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(most problems taken from J. J. Binney's 2006 course)

S7: Classical mechanics – problem set 1

1. A particle is confined to move under gravity along a smooth wire that passes through two rings at $(x, y, z) = (0, 0, h)$ and $(X, 0, 0)$. The particle starts at rest from the first, upper ring. Using conservation of energy, show that the time for the particle to travel from the upper to lower ring is given by

$$T[z(x)] = \int_0^X \left[\frac{1+z'^2}{2g(h-z)}\right]^{1/2} \mathrm{d}x, \tag{1-1}$$

where $z(x)$ is the height of the wire as a function of horizontal position x. Find the shape $z(x)$ that extremizes $T[z(x)]$. [Hint: integrals of the form

$$\int \left(\frac{A-t}{B+t}\right)^{1/2} \mathrm{d}t \tag{1-2}$$

can be solved by substituting $t = A - (B + A)\sin^2\theta$.]

Ans: By conservation of energy, the particle moves with speed $v^2(z) = 2g(h - z)$. We then have that

$$v^2 = \dot{x}^2 + \dot{z}^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 \left[1 + z'^2\right] = 2g(h-z), \tag{1-3}$$

where $z' = \mathrm{d}z/\mathrm{d}x$. Taking the square root of both sides and rearranging,

$$\mathrm{d}t = \left[\frac{1+z'^2}{2g(h-z)}\right]^{1/2} \mathrm{d}x, \tag{1-4}$$

so that the time taken to travel from $x = 0$ to $x = X$ along path $z(x)$ is given by $\tau[z(x)] = \int_0^X L(z, z', x)\,\mathrm{d}x$ with

$$L(z, z', x) = \left[\frac{1+z'^2}{2g(h-z)}\right]^{1/2}. \tag{1-5}$$

This is a functional of the form we've encountered in lectures, but with t replaced by x and $x(t)$ by $z(x)$. Since L does not depend explicitly on x, the path $z(x)$ that extremizes $\tau[z(x)]$ satisfies

$$\text{constant} = z'\frac{\partial L}{\partial z'} - L = -\frac{L}{1+z'^2} = -\left[\frac{1}{2g(h-z)(1+z'^2)}\right]^{1/2}. \tag{1-6}$$

Rearranging and integrating,

$$\int_h^{z(x)} \left(\frac{h-z}{A+z}\right)^{1/2} \mathrm{d}z = x, \tag{1-7}$$

where A is a constant and we have used the bc $z(x = 0) = h$. Substituting $z = h - (A + h)\sin^2\theta$ and integrating,

$$x = (A + h)\left[\theta - \tfrac{1}{2}\sin 2\theta\right]. \tag{1-8}$$

Writing $\phi \equiv 2\theta$ this becomes

$$\begin{aligned} x &= \tfrac{A+h}{2}\left[\phi - \sin\phi\right], \\ z &= h - \tfrac{A+h}{2}\left[1 - \cos\phi\right], \end{aligned} \tag{1-9}$$

which is a cycloid with $(x, z) = (0, h)$ at its cusp. The constant A is determined by the condition that the curve pass through the endpoint $(X, 0)$.

2. Write down the Lagrangian for the motion of a particle of mass m in a potential $\Phi(R,\phi)$ and obtain the equations of motion in plane-polar co-ordinates (R,ϕ). Show that if Φ does not explicitly depend on ϕ then the generalized momentum $p_\phi \equiv \partial L/\partial\dot\phi$ is a constant of the motion and interpret this result physically.

Ans: The radial speed is $\dot R$ and the angular speed $R\dot\phi$, so $T = \frac{1}{2}m[\dot R^2 + R^2\dot\phi^2]$. The Lagrangian $L = T - V = \frac{1}{2}m[\dot R^2 + R^2\dot\phi^2] - m\Phi(R,\phi)$, for which the EL equations become

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(m\dot R) &= mR\dot\phi^2 - m\frac{\partial\Phi}{\partial R}, \\ \frac{\mathrm{d}}{\mathrm{d}t}(mR^2\dot\phi) &= -m\frac{\partial\Phi}{\partial\phi}. \end{aligned} \tag{2-1}$$

The second of these shows that the generalized momentum $p_\phi \equiv mR^2\dot\phi$ is conserved if Φ does not depend on ϕ; the angular momentum of the particle is constant if there are no torques acting on it.

Obtain the Lagrangian in terms of the variables $u \equiv 1/R$ and ϕ. Show that if $\Phi(R) = -\alpha/R$ the EL equations give

$$u(\phi) = A\cos(\phi-\phi_0) + B, \tag{2-2}$$

where A, B and ϕ_0 are arbitrary constants. Show that the orbit is an ellipse if $B > A$ and a parabola or hyperbola otherwise.

Ans: For $R = 1/u$, $\dot R = -\dot u/u^2$ and the Lagrangian becomes

$$L = \tfrac{1}{2}m\left[\frac{\dot u^2}{u^4} + \frac{\dot\phi^2}{u^2}\right] - m\Phi(u). \tag{2-3}$$

The EL equation for ϕ,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot\phi}{u^2}\right) = 0 \quad\Rightarrow\quad \frac{\dot\phi}{u^2} = h,\ \text{a constant.} \tag{2-4}$$

The EL equation for u is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\dot u}{u^4}\right) + 2m\frac{\dot u^2}{u^5} + m\frac{\dot\phi^2}{u^3} - m\alpha = 0. \tag{2-5}$$

We can eliminate t from this by substituting $\dot\phi = hu^2$ and $\mathrm{d}t = \mathrm{d}\phi/hu^2$ to obtain

$$hu^2\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{m}{u^4}hu^2\frac{\mathrm{d}u}{\mathrm{d}\phi}\right) + \frac{2m}{u^5}h^2u^4\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + mh^2u - m\alpha = 0. \tag{2-6}$$

Expanding the derivative in the first term and simplifying,

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{\alpha}{h^2}, \tag{2-7}$$

for which the general solution is

$$u(\phi) = A\cos(\phi-\phi_0) + \alpha/h^2, \tag{2-8}$$

where A and ϕ_0 are constants. If $B = \alpha/h^2 > A$ then $u > 0$ and so the orbit is bound. To find the general shape of the orbit, let $x = r\cos(\phi-\phi_0)$ and $y = r\sin(\phi-\phi_0)$. Then, from the general solution above with $B = \alpha/h^2$,

$$\frac{1}{r} = A\frac{x}{r} + \frac{\alpha}{h^2} \quad\Rightarrow\quad 1 - Ax = Br \quad\Rightarrow\quad (1-Ax)^2 = B^2\left(x^2+y^2\right). \tag{2-9}$$

Rearranging gives $(B^2 - A^2)x^2 + 2Ax + By^2 = 1$, which is the equation of a conic section – an ellipse if $B > A$, hyperbola if $B < A$ or parabola if $B = A$.

3. A particle of mass m slides inside a smooth straight tube OA. The particle is connected to point O by a light spring of natural length a and spring constant mk/a. The system rotates in a horizontal plane with constant angular velocity ω about a fixed vertical axis through O. Find the distance r of the particle from O at time t for the case when $\omega^2 < k/a$, if $r = a$ and $\dot{r} = 0$ at $t = 0$. Show also for this case that the maximum value of the reaction of the tube on the particle is $2ma\omega^3/b$, where $b^2 = k/a - \omega^2$.

Ans: $T = \frac{1}{2}m(\dot{r}^2 + \omega^2 r^2)$ and $V(r) = \frac{1}{2}(mk/a)(r-a)^2$, so the Lagrangian

$$L(r,\dot{r}) = T - V = \tfrac{1}{2}m(\dot{r}^2 + \omega^2 r^2) - \tfrac{1}{2}\frac{mk}{a}(r-a)^2. \tag{3-1}$$

The equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}m\dot{r} - m\omega^2 r + \frac{mk}{a}(r-a) = 0 \quad\Rightarrow\quad \ddot{r} + \Omega^2 r = k, \tag{3-2}$$

with $\Omega^2 \equiv k/a - \omega^2$ ($= b^2$ in the question). The general solution is

$$r = A\cos\Omega t + B\sin\Omega t + \frac{k}{\Omega^2}. \tag{3-3}$$

Choosing the integration constants A and B to satisfy $r = a$ and $\dot{r} = 0$ at $t = 0$ gives

$$r = \left(a - \frac{k}{\Omega^2}\right)\cos\Omega t + \frac{k}{\Omega^2}. \tag{3-4}$$

The angular momentum $mr^2\omega$ of the particle varies as it oscillates in and out. Therefore the particle feels a torque fr, where f is the reaction of the tube on the particle and

$$fr = \frac{\mathrm{d}}{\mathrm{d}t}mr^2\omega = 2mr\dot{r}\omega \quad\Rightarrow\quad f = 2m\omega\dot{r}. \tag{3-5}$$

$\dot{r}$ is maximum at $\Omega t = \pi/2$ with value $\dot{r}_{\max} = \Omega(a - k/\Omega^2)$, so $f_{\max} = 2ma\omega^3/\Omega$.

Ans: (alternative) In terms of (non-rotating) polar co-ordinates (r,ϕ), the particle has Lagrangian

$$L(r,\phi,\dot{r},\dot{\phi}) = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \tfrac{1}{2}\frac{mk}{a}(r-a)^2, \tag{3-6}$$

but is subject to the holonomic constraint $\phi - \omega t = 0$. So, the augmented Lagrangian

$$\begin{aligned} L' &= L + \lambda(\phi - \omega t) \\ &= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \tfrac{1}{2}\frac{mk}{a}(r-a)^2 + \lambda(\phi - \omega t), \end{aligned} \tag{3-7}$$

for which the EL equations are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}m\dot{r} - mr\dot{\phi}^2 + \frac{mk}{a}(r-a) &= 0 \quad (r) \\ \frac{\mathrm{d}}{\mathrm{d}t}mr^2\dot{\phi} &= \lambda \quad (\phi) \\ \phi - \omega t &= 0 \quad (\lambda). \end{aligned} \tag{3-8}$$

The first and last of these taken together reduce to equation (3-2) above and the derivation of $r(t)$ proceeds as from there. To obtain the reaction force, use the fact that the (generalized) constraint force is given in this case by $\partial L/\partial\phi = \lambda$. By the second and third equations of (3-8), $\lambda = \frac{\mathrm{d}}{\mathrm{d}t}mr^2\omega$, a torque, and the reaction force follows from (3-5).

4. Write down the Euler equations for a free rigid body in terms of its principal moments of inertia I_1, I_2, I_3 and the angular velocity $\boldsymbol{\Omega}$ in the body frame. If the body is rotationally symmetric about its z axis show that Ω_3 is a constant of the motion and that $\boldsymbol{\Omega}$ precesses about the z axis with angular frequency $\Omega_3(I_3 - I_1)/I_1$. What is the period of this precession for the earth, which has $(I_3 - I_1)/I_1 = 0.00327$?

Ans: Euler's equation is $\dot{\boldsymbol{J}} + \boldsymbol{\Omega} \times \boldsymbol{J} = 0$, where $\boldsymbol{J}$ and $\boldsymbol{\Omega}$ are the angular momentum and angular velocity referred to axes that co-rotate with the body. If we choose these axes to be the body's principal axes, then $\boldsymbol{J} = (I_1\Omega_1, I_2\Omega_2, I_3\Omega_3)$ and Euler's equation becomes

$$
\begin{aligned}
I_1 \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} &= (I_2 - I_3)\Omega_2\Omega_3, \\
I_2 \frac{\mathrm{d}\Omega_2}{\mathrm{d}t} &= (I_3 - I_1)\Omega_3\Omega_1, \\
I_3 \frac{\mathrm{d}\Omega_3}{\mathrm{d}t} &= (I_1 - I_2)\Omega_1\Omega_2.
\end{aligned}
\tag{4-1}
$$

Recall that

$$
\begin{aligned}
I_1 &= \int \rho(\boldsymbol{x})(y^2 + z^2)\, \mathrm{d}^3\boldsymbol{x} \\
I_2 &= \int \rho(\boldsymbol{x})(x^2 + z^2)\, \mathrm{d}^3\boldsymbol{x} \\
I_3 &= \int \rho(\boldsymbol{x})(x^2 + y^2)\, \mathrm{d}^3\boldsymbol{x},
\end{aligned}
\tag{4-2}
$$

so $I_1 = I_2$ if the body is rotationally symmetric about the z axis. Then (4-1) implies that $\dot{\Omega}_3 = 0$ and the first two of (4-1) can be rewritten

$$
\dot{\Omega}_1 = -\Omega_\mathrm{p}\Omega_2, \quad \dot{\Omega}_2 = \Omega_\mathrm{p}\Omega_1,
\tag{4-3}
$$

with $\Omega_\mathrm{p} \equiv \Omega_3(I_3 - I_1)/I_1$. The general solution is

$$
\begin{aligned}
\Omega_1 &= A\cos(\Omega_\mathrm{p} + \phi_0) \\
\Omega_2 &= A\sin(\Omega_\mathrm{p} + \phi_0),
\end{aligned}
\tag{4-4}
$$

where the amplitude A and phase ϕ_0 are constants of integration. So, $\boldsymbol{\Omega}$ (and therefore $\boldsymbol{J}$) precesses about the body's z axis. For the earth, $\Omega_3 = (1\ \text{day})^{-1}$, so that the precession frequency should be $\Omega_\mathrm{p} = 0.00327\,\mathrm{day}^{-1}$, corresponding to a period of about 300 days. In fact, the observed period is 433 days.

5. A heavy symmetric top rotating about a fixed point has Lagrangian

$$L = \tfrac{1}{2}I_1\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + \tfrac{1}{2}I_3\left(\dot\psi + \dot\phi\cos\theta\right)^2 - mgl\cos\theta, \tag{5-1}$$

where $I_1 = I_2$ and I_3 are its principal moments of inertia and (ϕ,θ,ψ) are the usual Euler angles. Write down two conserved momenta and hence show that θ obeys the equation $I_1\ddot\theta = -\partial V_{\rm eff}/\partial\theta$, where

$$V_{\rm eff}(\theta) = \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + mgl\cos\theta. \tag{5-2}$$

Suppose that the top is released with initial conditions $\theta=\theta_0$, $\dot\phi = 0$ and $\dot\psi = \psi_0 \gg \sqrt{mglI_1}/I_3$. Show that it nutates about $\theta\simeq\theta_0$ with frequency $I_3\dot\psi_0/I_1$.

Ans: Since ϕ and ψ are cyclic co-ordinates ($\partial L/\partial\phi = \partial L/\partial\psi = 0$), the conserved momenta are clearly

$$\begin{aligned} p_\phi &= \frac{\partial L}{\partial\dot\phi} = \dot\phi(I_1\sin^2\theta + I_3\cos^2\theta) + \dot\psi I_3\cos\theta,\\ p_\psi &= \frac{\partial L}{\partial\dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta). \end{aligned} \tag{5-3}$$

The velocity $\dot\phi$ can be written in terms of these momenta as

$$\dot\phi = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}. \tag{5-4}$$

There are at least two ways of proceeding from here. The first way is to use the EL equation for θ to obtain $I_1\ddot\theta$ in terms of $(\theta,\dot\psi,\dot\phi)$, then eliminate $\dot\psi$ and $\dot\phi$ in that expression in favour of p_ψ and p_ϕ, finally showing that this gives something equal to (minus) the derivative of the given $V_{\rm eff}$. This is long!
The second, simpler, way is to find the Hamiltonian corresponding to the given L, noting that L is of the form $\frac{1}{2}\sum_{ij}\dot q_i A_{ij}\dot q_j - V(\boldsymbol{q})$, where the matrix elements $A_{ij} = A_{ji}$ are functions of the generalized coordinates $\boldsymbol{q} = (\theta,\phi,\psi)$. Then $p_i = \partial L/\partial\dot q_i = \sum_j A_{ij}\dot q_j$, so that $H = \boldsymbol{p}\cdot\dot{\boldsymbol{q}} - L = \frac{1}{2}\sum_{ij}\dot q_i A_{ij}\dot q_j + V(\boldsymbol{q})$, in which $\dot q_i$ is understood to be a function of $\boldsymbol{q}$ and $\boldsymbol{p}$. Therefore

$$\begin{aligned} H(\theta,p_\theta,p_\phi,p_\psi) &= \tfrac{1}{2}I_1\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + \tfrac{1}{2}I_3\left(\dot\psi + \dot\phi\cos\theta\right)^2 + mgl\cos\theta\\ &= \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3^2} + mgl\cos\theta, \end{aligned} \tag{5-5}$$

using (5-4) for $\dot\phi$, (5-3) for $(\dot\psi + \dot\phi\cos\theta)$ and the definition $p_\theta \equiv \partial L/\partial\dot\theta = I_1\dot\theta$ for $\dot\theta$. Hamilton's equation $\dot p_\theta = -\partial H/\partial\theta$ when written out in terms of $\dot\theta$ becomes $I_1\ddot\theta = -{\rm d}V_{\rm eff}/d\theta$.
For the initial conditions $\theta=\theta_0$, $\dot\phi=0$, $\dot\psi=\dot\psi_0 \gg \sqrt{mglI_1}/I_3$ we have $p_\phi = \dot\psi_0 I_3\cos\theta_0$, $p_\psi = \dot\psi_0 I_3$, so that

$$V_{\rm eff}(\theta) \approx \frac{I_3^2\dot\psi_0^2(\cos\theta_0 - \cos\theta)^2}{2I_1\sin^2\theta}, \tag{5-6}$$

the effect of the $mgl\cos\theta$ term being negligible as $\dot\psi_0 \gg \sqrt{mglI_1}/I_3$. Since the local minimum of $V_{\rm eff}$ is very close to θ_0, we may expand $V_{\rm eff}$ as a Taylor series about θ_0,

$$V_{\rm eff}(\theta_0+\Delta\theta) - V_{\rm eff}(\theta_0) \approx \frac{I_3^2\dot\psi_0^2(\cos\theta_0 - \cos\theta_0\cos\Delta\theta + \sin\theta_0\sin\Delta\theta)^2}{2I_1(\sin\theta_0\cos\Delta\theta + \cos\theta_0\sin\Delta\theta)^2} \approx \tfrac{1}{2}\left(\frac{I_3\dot\psi_0}{I_1}\right)^2(\Delta\theta)^2. \tag{5-7}$$

So, for small departures from equilibrium, θ oscillates about θ_0 with frequency $I_3\dot\psi_0/I_1$.

6. A particle of mass m_1 hangs by a light string of length l from a rigid support, and a second mass, m_2, hangs by an identical string from m_1. The angles with the (downward) vertical of the strings supporting m_1 and m_2 are θ_1 and θ_2, respectively. Write down the Lagrangian $L(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$ of the system. Hence show that the frequencies of the two normal modes of oscillation about the equilibrium $\theta_1 = \theta_2 = 0$ are

$$\omega_\pm^2 = \frac{g}{l}\frac{m_1+m_2}{m_1}\left[1 \pm \sqrt{\frac{m_2}{m_1+m_2}}\right]. \tag{6-1}$$

Describe the motion in each of the normal modes in the cases (a) $m_1 \gg m_2$, and (b) $m_2 \gg m_1$.

Ans: Mass m_1 moves with speed $l\dot\theta_1$ and has potential energy $-m_1 gl\cos\theta_1$. Mass m_2 has position $(x,y) = (l\sin\theta_1 + l\sin\theta_2, l\cos\theta_1 + l\cos\theta_2)$, so its potential potential energy is $-m_2 gl(\cos\theta_1 + \cos\theta_2)$ and, for $\theta_1, \theta_2 \simeq 0$, its speed is dominated by its horizontal component of velocity, $l\dot\theta_1 + l\dot\theta_2$. Expanding $L = T - V$ about $\theta_1 = \theta_2 = 0$,

$$L \approx \tfrac{1}{2}m_1 l^2\dot\theta_1^2 + \tfrac{1}{2}m_2 l^2(\dot\theta_1 + \dot\theta_2)^2 - \tfrac{1}{2}m_1 gl\theta_1^2 - \tfrac{1}{2}m_2 gl\theta_1^2 - \tfrac{1}{2}m_2 gl\theta_2^2. \tag{6-2}$$

The linearized equations of motion are then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[m_1 l^2\dot\theta_1 + m_2 l^2(\dot\theta_1 + \theta_2)\right] + (m_1+m_2)gl\theta_1 &= 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[m_2 l^2(\dot\theta_1 + \dot\theta_2)\right] + m_2 gl\theta_2 &= 0. \end{aligned} \tag{6-3}$$

Substituting $\theta_i = \Theta_i \mathrm{e}^{\mathrm{i}\omega t}$, these become the eigenvalue equation

$$\begin{pmatrix} -\omega^2(m_1+m_2)l^2 + (m_1+m_2)gl & -\omega^2 m_2 l^2 \\ -\omega^2 m_2 l^2 & -\omega^2 m_2 l^2 + m_2 gl \end{pmatrix}\begin{pmatrix}\Theta_1 \\ \Theta_2\end{pmatrix} = 0. \tag{6-4}$$

Taking the determinant,

$$\begin{aligned} & (m_1+m_2)(gl - \omega^2 l^2)m_2(gl - \omega^2 l^2) = (m_2\omega^2 l^2)^2 \\ \Rightarrow\quad & \sqrt{\frac{m_1+m_2}{m_2}}(gl - \omega^2 l^2) = \pm\omega^2 l^2 \\ \Rightarrow\quad & gl = \omega^2 l^2\left(1 \pm \sqrt{\frac{m_2}{m_1+m_2}}\right) \\ \Rightarrow\quad & \omega^2 = \frac{g}{l}\frac{1}{1 \pm \sqrt{\frac{m_2}{m_1+m_2}}} = \frac{g}{l}\left[\frac{1 \mp \sqrt{\frac{m_2}{m_1+m_2}}}{1 - \frac{m_2}{m_1+m_2}}\right] = \frac{g}{l}\frac{m_1+m_2}{m_1}\left[1 \mp \sqrt{\frac{m_2}{m_1+m_2}}\right]. \end{aligned} \tag{6-5}$$

Case (a), $m_1 \gg m_2$. Both frequencies $\approx \sqrt{g/l}$ because the upper, heavier mass swings without disturbance from the second.
Case (b), $m_2 \gg m_1$. To first order in m_1/m_2,

$$\sqrt{\frac{m_2}{m_1+m_2}} = \left[1 - \frac{m_1}{m_1+m_2}\right]^{1/2} \simeq 1 - \frac{m_1}{2m_2}. \tag{6-6}$$

One frequency is now very high (the light mass m_1 on the taught string) and the other is $\approx \sqrt{g/2l}$ (mass on a string of length $2l$).

7. A circular hoop of mass m and radius a hangs from a point on its circumference and is free to oscillate in its own plane. A bead of mass m can slide without friction around the hoop. Choose a set of generalized co-ordinates and write down the Lagrangian for the system. Show that the natural frequencies of small oscillations about equilibrium are $\omega_1 = \sqrt{2g/a}$ and $\omega_2 = \sqrt{g/2a}$.

Ans: See diagram below for co-ordinate system. The hoop's moment of inertia about its centre of mass $I_0 = ma^2$. By the parallel axis theorem, its moment of inertia about the pivot point is $I_\mathrm{p} = I_0 + ma^2 = 2ma^2$. The kinetic energy of the hoop is then $\frac{1}{2}I_\mathrm{p}\dot\theta^2 = ma^2\dot\theta^2$ and its PE is $-mga\cos\theta$. The bead's horizontal offset from the pivot is $a\sin\theta + a\sin\phi$ and its vertical offset is $a\cos\theta + a\cos\phi$. So for $\theta, \phi \simeq 0$ the bead has KE $\simeq \frac{1}{2}ma^2(\dot\theta + \dot\phi)^2$ and PE $-mga\cos\theta - mga\cos\phi$. To second order in θ, ϕ, the Lagrangian of bead+hoop is

$$L = \tfrac{1}{2}ma^2\left[2\dot\theta^2 + (\dot\theta + \dot\phi)^2\right] + \tfrac{1}{2}mga(2\theta^2 + \phi^2). \tag{7-1}$$

The equations of motion are

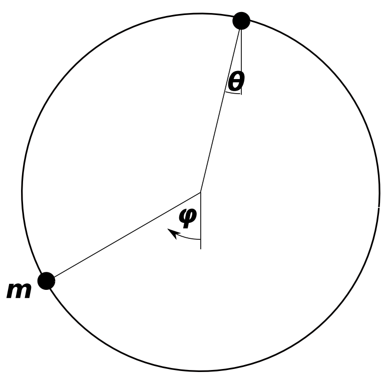
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[2ma^2\dot\theta + ma^2(\dot\theta + \dot\phi)\right] + 2mga\theta &= 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[ma^2(\dot\theta + \dot\phi)\right] + mga\phi &= 0. \end{aligned} \tag{7-2}$$

Substituting $\theta = \Theta e^{i\omega t}$, $\phi = \Phi e^{i\omega t}$ gives the eigenvalue equation

$$\begin{pmatrix} -3\omega^2 + 2\frac{g}{a} & -\omega^2 \\ -\omega^2 & -\omega^2 + \frac{g}{a} \end{pmatrix} \begin{pmatrix} \Theta \\ \Phi \end{pmatrix} = 0. \tag{7-3}$$

Taking the determinant of both sides,

$$\begin{aligned} \left(2\frac{g}{a} - 3\omega^2\right)\left(\frac{g}{a} - \omega^2\right) - \omega^4 = 0 \quad &\Rightarrow \quad 2\omega^4 - 5\omega^2\frac{g}{a} + 2\left(\frac{g}{a}\right)^2 = 0 \\ &\Rightarrow \quad \left(\omega^2 - \frac{g}{2a}\right)\left(\omega^2 - \frac{2g}{a}\right) = 0. \end{aligned} \tag{7-4}$$



8. The (X, Y, Z) frame rotates with angular speed $\boldsymbol{\omega} = \omega \boldsymbol{k}$. A particle of mass m moves in the potential

$$V(X, Y, Z) = \tfrac{1}{2}m(\omega_X^2 X^2 + \omega_Y^2 Y^2 + \omega_Z^2 Z^2). \tag{8-1}$$

By solving for the frequencies of the particle's normal modes about the equilibrium $X = Y = Z = 0$, show that the motion is unstable if $\omega_X < \omega < \omega_Y$.

Ans:

$$\begin{aligned} L &= \tfrac{1}{2}m\,[\dot{r} + \omega \boldsymbol{k} \times \boldsymbol{r}]^2 - V(\boldsymbol{r}) \\ &= \tfrac{1}{2}m\left[(\dot{X} - \omega Y)^2 + (\dot{Y} + \omega X)^2 + \dot{Z}^2\right] - \tfrac{1}{2}m\left[\omega_X^2 X^2 + \omega_Y^2 Y^2 + \omega_Z^2 Z^2\right]. \end{aligned} \tag{8-2}$$

Notice that $\partial^2 L/\partial X \partial \dot{Y} \neq 0$, so the matrix C_{ij} in the lectures is non-zero. Nevertheless, we can proceed using the same ideas as before. The equations of motion are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} m(\dot{X} - \omega Y) - m(\dot{Y} + \omega X)\omega + m\omega_X^2 X &= 0, \\ \frac{\mathrm{d}}{\mathrm{d}t} m(\dot{Y} + \omega X) + m(\dot{X} - \omega Y)\omega + m\omega_Y^2 Y &= 0, \\ \frac{\mathrm{d}}{\mathrm{d}t} m\dot{Z} + m\omega_Z^2 Z &= 0. \end{aligned} \tag{8-3}$$

The Z motion decouples from the (X, Y) motion, so one normal frequency is ω_Z.
Letting $X = X_0 \mathrm{e}^{\mathrm{i}\Omega t}$, $Y = Y_0 \mathrm{e}^{\mathrm{i}\Omega t}$, the equations of motion become

$$\begin{pmatrix} -\Omega^2 - \omega^2 + \omega_X^2 & -2\mathrm{i}\Omega\omega \\ 2\mathrm{i}\Omega\omega & -\Omega^2 - \omega^2 + \omega_Y^2 \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = 0. \tag{8-4}$$

The determinant

$$(\Omega^2 + \omega^2 - \omega_X^2)(\Omega^2 + \omega^2 - \omega_Y^2) - 4\Omega^2\omega^2 = 0, \tag{8-5}$$

which, after rearranging and using the usual formula for quadratics, has solution

$$\begin{aligned} 2\Omega^2 &= 2\omega^2 + \omega_X^2 + \omega_Y^2 \pm \sqrt{(2\omega^2 + \omega_X^2 + \omega_Y^2)^2 - 4(\omega^2 - \omega_X^2)(\omega^2 - \omega_Y^2)} \\ &= B \pm \sqrt{B^2 - 4(\omega^2 - \omega_X^2)(\omega^2 - \omega_Y^2)}. \end{aligned} \tag{8-6}$$

If $\omega_X < \omega$ and $\omega < \omega_Y$, then $-4(\omega^2 - \omega_X^2)(\omega^2 - \omega_Y^2) > 0$ and one of the two roots of Ω^2 is negative and so the motion is unstable.

9. What is meant by the terms *symmetry principle* and *conservation law* as used in classical dynamics? Give simple examples to illustrate the symmetries underlying the conservation of linear and angular momenta.

A system with three degrees of freedom described by co-ordinates q_1, q_2, q_3 has Lagrangian

$$L = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - \tfrac{1}{2}(q_1^2 + q_2^2 + q_3^2) - \alpha(q_2 q_3 + q_3 q_1 + q_1 q_2), \tag{9-1}$$

where $0 < \alpha < \frac{1}{2}$. Show that L is invariant under infinitesimal rotations about the $(1, 1, 1)$ axis in q-space, and hence find a constant of motion other than the total energy. Verify from the equation of motion that it is indeed constant.

Ans: The first two terms in the Lagrangian are clearly invariant under rotations about *any* axis, so we merely have to show that $L' = q_2 q_3 + q_3 q_1 + q_1 q_2$ is invariant under rotations about $\boldsymbol{n} = (1, 1, 1)$.
The change in L' under such a rotation

$$\begin{aligned} \delta L' &= L'(\boldsymbol{q} + \epsilon \boldsymbol{n} \times \boldsymbol{q}) - L'(\boldsymbol{q}) \\ &= \epsilon \frac{\partial L'}{\partial \boldsymbol{q}} \cdot (\boldsymbol{n} \times \boldsymbol{q}) + O(\epsilon^2) \\ &= \epsilon(q_3 + q_2, q_3 + q_1, q_2 + q_1) \cdot (q_3 - q_2, q_1 - q_3, q_2 - q_1) \\ &= \epsilon \left[(q_3^2 - q_2^2) + (q_1^2 - q_3^2) + (q_2^2 - q_1^2)\right] \\ &= 0. \end{aligned} \tag{9-2}$$

By Noether's theorem, the corresponding constant of the motion

$$\begin{aligned} C = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot (\boldsymbol{n} \times \boldsymbol{q}) &= (\dot{q}_1, \dot{q}_2, \dot{q}_3) \cdot (q_3 - q_2, q_1 - q_3, q_2 - q_1) \\ &= \dot{q}_1(q_3 - q_2) + \dot{q}_2(q_1 - q_3) + \dot{q}_3(q_2 - q_1). \end{aligned} \tag{9-3}$$

To show that C is indeed constant, differentiate (9-3) with respect to time,

$$\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}t} &= \ddot{q}_1(q_3 - q_2) + \ddot{q}_2(q_1 - q_3) + \ddot{q}_3(q_2 - q_1) + \dot{q}_1(\dot{q}_3 - \dot{q}_2) + \dot{q}_2(\dot{q}_1 - \dot{q}_3) + \dot{q}_3(\dot{q}_2 - \dot{q}_1) \\ &= \ddot{q}_1(q_3 - q_2) + \ddot{q}_2(q_1 - q_3) + \ddot{q}_3(q_2 - q_1), \end{aligned} \tag{9-4}$$

and take $\ddot{q}_i$ from the equations of motion,

$$\begin{aligned} \ddot{q}_1 + q_1 + \alpha(q_3 + q_2) &= 0, \\ \ddot{q}_2 + q_2 + \alpha(q_1 + q_3) &= 0, \\ \ddot{q}_3 + q_3 + \alpha(q_2 + q_1) &= 0, \end{aligned} \tag{9-5}$$

to obtain

$$\begin{aligned} -\frac{\mathrm{d}C}{\mathrm{d}t} &= (q_1 + \alpha(q_3 + q_2))(q_3 - q_2) + (q_2 + \alpha(q_1 + q_3))(q_1 - q_3) + (q_3 + \alpha(q_2 + q_1))(q_2 - q_1) \\ &= 0. \end{aligned} \tag{9-6}$$

10. A particle with position co-ordinates $\boldsymbol{r}$ moves in a central potential $V(r)$. Find all potential functions $V(r)$ and corresponding functions $\alpha(r)$ for which the vector

$$\boldsymbol{K} = \dot{\boldsymbol{r}} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}}) + \alpha(r)\boldsymbol{r} \tag{10-1}$$

is conserved.

Ans: $\boldsymbol{K}$ conserved $\Rightarrow \dot{\boldsymbol{K}} = 0$. So,

$$0 = \ddot{\boldsymbol{r}} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}}) + \dot{\boldsymbol{r}} \times \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times \dot{\boldsymbol{r}}) + \alpha' \frac{\boldsymbol{r} \cdot \dot{\boldsymbol{r}}}{r}\boldsymbol{r} + \alpha\dot{\boldsymbol{r}}, \tag{10-2}$$

where $\alpha' \equiv \mathrm{d}\alpha/\mathrm{d}r$ and we have used $\mathrm{d}r/\mathrm{d}t = \boldsymbol{r} \cdot \dot{\boldsymbol{r}}/r$.
The potential is spherically symmetric, so $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times \dot{\boldsymbol{r}}) = 0$ and from the equations of motion we have that $\ddot{\boldsymbol{r}} = -\partial V/\partial \boldsymbol{r} = -V'\boldsymbol{r}/r$. Then

$$\begin{aligned} 0 &= -\frac{V'}{r}\boldsymbol{r} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}}) + \frac{\alpha'}{r}(\boldsymbol{r} \cdot \dot{\boldsymbol{r}})\boldsymbol{r} + \alpha\dot{\boldsymbol{r}} \\ &= -\frac{V'}{r}\left[(\boldsymbol{r} \cdot \dot{\boldsymbol{r}})\boldsymbol{r} - r^2\dot{\boldsymbol{r}}\right] + \frac{\alpha'}{r}(\boldsymbol{r} \cdot \dot{\boldsymbol{r}})\boldsymbol{r} + \alpha\dot{\boldsymbol{r}} \end{aligned} \tag{10-3}$$

Multiplying by r and writing out the i^{th} component of the equation in tensor notation,

$$\begin{aligned} 0 &= -V'\left[\sum_j r_j\dot{r}_j r_i - r^2\dot{r}_i\right] + \alpha'\sum_j r_j\dot{r}_j r_i + \alpha r\dot{r}_i \\ &= -V'\left[\sum_j r_j\dot{r}_j r_i - \sum_j r^2\dot{r}_j\delta_{ij}\right] + \alpha'\sum_j r_j\dot{r}_j r_i + \sum_j \alpha r\dot{r}_j\delta_{ij} \\ &= \sum_j \dot{r}_j\left[-V' r_i r_j + r^2 V'\delta_{ij} + \alpha' r_i r_j + \alpha r\delta_{ij}\right]. \end{aligned} \tag{10-4}$$

We are free to choose each $\dot{r}_j$ in our ICs, so the contents of the square bracket must vanish. We are also free to choose each r_i, so if the $[\cdots]$ is to vanish for any trajectory the factors multiplying r_{ij} and δ_{ij} must vanish separately. So,

$$r^2V' + \alpha r = 0 \quad \text{and} \quad -V' + \alpha' = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\alpha}{\alpha} = -\frac{\mathrm{d}r}{r} \quad \Rightarrow \quad \alpha = A/r \quad \Rightarrow \quad V = -A/r, \tag{10-5}$$

where A is a constant.

Find also the potentials $V(r)$ and functions $\beta(r)$ for which the components of the matrix

$$Q_{ij} \equiv \dot{r}_i\dot{r}_j + \beta(r)r_i r_j \tag{10-6}$$

are constants of the motion, where r_i, $\dot{r}_i$ $(i = 1, 2, 3)$ are the components of position and velocity of the particle along any three independent fixed axes.

Ans:

$$\begin{aligned} 0 = \dot{Q}_{ij} &= \ddot{r}_i\dot{r}_j + \dot{r}_i\ddot{r}_j + \beta'\frac{\boldsymbol{r} \cdot \dot{\boldsymbol{r}}}{r} r_i r_j + \beta(\dot{r}_i r_j + r_i\dot{r}_j) \\ &= -\frac{V'}{r}(r_i\dot{r}_j + \dot{r}_i r_j) + \frac{\beta'}{r}\boldsymbol{r} \cdot \dot{\boldsymbol{r}} r_i r_j + \beta(\dot{r}_i r_j + r_i\dot{r}_j) \end{aligned} \tag{10-7}$$

Now expand $\boldsymbol{r} \cdot \dot{\boldsymbol{r}} = \sum_k r_k\dot{r}_k$ and introduce factors δ_{ik} or δ_{jk} into the other terms to pull out a common $\dot{r}_k$ factor. Independence of each r_i and $\dot{r}_k$ in ICs gives $0 = \beta - V'/r$ and $0 = \beta'$, so that $V = \frac{1}{2}\beta r^2$, where β is a constant.