Appendix: Kac and Other Resources for TQFTs

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Medium Material

Working out the details of a TQFT is an often tedious task and except in the simplest cases, one does not want to go through the pain of doing this. Here we explain the use of one particularly convenient way to find data for many TQFTs. At the end of this chapter we list a number of other useful resources and references.

38.1 Kac

Perhaps the most useful single resource I have found for obtaining data about TQFTs is a computer program called *Kac* written by A. N. Schellekens. The complicated part of the algorithm is described by Fuchs et al. [1996]. More details are given on the project webpage.

The progam can be downloaded from the webpage

`https://www.nikhef.nl/~t58/Site/Kac.html`

also mirrored on my web page.

While the program has many capabilities (and I encourage you to RTFM[1]), and one can type `Help` at the command prompt to get some of the information in the manual too. Nonetheless, it is probably useful to give here some annotated examples of how it works.

[1] Read the Frikkin Manual

Note that the program uses Dynkin diagram (Cartan) notation for describing Lie algebras. The correspondence is given by

$$\begin{aligned} A_r &= su(r+1) \\ B_r &= so(2r+1) \\ C_r &= sp(2r) \\ D_r &= so(2r) \end{aligned} \tag{38.1}$$

One can also use the $E6, E7, E8, F4$ and $G2$ Lie algebras.

Here we present some annotated sessions with Kac.

```
MYLINUXBOX$Kac
Kac (on MYLINUXBOX), version 8.05468, compiled on Sep  1 2016, at 16:27:29
Started Sun 30 Aug 05:19:53 BST 2020
Non-interactive mode; Assuming default answer: OK
> tensor
```

The tensor command tells the program that we might be tensoring together multiple theories.

```
> g a 1 2
```

This inputs the group (`g` for group) with the Cartan notation `a 1`, or A_1 which is $su(2)$ as given by the correspondence Eq. 38.1 above, and the 2 indicates level 2. So we are asking it to compute information about $SU(2)_2$.

```
> display
CFT {A1:2}; 3 primaries (2 simple currents)
 Lbl  Comb.  Weights     Wts.  F.l.  F.m.
   0  {0}    0.0000000   0     -     1
   1  {1}    0.5000000   1/2   -     1
   2  {2}    0.1875000   3/16  -     1
```

The fields are numbered 0,1,2, and we see their corresponding weights $h = 0, 1/2, 3/16$. The simple currents are always listed first. Recall that twist factors are given by $\theta = e^{2\pi i h}$. Note also that the weights are only correct modulo one. We can then ask for quantities like the fusion rules, or the S-matrix[2].

[2]Caution that the S-matrix $S(a,b) = S_{a\bar{b}}$ in our notation. See margin note 7 from chapter 17.

```
> fusion
(0) x (0) =  (0)

(0) x (1) =  (1)

(0) x (2) =  (2)

(1) x (1) =  (0)

(1) x (2) =  (2)

(2) x (2) =  (0)  + (1)

> S
S(0,0) =      0.50000000
S(0,1) =      0.50000000
S(0,2) =      0.70710678
S(1,1) =      0.50000000
S(1,2) =     -0.70710678
S(2,2) =      0.00000000
```

We can also get the central charge or the Frobenius-Schur indicator of, say, the second particle,

```
> Browse Central
Central charge              1.500000000000000

> Get Schur 2
-1
```

If we had wanted to look at the opposite chirality theory, we use `h` rather than `g`. To wipe the memory of the program and return to tensor mode we use `reset tensor`. So for example, we have

```
> reset tensor
> h a 1 2
> display
CFT {A1:2}; 3 primaries (2 simple currents)
 Lbl  Comb.  Weights             Wts.           F.l.  F.m.
   0  {0}    0.0000000  (mod 1)  0              -     1
   1  {1}    0.5000000  (mod 1)  1/2 (mod 1)    -     1
   2  {2}    0.8125000  (mod 1)  13/16 (mod 1)  -     1
```

Note that the weight of the 2 field is $13/16 = -3/16$ mod 1 so this is the opposite chirality version of $SU(2)_2$ which we write as $\overline{SU(2)_2}$.

The program can also handle $U(1)$, or abelian, Chern-Simons theory[3]. We produce these $U(1)$ theories using the code `g u` followed by the level as follows

[3]See the comment 9 in chapter 5. Be cautioned that there are multiple conventions as to how the subscript of a $U(1)$ theory is labled which differ by a factor of 2.

```
> reset tensor
> g u 4
> display
CFT {U4:0}; 4 primaries (4 simple currents)
 Lbl  Comb.  Weights    Wts.  F.l.  F.m.
   0  {0}    0.0000000  0     -     1
   1  {1}    0.1250000  1/8   -     1
   2  {2}    0.5000000  1/2   -     1
   3  {3}    0.1250000  1/8   -     1
```

The level here (4 in this case) is simply the constant k in Eq. 5.1. The program will caution you if you put in an odd level since that will return a non-modular theory as we found in section 4.3.2.

The program can generate minimal CFT models with N supersymmetries. If we want the usual Virasoro minimal series, we set $N = 0$. So for example, we have the first minimal CFT

```
> reset tensor
> g minimal 0 1
> display
CFT {minimal N=0 1}; 3 primaries (2 simple currents)
 Lbl  Comb.  Weights     Wts.  F.l.  F.m.
   0  {0}    0.0000000   0     -     1
   1  {1}    0.5000000   1/2   -     1
   2  {2}    0.0625000   1/16  -     1
```

which we recognize as the Ising model. The second Virasoro minimal model (the tricritcal Ising model) would be

```
> g minimal 0 2
```

and so forth. See the `Help` notes for the $N = 1, 2$ minimal models.

The program can handle condensation, as well as splitting. Let us consider the example used in section 25.4 of $SU(2)_4$. We first produce the $SU(2)_4$ theory

```
> reset tensor
> g a 1 4
> display
CFT {A1:4}; 5 primaries (2 simple currents)
 Lbl  Comb.  Weights     Wts.  F.l.  F.m.
   0  {0}    0.0000000   0     -     1
   1  {1}    1.0000000   1     -     1
   2  {2}    0.1250000   1/8   -     1
   3  {3}    0.6250000   5/8   -     1
   4  {4}    0.3333333   1/3   -     1
```

Note that one of the simple currents is a boson (integer weight). To condense it we issue the command `current` and the name of the field we want to condense.

```
> current 1
> display
CFT {A1:4}; 3 primaries
 Lbl  Comb.  Weights     Wts.  F.l.  F.m.
   0  {0}    0.0000000   0     -     1
   1  {4}    0.3333333   1/3   0     1
   2  {4}    0.3333333   1/3   1     1
```

Which correctly splits the 4-particle as we discussed in section 25.4.

To generate product theories, we just input several theories in a row. For example, to look at a product theory, $SU(2)_2 \times \overline{SU(2)_1} \times \overline{SU(2)_1}$

we write

```
> reset tensor
> g a 1 2
> h a 1 1
> h a 1 1
> display
CFT {A1:2_A1:1_A1:1}; 12 primaries (8 simple currents)
 Lbl  Comb.    Weights              Wts.           F.l.  F.m.
   0  {0,0,0}  0.0000000  (mod 1)  0              -     1
   1  {0,0,1}  0.7500000  (mod 1)  3/4 (mod 1)    -     1
   2  {0,1,0}  0.7500000  (mod 1)  3/4 (mod 1)    -     1
   3  {0,1,1}  0.5000000  (mod 1)  1/2 (mod 1)    -     1
   4  {1,0,0}  0.5000000  (mod 1)  1/2 (mod 1)    -     1
   5  {1,0,1}  0.2500000  (mod 1)  1/4 (mod 1)    -     1
   6  {1,1,0}  0.2500000  (mod 1)  1/4 (mod 1)    -     1
   7  {1,1,1}  0.0000000  (mod 1)  0 (mod 1)      -     1
   8  {2,0,0}  0.1875000  (mod 1)  3/16 (mod 1)   -     1
   9  {2,0,1}  0.9375000  (mod 1)  15/16 (mod 1)  -     1
  10  {2,1,0}  0.9375000  (mod 1)  15/16 (mod 1)  -     1
  11  {2,1,1}  0.6875000  (mod 1)  11/16 (mod 1)  -     1
```

Since $SU(2)_2$ has 3 fields, and each $\overline{SU(2)_1}$ has 2 fields, the product of these three theories has 12 fields. The second column of the output shows how each field is constructed from the constituent factors. For example, the output field labeled 9 in the far left column comes from the 2 field of $SU(2)$, the 0 field from the first $\overline{SU(2)_1}$ and the 1 field from the second $\overline{SU(2)_1}$.

Let us now construct the coset $SU(2)_2/(SU(2)_1 \times SU(2)_1)$. Recall from section 25.6 that one can construct this coset by starting with $SU(2)_2 \times \overline{SU(2)_1} \times \overline{SU(2)_1}$ and condensing all possible simple current bosons. Notice in the above output that there are 8 simple currents, and the one labeled 7 or `{1,1,1}` is a boson. We thus issue the command

```
> current 1 1 1
> display
CFT {A1:2_A1:1_A1:1}; 3 primaries (2 simple currents)
 Lbl  Comb.    Weights              Wts.           F.l.  F.m.
   0  {0,0,0}  0.0000000  (mod 1)  0              -     1
   1  {0,1,1}  0.5000000  (mod 1)  1/2 (mod 1)    -     1
   2  {2,0,1}  0.9375000  (mod 1)  15/16 (mod 1)  -     1
```

giving us the result that this coset is actually $\overline{\text{Ising}}$.

38.2 Other Resources

Note that many of the following references give only the so-called "modular data" for TQFTs — meaning the S-matrices (which imply the fusion rules via the Verlinde formula, Eq. 17.13) and the twist factors θ_a. However, it has recently been established that there can be cases where more than one modular TQFT can share the same modular data (Mignard and Schauenburg [2017][4]). However the simplest such case known where the modular data does not uniquely define the TQFT has 49 different particle types and for all simple TQFTs the modular data is, at least in principle, full information.

[4]See also Bonderson et al. [2019] and Delaney and Tran [2018] for discussion of what additional data might be added to make the TQFT unique

- A useful reference on conformal field theory, including WZW theories (which give you the content of the corresponding Chern-Simons theory) is given by Di Francesco et al. [1997].
- Many details of the simplest few modular tensor categories on the periodic table are given by Rowell et al. [2009]; A discussion for fermionic models is given by Bruillard et al. [2017, 2020]. See also Lan et al. [2017, 2016]; Wen [2015].
- Some nice data for some simple categories is given by Bonderson [2007]. This includes, for example, the F-matrices for $SU(2)_k$ and a number of other simple theories.
- F-matrices for many more complicated theories are given by Ardonne and Slingerland [2010]. Mathematica code based on this is available at

`https://github.com/ardonne/affine-lie-algebra-tensor-category`

- Online databases of vertex algebras, modular categories, fusion rings etc are given at

  `https://www.math.ksu.edu/~gerald/voas/`

  `http://www.thphys.nuim.ie/AnyonWiki/index.php/Main_Page`

- An online database of modular data for (twisted and untwisted) gauge theories (i.e., quantum doubles and twisted quantum doubles) is given by

  `https://tqft.net/web/research/students/AngusGruen/`

  Just to make the rest of us feel bad, this was a bachelor's thesis!