J. Magorrian MT 2009, borrowing from J. J. Binney's 2006 course

S7: Classical mechanics – problem set 2

1. Show that if the Hamiltonian is indepdent of a generalized co-ordinate q_0, then the conjugate momentum p_0 is a constant of motion. Such co-ordinates are called **cyclic co-ordinates**. Give two examples of physical systems that have a cyclic co-ordinate.

2. A dynamical system has generalized co-ordinates q_i and generalized momenta p_i. Verify the following properties of the Poisson brackets:

$$[q_i, q_j] = [p_i, p_j] = 0, \quad [q_i, p_j] = \delta_{ij}. \tag{2-1}$$

If $\boldsymbol{p}$ is the momentum conjugate to a position vector $\boldsymbol{r}$ and $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, evaluate $[L_x, L_y]$, $[L_y, L_x]$ and $[L_x, L_x]$.

The Lagrangian of a particle of mass m and charge e in a uniform magnetic field $\boldsymbol{B}$ and electrostatic potential ϕ is

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 + \frac{1}{2}e\dot{\boldsymbol{r}} \cdot (\boldsymbol{B} \times \boldsymbol{r}) - e\phi. \tag{2-2}$$

Derive the corresponding Hamiltonian and verify that the rate of change of $m\dot{\boldsymbol{r}}$ equals the Lorentz force.

Show that the momentum component along $\boldsymbol{B}$ and the sum of the squares of the momentum components are all constants of motion when $\phi = 0$. Find another constant of motion associated with time translation symmetry.

3. Let p and q be canonically conjugate co-ordinates and let $f(p, q)$ and $g(p, q)$ be functions on phase space. Define the Poisson bracket $[f, g]$. Let $H(p, q)$ be the Hamiltonian that governs the system's dynamics. Write down the equations of motion of p and q in terms of H and the Poisson bracket.

In a galaxy the density of stars in phase space is $f(\boldsymbol{q}, \boldsymbol{p}, t)$, where $\boldsymbol{q}$ and $\boldsymbol{p}$ each have three components. When evaluated at the location $(\boldsymbol{q}(t), \boldsymbol{p}(t))$ of any given star, f is time-independent. Show that f consequently satisfies

$$\frac{\partial f}{\partial t} + [f, H] = 0, \tag{3-1}$$

where H is the Hamiltonian that governs the motion of every star.

Consider motion in a circular razor-thin galaxy in which the potential of any star is given by the function $V(R)$, where R is a radial co-ordinate. Express H in terms of plane polar co-ordinates (R, ϕ) and their conjugate momenta, with the origin coinciding with the galaxy's centre. Hence, or otherwise, show that in this system f satisfies the equation

$$\frac{\partial f}{\partial t} + \frac{p_R}{m}\frac{\partial f}{\partial R} + \frac{p_\phi}{mR^2}\frac{\partial f}{\partial \phi} - \left(\frac{\partial V}{\partial R} - \frac{p_\phi^2}{mR^3}\right)\frac{\partial f}{\partial p_R} = 0, \tag{3-2}$$

where m is the mass of the star.

4. Show that in spherical polar co-ordinates the Hamiltonian of a particle of mass m moving in a potential $V(\boldsymbol{x})$ is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + V(\boldsymbol{x}). \tag{4-1}$$

Show that $p_\phi = \text{constant}$ when $\partial V/\partial\phi \equiv 0$ and interpret this result physically.

Given that V depends only on r, show that $[H, K] = 0$, where $K \equiv p_\theta^2 + p_\phi^2/\sin^2\theta$. By expressing K as a function of $\dot\theta$ and $\dot\phi$ interpret this result physically.

Consider circular motion with angular momentum h in a spherical potential $V(r)$. Evaulate $p_\theta(\theta)$ when the orbit's plane is inclined by ψ to the equatorial plane. Show that $p_\theta = 0$ when $\sin\theta = \pm\cos\psi$ and interpret this result physically.

5. Oblate spheroidal co-ordinates (u, v, ϕ) are related to regular cylindrical polars (R, z, ϕ) by

$$R = \Delta \cosh u \cos v; \quad z = \Delta \sinh u \sin v. \tag{5-1}$$

For a particle of mass m show that the momenta conjugate to these co-ordinates are

$$\begin{aligned} p_u &= m\Delta^2(\cosh^2 u - \cos^2 v)\dot{u}, \\ p_v &= m\Delta^2(\cosh^2 u - \cos^2 v)\dot{v}, \\ p_\phi &= m\Delta^2 \cosh^2 u \cos^2 v\dot{\phi}. \end{aligned} \tag{5-2}$$

Hence show that the Hamiltonian for motion in a potential $\Phi(u, v)$ is

$$H = \frac{p_u^2 + p_v^2}{2m\Delta^2(\cosh^2 u - \cos^2 v)} + \frac{p_\phi^2}{2m\Delta^2 \cosh^2 u \cos^2 v} + \Phi. \tag{5-3}$$

Show that $[H, p_\phi] = 0$ and hence that p_ϕ is a constant of motion. Identify it physically.

6. A particle of mass m and charge Q moves in the equatorial plane $\theta = \pi/2$ of a magnetic dipole. Given that the dipole has vector potential

$$\boldsymbol{A} = \frac{\mu \sin\theta}{4\pi r^2}\hat{\boldsymbol{e}}_\phi, \tag{6-1}$$

evaluate the Hamiltonian $H(p_r, p_\phi, r, \phi)$ of the system.

The particle approaches the dipole from infinity at speed v and impact parameter b. Show that p_ϕ and the particle's speed are constants of motion.

Show further that for $Q\mu > 0$ the distance of closest approach to the dipole is

$$D = \tfrac{1}{2}\begin{cases} b + \sqrt{b^2 - a^2} & \text{for } \dot{\phi} > 0, \\ b + \sqrt{b^2 + a^2} & \text{for } \dot{\phi} < 0, \end{cases} \tag{6-2}$$

where $a^2 \equiv \mu Q/\pi m v$.

7. An axisymmetric top has Lagrangian

$$L = \tfrac{1}{2}I_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - mga\cos\theta, \tag{7-1}$$

where (θ, ϕ, ψ) are the usual Euler angles. Show that the top's Hamiltonian

$$H = \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3} + mga\cos\theta. \tag{7-2}$$

Using Hamilton's equations or otherwise show that the top will precess steadily at fixed inclination to the vertical provided θ satisfies

$$0 = mga + \frac{(p_\phi - p_\psi\cos\theta)(p_\phi\cos\theta - p_\psi)}{I_1\sin^4\theta}. \tag{7-3}$$

8. A point charge q is placed at the origin in the magnetic field generated by a spatially confined current distribution. Given that

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{r^3} \tag{8-1}$$

and $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ with $\nabla\cdot\boldsymbol{A} = 0$, show that the field's momentum

$$\boldsymbol{P} \equiv \epsilon_0\int \boldsymbol{E}\times\boldsymbol{B}\,\mathrm{d}^3\boldsymbol{x} = q\boldsymbol{A}(0). \tag{8-2}$$

Use this result to intrepret the formula for the canonical momentum of a charged particle in an electromagnetic field. [Hint: use $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ and then index notation (easy) or vector identities (not so easy) to expand $\boldsymbol{E}\times\boldsymbol{B}$ into a sum of two terms. To each term apply the tensor form of Gauss's theorem, which states that $\int \mathrm{d}^3\boldsymbol{x}\nabla_i\boldsymbol{T} = \oint \mathrm{d}^2 S_i\boldsymbol{T}$, no matter how many indices the tensor $\boldsymbol{T}$ carries. In one term you can make use of $\nabla\cdot\boldsymbol{A} = 0$ and in the other $\nabla^2 r^{-1} = -4\pi\delta^3(\boldsymbol{r})$.]