J. Magorrian MT 2009, borrowing from J. J. Binney's 2006 course

S7: Classical mechanics – problem set 2

1. Show that if the Hamiltonian is indepdent of a generalized co-ordinate q_0, then the conjugate momentum p_0 is a constant of motion. Such co-ordinates are called **cyclic co-ordinates**. Give two examples of physical systems that have a cyclic co-ordinate.

Ans: Using $\dot{p}_i = -\partial H/\partial q_i$ it's obvious that if H doesn't depend explicitly on, say, q_1, then p_1 is conserved. Examples: p_ϕ is conserved in axisymmetric potential $V(R, z)$; p_z is conserved for motion in a magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{k}}$; p_x, p_y, p_z are conserved for free particles, etc.

2. A dynamical system has generalized co-ordinates q_i and generalized momenta p_i. Verify the following properties of the Poisson brackets:

$$[q_i, q_j] = [p_i, p_j] = 0, \quad [q_i, p_j] = \delta_{ij}. \tag{2-1}$$

Ans: From definition of PB

$$[q_i, p_j] = \sum_k \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \sum_k \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} = \sum_k \delta_{ik}\delta_{jk} - 0 = \delta_{ij}. \tag{2-2}$$

Similarly, $[q_i, q_j]$ and $[p_i, p_j]$ are obviously zero.

If $\boldsymbol{p}$ is the momentum conjugate to a position vector $\boldsymbol{r}$ and $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, evaluate $[L_x, L_y]$, $[L_y, L_x]$ and $[L_x, L_x]$.

Ans: By antisymmetry of PBs, $[L_x, L_x] = 0$ and $[L_x, L_y] = -[L_y, L_x]$. So we need only calculate $[L_x, L_y] = [yp_z - zp_y, zp_z - xp_z]$. One way to do this is to use the linearity, antisymmetry and chain rule for PBs to reduce the expression to something involving the canonical commutation relations (see lectures). Another is to apply the definition of PB directly:

$$\begin{aligned}
[L_x, L_y] &= \frac{\partial}{\partial x}(yp_z - zp_y)\frac{\partial}{\partial p_x}(zp_z - xp_z) - \frac{\partial}{\partial p_x}(yp_z - zp_y)\frac{\partial}{\partial x}(zp_z - xp_z) \\
&\quad + \frac{\partial}{\partial y}(yp_z - zp_y)\frac{\partial}{\partial p_y}(zp_z - xp_z) - \frac{\partial}{\partial p_y}(yp_z - zp_y)\frac{\partial}{\partial y}(zp_z - xp_z) \\
&\quad + \frac{\partial}{\partial z}(yp_z - zp_y)\frac{\partial}{\partial p_z}(zp_z - xp_z) - \frac{\partial}{\partial p_z}(yp_z - zp_y)\frac{\partial}{\partial z}(zp_z - xp_z) \\
&= (-p_y)(-x) - yp_x = L_z.
\end{aligned} \tag{2-3}$$

This is the overly cautious way of writing out. A more sensible answer would point out that L_x is independent of x and p_x and L_y is independent y and p_y, so the first two lines above must clearly be zero and we need only consider the third.

The Lagrangian of a particle of mass m and charge e in a uniform magnetic field $\boldsymbol{B}$ and electrostatic potential ϕ is

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 + \frac{1}{2}e\dot{\boldsymbol{r}} \cdot (\boldsymbol{B} \times \boldsymbol{r}) - e\phi. \tag{2-4}$$

Derive the corresponding Hamiltonian and verify that the rate of change of $m\dot{\boldsymbol{r}}$ equals the Lorentz force.

> **Ans:** Momentum
>
> $$\boldsymbol{p} \equiv \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = m\dot{\boldsymbol{r}} + \tfrac{1}{2}e(\boldsymbol{B} \times \boldsymbol{r}). \tag{2-5}$$
>
> Hamiltonian is Legendre transform w.r.t. $\dot{\boldsymbol{r}}$:
>
> $$\begin{aligned} H &= \boldsymbol{p} \cdot \dot{\boldsymbol{r}} - L = m\dot{\boldsymbol{r}}^2 + \tfrac{1}{2}e(\boldsymbol{B} \times \boldsymbol{r}) \cdot \dot{\boldsymbol{r}} - L \\ &= \tfrac{1}{2}m\dot{\boldsymbol{r}}^2 + e\phi = \tfrac{1}{2m}\left[\boldsymbol{p} - \tfrac{1}{2}e(\boldsymbol{B} \times \boldsymbol{r})\right]^2 + e\phi. \end{aligned} \tag{2-6}$$
>
> To verify that the rate of change of $m\dot{\boldsymbol{r}}$ is given by the Lorentz force, start with Hamilton's equation $\dot{\boldsymbol{p}} = -\partial H/\partial \boldsymbol{r}$:
>
> $$\begin{aligned} \text{(LHS)} \quad \dot{\boldsymbol{p}} &= \frac{\mathrm{d}}{\mathrm{d}t}m\dot{\boldsymbol{r}} + \tfrac{1}{2}e(\boldsymbol{B} \times \dot{\boldsymbol{r}}) \\ \text{(RHS)} \quad -\frac{\partial H}{\partial \boldsymbol{r}} &= -\tfrac{1}{m}\left[\boldsymbol{p} - \tfrac{1}{2}e(\boldsymbol{B} \times \boldsymbol{r})\right] \cdot \frac{\partial}{\partial \boldsymbol{r}}\left[-\tfrac{1}{2}e(\boldsymbol{B} \times \boldsymbol{r})\right] - e\frac{\partial \phi}{\partial \boldsymbol{r}} \end{aligned} \tag{2-7}$$
>
> We can expand the RHS using the following (remember that $\boldsymbol{r}$, $\boldsymbol{p}$ are independent):
>
> $$\begin{aligned} \boldsymbol{p} \cdot \frac{\partial}{\partial \boldsymbol{r}}(\boldsymbol{B} \times \boldsymbol{r}) &= \frac{\partial}{\partial \boldsymbol{r}}\left[\boldsymbol{p} \cdot (\boldsymbol{B} \times \boldsymbol{r})\right] = \frac{\partial}{\partial \boldsymbol{r}}\left[\boldsymbol{r} \cdot (\boldsymbol{p} \times \boldsymbol{B})\right] = \boldsymbol{p} \times \boldsymbol{B}, \\ (\boldsymbol{B} \times \boldsymbol{r}) \cdot \frac{\partial}{\partial \boldsymbol{r}}(\boldsymbol{B} \times \boldsymbol{r}) &= \tfrac{1}{2}\frac{\partial}{\partial \boldsymbol{r}}\left[(\boldsymbol{B} \times \boldsymbol{r})^2\right] = \tfrac{1}{2}\frac{\partial}{\partial \boldsymbol{r}}\left[B^2\boldsymbol{r}^2 - (\boldsymbol{B} \cdot \boldsymbol{r})^2\right] = B^2\boldsymbol{r} - (\boldsymbol{B} \cdot \boldsymbol{r})\boldsymbol{B} \\ &= (\boldsymbol{B} \times \boldsymbol{r}) \times \boldsymbol{B} \end{aligned} \tag{2-8}$$
>
> Using $\boldsymbol{p} = m\dot{\boldsymbol{r}} + \tfrac{1}{2}e(\boldsymbol{B} \times \boldsymbol{r})$ and (2-8) to expand $\dot{\boldsymbol{p}} = -\partial H/\partial \boldsymbol{r}$ gives
>
> $$\dot{\boldsymbol{p}} = \frac{\mathrm{d}}{\mathrm{d}t}m\dot{\boldsymbol{r}} + \frac{1}{2}e(\boldsymbol{B} \times \dot{\boldsymbol{r}}) = \frac{e}{2m}\left[m\dot{\boldsymbol{r}} + \frac{1}{2}e(\boldsymbol{B} \times \boldsymbol{r})\right] \times \boldsymbol{B} - \frac{e^2}{4m}(\boldsymbol{B} \times \boldsymbol{r}) \times \boldsymbol{B} - e\frac{\partial \phi}{\partial \boldsymbol{r}}, \tag{2-9}$$
>
> which simplifies to $\frac{\mathrm{d}}{\mathrm{d}t}m\dot{\boldsymbol{r}} = e\dot{\boldsymbol{r}} \times \boldsymbol{B} - e(\partial\phi/\partial \boldsymbol{r})$ as expected.

Show that the momentum component along $\boldsymbol{B}$ and the sum of the squares of the momentum components are all constants of motion when $\phi = 0$. Find another constant of motion associated with time translation symmetry.

> **Ans:** Notice ambiguity! "Momentum" could mean either $\boldsymbol{p}$ or $m\dot{\boldsymbol{r}}$. We take the latter. Dot (2-9) with $\boldsymbol{B}$:
>
> $$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \dot{\boldsymbol{p}} = -e\boldsymbol{B} \cdot \frac{\partial \phi}{\partial \boldsymbol{r}} = 0 \quad \text{if } \phi \text{ constant.} \tag{2-10}$$
>
> So component of $\boldsymbol{p}$ along $\boldsymbol{B}$ is conserved. So too is component of $m\dot{\boldsymbol{r}} = \boldsymbol{p} - \tfrac{1}{2}e(\boldsymbol{B} \times \boldsymbol{r})$. To show other two components are constant, assume $\phi = 0$ and dot Lorentz force equation with $m\dot{\boldsymbol{r}}$:
>
> $$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(m^2\dot{\boldsymbol{r}}^2) = m\dot{\boldsymbol{r}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}m\dot{\boldsymbol{r}} = (e\dot{\boldsymbol{r}} \times \boldsymbol{B}) \cdot \dot{\boldsymbol{r}} = 0. \tag{2-11}$$
>
> So $(m\dot{\boldsymbol{r}})^2$ is conserved. Since $\boldsymbol{B} \cdot m\dot{\boldsymbol{r}}$ is along conserved, must have that sum-square of other two components of $m\dot{\boldsymbol{r}}$ (but not $\boldsymbol{p}$) is conserved too.
> The constant of motion associated with time translation symmetry is H itself.

3. Let p and q be canonically conjugate co-ordinates and let $f(p,q)$ and $g(p,q)$ be functions on phase space. Define the Poisson bracket $[f,g]$. Let $H(p,q)$ be the Hamiltonian that governs the system's dynamics. Write down the equations of motion of p and q in terms of H and the Poisson bracket.

> **Ans:** Definition of PB:
>
> $$[f,g] = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}. \tag{3-1}$$
>
> Hamilton's equations are $\dot q = [q,H]$, $\dot p = [p,H]$.

In a galaxy the density of stars in phase space is $f(\boldsymbol{q},\boldsymbol{p},t)$, where $\boldsymbol{q}$ and $\boldsymbol{p}$ each have three components. When evaluated at the location $(\boldsymbol{q}(t),\boldsymbol{p}(t))$ of any given star, f is time-independent. Show that f consequently satisfies

$$\frac{\partial f}{\partial t} + [f,H] = 0, \tag{3-2}$$

where H is the Hamiltonian that governs the motion of every star.

> **Ans:** f is constant along orbits so $\mathrm{d}f/\mathrm{d}t = 0$. Equation (3-2) follows on using the chain rule to write $\mathrm{d}f/\mathrm{d}t = \partial f/\partial t + \dot q(\partial f/\partial q) + \dot p(\partial f/\partial q)$ and then substituting for $(\dot q, \dot p)$ from Hamilton's equations.

Consider motion in a circular razor-thin galaxy in which the potential of any star is given by the function $V(R)$, where R is a radial co-ordinate. Express H in terms of plane polar co-ordinates (R,ϕ) and their conjugate momenta, with the origin coinciding with the galaxy's centre. Hence, or otherwise, show that in this system f satisfies the equation

$$\frac{\partial f}{\partial t} + \frac{p_R}{m}\frac{\partial f}{\partial R} + \frac{p_\phi}{mR^2}\frac{\partial f}{\partial \phi} - \left(\frac{\partial V}{\partial R} - \frac{p_\phi^2}{mR^3}\right)\frac{\partial f}{\partial p_R} = 0, \tag{3-3}$$

where m is the mass of the star.

> **Ans:** Standard procedure: Write down Lagrangian L in terms of plane-polar co-ordinates; this L defines momenta conjugate to (R,ϕ) through $\boldsymbol{p} \equiv \partial L/\partial \dot{\boldsymbol{q}}$; take Legendre transform of L to get H.
> The Lagrangian
>
> $$L = \frac{1}{2}m\left(\dot R^2 + R^2\dot\phi^2\right) - V(R), \tag{3-4}$$
>
> so that the momenta $p_R = m\dot R$ and $p_\phi = mR^2\dot\phi$. Taking the Legendre transform,
>
> $$\begin{aligned} H &= p_R\dot R + p_\phi\dot\phi - \frac{1}{2}m(\dot R^2 + R^2\dot\phi^2) + V(R) \\ &= \frac{1}{2m}\left[p_R^2 + \frac{p_\phi^2}{R^2}\right] + V(R), \end{aligned} \tag{3-5}$$
>
> where the generalized velocities $\dot R$ and $\dot \phi$ have been expressed in terms of the phase-space co-ordinates (R,ϕ,p_R,p_ϕ).
> To obtain (3-3) start from (3-2), but writing out the $[f,H]$ explicitly:
>
> $$0 = \frac{\partial f}{\partial t} + [f,H] = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial R}\frac{\partial H}{\partial p_R} - \frac{\partial f}{\partial p_R}\frac{\partial H}{\partial R} + \frac{\partial f}{\partial \phi}\frac{\partial H}{\partial p_\phi} - \frac{\partial f}{\partial p_\phi}\frac{\partial H}{\partial \phi}, \tag{3-6}$$
>
> from which the required result follows.

4. Show that in spherical polar co-ordinates the Hamiltonian of a particle of mass m moving in a potential $V(\boldsymbol{x})$ is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(\boldsymbol{x}). \tag{4-1}$$

Show that $p_\phi = \text{constant}$ when $\partial V/\partial\phi \equiv 0$ and interpret this result physically.

Ans: Starting from $x = r\sin\theta\cos\phi$ etc, we can show that the particle's velocity satisfies $\dot{\boldsymbol{x}}^2 = \dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2$ and so the Lagrangian

$$L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 - V = \frac{1}{2}m\left[\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2\right] - V. \tag{4-2}$$

Using $p_i \equiv \partial L/\partial q_i$, the momenta

$$p_r = m\dot{r}, \quad p_\theta = mr^2\dot{\theta}, \quad p_\phi = mr^2\sin^2\theta\dot{\phi}. \tag{4-3}$$

Taking the Legendre transform of L,

$$\begin{aligned} H = \boldsymbol{p}\cdot\dot{\boldsymbol{q}} - L &= m\dot{r}^2 + mr^2\dot{\theta}^2 + mr^2\sin^2\theta\dot{\phi}^2 - L \\ &= \frac{1}{2}m\left[\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2\right] + V(r,\theta,\phi), \end{aligned} \tag{4-4}$$

from which (4-1) follows on using (4-3) to express the generalized velocities $(\dot{r},\dot{\theta},\dot{\phi})$ in terms of phase-space co-ordinates $(r,\theta,\phi,p_r,p_\theta,p_\phi)$.
If V does not depend on ϕ then $\dot{p}_\phi = -\partial H/\partial\phi = -\partial V/\partial\phi = 0$ and so p_ϕ (the angular momentum about the z axis) is conserved.

Given that V depends only on r, show that $[H,K] = 0$, where $K \equiv p_\theta^2 + p_\phi^2/\sin^2\theta$. By expressing K as a function of $\dot{\theta}$ and $\dot{\phi}$ interpret this result physically.

Ans: One way of showing $[H,K] = 0$ is by writing out the six terms in the Poisson bracket explicitly. Alternatively, note that the Hamiltonian (4-1) can be written $H = p_r^2/2m + K/2mr^2 + V$ and so

$$[H,K] = \frac{1}{2m}[p_r^2, K] + \frac{1}{m}\left[\frac{K}{r^2}, K\right] + [V,K]. \tag{4-5}$$

The first term vanishes because K does not depend on r. Similarly, the final term vanishes because $V = V(r)$ and K does not depend on p_r. Using the chain rule for PBs, the middle term

$$[K/r^2, K] = [K,K]\cdot\frac{1}{r^2} + [1/r^2, K]K, \tag{4-6}$$

but clearly $[K,K] = 0$ and $[1/r^2, K] = 0$ since K is independent of p_r. Therefore all terms in (4-5) vanish and so $[H,K] = 0$, meaning that K is a constant of motion.
Writing K out in terms of generalized velocities $\dot{\theta},\dot{\phi}$,

$$K = m^2r^2\left[(r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2\right] = m^2r^2v_{\text{tangential}}^2, \tag{4-7}$$

which is the square of the total angular momentum. It vanishes because the potential is spherically symmetric.

Consider circular motion with angular momentum h in a spherical potential $V(r)$. Evaulate $p_\theta(\theta)$ when the orbit's plane is inclined by ψ to the equatorial plane. Show that $p_\theta = 0$ when $\sin\theta = \pm\cos\psi$ and interpret this result physically.

Ans: The orbit is inclined at an angle ψ, so $p_\phi = h\cos\psi$. Using $h^2 = K = p_\theta^2 + p_\phi^2/\sin^2\theta$, we have that

$$p_\theta^2 = h^2\left(1 - \frac{\cos^2\psi}{\sin^2\theta}\right), \tag{4-8}$$

which tends to zero as $\sin\theta \to \pm\cos\psi$ – the particle is at its turning point in the (R,z) plane, which is where both p_θ and $\dot{\theta}$ are zero.

5. Oblate spheroidal co-ordinates (u, v, ϕ) are related to regular cylindrical polars (R, z, ϕ) by

$$R = \Delta \cosh u \cos v; \quad z = \Delta \sinh u \sin v. \tag{5-1}$$

For a particle of mass m show that the momenta conjugate to these co-ordinates are

$$\begin{aligned} p_u &= m\Delta^2(\cosh^2 u - \cos^2 v)\dot{u}, \\ p_v &= m\Delta^2(\cosh^2 u - \cos^2 v)\dot{v}, \\ p_\phi &= m\Delta^2 \cosh^2 u \cos^2 v\dot{\phi}. \end{aligned} \tag{5-2}$$

Hence show that the Hamiltonian for motion in a potential $\Phi(u, v)$ is

$$H = \frac{p_u^2 + p_v^2}{2m\Delta^2(\cosh^2 u - \cos^2 v)} + \frac{p_\phi^2}{2m\Delta^2 \cosh^2 u \cos^2 v} + \Phi. \tag{5-3}$$

Ans: Start from $L = \frac{1}{2}m[\dot{R}^2 + (R\dot{\phi})^2 + \dot{z}^2] - \Phi$. Have that

$$\begin{aligned} R &= \Delta \cosh u \cos v \\ z &= \Delta \sinh u \sin v \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{R} &= \dot{u}\Delta \sinh u \cos v - \dot{v}\Delta \cosh u \sin v \\ \dot{z} &= \dot{u}\Delta \cosh u \sin v + \dot{v}\Delta \sinh u \cos v \end{aligned} \tag{5-4}$$

So

$$\begin{aligned} \dot{R}^2 + \dot{z}^2 &= \Delta^2\dot{u}^2[\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v] + \Delta^2\dot{v}^2[\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v] \\ &= \Delta^2\dot{u}^2[(\cosh^2 u - 1)(1 - \sin^2 v) + \cosh^2 u \sin^2 v] + \Delta^2\dot{v}^2[\cosh^2 u(1 - \cos^2 v) + \sinh^2 u \cos^2 v] \\ &= \Delta^2 \left[\dot{u}^2 + \dot{v}^2\right](\cosh^2 u - \cos^2 v) \end{aligned} \tag{5-5}$$

and

$$L = \tfrac{1}{2}m\Delta^2 \left[(\cosh^2 u - \cos^2 v)(\dot{u}^2 + \dot{v}^2) + \cosh^2 u \cos^2 v\dot{\phi}^2\right] - \Phi. \tag{5-6}$$

The momenta (5-2) drop out using $p_u \equiv \partial L/\partial \dot{u}$ etc. Taking the Legendre transform of L w.r.t. $(\dot{u}, \dot{v}, \dot{\phi})$ we have that

$$\begin{aligned} H = p_u\dot{u} + p_v\dot{v} + p_\phi\dot{\phi} - L = &\frac{p_u^2}{m\Delta^2(\cosh^2 u - \cos^2 v)} + \frac{p_v^2}{m\Delta^2(\cosh^2 u - \cos^2 v)} + \frac{p_\phi^2}{m\Delta^2 \cosh^2 u \cos^2 v} \\ &- \frac{1}{2}m\Delta^2 \left[\frac{p_u^2 + p_v^2}{(m\Delta^2)^2(\cosh^2 u - \cos^2 v)} + \left(\frac{p_\phi}{m\Delta^2 \cosh u \cos v}\right)^2\right] + \Phi, \end{aligned} \tag{5-7}$$

using (5-2) to obtain $\dot{u} = p_u/m\Delta^2(\cosh^2 u - cos^2 v)$ etc. Simplifying gives the required expression.

Show that $[H, p_\phi] = 0$ and hence that p_ϕ is a constant of motion. Identify it physically.

Ans:

$$[H, p_\phi] = \frac{\partial H}{\partial u}\underbrace{\frac{\partial p_\phi}{\partial p_u}}_{0} - \underbrace{\frac{\partial p_\phi}{\partial u}}_{0}\frac{\partial H}{\partial p_u} + \frac{\partial H}{\partial v}\underbrace{\frac{\partial p_\phi}{\partial p_v}}_{0} - \underbrace{\frac{\partial p_\phi}{\partial v}}_{0}\frac{\partial H}{\partial p_v} + \frac{\partial H}{\partial \phi}\underbrace{\frac{\partial p_\phi}{\partial p_\phi}}_{1} - \underbrace{\frac{\partial p_\phi}{\partial \phi}}_{0}\frac{\partial H}{\partial p_\phi}, \tag{5-8}$$

all but one of the terms being zero since $\partial w_i/\partial w_j = \delta_{ij}$, where $\boldsymbol{w} \equiv (u, v, \phi, p_u, p_v, p_\phi)$. (Remember that phase-space co-ordinates are independent of one another!) The remaining term involves $\partial H/\partial \phi$ which vanishes, since the potential – and therefore H – does not depend on ϕ. The conserved momentum, p_ϕ, is the angular momentum about the symmetry axis.

6. A particle of mass m and charge Q moves in the equatorial plane $\theta = \pi/2$ of a magnetic dipole. Given that the dipole has vector potential

$$\boldsymbol{A} = \frac{\mu \sin\theta}{4\pi r^2}\hat{\boldsymbol{e}}_\phi, \tag{6-1}$$

evaluate the Hamiltonian $H(p_r, p_\phi, r, \phi)$ of the system.

Ans: You might be tempted to use $H = \frac{(\boldsymbol{p}-Q\boldsymbol{A})^2}{2m} + Q\phi$, but that's for Cartesian $\boldsymbol{p}$. One possibility is to make a canonical map to new polar co-ords, but it's simpler to go back to basics and rederive H from L.

$$\begin{aligned} L &= \tfrac{1}{2}m\dot{\boldsymbol{x}}^2 + Q(\dot{\boldsymbol{x}} \cdot \boldsymbol{A} - \Phi) \\ &= \tfrac{1}{2}m[\dot{r}^2 + r^2\dot{\phi}^2] + Q(r\dot{\phi})\frac{\mu}{4\pi r^2}. \end{aligned} \tag{6-2}$$

$$p_r = m\dot{r}, \quad p_\phi = mr^2\dot{\phi} + \frac{Q\mu}{4\pi r} \quad \Rightarrow \quad \dot{\phi} = \left(p_\phi - \frac{Q\mu}{4\pi r}\right) \Big/ mr^2. \tag{6-3}$$

Take LT of L,

$$\begin{aligned} H = p_r\dot{r} + p_\phi\dot{\phi} - L &= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) \\ &= \frac{1}{2m}\left[p_r^2 + \frac{1}{r^2}\left(p_\phi - \frac{Q\mu}{4\pi r}\right)^2\right]. \end{aligned} \tag{6-4}$$

The particle approaches the dipole from infinity at speed v and impact parameter b. Show that p_ϕ and the particle's speed are constants of motion.

Ans: H does not depend explicitly on ϕ, so $p_\phi = \text{const}$. We know that H is conserved, but from (6-4) $H = \frac{1}{2}mv^2$ and so the speed v is constant.

Show further that for $Q\mu > 0$ the distance of closest approach to the dipole is

$$D = \tfrac{1}{2}\begin{cases} b + \sqrt{b^2 - a^2} & \text{for } \dot{\phi} > 0, \\ b + \sqrt{b^2 + a^2} & \text{for } \dot{\phi} < 0, \end{cases} \tag{6-5}$$

where $a^2 \equiv \mu Q/\pi m v$.

Ans: From the ICs we have $p_\phi = \pm mbv$ (sign depends on initial $\dot{\phi}$) and $H = \frac{1}{2}mv^2$. H is constant along the orbit. Therefore, equating H at infinity with H at pericentre (radius r_0 for which $p_r = 0$), we find that

$$\begin{aligned} \frac{1}{2}mv^2 &= \frac{1}{2mr_0^2}\left(\pm mbv - \frac{Q\mu}{4\pi r_0}\right)^2 \\ \Rightarrow \quad v &= \frac{1}{r_0}\left|\pm bv - \frac{a^2 v}{4r_0}\right| \\ \Rightarrow \quad r_0^2 &= \left|\pm br_0 - \frac{1}{4}a^2\right|, \end{aligned} \tag{6-6}$$

using the fact that $v > 0$ when going from the first to the second line.
First let us investigate the case $\dot{\phi} < 0$. Then (6-6) becomes $r_0^2 = |-br_0 - \frac{1}{4}a^2| = br_0 + \frac{1}{4}a^2$. The solutions to this are given by $2r_0 = b \pm \sqrt{b^2 + a^2}$. We choose the larger of the two solutions because we seek the turning point for a particle that approaches from infinity. Therefore $r_0 = \frac{1}{2}(b + \sqrt{b^2 + a^2})$ when the initial $\dot{\phi} < 0$.
The case $\dot{\phi} > 0$ is slightly more complicated. We need to solve $r_0^2 = |br_0 - \frac{1}{4}a^2|$. The solutions are $2r_0 = b \pm \sqrt{b^2 - a^2}$ (assuming $br_0 > \frac{1}{4}a^2$) and $2r_0 = -b + \sqrt{b^2 + a^2}$ (for $br_0 < \frac{1}{4}a^2$). The largest value of r_0 is therefore $\frac{1}{2}(b + \sqrt{b^2 - a^2})$, provided $a^2 < 4br_0$.

7. An axisymmetric top has Lagrangian

$$L = \tfrac{1}{2}I_1(\dot\phi^2 \sin^2\theta + \dot\theta^2) + \tfrac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 - mga\cos\theta, \tag{7-1}$$

where (θ, ϕ, ψ) are the usual Euler angles. Show that the top's Hamiltonian

$$H = \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3} + mga\cos\theta. \tag{7-2}$$

Using Hamilton's equations or otherwise show that the top will precess steadily at fixed inclination to the vertical provided θ satisfies

$$0 = mga + \frac{(p_\phi - p_\psi\cos\theta)(p_\phi\cos\theta - p_\psi)}{I_1\sin^4\theta}. \tag{7-3}$$

Ans: The Hamiltonian is derived in the lecture notes. For the top to precess steadily at fixed inclination we require that $\dot\theta = \dot p_\theta = 0$. Using Hamilton's equation for the rate of change of $p_\theta \equiv I_1\dot\theta$,

$$\begin{aligned}
0 = -\frac{\partial H}{\partial\theta} &= \frac{(p_\phi - p_\psi\cos\theta)}{I_1\sin^2\theta}p_\psi\sin\theta - \frac{(p_\phi - p_\psi\cos\theta)^2}{I_1\sin^3\theta}\cos\theta - mga\sin\theta \\
&= \frac{(p_\phi - p_\psi\cos\theta)[p_\psi\sin^2\theta - (p_\phi - p_\psi\cos\theta)\cos\theta]}{I_1\sin^3\theta} - mga\sin\theta \\
&= \frac{(p_\phi - p_\psi\cos\theta)[p_\psi - p_\phi\cos\theta]}{I_1\sin^3\theta} - mga\sin\theta.
\end{aligned} \tag{7-4}$$

8. A point charge q is placed at the origin in the magnetic field generated by a spatially confined current distribution. Given that

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{r}}{r^3} \tag{8-1}$$

and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ with $\nabla \cdot \boldsymbol{A} = 0$, show that the field's momentum

$$\boldsymbol{P} \equiv \epsilon_0 \int \boldsymbol{E} \times \boldsymbol{B}\, \mathrm{d}^3\boldsymbol{x} = q\boldsymbol{A}(0). \tag{8-2}$$

Use this result to intrepret the formula for the canonical momentum of a charged particle in an electromagnetic field. [Hint: use $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and then index notation (easy) or vector identities (not so easy) to expand $\boldsymbol{E} \times \boldsymbol{B}$ into a sum of two terms. To each term apply the tensor form of Gauss's theorem, which states that $\int \mathrm{d}^3\boldsymbol{x}\nabla_i \boldsymbol{T} = \oint \mathrm{d}^2 S_i \boldsymbol{T}$, no matter how many indices the tensor $\boldsymbol{T}$ carries. In one term you can make use of $\nabla \cdot \boldsymbol{A} = 0$ and in the other $\nabla^2 r^{-1} = -4\pi\delta^3(\boldsymbol{r})$.]

Ans: Easy to show that

$$\boldsymbol{P} = -\frac{q}{4\pi}\int \mathrm{d}^3\boldsymbol{r}\left(\nabla\frac{1}{r}\right) \times (\nabla \times \boldsymbol{A}). \tag{8-3}$$

Permutation tensor (Levi-Civita symbol)

$$\begin{aligned} &\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \quad \text{even perm } (1,2,3) \\ &\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1 \quad \text{odd perm } (1,2,3) \\ &\qquad\qquad \text{all other} = 0. \end{aligned} \tag{8-4}$$

Handy because (summation convention)

$$\begin{aligned} (\boldsymbol{a} \times \boldsymbol{b})_i &= \epsilon_{ijk} a_j b_k \\ (\nabla \times \boldsymbol{a})_i &= \epsilon_{ijk}\partial_j a_k = \epsilon_{ijk}\frac{\partial}{\partial x_j} a_k \end{aligned} \tag{8-5}$$

Useful identity (contract over middle index)

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im} \tag{8-6}$$

Then

$$\begin{aligned} \left[\left(\nabla\frac{1}{r}\right) \times (\nabla \times \boldsymbol{A})\right]_i &= \epsilon_{ijk}\left(\partial_j\frac{1}{r}\right)(\epsilon_{klm}\partial_l A_m) \\ &= (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im})\left(\partial_j\frac{1}{r}\right)\partial_l A_m \\ &= \left(\partial_j\frac{1}{r}\right)\partial_i A_j - \left(\partial_j\frac{1}{r}\right)\partial_j A_i. \end{aligned} \tag{8-7}$$

Integrating by parts (strictly, Gauss' theorem)

$$\begin{aligned} \int \mathrm{d}^3\boldsymbol{x}\overbrace{\left(\partial_j\frac{1}{r}\right)}^{\text{integrate}}\partial_i A_j &= \oint \mathrm{d}^2 S_j\left(\frac{1}{r}\partial_i A_j\right) - \int \mathrm{d}^3\boldsymbol{x}\frac{1}{r}\partial_i\overbrace{\partial_j A_j}^{\nabla\cdot\boldsymbol{A}=0} \\ &= 0. \\ \int \mathrm{d}^3\boldsymbol{x}\left(\partial_j\frac{1}{r}\right)\underbrace{\partial_j A_i}_{\text{integrate}} &= \oint \mathrm{d}^2 S_j\left(\partial_j\frac{1}{r}\right)A_i - \int \mathrm{d}^3\boldsymbol{x}\underbrace{\left(\partial_j\partial_j\frac{1}{r}\right)}_{\nabla^2 r^{-1} = -4\pi\delta(\boldsymbol{r})} A_i \\ &= 4\pi A_i(0). \end{aligned} \tag{8-8}$$

To interpret, recall that the canonical momentum for a particle in a magnetic field $\boldsymbol{p} = m\dot{\boldsymbol{r}} + q\boldsymbol{A}$...