Riemannian Gauge Theory and Charge Uniqueness

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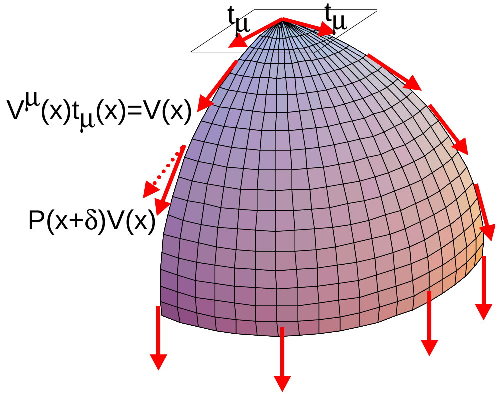
1. Geometry vs. Gauge Theory

- **Defining a gauge theory: A group $g(x)$**
  - **Matter fields:** $\phi'(x) = g(x)\phi(x)$.
  - **Gauge fields:** $A'_\mu = g\, A_\mu\, g^{-1} + (i/q)\, g\, \partial_\mu\, g^{-1}$
    - ◇ Defined to cancel extra terms in $\partial_\mu(g\,\phi)$
  - **Action: Lorentz & gauge invariant**
- **Similarities to General Relativity**

Object	General Relativity	Gauge Theory
Covariant Derivative	$\nabla_\mu V^\alpha = (\delta^\alpha_\sigma \partial_\mu + \Gamma^\alpha_{\mu\sigma}) V^\sigma$	$D_\mu \phi^a = (\delta^a_b \partial_\mu - iq A_\mu{}^a{}_b)\phi^b$
Connection Coefficient	$\Gamma^\alpha_{\mu\sigma}$	$-iq A_\mu{}^a{}_b$
Curvature Tensor	$R_{\mu\nu}{}^\lambda{}_\sigma V^\sigma = ([\nabla_\mu, \nabla_\nu] V)^\lambda$	$-i\, F_{\mu\nu}{}^a{}_b \phi^b = ([D_\mu, D_\nu]\phi)^a$

2. Embedding Geometry

- **Embedding is always possible (Nash (1956) [1])**



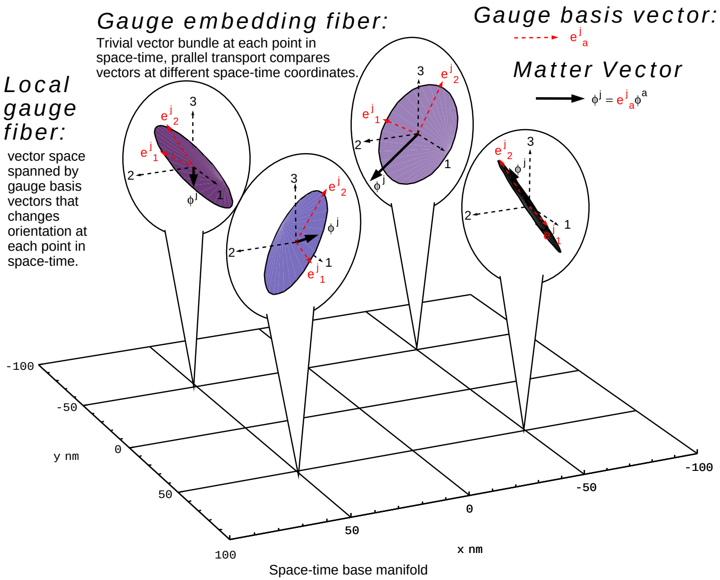
- **Dual** $t^\nu(t_\mu) = \delta^\mu_\nu$, **Projection** $P = t_\nu t^\nu(\;)$
- **Metric** $g_{\mu\nu} = \langle t_\mu, t_\nu \rangle$
- **The Covariant Derivative**

$$\begin{aligned}\nabla_\mu V &= P(\partial_\mu(V^\sigma t_\sigma)) \\ &= t_\nu t^\nu \left[t_\sigma \partial_\mu V^\sigma + V^\sigma \partial_\mu t_\sigma\right] \\ &= t_\nu \left[t^\nu(t_\sigma)\partial_\mu V^\sigma + V^\sigma t^\nu(\partial_\mu t_\sigma)\right] \\ &= t_\nu \left[\delta^\nu_\sigma \partial_\mu V^\sigma + V^\sigma t^\nu(\partial_\mu t_\sigma)\right]\end{aligned}$$

$$\Gamma^\nu_{\mu\sigma} = t^\nu(\partial_\mu t_\sigma)$$

3. Vector Bundle Gauge Theory

◇ *Based on:* Narasimhan and Ramanan (1961) [2, 3], Atiyah (1979) [4], Dubois-Violette and Georgelin (1979) [5], Cahill and Raghavan (1993) [6]

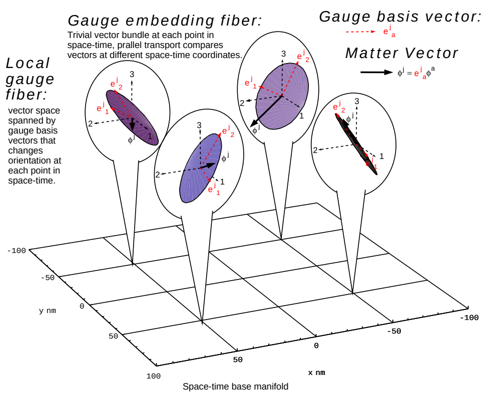


- Embed $F \subset \tilde{F}$
  - F = gauge vector bundle
  - $\tilde{F}$ = trivial embedding vector bundle.
- ϕ = Matter field (wave-function) on F.

- **Embedding is always possible (Narasimhan & Ramanan (1961) [2, 3])**
- **Dual** $e^b(e_a) = \delta^b_a$, **Projection** $P = e_b e^b(\ )$
- **Orthonormal basis vectors** $\delta_{\bar{a}b} = \langle e_a, e_b \rangle$
- **The Covariant Derivative**

$$
\begin{aligned}
D_\mu \phi &= P(\partial_\mu(\phi^a e_a)) \\
&= e_b e^b \left[e_\sigma \partial_\mu \phi^a + \phi^a \partial_\sigma e_a \right] \\
&= e_b \left[e^\nu(e_a) \partial_\mu \phi^a + \phi^a e^b(\partial_\mu e_a) \right] \\
&= e_b \left[\delta^b_a \partial_\mu \phi^a + \phi^a e^b(\partial_\mu e_a) \right]
\end{aligned}
$$

$$
-iA_\mu{}^a{}_b = e^a(\partial_\mu e_b) \quad \in \quad u(n)
$$



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Geometry vs. Gauge Theory

Embedding Geometry

Vector Bundle Gauge Theory

Charge Uniqueness

Multiple Charges

Conclusions

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- We have Represented gauge theory geometrically
  - ◇ *Based on:* Narasimhan and Ramanan (1961) [2, 3], Atiyah (1979) [4], Dubois-Violette and Georgelin (1979) [5], Cahill and Raghavan (1993) [6]

  - Covariant derivative

Riemannian Geometry	$P(\partial_\mu V) = t_\alpha \left[\delta^\alpha_\sigma \partial_\mu + t^\alpha(\partial_\mu t_\sigma)\right] V^\sigma$
Riemannian Gauge Theory	$P(\partial_\mu \phi) = e_a \left[\delta^a_b \partial_\mu + e^a(\partial_\mu e_b)\right] \phi^b$

  - The Connection Coefficient

Riemannian Geometry	$\Gamma^\alpha_{\mu\sigma} = t^\alpha(\partial_\mu t_\sigma)$
Riemannian Gauge Theory	$-iA_\mu{}^a{}_b = e^a(\partial_\mu e_b)$

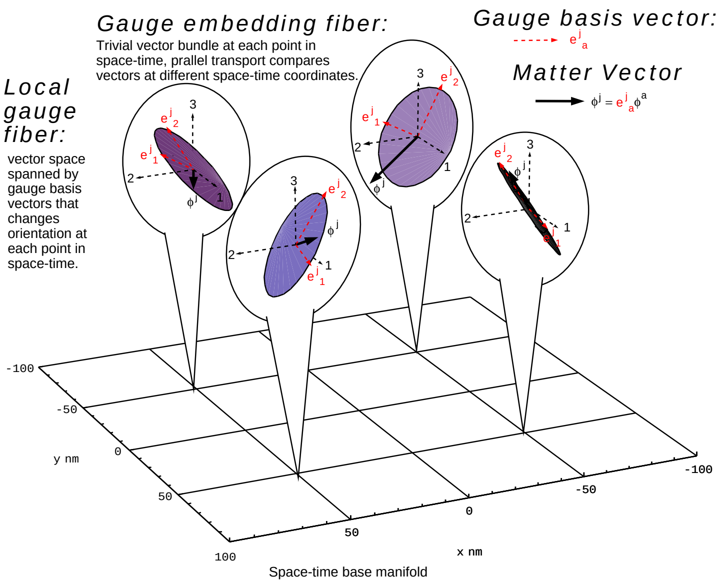
- What if we Define gauge theory geometrically with vectors on vector bundles?
  - We call this: Riemannian Gauge Theory

4. Charge Uniqueness

- **No adjustable parameter in covariant derivative**

$$P(\partial_\mu \phi) = e_a \left[\delta^a_b \partial_\mu + e^a (\partial_\mu e_b)\right] \phi^b$$

  - **Reason: Matter fields are vectors on a vector bundle.**
  - **Generalized Equivalence Principle**



5. Multiple Charges

- Steps to Simulating Multiple Charges in Riemannian Gauge Theory
  - Create two independent $U(1)$ gauge fields
    - ◇ $B_\mu = i\, b^1(\partial_\mu b_1)$ where b_1 spans $F^B = \mathbf{C}^1$
    - ◇ $C_\mu = i\, c^1(\partial_\mu c_1)$ where c_1 spans $F^C = \mathbf{C}^1$
  - b_1 and c_1 basis vectors for different $\mathbf{C}^1$ fibers.
  - Constrain the curvature of F^B and F^C:

$$B_\mu/q^B = C_\mu/q^C = A_\mu$$

  - Coupling is now

$$(\partial_\mu - iq^B A_\mu) \qquad (\partial_\mu - iq^C A_\mu)$$

- Shifted Charge Quantization Problem
  - All fields naturally couple with same charge
  - Why constrained curvature between fibers?
- Grand Unification provides one explanation for the constraint.

6. Conclusions

- Riemannian Gauge Theory
  - No adjustable parameter in covariant derivative.
  - $U(1)$ and $U(n)$ have natural single coupling to all fields

- Shifted Charge Quantization Problem
  - ***Old***: Why e^-, μ^-, τ^- ...same charge?
  - ***New***: Why e^- and q have different charges?

- Larger $GL(n, \mathbf{C})$ or $GL(n, \mathbf{R})$ gauge invariance

- For more information: /hep-th/0205250 [7]

References

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[2] **M. S. Narasimhan and S. Ramanan,** *Existence of universal connections*, ***Am. J. Math.*** **83 (1961) 563.**

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[6] **K. Cahill and S. Raghavan,** *Geometrical representations of gauge fields*, ***J. Phys.*** **A26 (1993) 7213–7217.**

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