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S7 and BT VII: Classical mechanics – problem set 3

1. Starting from the definition of a canonical map in terms of the Poincaré integral invariant, explain how functions of the form

(a) $F_1(q,Q,t)$ generate mappings $p=\frac{\partial F_1}{\partial q}$, $P=-\frac{\partial F_1}{\partial Q}$, $K=H+\frac{\partial F_1}{\partial t}$;

(b) $F_2(q,P,t)$ generate mappings $p=\frac{\partial F_2}{\partial q}$, $Q=\frac{\partial F_2}{\partial P}$, $K=H+\frac{\partial F_2}{\partial t}$;

(c) $F_3(p,Q,t)$ generate mappings $q=-\frac{\partial F_3}{\partial p}$, $P=-\frac{\partial F_3}{\partial Q}$, $K=H+\frac{\partial F_3}{\partial t}$;

(d) $F_4(p,P,t)$ generate mappings $q=-\frac{\partial F_4}{\partial p}$, $Q=\frac{\partial F_4}{\partial P}$, $K=H+\frac{\partial F_4}{\partial t}$.

Cases (c) and (d) are not covered in the notes, but you can derive them by analogy with how (b) is obtained from (a).

Ans: We require that $\oint_\gamma(P\mathrm{d}Q-K\mathrm{d}t)=\oint_\gamma(p\mathrm{d}q-H\mathrm{d}t)$ for any loop γ in extended phase space. For this to hold the integrands must be equal up to a total derivative:

$$P\mathrm{d}Q-K\mathrm{d}t+\mathrm{d}S=p\mathrm{d}q-H\mathrm{d}t, \tag{1-1}$$

where $S=S(P,Q,t)$. In the following we'll assume that all co-ordinate transformations are invertible.

(a) Assume that P can be expressed as $P=P(q,Q,t)$. Then we can take $S=F_1(q,Q,t)$, so that

$$\mathrm{d}S=\frac{\partial F_1}{\partial q}\mathrm{d}q+\frac{\partial F_1}{\partial Q}\mathrm{d}Q+\frac{\partial F_1}{\partial t}\mathrm{d}t. \tag{1-2}$$

Substitute into (1-1) and rearrange:

$$\left(P+\frac{\partial F_1}{\partial Q}\right)\mathrm{d}Q+\left(-K+\frac{\partial F_1}{\partial t}+H\right)\mathrm{d}t+\left(\frac{\partial F_1}{\partial q}-p\right)\mathrm{d}q=0. \tag{1-3}$$

The result follows on noting that $(\mathrm{d}q,\mathrm{d}Q,\mathrm{d}t)$ are allowed to vary independently – we're using (q,Q,t) as independent co-ordinates of points on our loop γ in extended phase space.

(b) The long way: apply a Legendre transform to $-S=-F_1(q,Q,t)$, replacing Q by $P=\partial(-F_1)/\partial Q$. This gives a new function $F_2(q,P,t)=QP-(-F_1(q,Q,t))$ in which $Q(q,P,t)$ is given implicitly by $P=\partial F_1/\partial Q$. Rearranging, $S=F_1=-QP+F_2(q,P,t)$. Therefore

$$\mathrm{d}S=-P\mathrm{d}Q-Q\mathrm{d}P+\frac{\partial F_2}{\partial q}\mathrm{d}q+\frac{\partial F_2}{\partial P}\mathrm{d}P+\frac{\partial F_2}{\partial t}\mathrm{d}t, \tag{1-4}$$

in which Q (and therefore $\mathrm{d}Q$) is to be regarded as a function of (q,P,t). Substituting into (1-1) and rearranging we have that

$$\left(-K+\frac{\partial F_2}{\partial t}+H\right)\mathrm{d}t+\left(\frac{\partial F_2}{\partial q}-p\right)\mathrm{d}q+\left(-Q+\frac{\partial F_2}{\partial P}\right)\mathrm{d}P=0, \tag{1-5}$$

from which the required result follows on noting that $(\mathrm{d}q,\mathrm{d}P,\mathrm{d}t)$ vary independently.
Quicker way of deriving the correct S is to note that simply taking $S=F_2(q,P,t)$ won't work because that would leave nothing to kill the first term $P\mathrm{d}Q$ of (1-1). So try $S=-QP+F_2(q,P,t)$ in which Q is regarded as a function $Q(q,P,t)$.

(c) Take $S=qp+F_3(p,Q,t)$, assuming $q=q(p,Q,t)$.

(d) $S=qp-QP+F_4(p,P,t)$, assuming $q=q(p,P,t)$ and $Q=Q(p,P,t)$.

2. A mechanical system has Hamiltonian $H = \frac{1}{2}(p^2 + \omega^2 q^2)$. By first eliminating $f(P)$, or otherwise, find a generating function $F_1(q, Q)$ for new phase-space co-ordinates (Q, P) in terms of which

$$\begin{aligned} p &= f(P)\cos Q, \\ q &= \frac{f(P)}{\omega}\sin Q. \end{aligned} \tag{2-1}$$

State any conditions that you need to apply to $f(P)$. What is the Hamiltonian in the new co-ordinates (Q, P)? What are Hamilton's equations for (Q, P)?

Ans: Substituting $f(P) = \omega q/\sin Q$ from the second equation into the first,

$$p = \omega q \frac{\cos Q}{\sin Q} = \frac{\partial F_1}{\partial q}. \tag{2-2}$$

Therefore $F_1(q, Q) = \frac{1}{2}\omega q^2 \cot Q + G(Q)$ for some function $G(Q)$. But we must also satisfy

$$P = -\frac{\partial F_1}{\partial Q} = \frac{1}{2}\omega q^2 \frac{1}{\sin^2 Q} - \frac{\partial G}{\partial Q}. \tag{2-3}$$

Substituting $q = (f(P)/\omega)\sin Q$ into this, we see that $G = 0$ and $f(P) = \pm\sqrt{2\omega P}$.
The transformation has no explicit time dependence, so $\partial F_1/\partial t = 0$. Then

$$K(Q, P) = H(q(Q, P), p(Q, P)) = \frac{1}{2}\left[(f(P)\cos Q)^2 + \omega^2\left(\frac{f(P)\sin Q}{\omega}\right)^2\right] = \frac{1}{2}f(P)^2 = \omega P. \tag{2-4}$$

Hamilton's equations are simply

$$\dot{P} = 0, \quad \dot{Q} = \omega. \tag{2-5}$$

This (P, Q) are **action–angle** co-ordinates for the SHO Hamiltonian and a more usually written (J, θ).

3. Find a generating function $F_2(q,P)$ for the mapping found in the previous question. (Do not expect your answer to be pretty.)

Ans: Brief statement of the problem: we need to find an $F_2(q,P)$ for which the implicit relations $p = \partial F_2/\partial q$, $Q = \partial F_2/\partial P$ produce the mapping

$$\begin{aligned} p &= \sqrt{2\omega P}\cos Q, \\ q &= \sqrt{\frac{2P}{\omega}}\sin Q. \end{aligned} \tag{3-1}$$

The least painful approach is to recall how F_2 is related to F_1. From Q1 we know that $F_1 = S = -PQ + F_2$. So,

$$\begin{aligned} F_2 &= PQ + F_1 \\ &= PQ + \tfrac{1}{2}\omega q^2 \cot Q, \end{aligned} \tag{3-2}$$

taking the expression for F_1 from Q2. We can use the second of (3-1) to obtain

$$\begin{aligned} \sin Q &= q\sqrt{\frac{\omega}{2P}} \\ \cos Q &= \pm\left(1 - q^2\frac{\omega}{2P}\right)^{1/2}, \end{aligned} \tag{3-3}$$

from which we have that

$$F_2 = P\sin^{-1}\left(q\sqrt{\frac{\omega}{2P}}\right) \pm q\sqrt{\frac{P\omega}{2}}\left(1 - \frac{\omega q^2}{2P}\right)^{1/2}. \tag{3-4}$$

An alternative is to solve the PDEs $p = \partial F_2/\partial q$, $Q = \partial F_2/\partial P$ for F_2 directly:

$$p = \pm\sqrt{2\omega P}\left[1 - \frac{\omega}{2P}q^2\right]^{1/2} = \frac{\partial F_2}{\partial q}. \tag{3-5}$$

Substituting $q\sqrt{\omega/2P} = \sin u$ and integrating gives

$$\begin{aligned} \pm F_2 &= \sqrt{2\omega P}\int\left[1 - \frac{\omega}{2P}q^2\right]^{1/2} \mathrm{d}q + G(P) \\ &= 2P\int \cos^2 u \,\mathrm{d}u + G(P) \\ &= P\int (1 + \cos 2u)\mathrm{d}u + G(P) \\ &= Pu + \tfrac{1}{2}P\sin 2u + G(P) \\ &= P\sin^{-1}\left(q\sqrt{\frac{\omega}{2P}}\right) \pm q\sqrt{\frac{P\omega}{2}}\left(1 - \frac{\omega q^2}{2P}\right)^{1/2} + G(P). \end{aligned} \tag{3-6}$$

Finally, we need to choose $G(P)$ to make $\partial F_2/\partial P = Q$. Checking that $G(P) = 0$ satisfies this condition is left as an exercise...

4. Show that the transformation

$$p = \mathrm{e}^{Q}$$
$$q = -P\mathrm{e}^{-Q} \tag{4-1}$$

is canonical.

Ans: Look at

$$p\mathrm{d}q - P\mathrm{d}Q = \mathrm{e}^{Q}\left(P\mathrm{e}^{-Q}\mathrm{d}Q - e^{-Q}\mathrm{d}P\right) - P\mathrm{d}Q = -\mathrm{d}P, \tag{4-2}$$

which is a total derivative. Therefore the transformation is canonical.
An alternative is to check Poisson brackets. Clearly we have that $[q, q] = [p, p] = 0$. The only step remaining is to check whether $[q, p] = 1$. We have that

$$[q, p] = \frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial p}{\partial Q}\frac{\partial q}{\partial P} = P\mathrm{e}^{Q} \times 0 - \mathrm{e}^{Q} \times \left(-\mathrm{e}^{-Q}\right) = 1. \tag{4-3}$$

5. A particle has Lagrangian $L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) = \frac{1}{2}\dot{\boldsymbol{x}}^2 - V(\boldsymbol{x}, t)$. Write down a Hamiltonian for the particle in terms of $(\boldsymbol{x}, \boldsymbol{p_x}, t)$.

The particle's co-ordinates in another frame are given by $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{\Delta}(t)$. Write down a Lagrangian $L(\boldsymbol{x}', \dot{\boldsymbol{x}}', t)$ and use it to construct a Hamiltonian in the new co-ordinates and momenta.

Find a generating function $F_2(\boldsymbol{x}, \boldsymbol{p}', t)$ for the mapping from $\boldsymbol{x}$ to $\boldsymbol{x}'$. Verify that under this mapping the original $H(\boldsymbol{x}, \boldsymbol{p_x}, t)$ transforms to the Hamiltonian you constructed from $L(\boldsymbol{x}', \dot{\boldsymbol{x}}', t)$.

Ans:

$$H(x, p_x, t) = \tfrac{1}{2}p_x^2 + V(x, t). \tag{5-1}$$

In the accelerated co-ordinates, we have

$$L(x', \dot{x}', t) = \tfrac{1}{2}(\dot{x}' + \dot{\Delta})^2 - V(x = x' + \Delta, t), \tag{5-2}$$

from which $p'_x = \dot{x}' + \dot{\Delta}$ and so

$$\begin{aligned} H(x', p'_x, t) &= \dot{x}'p'_x - L(x', \dot{x}', t) \\ &= \tfrac{1}{2}p'^2_x - p'_x\dot{\Delta} + V(x = x' + \Delta, t). \end{aligned} \tag{5-3}$$

The mapping generated by a function $F_2(x, p'_x, t)$ is $x' = \partial F_2/\partial p'_x$, $p_x = \partial F_2/\partial x$. So, try

$$x' = x - \Delta = \frac{\partial F_2}{\partial p'_x} \quad \Rightarrow \quad F_2(x, p'_x, t) = (x - \Delta)p'_x. \tag{5-4}$$

Check: $p_x = \partial F_2/\partial x = p'_x$, which is fine.
The transformed Hamiltonian

$$\begin{aligned} K(x', p'_x, t) &= H\Big(x(x', p'_x, t), p_x(x', p'_x, t), t\Big) + \frac{\partial F_2}{\partial t} \\ &= \tfrac{1}{2}p'^2_x + V(x = x' + \Delta, t) - \dot{\Delta}p_x. \end{aligned} \tag{5-5}$$

6. A particle moving in three dimensions has Hamiltonian $H(\boldsymbol{x}, \boldsymbol{p_x}, t) = \boldsymbol{p_x}^2/2m + V(\boldsymbol{x}, t)$. A mapping to new phase-space co-ordinates $(\boldsymbol{r}, \boldsymbol{p_r})$ is generated by the function

$$F_2(\boldsymbol{x}, \boldsymbol{p_r}, t) = \boldsymbol{p}_r^T B^T \boldsymbol{x}, \tag{6-1}$$

where $B(t)$ is a time-dependent rotation matrix ($BB^T = I$). Show that the Hamiltonian in the new co-ordinates is given by

$$K(\boldsymbol{r}, \boldsymbol{p_r}, t) = \frac{\boldsymbol{p_r}^2}{2m} + V(B\boldsymbol{r}, t) + \boldsymbol{p_r}^T \dot{B}^T \boldsymbol{x}. \tag{6-2}$$

Show further that $\boldsymbol{p_r}^T \dot{B}^T \boldsymbol{x}$ can be written as $-\boldsymbol{p_r} \cdot (\boldsymbol{\Omega} \times \boldsymbol{r})$ and explain how to obtain $\boldsymbol{\Omega}$.

Ans: It is safest to carry out the following manipulations using index notation. We have that $F_2 = p_{\boldsymbol{r}i} B_{ji} x_j$ and that the transformation is given implicitly by

$$\begin{aligned} r_k &= \frac{\partial F_2}{\partial p_{\boldsymbol{r}k}} = \delta_{ik} B_{ji} x_j = B_{jk} x_j \\ p_{\boldsymbol{x}k} &= \frac{\partial F_2}{\partial x_k} = p_{\boldsymbol{r}i} B_{ji} \delta_{jk} = p_{\boldsymbol{r}i} B_{ki}. \end{aligned} \tag{6-3}$$

Therefore $\boldsymbol{r} = B^T \boldsymbol{x}$ (so $\boldsymbol{x} = B\boldsymbol{r}$) and $\boldsymbol{p_x} = B\boldsymbol{p_r}$. The transformed Hamiltonian

$$\begin{aligned} K = H + \frac{\partial F_2}{\partial t} &= \tfrac{1}{2m}\boldsymbol{p_x}^2 + V(\boldsymbol{x}, t) \\ &= \tfrac{1}{2m}(B\boldsymbol{p_r})^2 + V(B\boldsymbol{r}, t) + \boldsymbol{p_r}^T \dot{B}^T \boldsymbol{x} \\ &= \tfrac{1}{2m}\boldsymbol{p_r}^2 + V(B\boldsymbol{r}, t) + \boldsymbol{p_r}^T \dot{B}^T \boldsymbol{x}, \end{aligned} \tag{6-4}$$

as required. To express the final term above in something more friendly, recall from the notes that

$$\begin{aligned} \dot{\boldsymbol{x}} &= \dot{B}\boldsymbol{r} + B\dot{\boldsymbol{r}} \\ &= \dot{B}B^T \boldsymbol{x} + B\dot{\boldsymbol{r}} \\ &= \boldsymbol{\omega} \times \boldsymbol{x} + B\dot{\boldsymbol{r}}, \end{aligned} \tag{6-5}$$

because $0 = \frac{\mathrm{d}}{dt}(BB^T) = \dot{B}B^T + B\dot{B}^T = \dot{B}B^T + (\dot{B}B^T)^T$, meaning that $\dot{B}B^T$ is skew symmetric and so $\dot{B}B^T\boldsymbol{x}$ can be written as $\boldsymbol{\omega} \times \boldsymbol{x}$ for some choice of $\boldsymbol{\omega}$. Therefore $(\dot{B}B^T)^T \boldsymbol{x} = B\dot{B}^T \boldsymbol{x} = -\boldsymbol{\omega} \times \boldsymbol{x}$. Premultiplying $B\dot{B}^T\boldsymbol{x} = -\boldsymbol{\omega} \times \boldsymbol{x}$ by B^T gives

$$\dot{B}^T \boldsymbol{x} = -B^T(\boldsymbol{\omega} \times \boldsymbol{x}) = -B^T\boldsymbol{\omega} \times B^T\boldsymbol{x} = -\boldsymbol{\Omega} \times \boldsymbol{r}, \tag{6-6}$$

where $\boldsymbol{\Omega} = B^T\boldsymbol{\omega}$. Thefore $\boldsymbol{p_r}^T \dot{B}^T \boldsymbol{x} = -\boldsymbol{p_r} \cdot (\boldsymbol{\Omega} \times \boldsymbol{r})$, as required.
The simplest way of finding $\boldsymbol{\Omega}$ is from the relation $\dot{B}B^T\boldsymbol{x} = \boldsymbol{\omega} \times \boldsymbol{x}$, which can be written $\dot{B}\boldsymbol{r} = B(\boldsymbol{\Omega} \times \boldsymbol{r})$, or $B^T\dot{B}\boldsymbol{r} = \boldsymbol{\Omega} \times \boldsymbol{r}$. Then $\boldsymbol{\Omega}$ can be found by writing out the RHS as a skew-symmetric matrix acting on $\boldsymbol{r}$,

$$\boldsymbol{\Omega} \times \boldsymbol{r} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}. \tag{6-7}$$

The components of $\boldsymbol{\Omega}$ follow by writing out $B^T\dot{B}$ explicitly and comparing to this.