These are work-in-progress notes for the second-year course on mathematical methods. The most up-to-date version is available from http://eww.tphys.ox.ac.uk/people/JohnMagorrian/maths.

7 Linear operators on functions

In this section we consider fairly general linear second-order linear differential operators of the form

$$A_u = \frac{1}{w(x)} \left[ p_0(x) \frac{d^2 u}{dx^2} + p_1(x) \frac{du}{dx} + p_2(x) u \right],$$

(7.1)

where • can be replaced by any sufficiently smooth function drawn from $L^2_w(a,b)$, and the $p_i(x)$ are smooth real functions $p_i: [a,b] \to \mathbb{R}$ with $p_0(x) \neq 0$ for any $x \in (a,b)$. Motivated by the pleasing properties of Hermitian operators in finite-dimensional vector spaces, we first show that many of the ordinary differential equations encountered in physics problems can be cast as eigenvalue equations, $Af = f$, where $A$ is an operator of the form (7.1) that is also Hermitian. We discuss various properties of the solutions to such equations, before giving more detailed examples of how to find explicit solutions.

7.1 Hermitian operators

Under what conditions is the operator $A$ defined by (7.1) Hermitian? Recall from §3.4 that the dual $A^\dagger$ to an operator $A$ is defined by requiring that $(u | A | v) = (v | A^\dagger | u)^*$ for all choices of $(u) and $(v). If $A$ is Hermitian then $A^\dagger = A$, which is equivalent to the condition that

$$(u | A | v) = (v | A | u)^*$$

(7.2)

for any $(u), (v). Taking $A$ from (7.1) and using the inner product defined by (4.2), this condition becomes

$$\int_a^b dx \, u^* \left[ p_0 \frac{d^2 u}{dx^2} + p_1 \frac{du}{dx} + p_2 u \right] = \left\{ \int_a^b dx \, v^* \left[ p_0 \frac{d^2 v}{dx^2} + p_1 \frac{du}{dx} + p_2 v \right] \right\}.$$  

(7.3)

The LHS is the sum of the three terms

$$\int_a^b dx \, u^* p_0 v'' = \int_a^b dx \, (u^* p_0) v'' = \int_a^b dx \, (u^* p_0) v'' - \int_a^b dx \, (u^* p_0) v'' = \int_a^b dx \,(u^* p_0) v''$$

and

$$\int_a^b dx \, u^* p_1 v'.$$

(7.4)

which, when added together, give

$$\text{LHS} = (u^* p_0 v'' + (u^* p_0) v'' + \int_a^b dx \, (u^* p_0) v' - (p_1 u^* + p_2 u^*) v).$$

(7.5)

The RHS is simply

$$\text{RHS} = \int_a^b dx \, v \left[ p_0 (u^*)'' + p_1 (u^*)' + p_2 u^* \right].$$

(7.6)

Two conditions must be satisfied if the LHS is to equal the RHS. First, the two integrals must be equal, which requires that $p_1 = 0$. Second, the boundary term that appears in the LHS must vanish. Setting $p_1 = 0$, this condition on the boundary term becomes

$$0 = [u^* p_0 v'' - (u^* p_0) v''].$$  

(7.7)

A sufficient condition for this to hold is

$$0 = [u^* p_0 v''].$$  

(7.8)

In summary, the operator

$$A_{SL} = \frac{1}{w(x)} \left[ \frac{d}{dx} \left( p(x) \frac{dw}{dx} \right) + q(x) \right].$$  

(7.9)

is Hermitian provided it is restricted to functions $u(x), v(x)$ that satisfy the boundary condition

$$[u^* w \frac{dw}{dx}]_a^b = 0.$$

(7.10)

Here $p(x)$ and $q(x)$ are smooth, real functions with $p(x) \neq 0$ for $x \in (a,b)$. The ‘SL’ subscript in $A_{SL}$ stands for ‘Sturm–Liouville’.

Exercise: Show that any differential operator

$$A = p_0(x) \frac{d^2}{dx^2} + p_1(x) \frac{d}{dx} + p_2(x)$$

(7.11)

can be written in Sturm–Liouville form (7.9) by choosing

$$\begin{align*}
p(x) &= \exp \int_a^b \frac{p_0(t)}{p_0(x)} dt, \\
w(x) &= \frac{\frac{d}{dx}}{p_0(x)} = \frac{-1}{p_0(x)} \exp \int_a^b \frac{p_0(t)}{p_0(x)} dt. \\
q(x) &= p_2(x) w(x).
\end{align*}$$

(7.12)

Therefore any operator of the form (7.1) can be taken to be Hermitian provided the boundary conditions (7.10) are satisfied. The condition that the $p_1(x)$ are real with $p_0(x) > 0$ ensures that $w(x) > 0$ for $x \in (a,b)$. What must we do to ensure $w(x) > 0$ if $p_0(x) < 0$ for $x \in (a,b)$?

7.2 Eigenfunctions

The eigenvalue equation,

$$A_{SL} \psi_n(x) = \lambda_n \psi_n(x),$$

(7.13)

in which $A_{SL}$ is given by (7.9) and the eigenfunctions are assumed to satisfy the boundary conditions (7.10), is known as a Sturm–Liouville equation.

NB: Many books omit the $1/w(x)$ factor in the definition of the operators (7.1) and (7.9), only to have the weight function reappear in the Sturm–Liouville equation, which becomes

$$A_{SL} \psi_n = \lambda_n w(x) \psi_n(x).$$
Some eigenproperties of $A_{SL}$:

(i) The eigenvalues $\lambda_n$ are real.

(ii) If $\lambda_i \neq \lambda_j$ the corresponding eigenfunctions $e_i(x)$, $e_j(x)$ are orthogonal:

$$\langle e_i | e_j \rangle \equiv \int_a^b e_i^*(x) e_j(x) w(x) \, dx = 0. \quad (7.14)$$

If $\lambda_i = \lambda_j \; (j \neq i)$ the eigenfunctions $e_i(x)$ and $e_j(x)$ are LI and we can use the Gram-Schmidt procedure to make them orthogonal.

(iii) The eigenfunctions are a complete set (basis): any function $f \in L^2_{\omega}(a,b)$ can be expressed as the generalised Fourier series

$$f(x) = \sum_j a_j e_j(x),$$

with $a_j = \langle c_j | f \rangle = \int_a^b e_j^*(x) f(x) w(x) \, dx$, \quad (7.15)

assuming the eigenfunctions $e_j(x)$ have been normalised so that $\langle e_j | e_j \rangle = 1$. If the weight of tradition means that $\langle e_i | e_i \rangle = 1$, we need to divide the RHS of the expression for $a_j$ in (7.15) by $\langle e_i | e_i \rangle$.

(iv) If the interval $[a,b]$ is of finite length then the eigenvalues $\lambda_n$ are discrete.

By construction $A_{SL}$ satisfies the condition (7.2), which means that the proofs of (i) and (ii) are identical to the corresponding proofs for finite-dimensional Hermite operators in §3.11. You do not need to be able to prove (iii) and (iv), but should remember the results.

### 7.3 Examples

#### Simple harmonic oscillator

A simple but important example of a Sturm-Liouville equation is

$$\frac{d^2 e_n}{dx^2} = \lambda_n e_n. \quad (7.16)$$

The LHS involves $A_{SL} \; (\text{equation 7.9})$ with $p(x) = w(x) = 1$ and $q(x) = 0$. There are two LI solutions to (7.16) for any choice of $\lambda_n$: $e_n^1(x) = \exp(\sqrt{\lambda_n} x)$ and $e_n^2(x) = \exp(-\sqrt{\lambda_n} x)$. We need to construct eigenfunctions from these that also satisfy the boundary condition (7.10). Taking $a = -\pi$, $b = \pi$, the boundary condition becomes

$$e_n \left. \frac{d e_n}{d x} \right|_x = 0. \quad (7.17)$$

Substituting $e_n(x) = e_n^k(x)$ and $r_n(x) = e_n^j(x)$ into (7.17) we see that this condition is satisfied if $\lambda_n = -k^2$, where $k \in \mathbb{Z}$. So, the (normalised) eigenfunctions of the Sturm-Liouville equation (7.16) are the familiar

$$e_n(x) = \frac{1}{\sqrt{2\pi}} \exp(k x), \quad k = 0, \pm 1, \pm 2, \ldots, \quad (7.18)$$

that we encountered in §4.4. We have already seen that these eigenfunctions are orthogonal and complete.

#### Legendre’s differential equation

Solving Laplace’s equation $\nabla^2 V = 0$ in spherical polar coordinates for axisymmetric systems ($\Theta_\theta = 0$) by substituting $V(r, \theta) = R(r) \Theta(\theta)$ and separating variables leads to the following equation for $\Theta(\theta)$:

$$\frac{d}{d \theta} \left[ \sin \theta \frac{d \Theta}{d \theta} \right] + \lambda \Theta = 0, \quad \Theta = \Theta(\theta) \quad (7.19)$$

where $\lambda$ is a separation constant. Substituting $x = \cos \theta$, this becomes

$$\left[ \frac{d}{dx} \left( 1 - x^2 \right) \frac{d}{dx} \right] \Theta = \lambda \Theta, \quad (7.20)$$

which is known as Legendre’s differential equation. It is a Sturm-Liouville equation (7.13),

$$A_{SL} \Theta = \lambda \Phi, \quad (7.21)$$

with $w(x) = 1$, $p(x) = 1 - x^2$ and $q(x) = 0$ in $A_{SL}$ \; (equation 7.9). The natural limits on $x = \cos \theta$ are $a = -1$ and $b = 1$. Notice then that $p(-1) = p(1) = 0$ and so the boundary condition (7.10) is naturally satisfied for any $\Theta(x)$. The eigenfunctions of this $A_{SL}$ are Legendre polynomials, $\Theta_n(x) = \Theta_n(\theta)$, with corresponding eigenvalues $\lambda_n = -l(l+1)$, where $l = 0, 1, 2, \ldots$. Section 8.1 explains how to explain explicit expressions for $\Theta(x)$ and why the eigenvalues have the form $\lambda_n = -l(l+1)$.

The table below lists some common examples of Sturm-Liouville equations. Notice that in most cases $p(x) = b(x) = 0$, which means that the boundary condition (7.10) is automatically satisfied.

<table>
<thead>
<tr>
<th>name</th>
<th>equation</th>
<th>$p(x)$</th>
<th>$q(x)$</th>
<th>$w(x)$</th>
<th>$a$</th>
<th>$b$</th>
<th>$\lambda_n$</th>
<th>$e_n(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SHO</td>
<td>$(1 - x^2)p(x) + q(x) = 0$</td>
<td>1</td>
<td>$-x^2$</td>
<td>1</td>
<td>$-\pi$</td>
<td>$\pi$</td>
<td>$-\pi^2$</td>
<td>$P_n^1(x)$</td>
</tr>
<tr>
<td>Legendre</td>
<td>$(1 - x^2)p(x) + q(x) = 0$</td>
<td>1</td>
<td>$-x^2$</td>
<td>1</td>
<td>$-\pi$</td>
<td>$\pi$</td>
<td>$-\pi^2$</td>
<td>$P_n^2(x)$</td>
</tr>
<tr>
<td>Laguerre</td>
<td>$x \frac{d}{dx} + q(x) = 0$</td>
<td>$x e^{-x}$</td>
<td>$e^{-x}$</td>
<td>0</td>
<td>$\infty$</td>
<td>$-n$</td>
<td>$\lambda_n(x)$</td>
<td></td>
</tr>
<tr>
<td>Hermite</td>
<td>$x \frac{d}{dx} + q(x) = 0$</td>
<td>$x e^{-x}$</td>
<td>$e^{-x}$</td>
<td>$\infty$</td>
<td>$-\infty$</td>
<td>$-2n$</td>
<td>$H_n(x)$</td>
<td></td>
</tr>
<tr>
<td>Bessel</td>
<td>$x^2 \frac{d}{dx} + x q(x) + (x^2 - \nu^2) u = 0$</td>
<td>$x$</td>
<td>$x - \frac{\nu^2}{2}$</td>
<td>$x$</td>
<td>$\nu$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table: Some examples of ODEs in Sturm-Liouville form, along with their eigenvalues $\lambda_n$ and eigenfunctions $e_n(x)$, where $n = 0, 1, 2, \ldots$. Explicit expressions for many of the eigenfunctions listed here are obtained in the next section. Bessel’s equation is peculiar – see §8.4.

### 7.4 Construction of eigenfunctions

In the next section we explain how to find explicit expressions for the eigenvalues and eigenfunctions of specific Sturm-Liouville equations by using the following methods.

(i) **Series solution** is the most general and direct method. Substituting $e_n(x + a) = x^2 \sum_{k=0}^\infty c_k (x + a)^k$ into the Sturm-Liouville equation and equating powers of $x$ leads to a recurrence relation for the series coefficients $c_k$. The series converges only for certain values of $\lambda$. This gives the spectrum of eigenvalues $\lambda_n$ and the associated eigenvectors $e_n(x)$.

(ii) **Rodrigues’ formula** works in some special cases. In particular, if (i) $q(x) = 0$ and (ii) $s(x) \equiv p(x)/w(x)$ is a quadratic (at most) polynomial with real roots, then the eigenfunctions are given by

$$u_n(x) \propto \frac{1}{K_n(w(x)^{1/2})} \left( \frac{d}{dx} \right)^n [w(x)^{1/2}], \quad (7.22)$$

provided $u \propto x$ and $w(s(x)) = w(b(s(x))) = 0$. The $K_n$ are normalisation constants. Each such $u_n(x)$ is an $n$th-order polynomial that satisfies the Sturm-Liouville equation (7.13) with eigenvalue

$$\lambda_n = n \left( \frac{1}{2} \right) (n-1) s'' + K_n u_n \quad (7.23)$$

for $n = 0, 1, 2, \ldots$. See DK III10.2 for more on this method.
Generating functions are by far the most powerful method, but are much less direct. For example, Legendre polynomials can be defined by the function
\[ G(x,t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{k=0}^{\infty} P_k(x)t^k, \]
(7.24)
in which each \( P_k(x) \) is defined to be the coefficient of \( t^k \) in the Taylor-series expansion of \( (1 - 2xt + t^2)^{-1/2} \). Legendre’s differential equation can actually be derived from this \( G(x,t) \), as can various useful recurrence relations among the \( P_l \).

8 Detailed examples of solving Sturm–Liouville problems

This section explains how to derive explicit expressions for the eigenfunctions and eigenvalues of some of the more frequently encountered Sturm–Liouville equations (7.13). You need only be able to identify the particular ODEs involved and be broadly familiar with the properties of the solutions and how they are obtained.

The methods presented here can easily be applied to other Sturm–Liouville problems.

8.1 Legendre’s differential equation

Legendre’s differential equation is
\[ (1 - x^2)u'' - 2xu' = \lambda u. \]  
(8.1)
This is a Sturm–Liouville equation with \( p(x) = 1 - x^2 \), \( q(x) = 0 \) and \( w(x) = 1 \). The boundaries on \( x \) are \( a = -1 \) and \( b = 1 \). It is a special case of the associated Legendre equation to be discussed later.

Series solution Let us look for solutions of the form
\[ u(x) = \sum_{k=0}^{\infty} a_k x^k, \]  
(8.2)
where the coefficients \( a_k \) depend on the choice of the eigenvalue \( \lambda \) in (8.1). Substituting the series (8.2) into the differential equation (8.1) gives
\[ (1 - x^2) \sum_{k=2}^{\infty} (k(k-1))a_k x^{k-2} - 2 \sum_{k=0}^{\infty} ka_k x^k = \lambda \sum_{k=0}^{\infty} a_k x^k. \]  
(8.3)
Changing the index label of the first sum from \( k \) to \( j = k - 2 \), this becomes
\[ \sum_{j=0}^{\infty} ((j+2)(j+1)a_{j+2} x^j - \sum_{k=0}^{\infty} k(k-1)a_k x^k - 2 \sum_{k=0}^{\infty} ka_k x^k = \lambda \sum_{k=0}^{\infty} a_k x^k, \]  
(8.4)
in which all terms involve \( x \) raised to the power of some summation index. Notice too that all four sums have the same limits (0 and \( \infty \)) on their index. Rewriting all four sums so that they use a common label \( k \) for their index and gathering together into a single sum, we have that
\[ \sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} - k(k+1) + \lambda \right] a_k x^k = 0. \]  
(8.5)
The quantity in curly brackets here must be zero because the \( x^k \) are LI. So, we have derived a recurrence relation,
\[ a_{k+2} = \frac{k(k+1) + \lambda}{(k+2)(k+1)} a_k, \]  
(8.6)
for the coefficients of the series solution (8.2). Notice that \( a_{k+2}/a_k \to 1 \) as \( k \to \infty \), which means that the series will in general diverge. The only way of avoiding this is if the series terminates: there must be some value of \( k \) for which the coefficient
\[ \frac{k(k+1) + \lambda}{(k+2)(k+1)} = 0, \]  
(8.7)
so that \( a_{k+2} = a_{k+4} = \cdots = 0 \). This means that the eigenvalues \( \lambda \) must be of the form \( \lambda = -(l+1) \), where \( l = 0 \) or 1 or 2 or 3 or ....
If we choose \( l \) to be an even (odd) number, then all of the odd- (even-)numbered \( u_n \) must be zero to avoid a divergent series. Therefore, if \( l \) is even (odd) the eigenfunctions are even (odd) functions of \( x \). The most natural starting point for the recurrence relation is \( (u_0, u_1) = (1, 0) \) if \( l \) is even, or \( (u_0, u_1) = (0, 1) \) if \( l \) is odd. The first few \( u_n(x) \) constructed in this way are
\[
\begin{align*}
  u_0(x) &= 1, \\
  u_1(x) &= x, \\
  u_2(x) &= -3x^2 + 1, \\
  u_3(x) &= -\frac{5}{2}x^3 + x.
\end{align*}
\]  
(8.8)

If we normalise these \( u_n(x) \) so that \( u_0(1) = 1 \) we obtain the first few Legendre polynomials,
\[
P_0(x) = 1, \\
P_1(x) = x, \\
P_2(x) = \frac{1}{2}(3x^2 - 1), \\
P_3(x) = \frac{1}{2}(5x^3 - 3x).
\]  
(8.9)

Note that this weighty historical convention that \( P_{-1/2}(\cos \theta) = \sqrt{2/\pi} \) here that the associated Legendre functions satisfy the orthogonality relation
\[
\int_{-1}^{1} P_m^l(x)P_n^l(x) \, dx = \frac{2}{2l + 1} \delta_{mn}.
\]  
(8.10)

The \( P_l(x) \) are a basis for the space \( L^2_{\text{tor}}(-1, 1) \) because they are eigenfunctions of a Sturm–Liouville operator (7.9) that has \( a = -1, b = 1 \) and \( w(x) = 1 \). Therefore we can express any well-behaved \( f : [-1, 1] \to \mathbb{C} \) as a Fourier–Legendre series,
\[
f(x) = \sum_{l=0}^{\infty} a_l P_l(x)
\]  
(8.11)

with coefficients
\[
a_l = \frac{2l + 1}{2} \int_{-1}^{1} P_l(x)^* f(x) \, dx.
\]  
(8.12)

[The \( * \) in the integrands of (8.12) and (8.10) is redundant because the \( P_l \) are real, but I leave it in to remind you that all this is making use of the inner product (4.2) that appears in the generalised Fourier series (4.7).] We have already encountered this series in §4.3, when we used the Gram–Schmidt procedure to construct an orthonormal basis for this space starting from the list of monomials \( 1, x, x^2, \ldots \). The \( s_n(x) \) we constructed there happen to be normalised Legendre polynomials.

### Rodrigues’ formula

Legendre’s differential equation satisfies the conditions for Rodrigues’ formula (equation 7.23) to apply. \( q(x) = 0; s(x) = p(x)/w(x) = 1 - x^2 \) is quadratic with real roots and with \( s(-1) = s(1) = 0 \); the first \( P_l(x) \) \( \neq \) \( 0 \). Substituting this \( s(x) \) and \( w(x) = 1 \) into equation (7.23) we obtain Rodrigues’ formula for the \( P_l(x) \).
\[
P_l(x) = \frac{1}{2l + 1} \frac{d^l}{dx^l} (x^2 - 1)^l.
\]  
(8.13)

which satisfies Legendre’s differential equation (8.1) with eigenvalue \( \lambda_l = -l(l + 1) \). The prefactor in this expression is chosen to maintain the convention that \( P_l(1) = 1 \).

### Generating function

Finally, we note that \( P_l(x) \) can be defined in terms of the generating function
\[
G(t, t) = \frac{1}{(1 - 2xt + t^2)^{1/2}} = \sum_{l=0}^{\infty} P_l(x)t^l.
\]  
(8.14)

Carrying out a Taylor expansion, we have that
\[
[1 - 2xt + t^2]^{-1/2} = 1 - \frac{1}{2} (-2xt + t^2) + \frac{3}{2} (-2xt + t^2)^2 - \frac{3 
3 \sum_2 (-2xt + t^2)^3 + \ldots
\]  
(8.15)
in agreement with the first few \( P_l(x) \) found above in (8.9). The innocent-looking expression (8.14) also encodes Legendre’s differential equation, the normalisation properties of the \( P_l(x) \), recurrence relations and much more. See homework for some examples.

## 8.2 Associated Legendre equation

The associated Legendre equation,
\[
(1 - x^2) \frac{d^2 u}{dx^2} - 2x \frac{du}{dx} - \frac{m^2}{1 - x^2} u = \lambda u,
\]  
(8.16)

appears when using separation of variables to solve equations that involve the Laplacian \( \nabla^2 \) in spherical polar coordinates \((r, \theta, \phi)\). The variable \( x = \cos \theta \), so that the natural boundaries are \( a = -1 \), \( b = 1 \). The equation is an example of a Sturm–Liouville problem with \( p(x) = 1 - x^2 \), \( q(x) = -m^2/(1 - x^2) \) and \( w(x) = 1 \). Legendre’s differential equation corresponds to the special case \( m = 0 \).

The eigenfunctions that satisfy (8.16) are the associated Legendre functions.
\[
P_l^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad (m \geq 0).
\]  
(8.17)

The eigenvalue corresponding to the eigenfunction \( P_l^m(x) \) is \( \lambda_l = -l(l + 1) \), where \( l = 0, 1, 2, \ldots \).

One way of showing this is by differentiating Legendre’s equation (8.1),
\[
(1 - x^2) \frac{d^2 P_l^m}{dx^2} - 2x \frac{dP_l^m}{dx} - \frac{m^2}{1 - x^2} \frac{d^m}{dx^m} P_l(x) = \lambda_l P_l(x),
\]  
(8.18)

\( m \) times to obtain
\[
(1 - x^2)^{m/2} \frac{d^m}{dx^m} \left( (1 - x^2) \frac{d^2 P_l^m}{dx^2} - 2x \frac{dP_l^m}{dx} - \lambda_l P_l(x) \right) = \frac{d^m}{dx^m} \left( -\frac{m^2}{1 - x^2} \frac{d^m}{dx^m} P_l(x) \right),
\]  
(8.19)

where we have introduced
\[
\frac{d^m}{dx^m} P_l(x) = \frac{d^m}{dx^m} P_l(x).
\]  
(8.20)

If we now rewrite (8.19) in terms of \( v(x) \),
\[
v(x) = (1 - x^2)^{m/2} u(x),
\]  
(8.21)

the result, after much manipulation, is
\[
(1 - x^2)^{m/2} \frac{d^m}{dx^m} \left( (1 - x^2) \frac{d^2 v}{dx^2} - 2x \frac{dv}{dx} - \lambda_l v \right) = 0,
\]  
(8.22)

which is associated Legendre equation (8.18). This proves the statement around equation (8.17).

We have so far assumed that \( m \geq 0 \). Using Rodrigues’ formula (8.13) for \( P_l(x) \) in (8.17), we have that
\[
P_l^m(x) = \frac{1}{2^m} \frac{(1 - x^2)^{m/2}}{x} \frac{d^m}{dx^m} \left( x^2 - 1 \right)^2
\]  
(8.23)

which is used to define \( P_l^m(x) \) for both positive and negative \( m \). Clearly \( P_l^m(x) = 0 \) unless \( |m| \leq l \).

### Exercises

By applying Leibnitz’ formula to \((x + 1)^l(x - 1)^l\), show that
\[
P_l^{-m}(x) = (-1)^m \frac{((l - m))!}{((l + m))!} P_l^m(x).
\]  
(8.24)

These \( P_l^m(x) \) are used in the definition of spherical harmonics, which are a natural basis for two-dimensional functions \( f(\theta, \phi) \) defined on the surface of a sphere (see §10.4 later). For reference, we note here that the associated Legendre functions satisfy the orthogonality relation
\[
\int_{-1}^{1} P_l^m(x)P_l^n(x) \, dx = \frac{2}{2l + 1} \frac{(l + m)!}{(l - m)!} \delta_{mn}.
\]  
(8.25)
The fact that the $H_n(x)$ are eigenfunctions of (8.26) with eigenvalue $\lambda = -2n$ is encoded in the generating function (8.33), as are various recurrence relations among the $H_n(x)$. For reference, the first few $H_n(x)$ extracted by expanding the generating function as a power series in $t$ are

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12,$$

in agreement (apart from normalisation) with the results (8.32) obtained from the recurrence relation.

**Rodrigues’ formula** Another way of defining Hermite polynomials is by the Rodrigues’ formula

$$H_n(x) = (-1)^n e^{-x^2} \frac{d^n}{dx^n} \left( e^{x^2} \right).$$

**Exercise:** Show that (8.36) follows from the generating function (8.33) by differentiating $G(x,t)$ $n$ times with respect to $t$ and then setting $t = 0$.

The completeness property of Hermite polynomials means that they are a basis for the space of functions defined on the real line with weight function $e^{-x^2}$. Any well-behaved function $f : \mathbb{R} \to \mathbb{C}$ can be expressed as the generalised Fourier series

$$f(x) = \sum_{n=0}^\infty a_n H_n(x),$$

where the coefficients

$$a_n = \frac{1}{2^{n+1} \sqrt{n!}} \int_{-\infty}^{\infty} H_n(x)f(x)e^{-x^2} \, dx,$$

the prefactor coming from the orthogonality relation (8.34). The $*$ in the integrand here is redundant, but is left in as a reminder that equations (8.37) and (8.38) are simply special cases of equation (4.7).

**8.4 Bessel’s equation**

Bessel’s equation,

$$x^2 u'' + xu' + (\nu^2 - x^2) u = 0,$$

often appears in problems involving cylindrical polar coordinates ($R, \phi, z$), with the variable $x$ being some multiple of the radius $R$ and the constant $\nu$ set by the details of the problem. Equation (8.39) can be squeezed into Sturm–Liouville form by introducing a new independent variable $\tilde{x} = x/\lambda$, in which case it can be written as

$$\frac{1}{\tilde{x}} \left( \frac{d}{d \tilde{x}} \left( \tilde{x} \frac{d}{d \tilde{x}} \right) - \nu^2 \right) u = -\lambda u,$$

so that $w(\tilde{x}) = \tilde{x}$, $p(\tilde{x}) = \tilde{x}$, $q(\tilde{x}) = -\nu^2/2$ and the definition of the independent variable $\tilde{x} = x/\lambda$ depends on the eigenvalue $-\lambda^2$. Unlike the other examples in this section, the boundary conditions in applications of Bessel’s equation normally depend on the details of the problem (see §11 later for an example).

**Series solution** For simplicity we consider only the case where $\nu = m \geq 0$, a non-negative integer. Substituting $u(x) = \sum_{n=0}^\infty a_n x^n$ into (8.39) gives

$$\sum_{n=0}^\infty n(n-1)a_n x^{n-2} + \sum_{n=0}^\infty 2n a_n x^n + \sum_{n=0}^\infty a_n x^{n+2} - m^2 \sum_{n=0}^\infty a_n x^n = 0,$$

Writing $k = n$ in the first, second and fourth sums and $k = n+2$ in the third, this becomes

$$\sum_{k=0}^\infty k(k-1)a_k x^k + \sum_{k=0}^\infty 2k a_k x^k + \sum_{k=0}^\infty a_{k-2} x^k - m^2 \sum_{k=0}^\infty a_k x^k = 0.$$
which, gathering together powers of $x^k$, is

$$-m^2a_0 + (1 - m^2)a_1 x + \sum_{k=2}^{\infty} \left\{ (k^2 - m^2)a_k + a_{k-2} \right\} x^k.$$  

(8.43)

As the $x^k$ are LI, we immediately have the recurrence relation

$$a_k = -\frac{a_{k-2}}{k^2 - m^2}$$  

(8.44)

which relates the even coefficients to one another and the odd coefficients to one another. Clearly, if $m = 0$ then $a_1 = 0$, all odd coefficients $a_{2k+1}$ vanish and we may choose $a_0 \neq 0$ to start the recurrence. If $m = 1$, then $a_0 = 0$, all even coefficients vanish and we may start with $a_1 \neq 0$. More generally, given $m \geq 0$ we need to set $a_0 = \cdots = a_{m-1} = 0$ and can then choose $a_m \neq 0$ to start. The conventional choice is $a_m = 1/(2^m m!)$. The resulting eigenfunctions are the (integer-order) Bessel functions,

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(m+n)!} \left( \frac{x}{2} \right)^{m+2n}.$$  

(8.45)

Generating function More generally, Bessel functions of integer order can be defined through

$$G(x, t) = \exp \left[ \frac{1}{2} \left( t - \frac{1}{t} \right) \right] = \sum_{n=-\infty}^{\infty} J_n(x)t^n.$$  

(8.46)

Further reading

Almost any book with “mathematical methods” and “physics” in its title will cover the topics we have merely skimmed over in this section. See, for example, RHB§ 18. Arfken & Weber’s *Mathematical Methods for Physicists* provides good overviews of how the various methods used to define special functions (differential equation, generating functions, Rodrigues and more) are related to one another.

The Sturm-Liouville equation is a second-order ODE, which of course has two LI solutions. Here we have focused on finding the “well-behaved” solutions that satisfy certain boundary conditions. (We did not explicitly state our assumptions about the boundary conditions for the solution to Bessel’s equation, but the form of the series we assumed was an implicit boundary condition.) The books mentioned above give more details on how to find the second, LI solutions that are less well behaved, but sometimes useful.