

Exercises on BCS Theory and Topological Matter(MMathPhys QCMT-2)

Question 1: BCS Theory

In this problem assume the interaction term of the Hamiltonian is given by the BCS form

$$H_{int} = -V_0 \sum_{\mathbf{k}, \mathbf{k}'}' c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger c_{\mathbf{k}\downarrow} c_{-\mathbf{k}\uparrow}$$

where the prime on the sum indicates that $\mathbf{k}, \mathbf{k}'$ is summed over only if its energy is within $\hbar\omega_D$ of the fermi surface.

(a) Using BCS theory calculate the order parameter defined as

$$\langle \Psi(\mathbf{r}) \rangle = \langle \hat{\psi}_\uparrow(\mathbf{r}) \hat{\psi}_\downarrow(\mathbf{r}) \rangle$$

To do this, first write this correlator in terms of expectations $\langle c_{\mathbf{k},\uparrow} c_{-\mathbf{k},\downarrow} \rangle$. Show that

$$\langle \Psi(\mathbf{r}) \rangle = \Delta / V_0$$

where Δ is the gap parameter defined by the gap equation

$$1 = \frac{V_0}{2} \sum_{\mathbf{k}}' \frac{1}{\sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + \Delta^2}}$$

You may assume zero temperature throughout this problem. Hint: Convert the c operators in the correlator into Bogoliubov operators.

(b) The order parameter in part (a) might be physically hard to justify if one insisted on a system with a fixed number of particles. Instead, one might prefer to work with the number conserving order parameter

$$\langle \Psi^\dagger(\mathbf{r}') \Psi(\mathbf{r}) \rangle$$

Give an expression for this order parameter as a function of $|\mathbf{r} - \mathbf{r}'|$.

Show that, in terms of your answer to part (a), we have

$$\langle \Psi^\dagger(\mathbf{r}') \Psi(\mathbf{r}) \rangle = \langle \Psi^\dagger(\mathbf{r}') \rangle \langle \Psi(\mathbf{r}) \rangle + C(\mathbf{r} - \mathbf{r}')$$

and determine an expression for $C(\mathbf{r} - \mathbf{r}')$. Show that C decays at large distance.

Question 2: About the Lowest Landau Level

If you have never before actually solved the problem of an electron in two dimensions in a magnetic field, it is worth doing. Even if you have done it before, it is worth doing again. (If you have done it many times, it might not be worth doing yet again).

Consider a two dimensional plane with a perpendicular magnetic field $\vec{B}$. Work in symmetric gauge $\vec{A} = \frac{1}{2}\vec{r} \times \vec{B}$.

(a) (This is the hard part, see below for hints if you need them.) Show that the single electron Hamiltonian can be rewritten as

$$H = \hbar\omega_c(a^\dagger a + \frac{1}{2}) \quad (1)$$

where $\omega_c = eB/mc$ and

$$a = \sqrt{2}\ell \left(\bar{\partial} + \frac{1}{4\ell^2}z \right) \quad (2)$$

with $z = x + iy$ and $\bar{\partial} = \partial/\partial\bar{z}$ with the overbar meaning complex conjugation. Here ℓ is the magnetic length $\ell = \sqrt{\hbar c/eB}$.

(b) Confirm that

$$[a, a^\dagger] = 1 \quad (3)$$

and therefore that the energy spectrum is that of the harmonic oscillator

$$E_n = \hbar\omega_c(n + \frac{1}{2}) \quad (4)$$

(c) Once you obtain Eq. 1, show that any wavefunction

$$\psi = f(z)e^{-|z|^2/4\ell^2} \quad (5)$$

with f any analytic function is an eigenstate with energy $E_0 = \frac{1}{2}\hbar\omega_c$. Show that an orthogonal basis of wavefunctions in the lowest Landau level (i.e., with eigenenergy E_0) is given by

$$\psi_m = N_m z^m e^{-|z|^2/4\ell^2} \quad (6)$$

where N_m is a normalization constant. Show that the maximum amplitude of the wavefunction ψ_m is a ring of radius $|z| = \ell\sqrt{2m}$ and calculate roughly how the amplitude of the wavefunction decays as the radius is changed away from this value.

(d) Defining further

$$b = \sqrt{2}\ell \left(\partial + \frac{1}{4\ell^2}\bar{z} \right) \quad (7)$$

with $\partial = \partial/\partial z$, Show that the operator b also has canonical commutations

$$[b, b^\dagger] = 1 \quad (8)$$

but both b and $b^\dagger$ commute with a and $a^\dagger$. Conclude that applying b or $b^\dagger$ to a wavefunction does not change the energy of the wavefunction.

(e) show that the $\hat{z}$ component of angular momentum (angular momentum perpendicular to the plane) is given by

$$L = \hat{z} \cdot (\vec{r} \times \vec{p}) = \hbar(b^\dagger b - a^\dagger a) \quad (9)$$

Conclude that applying b or $b^\dagger$ to a wavefunction changes its angular momentum, but not its energy.

Hints to part a

First, define the antisymmetric tensor ϵ_{ij} , so that the vector potential may be written as $A_i = \frac{1}{2}B\epsilon_{ij}r_j$. We have variables p_i and r_i that have canonical commutations (four scalar variables total). It is useful to work with a new basis of variables. Consider the coordinates

$$\pi_i^{(\alpha)} = p_i + \alpha \frac{\hbar}{2\ell^2} \epsilon_{ij} r_j \quad (10)$$

$$= \frac{\hbar}{\ell^2} \epsilon_{ij} \xi_j \quad (11)$$

defined for $\alpha = \pm 1$. Here $\alpha = +1$ gives the canonical momentum. Show that

$$\left[\pi_i^{(\alpha)}, \pi_j^{(\beta)} \right] = i\alpha\epsilon_{ij}\delta_{\alpha\beta} \frac{\hbar^2}{\ell^2} \quad (12)$$

The Hamiltonian

$$H = \frac{1}{2m} \left(p_i + \frac{e}{c} A_i \right) \left(p_i + \frac{e}{c} A_i \right) \quad (13)$$

can then be rewritten as

$$H = \frac{1}{2m} \pi_i^{(+1)} \pi_i^{(+1)} \quad (14)$$

with a sum on $i = \hat{x}, \hat{y}$ implied. Finally use

$$a = (-\pi_y^{(+1)} + i\pi_x^{(+1)}) \frac{\ell}{\sqrt{2}\hbar} \quad (15)$$

$$b = (\pi_y^{(-1)} + i\pi_x^{(-1)}) \frac{\ell}{\sqrt{2}\hbar} \quad (16)$$

to confirm that a and b are given by Eqs. 2 and 7 respectively. Finally confirm Eq. 1 by rewriting Eq. 14 using Eqs. 15 and 16.

A typical Place to get confused is the definition of ∂ . Note that

$$\partial z = \bar{\partial} \bar{z} = 1 \quad (17)$$

$$\bar{\partial} z = \partial \bar{z} = 0 \quad (18)$$

Question 3: Quantum Hall Conductivity vs Conductance

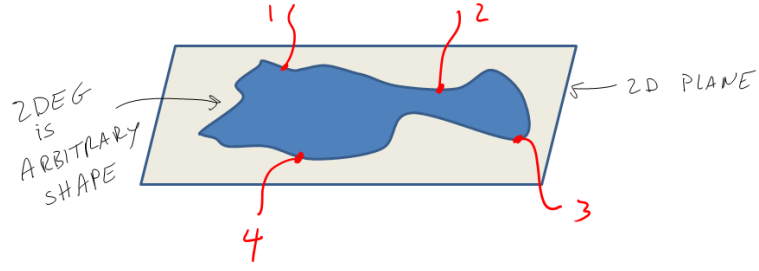


Figure 1: A 2DEG of arbitrary shape with contacts 1,2,3,4 attached on its perimeter in clockwise order

Consider a two dimensional electron gas (2DEG) of arbitrary shape in the plane with four contacts (1,2,3,4) attached at its perimeter in a clockwise order as shown in Fig. 1. The conductivity tensor σ_{ij} relates the electric field to the current via

$$j_i = \sigma_{ij} E_j \quad (19)$$

where indices i and j take values $\hat{x}$ and $\hat{y}$ (and sum over j is implied). Assume that this is a quantized hall system with quantized hall conductivity s . In other words, assume that

$$\sigma = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix} \quad (20)$$

Show that the following two statements are true independent of the shape of the sample.

(a) Suppose current I is run from contact 1 to contact 2, show that the voltage measured between contact 3 and 4 is zero.

(b) Suppose current I is run from contact 1 to contact 3, show that the voltage measured between contact 2 and 4 is $V = I/s$.

Note: The physical measurements proposed here measure the *conductance* of the sample, the microscopic quantity σ is the *conductivity*.

Question 4: Laughlin Plasma Analogy Consider the Laughlin wavefunction for N electrons at positions z_i

$$\Psi_m^0 = \mathcal{N} \prod_{1 \leq i < j \leq N} (z_i - z_j)^m \prod_{1 \leq i \leq N} e^{-|z_i|^2/4\ell^2} \quad (21)$$

with $\mathcal{N}$ a normalization constant. The probability of finding particles at positions $\{z_1, \dots, z_N\}$ is given by $|\Psi_m(z_1, \dots, z_N)|^2$.

Consider now N classical particles at temperature $\beta = \frac{1}{k_b T}$ in a plane interacting with logarithmic interactions $v(\vec{r}_i - \vec{r}_j)$ such that

$$\beta v(\vec{r}_i - \vec{r}_j) = -2m \log(|\vec{r}_i - \vec{r}_j|) \quad (22)$$

in the presence of a background potential u such that

$$\beta u(|\vec{r}|) = |\vec{r}|^2/(2\ell^2) \quad (23)$$

Note that this log interaction is “Coulombic” in 2d (i.e., $\nabla^2 v(\vec{r}) \propto \delta(\vec{r})$).

(a) Show that the probability that these classical particles will take positions $\{\vec{r}_1, \dots, \vec{r}_N\}$ is given by $|\Psi_m^0(z_1, \dots, z_N)|^2$ where $z_j = x_j + iy_j$ is the complex representation of position $\vec{r}_j$. Argue that the mean particle density is constant up to a radius of roughly $\ell\sqrt{Nm}$. (Hint: Note that u is a neutralizing background. What configuration of charge would fully screen this background?)

(b) Now consider the same Laughlin wavefunction, but now with M quasiholes inserted at positions $w_1, \dots, w_M$.

$$\Psi_m = \mathcal{N}(w_1, \dots, w_M) \left[\prod_{1 \leq i \leq N} \prod_{1 \leq \alpha \leq M} (z_i - w_\alpha) \right] \Psi_m^0 \quad (24)$$

where $\mathcal{N}$ is a normalization constant which may now depend on the positions of the quasiholes. Using the plasma analogy, show that the $w - z$ factor may be obtained by adding additional logarithmically interacting charges at positions w_i , with $1/m$ of the charge of each of the z particles

(c) Note that in this wavefunction the z ’s are physical parameters (and the wavefunction must be single-valued in z ’s), but the w ’s are just parameters of the wavefunction – and so the function $\mathcal{N}$ could be arbitrary — and is only fixed by normalization. Argue using the plasma analogy that in order for the wavefunction to remain normalized (with respect to integration over the z ’s) as the w ’s are varied, we must have

$$|\mathcal{N}(w_1, \dots, w_M)| = \mathcal{K} \prod_{1 \leq \alpha < \gamma \leq M} |w_\alpha - w_\gamma|^{1/m} \prod_{1 \leq \alpha \leq M} e^{-|w_\alpha|^2/(4m\ell^2)} \quad (25)$$

with $\mathcal{K}$ a constant. (Hint: a plasma will screen a charge).