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(most problems taken from J. J. Binney's 2006 course)

S7: Classical mechanics – problem set 1

1. A particle is confined to move under gravity along a smooth wire that passes through two rings at $(x, y, z) = (0, 0, h)$ and $(X, 0, 0)$. The particle starts at rest from the first, upper ring. Using conservation of energy, show that the time for the particle to travel from the upper to lower ring is given by

$$T[z(x)] = \int_0^X \left[\frac{1 + z'^2}{2g(h - z)}\right]^{1/2} \mathrm{d}x, \tag{1-1}$$

where $z(x)$ is the height of the wire as a function of horizontal position x. Find the shape $z(x)$ that extremizes $T[z(x)]$. [Hint: integrals of the form

$$\int \left(\frac{A - t}{B + t}\right)^{1/2} \mathrm{d}t \tag{1-2}$$

can be solved by substituting $t = A - (B + A)\sin^2\theta$.]

2. Write down the Lagrangian for the motion of a particle of mass m in a potential $\Phi(R, \phi)$ and obtain the equations of motion in plane-polar co-ordinates (R, ϕ). Show that if Φ does not explicitly depend on ϕ then the generalized momentum $p_\phi \equiv \partial L/\partial\dot\phi$ is a constant of the motion and interpret this result physically.

Obtain the Lagrangian in terms of the variables $u \equiv 1/R$ and ϕ. Show that if $\Phi(R) = -\alpha/R$ the EL equations give

$$u(\phi) = A\cos(\phi - \phi_0) + B, \tag{2-1}$$

where A, B and ϕ_0 are arbitrary constants. Show that the orbit is an ellipse if $B > A$ and a parabola or hyperbola otherwise.

3. A particle of mass m slides inside a smooth straight tube OA. The particle is connected to point O by a light spring of natural length a and spring constant mk/a. The system rotates in a horizontal plane with constant angular velocity ω about a fixed vertical axis through O. Find the distance r of the particle from O at time t for the case when $\omega^2 < k/a$, if $r = a$ and $\dot r = 0$ at $t = 0$. Show also for this case that the maximum value of the reaction of the tube on the particle is $2ma\omega^3/b$, where $b^2 = k/a - \omega^2$.

4. Write down the Euler equations for a free rigid body in terms of its principal moments of inertia I_1, I_2, I_3 and the angular velocity $\boldsymbol{\Omega}$ in the body frame. If the body is rotationally symmetric about its z axis show that Ω_3 is a constant of the motion and that $\boldsymbol{\Omega}$ precesses about the z axis with angular frequency $\Omega_3(I_3 - I_1)/I_1$. What is the period of this precession for the earth, which has $(I_3 - I_1)/I_1 = 0.00327$?

5. A heavy symmetric top rotating about a fixed point has Lagrangian

$$L = \tfrac{1}{2}I_1\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + \tfrac{1}{2}I_3\left(\dot\psi + \dot\phi\cos\theta\right)^2 - mgl\cos\theta, \tag{5-1}$$

where $I_1 = I_2$ and I_3 are its principal moments of inertia and (ϕ, θ, ψ) are the usual Euler angles. Write down two conserved momenta and hence show that θ obeys the equation $I_1\ddot\theta = -\partial V_{\rm eff}/\partial\theta$, where

$$V_{\rm eff}(\theta) = \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + mgl\cos\theta. \tag{5-2}$$

Suppose that the top is released with initial conditions $\theta = \theta_0$, $\dot\phi = 0$ and $\dot\psi = \psi_0 \gg \sqrt{mglI_1}/I_3$. Show that it nutates about $\theta \simeq \theta_0$ with frequency $I_3\dot\psi_0/I_1$.

6. A particle of mass m_1 hangs by a light string of length l from a rigid support, and a second mass, m_2, hangs by an identical string from m_1. The angles with the (downward) vertical of the strings supporting m_1 and m_2 are θ_1 and θ_2, respectively. Write down the Lagrangian $L(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ of the system. Hence show that the frequencies of the two normal modes of oscillation about the equilibrium $\theta_1 = \theta_2 = 0$ are

$$\omega_\pm^2 = \frac{g}{l}\frac{m_1+m_2}{m_1}\left[1 \pm \sqrt{\frac{m_2}{m_1+m_2}}\right]. \tag{6-1}$$

Describe the motion in each of the normal modes in the cases (a) $m_1 \gg m_2$, and (b) $m_2 \gg m_1$.

7. A circular hoop of mass m and radius a hangs from a point on its circumference and is free to oscillate in its own plane. A bead of mass m can slide without friction around the hoop. Choose a set of generalized co-ordinates and write down the Lagrangian for the system. Show that the natural frequencies of small oscillations about equilibrium are $\omega_1 = \sqrt{2g/a}$ and $\omega_2 = \sqrt{g/2a}$.

8. The (X, Y, Z) frame rotates with angular speed $\boldsymbol{\omega} = \omega\boldsymbol{k}$. A particle of mass m moves in the potential

$$V(X,Y,Z) = \tfrac{1}{2}m(\omega_X^2 X^2 + \omega_Y^2 Y^2 + \omega_Z^2 Z^2). \tag{8-1}$$

By solving for the frequencies of the particle's normal modes about the equilibrium $X = Y = Z = 0$, show that the motion is unstable if $\omega_X < \omega < \omega_Y$.

9. What is meant by the terms *symmetry principle* and *conservation law* as used in classical dynamics? Give simple examples to illustrate the symmetries underlying the conservation of linear and angular momenta.

A system with three degrees of freedom described by co-ordinates q_1, q_2, q_3 has Lagrangian

$$L = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - \tfrac{1}{2}(q_1^2 + q_2^2 + q_3^2) - \alpha(q_2 q_3 + q_3 q_1 + q_1 q_2), \tag{9-1}$$

where $0 < \alpha < \frac{1}{2}$. Show that L is invariant under infinitesimal rotations about the $(1,1,1)$ axis in q-space, and hence find a constant of motion other than the total energy. Verify from the equation of motion that it is indeed constant.

10. A particle with position co-ordinates $\boldsymbol{r}$ moves in a central potential $V(r)$. Find all potential functions $V(r)$ and corresponding functions $\alpha(r)$ for which the vector

$$\boldsymbol{K} = \dot{\boldsymbol{r}} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}}) + \alpha(r)\boldsymbol{r} \tag{10-1}$$

is conserved.

Find also the potentials $V(r)$ and functions $\beta(r)$ for which the components of the matrix

$$Q_{ij} \equiv \dot{r}_i \dot{r}_j + \beta(r) r_i r_j \tag{10-2}$$

are constants of the motion, where r_i, $\dot{r}_i$ $(i = 1, 2, 3)$ are the components of position and velocity of the particle along any three independent fixed axes.