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S7 and BT VII: Classical mechanics – problem set 3

1. Starting from the definition of a canonical map in terms of the Poincaré integral invariant, explain how functions of the form

(a) $F_1(q,Q,t)$ generate mappings $p = \frac{\partial F_1}{\partial q}$, $P = -\frac{\partial F_1}{\partial Q}$, $K = H + \frac{\partial F_1}{\partial t}$;
(b) $F_2(q,P,t)$ generate mappings $p = \frac{\partial F_2}{\partial q}$, $Q = \frac{\partial F_2}{\partial P}$, $K = H + \frac{\partial F_2}{\partial t}$;
(c) $F_3(p,Q,t)$ generate mappings $q = -\frac{\partial F_3}{\partial p}$, $P = -\frac{\partial F_3}{\partial Q}$, $K = H + \frac{\partial F_3}{\partial t}$;
(d) $F_4(p,P,t)$ generate mappings $q = -\frac{\partial F_4}{\partial p}$, $Q = \frac{\partial F_4}{\partial P}$, $K = H + \frac{\partial F_4}{\partial t}$.

Cases (c) and (d) are not covered in the notes, but you can derive them by analogy with how (b) is obtained from (a).

2. A mechanical system has Hamiltonian $H = \frac{1}{2}(p^2 + \omega^2 q^2)$. By first eliminating $f(P)$, or otherwise, find a generating function $F_1(q,Q)$ for new phase-space co-ordinates (Q,P) in terms of which

$$\begin{aligned} p &= f(P)\cos Q, \\ q &= \frac{f(P)}{\omega}\sin Q. \end{aligned} \tag{2-1}$$

State any conditions that you need to apply to $f(P)$. What is the Hamiltonian in the new co-ordinates (Q,P)? What are Hamilton's equations for (Q,P)?

3. Find a generating function $F_2(q,P)$ for the mapping found in the previous question. (Do not expect your answer to be pretty.)

4. Show that the transformation

$$\begin{aligned} p &= \mathrm{e}^{Q} \\ q &= -P\mathrm{e}^{-Q} \end{aligned} \tag{4-1}$$

is canonical.

5. A particle has Lagrangian $L(\boldsymbol{x},\dot{\boldsymbol{x}},t) = \frac{1}{2}\dot{\boldsymbol{x}}^2 - V(\boldsymbol{x},t)$. Write down a Hamiltonian for the particle in terms of $(\boldsymbol{x},\boldsymbol{p}_{\boldsymbol{x}},t)$.

The particle's co-ordinates in another frame are given by $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{\Delta}(t)$. Write down a Lagrangian $L(\boldsymbol{x}',\dot{\boldsymbol{x}}',t)$ and use it to construct a Hamiltonian in the new co-ordinates and momenta.

Find a generating function $F_2(\boldsymbol{x},\boldsymbol{p}',t)$ for the mapping from $\boldsymbol{x}$ to $\boldsymbol{x}'$. Verify that under this mapping the original $H(\boldsymbol{x},\boldsymbol{p}_{\boldsymbol{x}},t)$ transforms to the Hamiltonian you constructed from $L(\boldsymbol{x}',\dot{\boldsymbol{x}}',t)$.

6. A particle moving in three dimensions has Hamiltonian $H(\boldsymbol{x},\boldsymbol{p}_{\boldsymbol{x}},t) = \boldsymbol{p}_{\boldsymbol{x}}^2/2m + V(\boldsymbol{x},t)$. A mapping to new phase-space co-ordinates $(\boldsymbol{r},\boldsymbol{p}_{\boldsymbol{r}})$ is generated by the function

$$F_2(\boldsymbol{x},\boldsymbol{p}_{\boldsymbol{r}},t) = \boldsymbol{p}_{\boldsymbol{r}}^T B^T \boldsymbol{x}, \tag{6-1}$$

where $B(t)$ is a time-dependent rotation matrix ($BB^T = I$). Show that the Hamiltonian in the new co-ordinates is given by

$$K(\boldsymbol{r},\boldsymbol{p}_{\boldsymbol{r}},t) = \frac{\boldsymbol{p}_{\boldsymbol{r}}^2}{2m} + V(B\boldsymbol{r},t) + \boldsymbol{p}_{\boldsymbol{r}}^T \dot{B}^T \boldsymbol{x}. \tag{6-2}$$

Show further that $\boldsymbol{p}_{\boldsymbol{r}}^T \dot{B}^T \boldsymbol{x}$ can be written as $-\boldsymbol{p}_{\boldsymbol{r}} \cdot (\boldsymbol{\Omega} \times \boldsymbol{r})$ and explain how to obtain $\boldsymbol{\Omega}$.