosons. This particular phase of quantum Hall matter has been produced experimentally, but only in very small puddles so far and it has not been possible to measure braiding statistics as of yet.

Exercise 3.3 a,b from Book
Ising Anyons and Majorana Fermions

The most commonly discussed type of nonabelian anyon is the Ising anyon. Ising anyons occurs in the Moore-Read quantum Hall state ($\nu = 5/2$), as well as in any chiral p-wave superconductor and in recently experimentally relevant so called "Majorana" systems.

The nonabelian statistics of these anyons may be described in terms of Majorana fermions by attaching a Majorana operator to each anyon. The Hamiltonian for these Majoranas is zero — they are completely noninteracting.

In case you haven't seen them before, Majorana Fermions γ_j satisfy the anticommutation relati on

$$\{\gamma_i, \gamma_j\} \equiv \gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} \tag{1.1}$$

as well as being self conjugate $\gamma_i^\dagger = \gamma_i$.

(a) Show that the ground-state degeneracy of a system with $2N$ Majoranas is 2^N if the Hamiltonian is zero. Thus conclude that each *pair* of Ising anyons is a two-state system. Hint: Construct a regular (Dirac) fermion operator from two Majorana fermion operators. For example,

$$c^\dagger = \frac{1}{2}(\gamma_1 + i\gamma_2)$$

will then satisfy the usual fermion anti-commutation $\{c, c^\dagger\} = cc^\dagger + c^\dagger c = 1$. (If you haven't run into fermion creation operators yet, you might want to read up on this first!)

(b) When anyon i is exchanged clockwise with anyon j, the unitary transformation that occurs on t he ground state is

$$U_{ij} = \frac{e^{i\alpha}}{\sqrt{2}}\left[1 + \gamma_i \gamma_j\right] \qquad i < j. \tag{1.2}$$

for some real value of α. Show that these unitary operators form a representation of the braid group. In other words we must show that replacing σ_i with $U_{i,i+1}$ in the following two equations below yields equalities.

$$\sigma_i\,\sigma_{i+1}\,\sigma_i \quad = \quad \sigma_{i+1}\,\sigma_i\,\sigma_{i+1} \tag{1.3}$$

$$\sigma_i\,\sigma_j \quad = \quad \sigma_j\,\sigma_i \qquad \text{for} \qquad |i-j| > 1 \tag{1.4}$$

This representation is 2^N-dimensional since the ground-state degeneracy is 2^N.

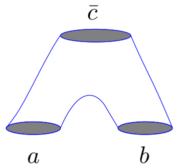


Fig. 1.5 A three-times punctured sphere is known as a "pair of pants".

Exercise 8.2a,b,c from Book.
Fusion and Ground-State Degeneracy

To determine the ground-state degeneracy of a 2-manifold in a (2+1)-dimensional TQFT one can cut the manifold into pieces and glue back together. One can think of the open "edges" or connecting tube-ends as each having a label given by one of the particle types (i.e., one of the anyons) of the theory. Really we are labeling each edge with a basis element of a possible Hilbert space. The labels on two tubes that have been connected together must match (label a on one tube fits into label $\bar{a}$ on another tube.) To calculate the ground-state degeneracy we must keep track of all possible ways that these assembled tubes could have been labeled. For example, when we assemble a torus from a cylinder, we must match the quantum number on one open end to the (opposite) quantum number on the opposite open end. The ground-state degeneracy is then just the number of different possible labels, or equivalently the number of different particle types.

For more complicated 2-d manifolds, we can decompose the manifold into so-called pants diagrams (see Fig. 1.5). When we sew together pants diagrams, we should include a factor of the fusions multiplicity N_{ab}^{c} for each pants which has its three tube edges labeled with a, b and $\bar{c}$.

(a) Show that the general formula for the ground-state degeneracy of an g-handled torus in terms of the N matrices can be written as follows

$$\mathrm{Dim}V(g\text{-handled torus}) = \mathrm{Tr}[M^{g-1}] \quad \text{where} \quad M_{cd} = \sum_{a,b} N_{ab}^{\bar{c}} N_{\bar{a}\bar{b}}^{d} \tag{1.5}$$

where the sum over a and b are over all particle types (including the identity).

(b) For the Fibonacci anyon model, find the ground-state degeneracy of a four-handled torus.

(c) Show that in the limit of large number of handles g the ground-state degeneracy scales as $\sim \mathcal{D}^{\,2g}$ where $\mathcal{D}^2 = \sum_a \mathsf{d}_a^2$.

Exercise 9.4 from Book
Fibonacci Pentagon

In the Fibonacci anyon model, there are two particle types which are usually called I and τ. The only nontrivial fusion rule is $\tau \times \tau = I + \tau$. With these fusion rules, the F-matrix is completely fixed up to a gauge freedom (corresponding to adding a phase to some of the kets). If we choose all elements of the F-matrix to be real, then the F-matrix is completely determined by the pentagon equations up to one sign (gauge) choice. Using the pentagon equation determine the F-matrix.

If you are stuck as to how to start, part of the calculation is given in Nayak et al Rev Mod Phys Review article from 2008.

1.2 Optional Problems (Easy)

[These problems will be discussed in class only if you find them hard.]

Exercise 4.1 from Book
Abelian Anyon Vacuum on a Two-Handle Torus
Show that the ground-state vacuum degeneracy on a two handle torus is m^2 for a system of abelian anyons with statistical angle $\vartheta = \pi p/m$ for integers p and m relatively prime. Hint: Consider what the independent nontrivial cycles i are on a two-handled torus and determine the commutation relations for operators T_i that create an anyon-antianyon pairs, takes one of the particles around cycle i and then reannhilate.

Exercise 8.3 from Book
Consistency of Fusion Rules
Show by using commutativity and associativity of fusion along with identity $N_{ab}^{c} = N_{\bar{a}\bar{b}}^{\bar{c}}$ that no anyon theory can have a particle a different from the vacuum I such that $a \times a = a$, meaning a fuses with a to form only a and nothing else.

Exercise 9.1 from Book
***F* Gauge Choice**
(a) Explain why in the Fibonacci theory, $[F_{\tau}^{\tau\tau\tau}]_{\tau\tau}$ is gauge independent but $[F_{\tau}^{\tau\tau\tau}]_{I\tau}$ is gauge dependent.
(b) Explain why in the Ising theory $[F_{\sigma}^{\psi\sigma\psi}]_{\sigma\sigma}$ is gauge independent, but $[F_{\psi}^{\sigma\psi\sigma}]_{\sigma\sigma}$ is gauge dependent.

Exercise 9.2 from Book *F*'s With the Vacuum Field *I*
Explain why $[F_{e}^{aIc}]_{ac} = [F_{d}^{abI}]_{db} = [F_{e}^{Ibc}]_{be} = 1$.

1.3 Optional Problems (The Best Ones)

[We hope to discuss these in class.]

Exercise 2.3 from Book
Reidemeister moves and the Kauffman Bracket
Show that the Kauffman bracket invariant is unchanged under application of Reidemeister move of type II and type III. Thus conclude that the Kauffman invariant is an invariant of regular isotopy.

Exercise 3.1 from Book
About the Braid Group
(a) Convince yourself geometrically that the defining relations of the braid group on M particles B_M are:

$$\sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1} \qquad 1 \le i \le M-2 \tag{1.6}$$
$$\sigma_i\,\sigma_j = \sigma_j\,\sigma_i \quad \text{for} \quad |i-j| > 1, \qquad 1 \le i,j \le M-1 \tag{1.7}$$

(b) Instead of thinking about particles on a plane, let us think about particles on the surface of a sphere. In this case, the braid group of M strands on the sphere is w ritten as $B_M(S^2)$. To think about braids on a sphere, it is useful to think of time as being the radial direction of the sphere, so that braids are drawn as in Fig. 1.6.

The braid generators on the sphere still obey Eqns. 1.6 and 1.7, but they also obey one additional identity

$$\sigma_1\sigma_2\ldots\sigma_{M-2}\sigma_{M-1}\sigma_{M-1}\sigma_{M-2}\ldots\sigma_2\sigma_1 = I \tag{1.8}$$

where I is the identity (or trivial) braid. What does this additional identity mean geometrically?

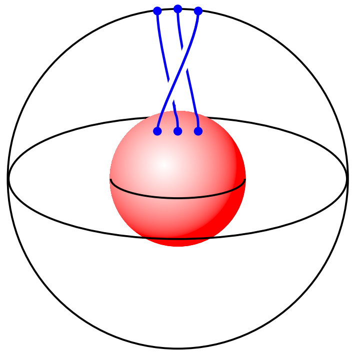


Fig. 1.6 An element of the braid group $B_3(S^2)$. The braid shown here (reading right to left meaning bottom to top in the braid) is $\sigma_2\sigma_1$.

Exercise 8.2d from Book
Fusion and Ground-State Degeneracy: More

(extension to above exercise 8.2 a,b,c)

(d) Generalize Eq. 1.5 to the case of a g-handled torus where there are also m particles on the surface of the manifold with quantum numbers $a_1, \ldots, a_m$.

Exercise 9.3 from Book
Ising Pentagon

Consider a system of Ising anyons. Given the fusion rules, F^{xyz}_w will be the 2 by 2 matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

in the case of $x = y = z = w = \sigma$ and is a simply a scalar otherwise. One might hope that these scalars can all be taken to be unity. Unfortunately this is not the case. By examining the pentagon equation,

$$[F^{fcd}_e]_{gl}[F^{abl}_e]_{fk} = \sum_h [F^{abc}_g]_{fh}[F^{ahd}_e]_{gk}[F^{bcd}_k]_{hl} \tag{1.9}$$

in the case of $a = b = c = \sigma$ and $d = f = \psi$ show that taking the scalar to always be unity is not consistent. Show further that choosing $[F^{\psi\sigma\psi}_\sigma]_{\sigma\sigma} = -1$ (and leaving the other scalars to be unity) allows a consistent solution of the pentagon equations for $a = b = c = \sigma$ and $d = f = \psi$.

1.4 Optional Problems (More)

[Solve these if you have time. We can discuss them after class if you are interested!]

Exercise 3.4 a,c from Book
Small Numbers of Anyons on a Sphere

On the plane, the braid group of two particles is an infinite group (the group of integers describing the number of twists!). However, this is not true on a sphere

(a) First review the problem "About the Braid Group" about braiding on a sphere. Now consider the case of two particles on a sphere. Determine the full structure of the braid g roup. Show it is a well known finite discrete group. What group is it?

(b) [Skip this part, it is hard!]

(c) Suppose we have two (or three) anyons on a sphere. Suppose the ground state is two-fold degenerate (or more generally N-fold degenerate for some finite N). Since the braid group is discrete, conclude that no type of anyon statistics can allow us to do arbitrary $SU(2)$ (or $SU(N)$) rotations on this degenerate ground state by braiding.

Exercise 5.2 from Book
Gauge Transforming the Chern-Simons Action

The Chern-Simons action on a manifold $\mathcal{M}$ is

$$S_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}} d^3x \; \epsilon^{\alpha\beta\gamma} \; \mathrm{Tr}\left[a_\alpha \partial_\beta a_\gamma + \frac{2}{3} a_\alpha a_\beta a_\gamma\right] \tag{1.10}$$

where a_α is a vector of fields valued in a Lie algebra and ϵ is the antisymmetric tensor $(\alpha, \beta, \gamma \in 0, 1, 2)$.

A gauge transformation on the Chern-Simons field is

$$a_\mu \to U^{-1} a_\mu U + U^{-1} \partial_\mu U$$

for arbitrary $U(x)$ in the Lie group (we are considering unitary representations of the Lie group).

Show that this gauge transformation results in

$$S_{CS} \to S_{CS} + 2\pi\nu k \tag{1.11}$$

where

$$\nu = \frac{1}{24\pi^2} \int_{\mathcal{M}} d^3x \; \epsilon^{\alpha\beta\gamma} \; \mathrm{Tr}\left[(U^{-1}\partial_\alpha U)(U^{-1}\partial_\beta U)(U^{-1}\partial_\gamma U)\right] \tag{1.12}$$

is known as the Pontryagin index, which can be shown to be an integer (you do not need to show this fact). Note that there will be an additional term that shows up which is a total derivative and will therefore vanish when integrated over the whole manifold $\mathcal{M}$.

Exercise 8.1 from Book
Quantum Dimension

Let N_{ab}^c be the fusion multiplicity matrices of a TQFT

$$a \times b = \sum_c N_{ab}^c \, c$$

meaning that N_{ab}^c is the number of distinct ways that a and b can fuse to c. (In many, or even most, theories of interest all N's are either 0 or 1).

The quantum dimension d_a of a particle a is defined as the largest eigenvalue of the matrix $[N_a]_b^c$ where this is now thought of as a two-dimensional matrix with a fixe d and b, c the indices.

Show that

$$\mathsf{d}_a \mathsf{d}_b = \sum_c N_{ab}^c \, \mathsf{d}_c$$

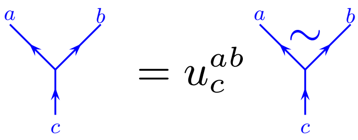
We will prove this formula algebraically later. However there is a simple and much more physical way to get to the result: Imagine fusing together M anyons of type a and M anyons of type b where M gets very large and determine the dimension of space that results. Then imagine fusing together $a \times b$ and do this M times and then fuse together all the results.

Exercise 9.6a from Book
Gauge Change

(a.i) Confirm that the F-matrix transforms under gauge change as

$$\widetilde{[F_e^{abc}]}_{df} = \frac{u_e^{af} u_f^{bc}}{u_d^{ab} u_e^{dc}} [F_e^{abc}]_{df}$$

where the u coefficients transform the vertices as



(a.ii) Show that a solution of the pentagon equation remains a solution under any gauge transformation.

Problem Set 2

2

More Fun!

2.1 Required: Due Fri Week 7, 11 pm

Exercise 10.6a,b,c,d from Book
Enforcing the Locality Constraint

The locality constraint shown in Fig. 2.1 turns out to be extremely powerful. In this exercise we will use this constraint to (almost) derive the possible values for the R-matrix for Fibonacci anyons given the known F-matrix.

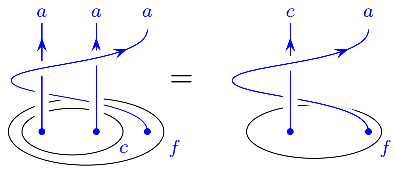


Fig. 2.1 The locality constraint.

Consider an anyon theory with Fibonacci fusion rules and Fibonacci F-matrix

$$F_\tau^{\tau\tau\tau} = F = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix} \tag{2.1}$$

where $\phi = (\sqrt{5}+1)/2$ is the golden mean (and the matrix notation has the first row/column being I and the second row/column being τ).

(a) [Easy] Confirm the locality constraint shown in Fig. 2.1 given the right-handed R-matrix

$$\begin{aligned} R_\tau^{\tau\tau} &= e^{+3\pi i/5} \\ R_I^{\tau\tau} &= e^{-4\pi i/5} \end{aligned} \tag{2.2}$$

(the left-handed R-matrix would have these values complex conjugated).

Make sure to confirm the equality for all three cases $f = I, c = \tau$ and $f = \tau, c = I$ and $f = \tau, c = \tau$.

Note that on the left of Fig. 2.1 is the braiding operation $\hat{O} = \hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_1\hat{\sigma}_2$. whereas the operation on the right is σ^2.

(b) Show that the locality constraint of Fig. 2.1 would also be satisfied by

$$R_I^{\tau\tau} \to -R_I^{\tau\tau} \qquad R_\tau^{\tau\tau} \to -R_\tau^{\tau\tau} \tag{2.3}$$

(c) In addition to right- and left-handed Fibonacci anyons and the two additional spurious solutions provided by Eq. 2.3, there are four additional possible sets of R-matrices that are consistent with the F-matrices of the Fibonacci theory given the locality constraint of Fig. 2.1. These additional solutions are all fairly trivial. Can you guess any of them?

If we cannot guess the additional possible R-matrices, we can derive them explicitly (and show that no others exist). Let us suppose that we do not know the values of the R-matrix elements $R_I^{\tau\tau}$ and $R_\tau^{\tau\tau}$.

(d) For the case of $f = I$ and $c = \tau$ show that Fig. 2.1 implies

$$[R_\tau^{\tau\tau}]^4 = [R_I^{\tau\tau}]^2 \tag{2.4}$$

Exercise 11.1 from Book
Ising Nonuniversality

The braiding matrices for Ising anyons are given by

$$\hat{\sigma}_1 = e^{-i\pi/8} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{2.5}$$

$$\hat{\sigma}_2 = \frac{e^{i\pi/8}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{2.6}$$

where the order of rows and columns are I, ψ.

Demonstrate that any multiplication of these matrices and their inverses will only produce a finite number of possible results. Thus conclude that Ising anyons are not universal for quantum computation. Hint: write the braiding matrices as $e^{i\alpha}U_j$ where U_j is unitary with unit determinant, i.e., is an element of $SU(2)$. Then note that any $SU(2)$ matrix can be thought of as a rotation $\exp(i\,\hat{n}\cdot\boldsymbol{\sigma}\,\theta/2)$ where here θ is an angle of rotation, $\hat{n}$ is the axis of rotation, and $\boldsymbol{\sigma}$ is the vector of Pauli spin matrices.

Exercise 13.4a,b,c,d of 2021 from Book =
Exercise 13.6a,b,c,d of 2022 from Book
Reidemeister Moves

(a) Use the R-matrix, and the completeness relationship, to derive the equivalence shown in Fig. 2.2

(b) How does the hexagon equation imply the equivalence shown in Fig. 2.3. Hint: This is very subtle, but is almost trivial.

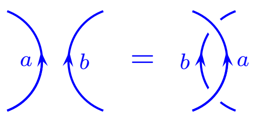


Fig. 2.2 "Reidemeister" Move II (The quotes are here because we have labeled the strands which is more general than the original Reidemeister definitions)

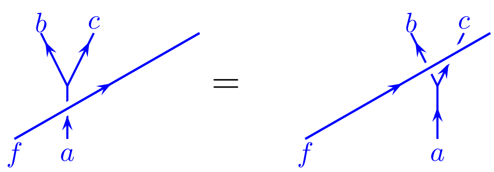


Fig. 2.3 This move is implied by the hexagon equations. (Similar with the straight line f going under the other two, and similar if the left-to-right slope of f is negative instead of positive.)

(c) Use Fig. 2.3 to show the equality of Fig. 2.4.

(d) Use the result of Fig. 2.3 along with completeness and the R-matrix to demonstrate

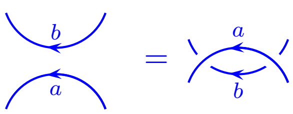


Fig. 2.4 "Reidemeister" Move II oriented sideways

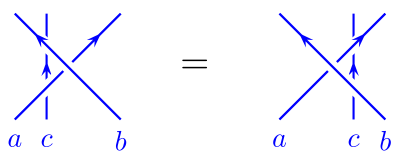


Fig. 2.5 "Reidemeister" Move III

This exercise shows that equalities like those shown in Fig. 2.2 and 2.5 are not independent assumptions but can be derived from the planar algebra and the definition of an R-matrix satisfying the hexagon equation.

2.2 Optional Problems (Easy)

[These problems will be discussed in class only if you find them hard.]

Exercise 10.1a,b,c from Book
Calculating Exchanges

As shown in the book (Eq. 10.1), given three anyons of type a. If anyons 1 and 2 are in fusion channel c and all three anyons together are in fusion channel f (see left of Fig. 2.1), the counterclockwise exchange of anyons 2 and 3 is given by

$$\hat{\sigma}_2|c;f\rangle = \sum_{g,z} \; [F_f^{aaa}]_{cg} \;\; R_g^{aa} \;\; [(F_f^{aaa})^{-1}]_{gz} \;\; |z;f\rangle \tag{2.7}$$

You should make sure you know where this expression comes from!

(a) Use Eq. 2.7 to confirm Eq. 2.5 and 2.6.

(b) Use Eq. 2.7 to confirm (given the information in Exercise 10.6a,b,c,d above) that

$$\hat{\sigma}_1 = \begin{pmatrix} e^{3\pi i/5} & & \\ & e^{-4\pi i/5} & \\ & & e^{3\pi i/5} \end{pmatrix} \tag{2.8}$$

$$\hat{\sigma}_2 = \begin{pmatrix} e^{3\pi i/5} & & \\ & \phi^{-1}e^{4\pi i/5} & \phi^{-1/2}e^{-3\pi i/5} \\ & \phi^{-1/2}e^{-3\pi i/5} & -\phi^{-1} \end{pmatrix} \tag{2.9}$$

where $\phi = (1+\sqrt{5})/2$ is the golden mean, and the order of rows/columns is N, I, τ. Here N means all three anyons fuse together to the vacuum ($f = I$), and whereas I and τ indicate that $c = I$ or τ with $f = \tau$.

(c) Confirm the braiding relation $\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_1 = \hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_2$ in both cases. What does this identity mean geometrically?

2.3 Optional Problems (The Best Ones)

[We hope to discuss these in class.]

Exercise 10.2
Ising Anyons Redux

In exercise 3.3 (Required in the first Homework set), we introduced a representation for the exchange matrices for Ising anyons which, for three anyons, would be of the form

$$\hat{\sigma}_1 = \frac{e^{i\alpha}}{\sqrt{2}}(1 + \gamma_1\gamma_2) \tag{2.10}$$

$$\hat{\sigma}_2 = \frac{e^{i\alpha}}{\sqrt{2}}(1 + \gamma_2\gamma_3) \tag{2.11}$$

where the γ's are Majorana operators defined by

$$\{\gamma_i, \gamma_j\} \equiv \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}$$

with $\gamma_i = \gamma_i^\dagger$.

Show that the exchange matrices in Eqs. 2.5-2.6 are equivalent to this representation. How does one represent the $|0\rangle$ and $|1\rangle$ states of the Hilbert space in this language? The answer may not be unique.

Exercise 10.4
Determinant and Trace of Braid Matrices

Consider a system of N-identical anyons with a total Hilbert space dimension D. The braid matrix $\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_{N-1}$ are all D-dimensional. Show that each of these matrices has the same determinant, and each of these matrices has the same trace. Hint: This is easy if you think about it right!

Exercise 13.1
Fibonacci Hexagon

Once the F-matrices for a TQFT are defined, the consistency of the R-matrix is enforced by the so-called hexagon equations. For the Fibonacci anyon theory, once the F-matrix is fixed as in Eq. 2.1, the R-matrices are defined up to complex conjugation (i.e., there is a right- and left-handed Fibonacci anyon theory — both are consistent). Derive these R-matrices. Confirm Eqs. 2.2 as one of the two solutions and show no other solutions exist.

Exercise 15.2 of 2021 from Book=
Exercise 15.4 of 2022 from Book
Using Geometric Moves I

(a) Using the allowed geometric moves such as Fig. 2.2, 2.4, 2.3, show the equivalence of the left and right of Fig. 2.6 (b) Similarly, show the equivalence of the left and right of Fig. 2.7. (c) Similarly show the equivalence of the middle two figures in Fig. 2.8.

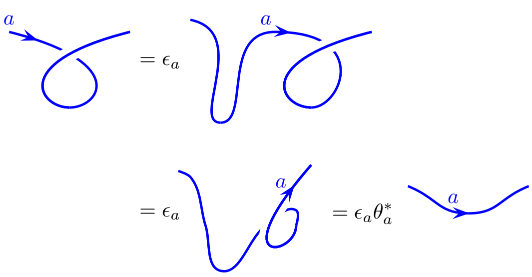


Fig. 2.6 This equality establishes $\theta_a \theta_a^* = 1$

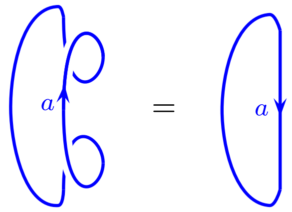


Fig. 2.7 The equality of these diagrams establishes $\theta_a = \theta_{\bar{a}}$.

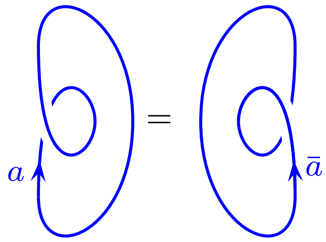


Fig. 2.8 A curl in a rope turned sideways gets a twist factor θ^* along with a zig-zag factor ϵ_a.

2.4 Optional Problems (More)

[Solve these if you have time. We can discuss them after class if you are interested!]

Exercise 10.5 from Book
Checking the Locality Constraint

[Easy] Consider Fig. 2.1. The braid on the left can be written as $\hat{b}_3 = \hat{\sigma}_2 \hat{\sigma}_1^2 \hat{\sigma}_2$.

(a) For the Fibonacci theory with $a = \tau$ check that the matrix $\hat{b}_3$ gives just a phase, which is dependent on the fusion channel c. I.e., show the matrix $\hat{b}_3$ is a diagonal matrix of complex phases. Show further that these phases are the same as the phases that would be accumulated for taking a single τ particle around the particle c.

(b) Consider the same braid for the Ising theory with $a = \sigma$. Show again that the result is a c-dependent phase.

[Hard] Consider the braid shown on the left of Fig. 2.9. The braid can be written as $\hat{b}_4 = \hat{\sigma}_3 \hat{\sigma}_2 \hat{\sigma}_1^2 \hat{\sigma}_2 \hat{\sigma}_3$.

(c) Consider Ising anyons where $a = \sigma$. Use the F- and R-matrices to calculate $\hat{\sigma}_3$. Since the fusion of three σ anyons always gives $c = \sigma$, show that $\hat{b}_4$ is just a phase times the identity matrix. Show further that this phase is the same as the phase accumulated by taking a single σ all the way around another σ.

(d) Consider Fibonacci anyons with $a = \tau$, Use the F- and R-matrices to calculate $\hat{\sigma}_3$. Check that $\hat{b}_4$ is a diagonal matrix of phases. Check that the phases match the two possible phases accumulated by wrapping a single τ all the way around a single particle c which can be I or τ.

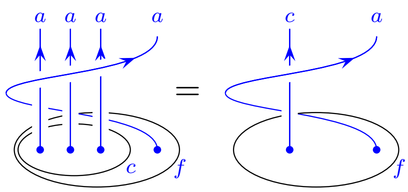
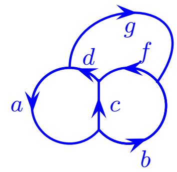


Fig. 2.9 The locality constraint.

Exercise 12.1
Evaluating Diagrams with F-matrices

(a) Evaluate the following diagram, writing the result in terms of F's.



(b) If we take the magnitude squared of this diagram and sum over all g, we should get one. Physically, why is this?

Exercise 13.2 from 2021 Book=
Exercise 13.3 from 2022 Book
Evaluation of a Diagram

Evaluate the diagram shown in Fig. 2.10 in terms of R's and F's. Hint: First reduce the diagram to that shown in exercise 2.9.

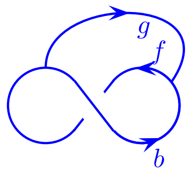


Fig. 2.10 Evaluate this diagram.

Problem Set 3

3

The Final Frontier!

3.1 Required: Due Fri Week 0 HT, 11 pm

Exercise 17.4
Evaluation of the S-link

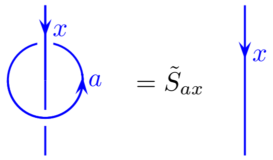
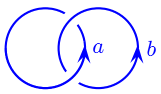


Fig. 3.1 Definition of the unlinking matrix

(a) Use the R-matrices and the ribbon identity to derive the value of the matrix $\tilde{S}_{ax}$ (see Fig. 3.1) in terms of fusion multiplicities, twist factors θ_a, and the quantum dimensions d_a.

(b) From your result show that



$$= \sum_c N_{a\bar{b}}^c \frac{\theta_c}{\theta_a \theta_b} \mathsf{d}_c$$

Note that this diagram differs from S_{ab} by a factor of $Z(S^3) = 1/\mathcal{D}$.

Exercise 17.5
Theories With One Nontrivial Particle

Consider an anyon theory with only the identity and one nontrivial particle type s having twist factor θ_s. The only possible fusion rules are $s \times s = I + ms$ for some integer m (the semion model is $m = 0$ the Fibonacci model is $m = 1$). Calculate d_s and $\mathcal{D}$ from the fusion rules. Use the result of Exercise 17.4b to calculate the S-matrix in terms of θ_s. Show that this matrix cannot be unitary for any $m > 1$. This justifies that on our "periodic table" there are only two types of theory with one nontrivial particle.

Exercise 22.2 in 2021 Book =
Exercise 24.2 in 2022 Book
Surgery on the Hopf Link [Not hard if you think about it right!]

Consider two linked rings, known as the Hopf link (See Fig. 3.2). Consider starting with S^3 and embedding the Hopf link within the S^3 with "blackboard framing" (i.e., don't introduce any additional twists when you embed it). Thicken both strands into solid tori and perform surgery on each of the two links. Argue that the resulting manifold is S^3.

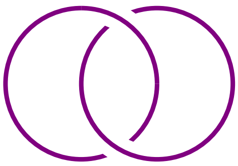


Fig. 3.2 A Hopf Link

Exercise 22.4 from 2021 Book=
Exercise 24.4 from 2022 Book
Evaluation of Borromean Ring Ω-Link

Use the Killing property of Ω to evaluate the Ω-link of Borromean rings shown in Fig. 3.3. Use this to establish $Z(T^3)$ = number of particle species. Note that the signature of the link is zero.

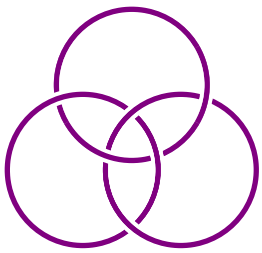


Fig. 3.3 Borromean Rings. Cutting any one strand disconnects the other two. Surgery on this link in S^3 creates the three-torus $S^1 \times S^1 \times S^1$.

Exercise 22.5 from 2021 Book=
Exercise 24.5 from 2022 Book
Product of Blow Up and Blow Down

Use the handle-slide and the killing property of Ω to prove that the diagram made of two oppositely twisted Ω loops, as shown in Fig. 3.4, gives the identity.

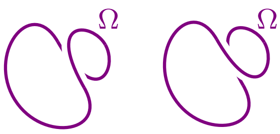


Fig. 3.4 The product of these two oppositely twisted Ω loops gives the identity.

Exercise 28.1 from 2021 Book=
Exercise 30.1 from 2022 Book
Toric Code S-matrix

Derive the S-matrix of the toric code by using modular transformation of the torus, as described in section 28.4 (2021 Book) or 30.4 (2022 Book). If you need help with this calculation look at the more general calculation given for the $\mathbb{Z}_N$ version of the toric code.

Exercise 28.2 from 2021 Book=
Exercise 30.2 from 2022 Book
Braiding Quasiparticles in Toric Code Loop Gas

(a) Use the graphical technique of section 28.5 (2021 Book) or 30.5 (2022 Book) to show that exchanging two f's gives a minus sign (i.e., confirm the details of the argument given there).

(b) Use similar techniques to show that exchanging two e particles gives no sign and exhanging two m particles gives no sign.

(c) Show that braiding an e particle or an f particle all the way around an m particle give a minus sign but braiding around the identity gives no sign.

(d) Show that braiding e all the way around f gives a minus sign.

3.2 Optional Problems (Easy)

[These problems will be discussed in class only if you find them hard.]

Exercise 15.4 from 2021 Book=
Exercise 15.6 from 2022 Book
Gauge Independence of Ribbon Identity
Show that the ribbon identity

$$R_c^{ba} R_c^{ab} = \frac{\theta_c}{\theta_a \theta_b}$$

is gauge independent.

Exercise 22.1 from 2021 Book=
Exercise 24.1 from 2022 Book
Surgery on a Loop
Beginning with the three-sphere S^3, consider the so-called "unknot" (a simple unknotted circle S^1 with no twists) embedded in this S^3. Thicken the circle into a solid torus ($S^1 \times D^2$) which has boundary $S^1 \times S^1$. Now perform surgery on this torus by excising the solid torus from the manifold S^3 and replacing it with another solid torus that has the longitude and meridian swi tched. I.e., replace $S^1 \times D^2$ with $D^2 \times S^1$. Note that both of the two solid tori have the same boundary $S^1 \times S^1$ so that the new torus can be smoothly sewed back in where the old one was removed. What is the new manifold you obtain? (This should be easy because it is in the book!).

3.3 Optional Problems (The Best Ones)

[We hope to discuss these in class.]

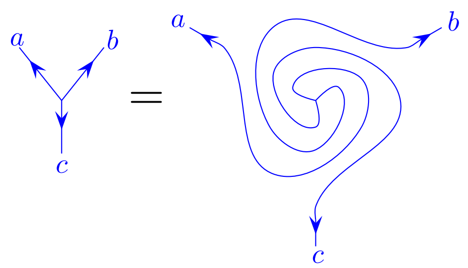


Fig. 3.5 The pivotal identity

Exercise 15.3 from 2021 Book=
Exercise 15.5 from 2022 Book
Using Geometric Moves II
Demonstrate the middle step of Fig. 3.6 by using allowed geometric moves such as Fig. 2.2 and Fig. 2.5 and Fig. 2.3. You may also need the pivotal identity.

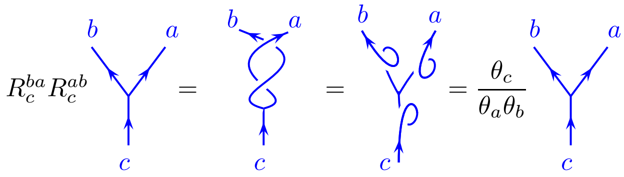


Fig. 3.6 Deriving the ribbon identity. The middle is the nonobvious geometric step.

Exercise 17.2
Using the Pivotal Property

Use the pivotal property to demonstrate the identity shown in Fig. 3.7. You should not assume full isotopy invariance. Nor should you assume $\epsilon = +1$ for any of the particles.

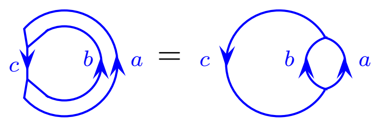


Fig. 3.7 This identity can be shown without full isotopy invariance by using the pivotal property.

3.4 Optional Problems (More)

[Solve these if you have time. We can discuss them after class if you are interested!]

Exercise 18.4 from 2021 Book=
Exercise 18.7 from 2022 Book
Evaluating Diagrams I

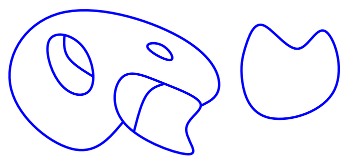


Fig. 3.8 A Fibonacci branching loop diagram allows intersections of loops, but no loop ends.

Show that evaluation of the diagram in Fig. 3.8 gives $-d_\tau^{9/2}$.

Exercise 18.5 from 2021 Book=
Exercise 18.8 from 2022 Book
Evaluating Diagrams II

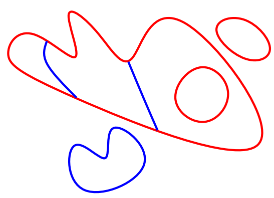


Fig. 3.9 A diagram with Ising fusion rules. Here σ is red and ψ is blue.

Show that evaluation of the diagram in Fig. 3.9 gives $d_\psi^2 d_\sigma^3 \kappa_\sigma$.

Exercise 22.3 from 2021 Book=
Exercise 24.3 from 2022 Book
Surgery on the Borromean Rings [Hard]

Consider the link shown in Fig. 3.3 known as the Borromean rings. Consider starting with S^3 and embedding the Borromean rings within the S^3 with "blackboard framing". Thicken all three strands into solid tori and perform surgery on each of the three links exactly as we did in the previous two problems. Show that one gets the three torus as a result. Hint 1: Think about the group of topologically different loops through the manifold starting and ending at the same point, the so-called "fundamental group" or first homotopy group. Hint 2: If we say a path around the meridian of one of the three Borromean rings (i.e., threading though the loop) is called a and the path around the meridian of the second ring is called b, then notice that the third ring is topologically equivalent to $aba^{-1}b^{-1}$. Hint 3: In some cases the fundamental group completely defines the manifold! (Don't try to prove this, just accept this as true in this particular case.)

Exercise 30.1 from 2021 Book=
Exercise 32.1 from 2022 Book
Fusing Quasiparticles in the Doubled-Semion Loop Gas

Use the graphical technique of section 28.5 (2021 Book) or 30.5 (2022 Book) to deduce the full fusion table for the doubled-semion model.

(a) First try using the $d = -1$ algebra (section 30.2, 2021 Book) or 32.2, 2022 Book).

(b) Now try the unitary version of this algebra (section 30.3, 2021 Book or 32.3 2022 Book)

Exercise 30.2 from 2021 Book=
Exercise 32.2 from 2022 Book
Braiding Quasiparticles in the Doubled-Semion Loop Gas

Use the graphical technique (as in Exercise 28.2/30.2 above for toric code) to calculate the R-matrix for the doubled-semion model.

(a) Show that wrapping an sL all the way around an sR gives no phase.

(b) Show that wrapping an m particle around either an sL or an sR gives a phase of -1.

(c) Show that wrapping an m particle around another m particle gives no phase.

(d) [Harder] Use the graphical technique to calculate the phase from exchanging two sL particles and the phase for exchanging two sR particles. In this part of this problem you will get an incorrect sign if you work with the non-unitary theory ($d < 0$).