Quantum Mechanics MT 2024: Problem Sheet 1

This problem sheet has been developed from previous problem sheets of (at least) James Binney, Stephen Blundell, Fabian Essler

Quantum Ideas

1.1 Give an example of a physical phenomenon that requires us to work with *probability amplitudes* rather than with probabilities.

1.2 Determine the kinetic energy of an electron that has de Broglie wavelength $\lambda = h/p$ of 0.1nm.

Dirac notation

1.3 Given the ket $|\psi\rangle = e^{i\pi/5}|a\rangle + e^{i\pi/4}|b\rangle$, express the bra $\langle\psi|$ as a linear combination of the bras $\langle a|$ and $\langle b|$.

1.4 An electron can be in one of two potential wells that are so close that it can 'tunnel' from one to the other. Its state vector can be written

$$|\psi\rangle = a|A\rangle + b|B\rangle, \tag{1}$$

where $|A\rangle$ is the state of being in the first well and $|B\rangle$ is the state of being in the second well and all kets are correctly normalised. What is the probability of finding the particle in the first well given that: (a) $a = i/2$; (b) $b = e^{i\pi}$; (c) $b = \frac{1}{3} + i/\sqrt{2}$?

1.5 What is the difference between 0, $\hat{0}$, $|0\rangle$ and $\hat{0}|\psi\rangle$, where $|\psi\rangle$ is any ket?

1.6 An electron can "tunnel" between potential wells that form a linear chain, so its state vector can be written as

$$|\psi\rangle = \sum_{n=-\infty}^{\infty} a_n|n\rangle, \tag{2}$$

where $|n\rangle$ is the state of being in the n^{th} well, where n increases from left to right. Let

$$a_n = \frac{1}{\sqrt{2}}\left(\frac{-i}{3}\right)^{\frac{|n|}{2}} e^{in\pi}. \tag{3}$$

(a) What is the probability of finding the electron in the n^{th} well?
(b) What is the probability of finding the electron in well 0 or anywhere to the right of it?

1.7 *A watched kettle never boils (proverb)*

A certain 2-state quantum system has two states $|\text{on}\rangle$ and $|\text{off}\rangle$. Whenever the system starts in initial condition $|\psi(t=0)\rangle = |\text{off}\rangle$, the subsequent time-evolution of this particular system is such that

$$|\psi(t)\rangle = \cos(\omega t)|\text{off}\rangle + \sin(\omega t)|\text{on}\rangle \tag{4}$$

If the system is measured at time T, what is the probability that the system is found in state $|\text{on}\rangle$?

If $T < \frac{\pi}{2\omega}$, does including additional earlier measurements at various times $t_i < T$ make it more or less likely that the final measurement at time T will now find the system in state $|\text{on}\rangle$? What happens in the limiting case where the system is measured continually up until time T?

Operators

1.8 What is a Hermitian operator? Given that $\hat{A}$ and $\hat{B}$ are Hermitian, which of the following operators are Hermitian (c is an arbitrary complex number):

$\hat{A} + \hat{B}$; $c\hat{A}$; $\hat{A}\hat{B}$; $\hat{A}\hat{B} + \hat{B}\hat{A}$?

1.9 Given that $\hat{A}$ and $\hat{B}$ are Hermitian operators, show that $i[\hat{A}, \hat{B}]$ is a Hermitian operator.

1.10 Given that for any two operators $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$, show that

$$(\hat{A}\hat{B}\hat{C}\hat{D})^\dagger = \hat{D}^\dagger\hat{C}^\dagger\hat{B}^\dagger\hat{A}^\dagger.$$

1.11 A three-state quantum system has a complete orthonormal set of states $|1\rangle, |2\rangle, |3\rangle$. With respect to this basis the operators $\hat{H}$ and $\hat{B}$ have matrices

$$\hat{H} = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \hat{B} = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω and b are real constants.

(a) Are $\hat{H}$ and $\hat{B}$ Hermitian?

(b) Write down the eigenvalues of $\hat{H}$ and find the eigenvalues of $\hat{B}$. Solve for the eigenvectors of both $\hat{H}$ and $\hat{B}$. Explain why neither matrix uniquely specifies its eigenvectors.

(c) Show that $\hat{H}$ and $\hat{B}$ commute. Give a basis of eigenvectors common to $\hat{H}$ and $\hat{B}$.

1.12 Let $\hat{Q}$ be the operator of an observable and let $|\psi\rangle$ be the ket giving the state of our system.

(a) What are the physical interpretations of $\langle\psi|\hat{Q}|\psi\rangle$ and $|\langle q_n|\psi\rangle|^2$, where $|q_n\rangle$ is the n^{th} eigenket of the observable $\hat{Q}$ and q_n is the corresponding eigenvalue?

(b) What is the operator $\sum_n |q_n\rangle\langle q_n|$, where the sum is over all eigenkets of $\hat{Q}$? What is the operator $\sum_n q_n|q_n\rangle\langle q_n|$?

Commutators : Manipulations

1.13 Show that

(a) $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

(b) $[\hat{A}\hat{B}\hat{C}, \hat{D}] = \hat{A}\hat{B}[\hat{C}, \hat{D}] + \hat{A}[\hat{B}, \hat{D}]\hat{C} + [\hat{A}, \hat{D}]\hat{B}\hat{C}$. Explain the similarity with the rule for differentiating a product.

(c) $[\hat{x}^n, \hat{p}] = \mathrm{i}\hbar n\hat{x}^{n-1}$ (you may use that $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$)

(d) $[f(\hat{x}), \hat{p}] = \mathrm{i}\hbar\frac{\mathrm{d}f}{\mathrm{d}x}$ for any analytic function $f(x)$.

1.14 Does it always follow that if a system is an eigenstate of $\hat{A}$ and $[\hat{A}, \hat{B}] = 0$ then the system will be in a eigenstate of $\hat{B}$? If not, give a counterexample.