Topological Phases of Matter: Problem Set # 3

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Problem 1 *Bose Vertex Operators*

In lecture we needed the following identity

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\ldots V_{\alpha_N}(z_N)\rangle = \prod_{i<j}(z_i - z_j)^{\alpha_i\alpha_j} \quad (1)$$

where

$$\sum_i \alpha_i = 0 \quad (2)$$

where the vertex operators are defined by

$$V_\alpha(z) =: e^{i\alpha\phi(z)} : \quad (3)$$

with ϕ a chiral bose field and colons meaning normal ordering.

(a) To get to this result, let us first show that for a bose operator a, such that $[a, a^\dagger] = 1$, we have

$$e^{\alpha a}e^{\beta a^\dagger} = e^{\beta a^\dagger}e^{\alpha a}e^{\alpha\beta} \quad (4)$$

(b) Thus derive

$$\langle V_{A_1}V_{A_2}\ldots V_{A_N}\rangle = e^{\sum_{i<j}\langle A_i A_j\rangle} \quad (5)$$

where

$$A_i = u_i a^\dagger + v_i a \quad (6)$$

and

$$V_{A_i} =: e^{A_i} := e^{u_i a^\dagger}e^{v_i a} \quad (7)$$

with the colons meaning normal ordering (all daggers moved to the left).

(c) Show that Eq. 5 remains true for any operators A_i that are sums of different bose modes a_k, i.e., if

$$A_i = \sum_k [u_i(k)a_k^\dagger + v_i(k)a_k] \quad (8)$$

Set $A_i = i\alpha_i\phi(z_i)$ such that $V_{A_i} = V_\alpha(z_i)$. If ϕ is a free massless chiral bose filed which can be written as the sum of fourier modes of bose operators such that

$$\langle\phi(z)\phi(w)\rangle = -\ln(z - w) \quad (9)$$

conclude that Eq. 1 holds.

Note: This result is not quite correct, as it fails to find the constraint Eq. 2 properly. The reason it fails is a subtlety which involves how one separates a bose field into two chiral components. (More detailed calculations that get this part right are given in the Big Yellow CFT book (P. Di Francesco, P. Mathieu, and D. Senechal) and in a different language in A. Tsvelik's book.)

There is, however, a quick way to see that the constraint must be true. Note that the lagrangian of a massless chiral bose field is

$$\mathcal{L} = \frac{1}{2\pi}\partial_x\phi(\partial_x + v\partial_t)\phi \tag{10}$$

which clearly must be invariant under the global transformation $\phi \rightarrow \phi + b$.

(d)Show that the correlator Eq. 1 (with Eq. 3) cannot be invariant under this transformation unless Eq. 2 is satisfied, or unless the value of the correlator is zero.

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Problem 2 $\mathbb{Z}_4$ Quantum Hall State

In this problem we intend to construct a quantum hall state from the the $\mathbb{Z}_4$ parafermion conformal field theory (Details of the CFT can be found in A. B. Zamolodchikov and V. A. Fateev, Soviet Physics JETP 62, 216 (1985), but we will not need too many of the details here).

The wavefunction we construct is known as the $\mathbb{Z}_4$ Read-Rezayi wavefunction (N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999)).

The $\mathbb{Z}_4$ parafermion conformal field theory has 10 fields with corresponding conformal weights (scaling dimension)

field	$\mathbf{1}$	ψ_1	ψ_2	ψ_3	σ_+	σ_-	ϵ	ρ	χ_+	χ_-
weight h	0	$\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{9}{16}$	$\frac{9}{16}$

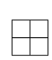
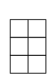
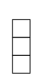
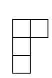
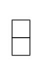
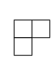
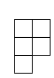
and the fusion table is given by

$\times$	$\mathbf{1}$	ψ_1	ψ_2	ψ_3	σ_+	σ_-	ϵ	ρ	χ_+	χ_-
$\mathbf{1}$	$\mathbf{1}$	ψ_1	ψ_2	ψ_3	σ_+	σ_-	ϵ	ρ	χ_+	χ_-
ψ_1	ψ_1	ψ_2	ψ_3	$\mathbf{1}$	χ_-	σ_+	ρ	ϵ	σ_-	χ_+
ψ_2	ψ_2	ψ_3	$\mathbf{1}$	ψ_1	χ_+	χ_-	ϵ	ρ	σ_+	σ_-
ψ_3	ψ_3	$\mathbf{1}$	ψ_1	ψ_2	σ_-	χ_+	ρ	ϵ	χ_-	σ_+
σ_+	σ_+	χ_-	χ_+	σ_-	$\psi_1+\rho$	$\mathbf{1}+\epsilon$	$\sigma_+ + \chi_+$	$\sigma_- + \chi_-$	$\psi_3+\rho$	$\psi_2+\epsilon$
σ_-	σ_-	σ_+	χ_-	χ_+	$\mathbf{1}+\epsilon$	$\psi_3+\rho$	$\sigma_- + \chi_-$	$\sigma_+ + \chi_+$	$\psi_2+\epsilon$	$\psi_1+\rho$
ϵ	ϵ	ρ	ϵ	ρ	$\sigma_+ + \chi_+$	$\sigma_- + \chi_-$	$\mathbf{1}+\psi_2+\epsilon$	$\psi_1+\psi_3+\rho$	$\sigma_+ + \chi_+$	$\sigma_- + \chi_-$
ρ	ρ	ϵ	ρ	ϵ	$\sigma_- + \chi_-$	$\sigma_+ + \chi_+$	$\psi_1+\psi_3+\rho$	$\mathbf{1}+\psi_2+\epsilon$	$\sigma_- + \chi_-$	$\sigma_+ + \chi_+$
χ_+	χ_+	σ_-	σ_+	χ_-	$\psi_3+\rho$	$\psi_2+\epsilon$	$\sigma_+ + \chi_+$	$\sigma_- + \chi_-$	$\psi_1+\rho$	$\mathbf{1}+\epsilon$
χ_-	χ_-	χ_+	σ_-	σ_+	$\psi_2+\epsilon$	$\psi_1+\rho$	$\sigma_- + \chi_-$	$\sigma_+ + \chi_+$	$\mathbf{1}+\epsilon$	$\psi_3+\rho$

If I have not made any mistake in typing this table, the fusion rules should be associative

$$(a \times b) \times c = a \times (b \times c) \tag{11}$$

Note of interest: These fusion rules may look mysterious, but in fact they are very closely related to the fusion rules of $SU(2)$ appropriately truncated (i.e., this is the $SU(2)_4$ WZW model). We can write each field as a young tableau with no more than 2 (for $SU(2)$) columns and no more than $4-1=3$ rows

field	$\mathbf{1}$	ψ_1	ψ_2	ψ_3	σ_+	σ_-	ϵ	ρ	χ_+	χ_-
tableau	empty									

The fusion rules are just a *slight* modification of the usual young tableau manipulations for $SU(2)$ where columns are removed if they have 4 boxes. (See the big yellow book for details).

Using the techniques discussed in lecture:

(a) Use the operator product expansion (dimension counting) to find the singularity as two ψ_1 fields come close together. I.e, find the exponent α in the relation

$$\lim_{z' \to z} \psi_1(z')\psi_1(z) \sim (z'-z)^{\alpha}\, \psi_2(z) \tag{12}$$

(b) Construct all possible "electron" fields by making a product of the ψ_1 field and a chiral bose vertex operator of the form

$$\psi_e(z) = \psi_1(z)e^{i\beta\phi(z)} \tag{13}$$

that give a single-valued and nonsingular wavefunction for the electron. (See Eq. 1, but ignore the sum condition Eq. 2) I.e., find all acceptable values of β. Consider both the case where the "electron" is a boson or a fermion. What filling fractions do these correspond to? (There are multiple allowable solutions for both bosons and fermions). Consider among the bosonic solution, the one solution of the highest density. The ground state wavefunction in this case is the highest density zero energy state of a 5-point delta function interaction. Show that the wavefunction does not vanish when 4 particles come to the same point, but does indeed vanish as 5 particles come to the same point.

(c) Given a choice of the electron field, construct all possible quasihole operators from all fields φ in the above table

$$\phi_{qh}(w) = \varphi(w)e^{i\kappa\phi(w)} \tag{14}$$

For each case, fix the values of κ by insisting that the wavefunction remain single-valued in the electron coordinates. Determine the quasihole with the lowest possible (nonzero) electric charge. What is this charge?

(d) Two such quasiholes can fuse together in two possible fusion channels. What is the monodromy in each of these channels. I.e, what phase is accumulated when the two quasiholes are transported around each other (assuming the Berry matrix is zero – which is a statement about wavefunctions being properly orthonormal – which we usually assume is true).

(e) Draw a Bratteli diagram (a tree) describing the possible fusion channels for many of these elementary particles. Label the number of paths in the diagram for up to 10 quasiholes. If there are 8 quasiparticles and the number of electrons is divisible by 4, what is the degeneracy of the ground state? If there are 4 quasiparticles and the number of electrons is $4m+2$ what is the degeneracy of the ground state?

(f) Construct a 5 by 5 transfer matrix and show how to calculate the ground state degeneracy in the presence of any number of quasiholes. Finding the largest eigenvalue of this matrix allows you to calculate the "quantum dimension" d which is the scaling

$$\text{Degeneracy} \sim d^{[\text{Number of Quasiholes}]} \tag{15}$$

in the limit of large number of quasiholes. While diagonalizing a 5 by 5 matrix seems horrid, this one can be solved in several easy ways (look for a trick or a nice factorization of the characteristic polynomial).

(g) Consider instead constructing a wavefunction from the ψ_2 field

$$\psi_e(z) = \psi_2(z) e^{i\beta\phi(z)} \tag{16}$$

What filling fraction does this correspond to (for bosons or fermions). In the highest density case, what are the properties of this wavefunction (how does it vanish as how many many electrons come to the same point).

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