

## S7 and BT VII: Classical mechanics

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These work-in-progress notes accompany the lectures for the Physics Short Option S7 / Physics & Philosophy BT VII course on classical mechanics. You can find the most up-to-date version at <http://www-thphys.physics.ox.ac.uk/user/JohnMagorrian/cm.pdf>.

Last updated: 22 Jan 2013. (Minor corrections/clarifications to Lagrangian mechanics)

**Syllabus** (Sections marked \* are not covered in the HT2013 course)

Calculus of variations: Euler–Lagrange equation, (variation subject to constraints)\*.

Lagrangian mechanics: principle of least action; generalized co-ordinates; configuration space. Application to motion in strange co-ordinate systems, particle in an electromagnetic field, normal modes, rigid bodies\*. Noether’s theorem and conservation laws.

Hamiltonian mechanics: Legendre transform; Hamilton’s equations; examples; principle of least action again\*; Liouville’s theorem\*; Poisson brackets; symmetries and conservation laws; canonical transformations. Hamilton–Jacobi equation\*.

### Recommended reading

T. W. B. Kibble & F. H. Berkshire, *Classical mechanics*, 5th ed. About £19.

The single most suitable book for this course.

L. D. Landau & E. M. Lifshitz, *Mechanics*. About £30.

First volume of the celebrated “Course of Theoretical Physics”. Succinct.

H. Goldstein, C. Poole & J. Safko, *Classical mechanics*, 3rd ed. About £50.

Covers more advanced topics too. Verbose.

### Supplementary reading

The following books are more difficult, but some might find them inspiring for a second pass at the subject.

V. I. Arnol’d, *Mathematical methods of classical mechanics*

Adopts a more elegant, more mathematically sophisticated approach than the other books listed here, but develops the maths along with the mechanics.

G. J. Sussman & J. Wisdom, *Structure and interpretation of classical mechanics*. About £45, but also freely available online.

Uses a modern, explicit “functional” notation and breaks everything down into baby steps suitable for a computer.

## 0 Some maths

### 0.0 Notation

Vectors:

$$\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (0.1)$$

(in 3d) or

$$\mathbf{x} = x_1\hat{\mathbf{x}}_1 + x_2\hat{\mathbf{x}}_2 + \cdots + x_n\hat{\mathbf{x}}_n \quad (0.2)$$

for the general case.

Gradients of function  $f(\mathbf{x}, \dot{\mathbf{x}})$ :

$$\begin{aligned} \frac{\partial f}{\partial \mathbf{x}} &\equiv \frac{\partial f}{\partial x_1}\hat{\mathbf{x}}_1 + \frac{\partial f}{\partial x_2}\hat{\mathbf{x}}_2 + \cdots, \\ \frac{\partial f}{\partial \dot{\mathbf{x}}} &\equiv \frac{\partial f}{\partial \dot{x}_1}\hat{\mathbf{x}}_1 + \frac{\partial f}{\partial \dot{x}_2}\hat{\mathbf{x}}_2 + \cdots. \end{aligned} \quad (0.3)$$

So,

$$\mathbf{p} \equiv \frac{\partial f}{\partial \dot{\mathbf{x}}} = p_1 \frac{\partial f}{\partial \dot{x}_1} + p_2 \frac{\partial f}{\partial \dot{x}_2} + \cdots. \quad (0.4)$$

### 0.1 An introduction to the calculus of variations

Recall that a function is simply a rule for mapping elements of one set (the function’s domain) to elements of another set (its range). A **functional** is a mapping from the set of all functions that satisfy some specified conditions (e.g., the set of all smooth maps from the real line to three-dimensional space) to the real numbers.

Examples include:

- the  $n^{\text{th}}$  moment

$$I_n[y] = \int_{x_0}^{x_1} x^n y(x) dx \quad (0.5)$$

of a one-dimensional function  $y(x)$  defined for  $x_0 < x < x_1$  and having  $y(x_0) = y(x_1) = 0$ ;

- the length of a curve  $\mathbf{x}(t)$  joining two fixed points  $\mathbf{x}(t_0)$  and  $\mathbf{x}(t_1)$  in  $n$ -dimensional space,

$$\mathcal{L}[\mathbf{x}] = \int_{t_0}^{t_1} |\dot{\mathbf{x}}| dt; \quad (0.6)$$

- the gravitational potential energy of a mass distribution  $\rho(\mathbf{x})$ ,

$$V[\rho] = -\frac{1}{2}G \int \frac{\rho(\mathbf{x})\rho(\mathbf{x}') d^3x d^3x'}{|\mathbf{x} - \mathbf{x}'|}. \quad (0.7)$$

For this course we need only consider functionals of the form

$$F[\mathbf{x}] = \int_{t_0}^{t_1} L(\mathbf{x}, \dot{\mathbf{x}}, t) dt \quad (0.8)$$

that eat smooth one-dimensional curves  $\mathbf{x}(t)$  with *fixed endpoints*  $\mathbf{x}(t_0) = \mathbf{x}_0$ ,  $\mathbf{x}(t_1) = \mathbf{x}_1$  in  $n$ -dimensional space. The first two examples above are of this form. The third is not. Internally, the functional runs over the curve, feeding the local values of  $(\mathbf{x}, \dot{\mathbf{x}}, t)$  to a function  $L$  and accumulating the results. Note that this  $L$  treats  $\mathbf{x}$ ,  $\dot{\mathbf{x}}$  and  $t$  as independent variables; it does not know that  $\dot{\mathbf{x}} = d\mathbf{x}/dt$ !

Now let's look at how the output of the functional changes when we distort the curve slightly from  $\mathbf{x}(t)$  to  $\mathbf{x}(t) + \mathbf{h}(t)$ . The variation of the curve  $\mathbf{h}(t)$  must be smooth and vanish at the endpoints in order that  $\mathbf{x} + \mathbf{h}$  be admissible to  $F$ , but is otherwise arbitrary. The **variation** or **differential** of the functional

$$\delta F[\mathbf{x}; \mathbf{h}] \equiv \lim_{\epsilon \rightarrow 0} \frac{F[\mathbf{x} + \epsilon \mathbf{h}] - F[\mathbf{x}]}{\epsilon}. \quad (0.9)$$

An **extremal** is a curve  $\mathbf{x}(t)$  for which  $\delta F[\mathbf{x}; \mathbf{h}] = 0$  for *all* admissible  $\mathbf{h}(t)$ . Finding these extremals (if they exist) is the business of the calculus of variations.

**Fundamental lemma of the calculus of variations** If a smooth curve  $\mathbf{f}(t)$ , defined on the range  $t_0 < t < t_1$ , satisfies

$$\int_{t_0}^{t_1} \mathbf{f}(t) \cdot \mathbf{h}(t) dt = 0 \quad (0.10)$$

for all continuous  $\mathbf{h}(t)$  having  $\mathbf{h}(t_0) = \mathbf{h}(t_1) = 0$ , then  $\mathbf{f}(t) \equiv 0$ .

**Proof** by contradiction: we show that if  $\mathbf{f} \neq 0$  then equation (0.10) would not be true for all  $\mathbf{h}(t)$ .

Suppose that there were some  $t_{\text{blip}}$  between  $t_0$  and  $t_1$  for which  $\mathbf{f}(t_{\text{blip}}) \neq 0$ . Then, because  $\mathbf{f}$  has no discontinuous jumps we can always find a small interval  $(t_{\text{left}}, t_{\text{right}})$  around this  $t_{\text{blip}}$  where  $\mathbf{f} \neq 0$ . Now consider the function

$$\mathbf{h}(t) = \mathbf{f}(t) \times \begin{cases} (t_{\text{right}} - t)(t - t_{\text{left}}), & \text{for } t_{\text{left}} < t < t_{\text{right}}, \\ 0, & \text{otherwise.} \end{cases} \quad (0.11)$$

This clearly satisfies the conditions of the lemma, but

$$\int_{t_0}^{t_1} \mathbf{f}(t) \cdot \mathbf{h}(t) dt = \int_{t_{\text{left}}}^{t_{\text{right}}} f^2(t)(t_{\text{right}} - t)(t - t_{\text{left}}) dt > 0, \quad (0.12)$$

since the integrand is positive between  $t_{\text{left}}$  and  $t_{\text{right}}$ . So, we've shown that if  $\mathbf{f} \neq 0$  anywhere then we can always find *some*  $\mathbf{h}(t)$  that makes  $\int \mathbf{f} \cdot \mathbf{h} dt \neq 0$ . Turning this around, if there is *no*  $\mathbf{h}$  for which the integral is non-zero, then we must have  $\mathbf{f} = 0$  between  $t_0$  and  $t_1$ .

Now we come to the key result of this section. Let  $F[\mathbf{x}]$  be a functional of the form

$$\int_{t_0}^{t_1} L(\mathbf{x}, \dot{\mathbf{x}}, t) dt, \quad (0.13)$$

defined on the set of smooth functions  $\mathbf{x}(t)$  satisfying boundary conditions  $\mathbf{x}(t_0) = \mathbf{x}_0$  and  $\mathbf{x}(t_1) = \mathbf{x}_1$ . Then a curve  $\mathbf{x}(t)$  is an extremal of  $F$  if and only if it satisfies the **Euler–Lagrange** equation,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \right) - \frac{\partial L}{\partial \mathbf{x}} = 0. \quad (0.14)$$

**Proof:** If  $\mathbf{x}$  is an extremal of  $F$  then for any variation  $\mathbf{h}$  we have

$$\begin{aligned} 0 = \delta F &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{t_0}^{t_1} (L(\mathbf{x} + \epsilon \mathbf{h}, \dot{\mathbf{x}} + \epsilon \dot{\mathbf{h}}, t) - L(\mathbf{x}, \dot{\mathbf{x}}, t)) dt \\ &= \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial \mathbf{x}} \cdot \mathbf{h} + \frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \dot{\mathbf{h}} \right) dt \\ &= \int_{t_0}^{t_1} \left[ \frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \right) \right] \cdot \mathbf{h} dt + \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \mathbf{h} \right) \Big|_{t_0}^{t_1}, \end{aligned} \quad (0.15)$$

where the last line follows from the previous one using integration by parts. The final term on the last line vanishes because the boundary conditions mean that  $\mathbf{h}(t_0) = \mathbf{h}(t_1) = 0$ . Thus (0.15) becomes

$$0 = \int_{t_0}^{t_1} \left[ \frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \right) \right] \cdot \mathbf{h} dt \quad (0.16)$$

for any smooth  $\mathbf{h}$ . Applying the fundamental lemma, our extremal curve  $\mathbf{x}(t)$  must satisfy the Euler–Lagrange equation (0.14). Conversely, if a curve  $\mathbf{x}(t)$  satisfies (0.14) then it is clear from (0.15) that it is an extremal of the functional  $F$ .

**Example: the shortest path between two points** Consider the set of smooth curves in the  $(t, x)$  plane that pass between two fixed points  $x(t_0) = x_0$  and  $x(t_1) = x_1$ . The path length of any such curve  $x(t)$  is given by the functional (0.13) with  $L = \sqrt{1 + \dot{x}^2}$ . The EL equation for this  $L$  is

$$\frac{d}{dt} \left( \frac{\dot{x}}{\sqrt{1 + \dot{x}^2}} \right) = 0, \quad (0.17)$$

since  $\partial L / \partial x = 0$  and  $\partial L / \partial \dot{x} = \dot{x} / \sqrt{1 + \dot{x}^2}$ . Therefore extremals satisfy  $\dot{x} = A$ , a constant. Integrating,  $x = At + B$ , with the constants  $A$  and  $B$  completely determined by the condition that the curve pass through the two fixed points.

**Important:** When writing down the EL equation, remember that  $\mathbf{x}$  and  $\dot{\mathbf{x}}$  are independent arguments of  $L$ . Use the fact that along extremals  $\mathbf{x}(t)$  satisfies  $\dot{\mathbf{x}} = d\mathbf{x}/dt$  only when solving (i.e., integrating) for  $\mathbf{x}(t)$ .

**Easy first integral when  $L$  does not depend explicitly on  $t$  (Beltrami identity)** Solving the EL equation often leads to lots of messy algebra. But if  $L = L(\mathbf{x}, \dot{\mathbf{x}})$ , the EL equation can be reduced to the first-order differential equation  $\dot{\mathbf{x}} \cdot (\partial L / \partial \dot{\mathbf{x}}) - L = \text{constant}$ . To see this, note that on solution paths

$$\frac{d}{dt} = \frac{\partial}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial}{\partial \dot{\mathbf{x}}} \cdot \frac{d\dot{\mathbf{x}}}{dt} = \frac{\partial}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial}{\partial \dot{\mathbf{x}}} \cdot \ddot{\mathbf{x}} \quad (0.18)$$

for any function  $f(\mathbf{x}, \dot{\mathbf{x}})$ . So,

$$\begin{aligned} \frac{d}{dt} \left[ \dot{\mathbf{x}} \cdot \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \right) - L \right] &= \left[ \ddot{\mathbf{x}} \cdot \frac{\partial L}{\partial \dot{\mathbf{x}}} + \dot{\mathbf{x}} \cdot \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \right) - \frac{\partial L}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} - \frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \ddot{\mathbf{x}} \right] \\ &= \dot{\mathbf{x}} \cdot \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \right) - \frac{\partial L}{\partial \mathbf{x}} \right] = 0. \end{aligned} \quad (0.19)$$

Look out for a more “physical” way of deriving this result later in the course!

**Example: minimal surface of revolution** Among all the curves that pass through the points  $(t_0, x_0)$  and  $(t_1, x_1)$ , find the one that generates the surface of minimum area when rotated about the  $t$ -axis. A physical example is a soap bubble drawn between two coaxial circular hoops.

The surface area generated by a curve  $x(t)$  is

$$2\pi \int_{t_0}^{t_1} x \sqrt{1 + \dot{x}^2} dt. \quad (0.20)$$

Comparing to equation (0.13), we see that  $L(x, \dot{x}) = 2\pi x \sqrt{1 + \dot{x}^2}$ , independent of  $t$ . Using the result above for general  $L(x, \dot{x})$ , the extremals of (0.20) satisfy

$$\frac{\dot{x}x}{\sqrt{1 + \dot{x}^2}} - x \sqrt{1 + \dot{x}^2} = A. \quad (0.21)$$

This is easily rearranged, via  $x = A\sqrt{1 + \dot{x}^2}$ , to give

$$A\dot{x} = \sqrt{x^2 - A^2}. \quad (0.22)$$

Integrating, the curve that extremizes the surface area of revolution is

$$x(t) = A \cosh \left( \frac{t + B}{A} \right), \quad (0.23)$$

where the constants  $A$  and  $B$  are chosen to satisfy the boundary conditions  $x(t_0) = x_0$  and  $x(t_1) = x_1$ . Depending on the choice of  $x_0$  and  $x_1$ , there can be zero, one or two solutions for  $(A, B)$ . Of course, in the zero-solution case an extremal does exist, but it is not smooth and therefore lies beyond the remit of the machinery developed above.

## 0.2 Variation subject to constraints\*

Sometimes it is necessary to find extremals of a functional  $F[\mathbf{x}]$  (equation (0.13)) subject to a constraint of the form

$$g(\mathbf{x}, t) = 0 \quad (0.24)$$

among the co-ordinates. From (0.15), the condition for a curve  $\mathbf{x}(t)$  to be an extremal is then

$$\delta F[\mathbf{x}, \mathbf{h}] = \int_{t_0}^{t_1} \left[ \frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \right) \right] \cdot \mathbf{h} dt = 0 \quad (0.25)$$

for any smooth  $\mathbf{h}(t)$  that satisfies

$$\mathbf{h} \cdot \nabla g = h_1 \frac{\partial g}{\partial x_1} + \dots + h_n \frac{\partial g}{\partial x_n} = 0, \quad (0.26)$$

since we must have  $g(\mathbf{x} + \mathbf{h}, t) = 0$ . This last condition means that we cannot use the fundamental lemma directly. Instead let us multiply (0.26) by an arbitrary function  $\lambda(t)$  and insert it into the integrand of (0.25). This combines the two conditions (0.25) and (0.26) into one:

$$0 = \int_{t_0}^{t_1} \left[ \frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \right) + \lambda \frac{\partial g}{\partial \mathbf{x}} \right] \cdot \mathbf{h} dt. \quad (0.27)$$

The function  $\lambda(t)$  is a **Lagrange multiplier**. Now suppose that, say,  $\partial g / \partial x_1 \neq 0$ . Then we may choose  $\lambda(t)$  to make

$$\left[ \frac{\partial L}{\partial x_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) + \lambda \frac{\partial g}{\partial x_i} \right] = 0 \quad (0.28)$$

for  $i = 1$ , so that the  $\dot{x}_1$  term in the integrand of (0.27) vanishes. We are then free to vary  $(h_2(t), \dots, h_n(t))$  independently as long as we choose  $h_1(t)$  to ensure that the constraint condition (0.26) holds. Using the fundamental lemma on (0.27) we see that the relation (0.28) must apply for  $i = 2, \dots, n$  as well as for  $i = 1$ . Therefore extremals of  $F[\mathbf{x}]$  subject to the constraint  $g = 0$  satisfy

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \right) - \frac{\partial L}{\partial \mathbf{x}} = \lambda \frac{\partial g}{\partial \mathbf{x}}. \quad (0.29)$$

This results in  $n + 1$  equations ( $n$  components of EL equation plus the constraint  $g = 0$ ) for  $n + 1$  unknowns ( $x_1, \dots, x_n$  and  $\lambda$ ). In practice one usually takes linear combinations of different components of the EL equation to eliminate  $\lambda(t)$ . For this reason  $\lambda$  is sometimes known as Lagrange's undetermined multiplier.

**Alternatively** Introduce a new co-ordinate  $\lambda$  and a new functional

$$G[\mathbf{x}, \lambda] \equiv \int_{t_0}^{t_1} \lambda(t) g(\mathbf{x}, t) dt \quad (0.30)$$

that acts on curves  $\{\mathbf{x}(t), \lambda(t)\}$  in this  $(n + 1)$ -dimensional space. Obviously, variations of  $\{\mathbf{x}(t), \lambda(t)\}$  that satisfy  $g = 0$  will have  $\delta G = 0$  too. Combining the two conditions  $\delta F = 0$  and  $\delta G = 0$  into one,

$$\delta(F + G) = \delta \int_{t_0}^{t_1} L' dt = 0 \quad (0.31)$$

with

$$L'(\{\mathbf{x}, \lambda\}, \{\dot{\mathbf{x}}, \dot{\lambda}\}, t) \equiv L(\mathbf{x}, \dot{\mathbf{x}}, t) + \lambda g(\mathbf{x}, t). \quad (0.32)$$

Writing down the  $\mathbf{x}$  and  $\lambda$  components of the EL equation for  $L'$ , we find that

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \right) - \frac{\partial L}{\partial \mathbf{x}} &= \lambda \frac{\partial g}{\partial \mathbf{x}}, \\ g &= 0. \end{aligned} \quad (0.33)$$

It is clear that a path  $\mathbf{x}(t)$  found by solving (0.33) satisfies the constraint  $g = 0$  and therefore  $G \equiv 0$  and so  $\delta G = 0$  too. Since  $\delta(F + G) = 0$  by construction, the path is an extremal of  $F$  too,  $\delta F = 0$ .

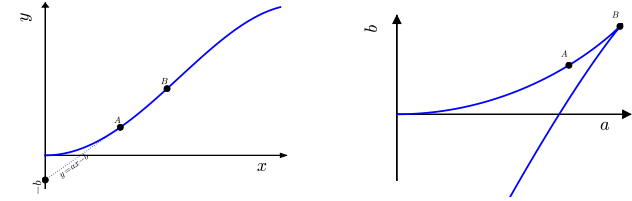
**Exercise:** At this point one might object that the EL equation for  $L'$  applies only to paths  $\{\mathbf{x}(t), \lambda(t)\}$  with fixed endpoints. The conditions on the functional (0.13) mean that we are given  $\mathbf{x}(t_0) = \mathbf{x}_0$  and  $\mathbf{x}(t_1) = \mathbf{x}_1$ . What do we know about  $\lambda(t_0)$  and  $\lambda(t_1)$ ?

## 0.3 Legendre transforms

Given a function  $f(x)$ , its Legendre transform  $g(p)$  is another function that encodes the same information as  $f(x)$  but in terms of  $p = df/dx$  instead of  $x$ . A necessary condition for the Legendre transform to exist is that the first derivative  $f'(x)$  be strictly monotonic, so that either  $f'' > 0$  everywhere or that  $f'' < 0$  everywhere.

Consider the set of (non-vertical) lines in the  $(x, y)$  plane,  $y = ax - b$ . Introduce another plane and to each line in the original plane assign a single point  $(a, b)$  in the new plane. The  $(a, b)$  plane is known as the **(projective) dual** of the original  $(x, y)$  plane. Since the relation  $b + y = ax$  still holds if we exchange  $(a, b)$  with  $(x, y)$ , it follows that the dual of the  $(a, b)$  plane is the original  $(x, y)$  plane; each plane is the dual of the other.

Now take a smooth curve  $y = f(x)$  in the original  $(x, y)$  plane. This curve traces out another curve in the dual  $(a, b)$  plane, the point  $(x, f(x))$  being mapped to  $a = f'(x)$ ,  $b = xf'(x) - f(x)$ . For example, the plots below show the curve  $y = f(x) = x \sin x$  (left) and its image (right) in the dual  $(a, b)$  space. The second derivative  $f''(x)$  changes sign at the point  $B$ .



If  $f(x)$  is convex ( $f'' > 0$ ) then  $a$  increases monotonically with  $x$  and we can define the **Legendre transform** of  $f(x)$  as

$$\begin{aligned} g(a) &\equiv xf'(x) - f(x) \\ &= xa - f(x), \end{aligned} \quad (0.34)$$

where  $x(a)$  is the point on the original curve where  $f'(x) = a$ . That is,  $b = g(a)$  is the dual to the curve  $y = f(x)$  and vice versa.

**Exercise:** For the higher-dimensional case in which hyperplanes  $y = \mathbf{a} \cdot \mathbf{x} - b$  map to points  $(\mathbf{a}, b)$  in the dual space, show that the Legendre transform of a function  $f(\mathbf{x})$  is  $g(\mathbf{a}) = \mathbf{x} \cdot \mathbf{a} - f(\mathbf{x})$ , where  $\mathbf{x}(\mathbf{a})$  is the point for which  $\nabla f = \mathbf{a}$ .

In particular, for later use note that the Legendre transform of a function  $L(\dot{\mathbf{q}})$  is given by  $H(\mathbf{p}) = \dot{\mathbf{q}} \cdot \mathbf{p} - L(\dot{\mathbf{q}})$ , where  $\dot{\mathbf{q}}$  in the RHS is expressed in terms of  $\mathbf{p} = \partial L / \partial \dot{\mathbf{q}}$ .

**Example from thermodynamics:** the Helmholtz free energy  $A(T, V, N) = U - TS$  is the Legendre transform of the internal energy  $U(S, V, N)$  with respect to  $T = \partial U / \partial S$ .

## 1 Lagrangian mechanics

### 1.1 Hamilton's principle of least action

Consider a particle of mass  $m$  whose potential energy  $V(\mathbf{x}; t)$  is independent of its velocity. Its equation of motion is

$$\frac{d}{dt} m\dot{\mathbf{x}} = -\frac{\partial V}{\partial \mathbf{x}}. \quad (1.1)$$

This is equivalent to the EL equation (0.14),

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{x}}} - \frac{\partial L}{\partial \mathbf{x}} = 0, \quad (1.2)$$

if we choose  $\partial L/\partial \dot{\mathbf{x}} = m\dot{\mathbf{x}}$  and  $\partial L/\partial \mathbf{x} = -\partial V/\partial \mathbf{x}$ , or  $L = \frac{1}{2}m\dot{\mathbf{x}}^2 - V$ . Now suppose we were given the instantaneous positions of the particle at times  $t_0$  and  $t_1$ . The results above imply that the path that the particle takes between these two fixed points is an extremal of the **action** integral,

$$S[\mathbf{x}] \equiv \int_{t_0}^{t_1} L(\mathbf{x}, \dot{\mathbf{x}}, t) dt, \quad (1.3)$$

where the **Lagrangian**  $L = T - V$  is the difference between the particle's kinetic and potential energies.

Now let us consider a system of  $N$  particles, having masses  $m_i$ , positions  $\mathbf{x}_i$  and for which the potential energy is  $V(\mathbf{x}_1, \dots, \mathbf{x}_N; t)$ . The latter includes the effects of inter-particle interactions, such as gravity or electrostatic repulsion, as well as any externally applied forces, but we assume that it does not depend on the particles' velocities. If we again take  $L(\{\mathbf{x}_i\}, \{\dot{\mathbf{x}}_i\}; t) = T - V$  to be the difference between the kinetic and potential energies of the whole system of particles, then the EL equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{x}}_i} - \frac{\partial L}{\partial \mathbf{x}_i} = 0 \quad (i = 1, \dots, N), \quad (1.4)$$

reduce to the standard Newtonian equations of motion:

$$\frac{d}{dt} m_i \dot{\mathbf{x}}_i = -\frac{\partial V}{\partial \mathbf{x}_i} \quad (i = 1, \dots, N). \quad (1.5)$$

We can think of the system of particles as moving in  $3N$ -dimensional configuration space. Given snapshots of the  $3N$  co-ordinates of the full system at times  $t_0$  and  $t_1$ , we see that the path the system traces out in configuration space at intermediate times is an extremal of the action

$$S(\{\mathbf{x}_i(t)\}) = \int_{t_0}^{t_1} L(\{\mathbf{x}_i\}, \{\dot{\mathbf{x}}_i\}, t) dt. \quad (1.6)$$

Notice that the condition for a curve to be an extremal of the action (1.3) is independent of the particular co-ordinate system we use to describe the curve. This means we can use any sensible co-ordinate system to parametrize the curves we feed in to the action integral and the EL equation will return the extremal curve (i.e., the equation of motion) in that co-ordinate system. We describe our mechanical system using a set of **generalized co-ordinates**,  $\mathbf{q}(t) \equiv (q_1(t), \dots, q_n(t))$ , that pin down the instantaneous position of the system in  $n$ -dimensional **configuration space**. We assume that there is no redundancy among the  $q_i$ , so that the system has  $n$  **degrees of freedom**. The system moves through configuration space with a **generalized velocity**  $\dot{\mathbf{q}} \equiv (\dot{q}_1, \dots, \dot{q}_n)$ .

Now suppose we know that  $\mathbf{q}(t_0) = \mathbf{q}_0$  and  $\mathbf{q}(t_1) = \mathbf{q}_1$ . Then the general form of Hamilton's principle of least action states that the path in configuration space the system takes between these two times is an extremal of the action integral

$$S[\mathbf{q}] \equiv \int_{t_0}^{t_1} L(\mathbf{q}, \dot{\mathbf{q}}, t) dt, \quad (1.7)$$

where the Lagrangian  $L$  is a function only of the generalized co-ordinates, the generalized velocities and time. Therefore, the **equation of motion** of the system is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = 0. \quad (1.8)$$

The quantity  $\mathbf{p} \equiv \partial L/\partial \dot{\mathbf{q}}$  is known as the **generalized momentum** of the system,  $\mathbf{F} = \partial L/\partial \mathbf{q}$  is the **generalized force**.

Some comments:

- (i) In this formulation we assume only that that  $L$  is some scalar function of  $(\mathbf{q}, \dot{\mathbf{q}}, t)$ , which we are free to choose in order to make the EL equations (1.8) match the true equations of motion of the system. For the common case in which the particles move in a velocity-independent potential  $V(\mathbf{q}, t)$  we know from the examples above that a suitable choice is  $L = T - V$ .
- (ii)  $L$  is not unique. For example, for any function  $\Lambda(\mathbf{q}, t)$  we can add  $d\Lambda/dt$  to  $L$  and still obtain the same equations of motion. (**Prove it!**)
- (iii) Different elements of  $\mathbf{q}$  can have different units. Therefore different elements of the generalized momentum  $\mathbf{p}$  and generalized forces  $\partial L/\partial \mathbf{q}$  can have different units too.
- (iv) If one has external (generalized) forces that are not accounted for in  $L$ , they can be added to the RHS of (1.8).

### 1.2 Why bother?

The most obvious advantage of the Lagrangian approach to mechanics over the elementary ‘‘Newtonian’’ approach is that it allows us to derive the equations of motion of many mechanical systems without the tedious task of resolving forces. As  $L$  is a scalar quantity we are free to use whatever co-ordinate system we like to label points in configuration space: we may express  $L$  in terms of those co-ordinates, turn the handle and obtain the equations of motion. It does not directly tell us how to solve the equations of motion though.

A related benefit of the Lagrangian approach is that it makes a deep connection between symmetries and conservation laws. Problems involving mechanical systems are often invariant under some continuous transformations (e.g., rotation about a particular axis or translation in a certain direction), which means that there is a corresponding constant of motion (see §1.6 below). This often suggests the most ‘‘natural’’ co-ordinate system to use for the problem, which can then help us to solve the equations of motion explicitly, or at least teach us something qualitative about the behaviour of solutions.

Lagrangian mechanics really comes into its own when modelling the motion of rigid bodies (§A.2 below). We probably won't have time to cover that topic during lectures though.

The methods we're applying to mechanical systems in this course can also be applied to other problems (see, e.g., the first two chapters of Goldstein). Much of modern, non-classical physics is derived from some form of action principle.

### 1.3 Equations of motion for some simple systems

**Simple pendulum** A bob of mass  $m$  is attached to one end of a rigid massless rod of length  $l$ . The other end of the rod is attached to a fixed point, about which the rod can rotate in a fixed vertical plane. The most natural parameter to use to describe the instantaneous configuration of this one-dimensional pendulum is the angle  $\theta$  the rod makes with the vertical. Since the potential energy  $V(\theta) = -mgl \cos \theta$  is independent of generalized velocity  $\dot{\theta}$ , we have that

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl \cos \theta, \quad (1.9)$$

so that  $p_\theta \equiv \partial L/\partial \dot{\theta} = ml^2\dot{\theta}$ . The equation of motion (1.8) for the system is then  $\ddot{\theta} + (g/l) \sin \theta = 0$ .

**Springy pendulum** Replace the rigid rod in the simple pendulum above with a massless spring of natural length  $l$  and spring constant  $\omega^2$ , so that when the string is extended or compressed to a length  $r$  its

potential energy  $V_{\text{spring}}(r) = \frac{1}{2}\omega^2(r-l)^2$ . The natural generalized co-ordinates to use for this system are  $(r, \theta)$ . A Lagrangian in these co-ordinates is

$$L = T - (V_{\text{grav}} + V_{\text{spring}}) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr \cos \theta - \frac{1}{2}\omega^2(r-l)^2, \quad (1.10)$$

which yields the equations of motion

$$\begin{aligned} \frac{d}{dt}m\dot{r} &= mr\dot{\theta}^2 + mg \cos \theta - \omega^2(r-l), \\ \frac{d}{dt}mr^2\dot{\theta} &= -mgr \sin \theta. \end{aligned} \quad (1.11)$$

**Spherical pendulum** Now let's return to our simple pendulum constructed from a rigid rod, but relax the constraint that the rod can rotate only in a fixed plane. The Cartesian co-ordinates of the location of the bob with respect to the pivot can be written as

$$\begin{aligned} x &= l \sin \theta \cos \phi \\ y &= l \sin \theta \sin \phi \\ z &= l \cos \theta, \end{aligned} \quad (1.12)$$

where  $(\phi, \theta)$  are the usual polar co-ordinates of a point on the surface of a sphere. We orient our co-ordinate system with the  $Oz$  axis pointing *downwards*, so that  $\theta$  is the angle the bob makes with the downwards vertical. Differentiating (1.12) with respect to time to find  $\dot{x}(\theta, \phi)$ , we have that the Lagrangian

$$\begin{aligned} L &= \frac{1}{2}m\dot{\mathbf{x}}^2 - V = \frac{1}{2}m[\dot{x}^2 + \dot{y}^2 + \dot{z}^2] + mgl \cos \theta \\ &= \frac{1}{2}ml^2[\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta] + mgl \cos \theta. \end{aligned} \quad (1.13)$$

The generalized momenta  $(p_\theta, p_\phi)$  conjugate to the generalized co-ordinates  $(\theta, \phi)$  are given by

$$p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}, \quad p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = ml^2 \sin^2 \theta \dot{\phi}. \quad (1.14)$$

Notice that these are both angular momenta. In particular,  $p_\phi$  is the angular momentum about the  $z$  axis and the EL equation for  $\phi$ ,  $\dot{p}_\phi = \partial L / \partial \phi = 0$ , tells us that  $p_\phi$  is conserved. The EL equation for  $\theta$  is

$$\begin{aligned} ml^2\ddot{\theta} &= ml^2\dot{\phi}^2 \sin \theta \cos \theta - mgl \sin \theta \\ &= \frac{p_\phi^2 \cos \theta}{ml^2 \sin^3 \theta} - mgl \sin \theta, \end{aligned} \quad (1.15)$$

where in the second line we have used our expression for the constant  $p_\phi = ml^2 \sin^2 \theta \dot{\phi}$  to eliminate  $\dot{\phi}$ .

**Exercise:** It is difficult to integrate equation (1.15) to obtain an explicit expression for  $\theta$  as a function of  $t$ . Using the fact that  $\ddot{\theta} = \dot{\theta}(d\dot{\theta}/d\theta)$ , explain how (1.15) can be used to obtain an expression for  $\dot{\theta}$  as a function of  $\theta$ . Show that the  $\theta$  motion reduces to motion in a one-dimensional **effective potential**

$$V_{\text{eff}}(\theta) = \frac{p_\phi^2}{2ml^2 \sin^2 \theta} - mgl \cos \theta, \quad (1.16)$$

and explain how to find the minimum and maximum values of  $\theta$  taken by the pendulum for a given set of initial conditions.

**Particle in a central field** The location of particle of mass  $m$  moving in three dimensions in a spherically symmetric gravitational potential  $\Phi(r)$  is most naturally expressed using spherical polar co-ordinates,  $\mathbf{q} = (r, \theta, \phi)$ , in terms of which

$$(x, y, z) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta). \quad (1.17)$$

Since  $V = m\Phi$  does not depend on  $\dot{\mathbf{q}} = (\dot{r}, \dot{\theta}, \dot{\phi})$ , the Lagrangian

$$L = T - V = \frac{1}{2}m \left[ \dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right] - V(r), \quad (1.18)$$

where the velocity  $\dot{\mathbf{x}}^2$  in the square brackets comes from differentiating the co-ordinate transform (1.17) with respect to  $t$ . The EL equation (1.8) gives the equations of motion

$$\dot{p}_r = m\dot{\theta}^2 + m\dot{\phi}^2 r \sin^2 \theta - \frac{dV}{dr}, \quad \dot{p}_\theta = mr^2 \sin \theta \cos \theta \dot{\phi}^2, \quad \dot{p}_\phi = 0, \quad (1.19)$$

where the components of the generalized momentum

$$p_r \equiv m\dot{r}, \quad p_\theta \equiv mr^2\dot{\theta}, \quad \text{and} \quad p_\phi \equiv mr^2 \sin^2 \theta \dot{\phi}. \quad (1.20)$$

$p_\phi$  is a constant because  $\dot{p}_\phi = 0$ . As the potential is spherically symmetric, we can orient our co-ordinate system so that  $\theta = \frac{\pi}{2}$  and  $\dot{\theta} = 0$  initially. Then  $p_\theta = \dot{p}_\theta = 0$ , showing that the motion remains confined to the plane  $\theta = \frac{\pi}{2}$ .

**Exercise:** Write down the Euler equation for  $r$ . In the equation you get, use (1.20) to express  $\dot{\theta}$  and  $\dot{\phi}$  in terms of the constants  $p_\theta$  and  $p_\phi$ . Show that this motion is identical to that obtained from the one-dimensional effective Lagrangian,

$$\begin{aligned} L(r, \dot{r}) &= \frac{1}{2}m\dot{r}^2 - V_{\text{eff}}(r), \\ \text{with } V_{\text{eff}}(r) &= V(r) + \frac{p_\phi^2}{2mr^2}. \end{aligned} \quad (1.21)$$

A common temptation is to try to save work by first eliminating  $\dot{\phi}$  and  $\dot{\theta}$  from  $L(r, \theta, \dot{r}, \dot{\theta}, \dot{\phi})$  and then to obtain the EL equations from the resulting "Lagrangian"  $L(r, \theta, \dot{r}, p_\theta, p_\phi)$ . This is wrong! Why?

If a co-ordinate  $q_i$  does not appear explicitly in  $L$ , then  $\partial L / \partial q_i = 0$  and the EL equation tells us that the corresponding momentum  $p_i \equiv \partial L / \partial \dot{q}_i$  is conserved. Such  $q_i$  are known as **cyclic** or **ignorable co-ordinates**.

## 1.4 Particle in a magnetic field

So far we have considered problems in which the Lagrangian can be written as  $L = T - V$ , where  $T$  is kinetic energy and  $V(\mathbf{q}, t)$  is the (velocity-independent) potential energy of the system. It turns out that these same methods can be used to describe more general systems.

Consider a particle of charge  $Q$  and mass  $m$  moving in an electromagnetic field. Its equation of motion is

$$\frac{d}{dt}m\dot{\mathbf{x}} = Q\mathbf{E} + Q\dot{\mathbf{x}} \times \mathbf{B}. \quad (1.22)$$

Since the Lorentz force  $\mathbf{F} = Q\dot{\mathbf{x}} \times \mathbf{B}$  does no work on the particle, it makes no contribution to either  $T$  or  $V$  and so we cannot derive (1.22) from a Lagrangian of the form  $L = T - V(\mathbf{x})$ . Nevertheless, one can still concoct a Lagrangian that produces this motion. Recall that we can express

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A} \quad (1.23)$$

in terms of an electrostatic potential  $\phi(\mathbf{x}, t)$  and a magnetic vector potential  $\mathbf{A}(\mathbf{x}, t)$ . Here we show that the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 + Q(\dot{\mathbf{x}} \cdot \mathbf{A} - \phi). \quad (1.24)$$

produces the equation of motion (1.22).

The EL equation for this  $L$  is

$$\frac{d}{dt}(m\dot{\mathbf{x}} + Q\mathbf{A}) - Q\nabla(\phi - \dot{\mathbf{x}} \cdot \mathbf{A}) = 0. \quad (1.25)$$

(Notice how the presence of the velocity-dependent force from the magnetic field means that the generalized momentum  $\mathbf{p} = m\dot{\mathbf{x}} + Q\mathbf{A} \neq m\dot{\mathbf{x}}$ .) The derivative with respect to time in the LHS of (1.25) is to be carried out along the curve  $\mathbf{x}(t)$ . Therefore, using the chain rule,

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \sum_j \frac{dx_j}{dt} \frac{\partial}{\partial x_j} A_i = \left[ \frac{\partial \mathbf{A}}{\partial t} + \left( \frac{d\mathbf{x}}{dt} \cdot \nabla \right) \mathbf{A} \right]_i, \quad (1.26)$$

in which I avoid writing  $d\mathbf{x}/dt$  as  $\dot{\mathbf{x}}$  because I want to emphasise that – for now –  $\mathbf{x}$  and  $\dot{\mathbf{x}}$  are independent. Substituting this into (1.25) and rearranging, we have that

$$\frac{d}{dt}m\dot{\mathbf{x}} - Q \left[ \frac{\partial \mathbf{A}}{\partial t} + \nabla\phi \right] - Q \left[ \left( \frac{d\mathbf{x}}{dt} \cdot \nabla \right) \mathbf{A} - \nabla(\dot{\mathbf{x}} \cdot \mathbf{A}) \right] = 0. \quad (1.27)$$

The expression inside the first square bracket is simply  $-\mathbf{E}$ . The expression inside the second is  $-\dot{\mathbf{x}} \times \mathbf{B}$ . To see this, use the vector identity

$$\nabla(\dot{\mathbf{x}} \cdot \mathbf{A}) = (\dot{\mathbf{x}} \cdot \nabla)\mathbf{A} + \underbrace{(\mathbf{A} \cdot \nabla)\dot{\mathbf{x}}}_0 + \dot{\mathbf{x}} \times \underbrace{(\nabla \times \mathbf{A})}_\mathbf{B} + \mathbf{A} \times \underbrace{(\nabla \times \dot{\mathbf{x}})}_0, \quad (1.28)$$

in which the second and fourth terms vanish because  $\dot{\mathbf{x}}$  and  $\mathbf{x}$  are independent variables:  $\partial\dot{x}_i/\partial x_j = 0$ . Finally, use the fact that  $\dot{\mathbf{x}} = d\mathbf{x}/dt$  along extremals and it is clear that the second bracket vanishes. Therefore the EL equations (1.25) for the Lagrangian (1.24) reduce to the familiar (1.22).

Here I have simply pulled this Lagrangian out of a hat, but when one looks at the problem in a proper, relativistically covariant way the action

$$S[x] = \int \left[ \frac{1}{2}m\dot{\mathbf{x}}^2 + Q(\dot{\mathbf{x}} \cdot \mathbf{A} - \phi) \right] dt \quad (1.29)$$

pops out naturally. As ever, adding a total derivative

$$\frac{d\Lambda}{dt} = \frac{\partial \Lambda}{\partial t} + \dot{\mathbf{x}} \cdot \nabla \Lambda \quad (1.30)$$

to the integrand of  $S$  (and thus to the Lagrangian (1.24)) has no effect on the extremal  $\mathbf{x}(t)$  obtained by solving  $\delta S = 0$ . This is equivalent to the gauge transformation

$$\phi \rightarrow \phi - \frac{\partial \Lambda}{\partial t}, \quad \mathbf{A} \rightarrow \mathbf{A} + \nabla \Lambda. \quad (1.31)$$

## 1.5 Motion in non-inertial co-ordinate systems

**Ant on a turntable** An ant finds itself on a turntable that rotates with constant angular velocity  $\Omega$ . The ant sets up cartesian  $(X, Y, Z)$  co-ordinates co-rotating with the turntable, so that the “lab” co-ordinates  $(x, y, z)$  of a point  $(X, Y, Z)$  on the turntable are given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos \Omega t & -\sin \Omega t & 0 \\ \sin \Omega t & \cos \Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \quad (1.32)$$

The Lagrangian of a particle in the ant’s co-ordinates

$$\begin{aligned} L &= T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V \\ &= \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + m\Omega(X\dot{Y} - \dot{X}Y) + \frac{1}{2}m\Omega^2(X^2 + Y^2) - V, \end{aligned} \quad (1.33)$$

the second line following from the first on differentiating the transformation matrix (1.32). The  $(X, Y)$  equations of motion for a free particle ( $V = 0$ ) on the turntable in the ant’s co-ordinate system are therefore

$$\frac{d}{dt}m\dot{X} = 2m\Omega\dot{Y} + m\Omega^2 X, \quad \frac{d}{dt}m\dot{Y} = -2m\Omega\dot{X} + m\Omega^2 Y. \quad (1.34)$$

Notice that these are simply  $\dot{x} = \dot{y} = 0$  in the co-rotating frame. Turning to the ant itself, if friction keeps it at rest ( $\dot{X} = \dot{Y} = 0$ ) with respect to the turntable, then it feels an outward force of magnitude  $mR\Omega^2$ , where  $R^2 = X^2 + Y^2$  (second term on RHS of each of (1.34)). This is reduced if the ant tries to walk against the rotation of the turntable (first terms on RHS), and vanishes completely if the ant runs around the circle  $R = \text{constant}$  with speed  $R\Omega$ .

Writing  $\mathbf{r} = (X, Y, Z)$  for the co-ordinates of a particle in the rotating frame ( $\mathbf{r}$  for rotating,  $\mathbf{x}$  for fixed) and  $\Omega \equiv (0, 0, \Omega)$ , the Lagrangian (1.33) can be expressed as

$$\begin{aligned} L(\mathbf{r}, \dot{\mathbf{r}}, t) &= \frac{1}{2}m\dot{\mathbf{r}}^2 + m\dot{\mathbf{r}} \cdot (\Omega \times \mathbf{r}) + \frac{1}{2}m(\Omega \times \mathbf{r})^2 - V(\mathbf{r}, t) \\ &= \frac{1}{2}m(\dot{\mathbf{r}} + \Omega \times \mathbf{r})^2 - V(\mathbf{r}, t). \end{aligned} \quad (1.35)$$

Instead of wrestling with the matrix (1.32), a simpler way of deriving (1.35) is to note that a particle moving with respect to the rotating  $\mathbf{r}$  frame with velocity  $\dot{\mathbf{r}}$  has in the  $\mathbf{x}$  frame a velocity whose magnitude

$$|\dot{\mathbf{x}}| = |\dot{\mathbf{r}} + \Omega \times \mathbf{r}|, \quad (1.36)$$

which, together with  $L = \frac{1}{2}m\dot{\mathbf{x}}^2 - V$ , gives (1.35) directly. Notice that equation (1.36) is a statement only about the *magnitude* of the vector  $\dot{\mathbf{x}}$ , not its direction; we show below that  $\dot{\mathbf{x}}$  and  $(\dot{\mathbf{r}} + \Omega \times \mathbf{r})$  are related by a rotation, as one might expect.

The equations of motion for the particle in the rotating frame are easy to obtain from (1.35). Making use of the relation  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ , the partial derivatives of  $L$  are found to be

$$\begin{aligned} \mathbf{p} \equiv \frac{\partial L}{\partial \dot{\mathbf{r}}} &= m\dot{\mathbf{r}} + m\Omega \times \mathbf{r}, \\ \frac{\partial L}{\partial \mathbf{r}} &= m\dot{\mathbf{r}} \times \Omega + m(\Omega \times \mathbf{r}) \times \Omega - \frac{\partial V}{\partial \mathbf{r}}. \end{aligned} \quad (1.37)$$

Therefore the equation of motion

$$\frac{d}{dt}m\dot{\mathbf{r}} = -m\dot{\Omega} \times \mathbf{r} - 2m\Omega \times \dot{\mathbf{r}} - m\Omega \times (\Omega \times \mathbf{r}) - \frac{\partial V}{\partial \mathbf{r}}, \quad (1.38)$$

showing that in this non-inertial, rotating frame the particle moves as if it were subject to three additional “pseudo-forces”: the inertial force of rotation  $-m\dot{\Omega} \times \mathbf{r}$ , the **Coriolis force**  $-2m\Omega \times \dot{\mathbf{r}}$  and the **centrifugal force**  $-m\Omega \times (\Omega \times \mathbf{r})$ .

**Exercise:** Show that in the northern hemisphere the Coriolis force deflects every body moving across the earth’s surface to the right and every falling body towards the East.

**Exercise:** In cosmology it is often useful to express the equations of motion of “dust” (stars, gas) in terms of co-moving co-ordinates,  $\mathbf{r}$ , which are related to “physical” co-ordinates,  $\mathbf{x}$ , through  $\mathbf{x} = a(t)\mathbf{r}$  where  $a(t)$  is the scale factor of the universe. Show that in these co-ordinates the motion of a dust particle satisfies

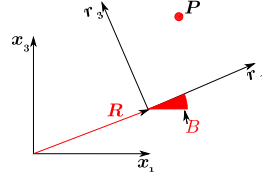
$$\ddot{\mathbf{r}} + 2\frac{\dot{a}}{a}\dot{\mathbf{r}} + \frac{\ddot{a}}{a}\mathbf{r} = -\frac{1}{a^2}\frac{\partial \Phi}{\partial \mathbf{r}}. \quad (1.39)$$

**More general moving co-ordinates** There is a more general way of dealing with moving frames. Consider a co-ordinate transformation of the form

$$\mathbf{x} = \mathbf{R} + \mathbf{B}\mathbf{r}, \quad (1.40)$$

in which the co-ordinates of a particle  $P$  in the “fixed”  $\mathbf{x}$  system are given in terms of those in the  $\mathbf{r}$  system by a rotation  $\mathbf{B}(t)$  followed by a translation  $\mathbf{R}(t)$ . Differentiating (1.40), the velocity of the particle in the  $\mathbf{x}$  frame is given by

$$\dot{\mathbf{x}} = \dot{\mathbf{R}} + \dot{\mathbf{B}}\mathbf{r} + \mathbf{B}\dot{\mathbf{r}}. \quad (1.41)$$



For example, a rock on the earth’s equator has co-ordinates  $\mathbf{r} = (R_{\oplus}, 0, 0)$ . Equations (1.40) and (1.41) give its co-ordinates and velocities in a frame centred on the sun and oriented with respect to the “fixed stars” if we choose  $\mathbf{R}(t)$  to be the location of the centre of the earth in this fixed frame and use  $\mathbf{B}(t)$  to describe the rotation of the earth about its axis.

**Pure rotation** Let us first consider the case  $\mathbf{R} = \dot{\mathbf{R}} = 0$  in which the (three-dimensional)  $\mathbf{x}$  and  $\mathbf{r}$  co-ordinate axes are related by a pure rotation, so that  $\mathbf{x} = \mathbf{B}\mathbf{r}$ . Since  $\mathbf{B}$  is a rotation,  $\mathbf{B}\mathbf{B}^T = \mathbf{I}$ , so  $\mathbf{B}^{-1} = \mathbf{B}^T$  and  $\mathbf{r} = \mathbf{B}^{-1}\mathbf{x} = \mathbf{B}^T\mathbf{x}$ . Substituting this into (1.41) gives

$$\dot{\mathbf{x}} = \dot{\mathbf{B}}\mathbf{B}^T\mathbf{x} + \mathbf{B}\dot{\mathbf{r}}. \quad (1.42)$$

To understand the effect of  $\dot{\mathbf{B}}\mathbf{B}^T$ , differentiate the relation  $\mathbf{B}\mathbf{B}^T = \mathbf{I}$  to obtain

$$\dot{\mathbf{B}}\mathbf{B}^T + \mathbf{B}\dot{\mathbf{B}}^T = 0 \quad \Rightarrow \quad \dot{\mathbf{B}}\mathbf{B}^T + (\dot{\mathbf{B}}\mathbf{B}^T)^T = 0. \quad (1.43)$$

Thus  $\dot{\mathbf{B}}\mathbf{B}^T$  is a skew-symmetric matrix. Now write out the expression  $\boldsymbol{\omega} \times \mathbf{x} = (\omega_2x_3 - \omega_3x_2, \omega_3x_1 - \omega_1x_3, \omega_1x_2 - \omega_2x_1)$  in matrix form. The result is

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (1.44)$$

So, by choosing  $\boldsymbol{\omega}$  appropriately, any skew-symmetric matrix can be represented by the operation  $\boldsymbol{\omega} \times \mathbf{x}$ . In particular, the relation (1.42) can be written as

$$\dot{\mathbf{x}} = \boldsymbol{\omega} \times \mathbf{x} + \mathbf{B}\dot{\mathbf{r}} \quad (1.45)$$

for some (possibly time-dependent)  $\boldsymbol{\omega}$ , which is an eigenvector of  $\dot{\mathbf{B}}\mathbf{B}^T$  with eigenvalue 0. This  $\boldsymbol{\omega}$  is the instantaneous angular velocity of the  $\mathbf{r}$  framewith respect to the  $\mathbf{x}$  frame. Using  $\mathbf{x} = \mathbf{B}\mathbf{r}$  and introducing  $\boldsymbol{\Omega} \equiv \mathbf{B}^{-1}\boldsymbol{\omega}$ , the instantaneous angular velocity in the  $\mathbf{r}$  frame, we have that

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{B}\boldsymbol{\Omega} \times \mathbf{B}\mathbf{r} + \mathbf{B}\dot{\mathbf{r}} \\ &= \mathbf{B}(\boldsymbol{\Omega} \times \mathbf{r} + \dot{\mathbf{r}}), \end{aligned} \quad (1.46)$$

because  $\mathbf{B}\mathbf{b} \times \mathbf{B}\mathbf{c} = \mathbf{B}(\mathbf{b} \times \mathbf{c})$ . Substituting this  $\dot{\mathbf{x}}$  into  $L = \frac{1}{2}m\dot{\mathbf{x}}^2 - V = \frac{1}{2}m(\dot{\mathbf{x}}^T \cdot \dot{\mathbf{x}}) - V$  gives the Lagrangian (1.35). So, the Lagrangian we derived earlier for the special case of a steady rotation  $\boldsymbol{\Omega}t$  about the  $\hat{\mathbf{x}}_3 = \hat{\mathbf{r}}_3$  axis holds even when the rotation axis and rotation rate change with time, provided we take  $\boldsymbol{\Omega}$  to be the instantaneous angular velocity in the rotating frame.

**Exercise:** Calculate  $\dot{\mathbf{B}}\mathbf{B}^T$  for each of the following matrices and find  $\boldsymbol{\omega}$  by comparing your results with equation (1.44). What is  $\boldsymbol{\Omega}$  in each case?

$$\mathbf{B}_1 = \begin{pmatrix} \cos \Omega t & -\sin \Omega t & 0 \\ \sin \Omega t & \cos \Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} \cos \Omega t & -\sin \Omega t & 0 \\ 0 & 0 & -1 \\ \sin \Omega t & \cos \Omega t & 0 \end{pmatrix}. \quad (1.47)$$

**Pure translation, no rotation** If the  $\mathbf{r}$  and  $\mathbf{x}$  co-ordinates are related by a pure translation, then  $\mathbf{B} = \mathbf{I}$  and equation (1.41) becomes

$$\begin{aligned} \dot{\mathbf{x}}^2 &= (\dot{\mathbf{R}} + \dot{\mathbf{r}})^2 = \dot{\mathbf{R}}^2 + 2\dot{\mathbf{R}} \cdot \dot{\mathbf{r}} + \dot{\mathbf{r}}^2 \\ &= \dot{\mathbf{R}}^2 + 2\frac{d}{dt}(\dot{\mathbf{R}} \cdot \mathbf{r}) - 2\ddot{\mathbf{R}} \cdot \mathbf{r} + \dot{\mathbf{r}}^2. \end{aligned} \quad (1.48)$$

A suitable Lagrangian is

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - m\ddot{\mathbf{R}} \cdot \mathbf{r} - V(\mathbf{r}, t), \quad (1.49)$$

dropping the first two terms from (1.48) because they contribute nothing to the equations of motion.

**General case – translation plus rotation** For the general case, we introduce an intermediate co-ordinate system  $\mathbf{x}'$  related to  $\mathbf{x}$  by a translation and to  $\mathbf{r}$  by a rotation:

$$\begin{aligned} \mathbf{x} &= \mathbf{R} + \mathbf{x}', \\ \mathbf{x}' &= \mathbf{B}\mathbf{r}. \end{aligned} \quad (1.50)$$

Using (1.49), the Lagrangian  $L(\mathbf{x}', \dot{\mathbf{x}}', t) = \frac{1}{2}m\dot{\mathbf{x}}'^2 - m\ddot{\mathbf{R}} \cdot \mathbf{x}' - V$ . Taking  $\dot{\mathbf{x}}' = \mathbf{B}(\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r})$  from (1.46), we have finally that

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}m(\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r})^2 - m\ddot{\mathbf{R}} \cdot (\mathbf{B}\mathbf{r}) - V. \quad (1.51)$$

**Exercise:** Let  $\mathbf{B}$  be a constant (time-independent) rotation matrix and choose  $m\ddot{\mathbf{R}} = -\partial V/\partial \mathbf{x}$ . Show that in this freely falling frame the equations of motion become  $\frac{d}{dt}m\dot{\mathbf{r}} = 0$ .

**Exercise:** Show for the case  $\dot{\mathbf{r}} = 0$  that

$$\begin{aligned} \dot{\mathbf{x}} - \dot{\mathbf{R}} &= \dot{\mathbf{B}}\mathbf{B}^T(\mathbf{x} - \mathbf{R}) \\ &= \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{R}). \end{aligned} \quad (1.52)$$

Explain why this means that  $\boldsymbol{\Omega}$  and  $\boldsymbol{\omega}$  are independent of the choice of  $\mathbf{R}$ .

## 1.6 Noether’s theorem

A **constant of motion** is any function  $C(\mathbf{q}, \dot{\mathbf{q}}, t)$  for which the total time derivative

$$\frac{dC}{dt} = \frac{\partial C}{\partial t} + \dot{\mathbf{q}} \cdot \frac{\partial C}{\partial \mathbf{q}} + \ddot{\mathbf{q}} \cdot \frac{\partial C}{\partial \dot{\mathbf{q}}} \quad (1.53)$$

vanishes along a trajectory  $\mathbf{q}(t)$  that satisfies the equations of motion. For example, if  $\partial L/\partial t = 0$  then we already know from the Beltrami identity (0.19) of §0.1 that

$$H(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}} \cdot \frac{\partial L}{\partial \dot{\mathbf{q}}} - L \quad (1.54)$$

is a constant of motion. Similarly, if  $L$  contains a cyclic co-ordinate  $q_i$  (one for which  $\partial L/\partial q_i = 0$ ), then the generalized momentum  $p_i = \partial L/\partial \dot{q}_i$  is a constant of the motion.

In general, a system with  $n$  degrees of freedom has  $2n - 1$  independent constants of motion. To see this, suppose that a system has  $(\mathbf{q}, \dot{\mathbf{q}})$  at some time  $t$ . Then one can in principle integrate the system forwards/backwards to some reference time,  $t_0$ . The values of  $\mathbf{q}$  and  $\dot{\mathbf{q}}$  at  $t_0$  are some complicated functions  $q_i(t_0) = f_i(\mathbf{q}, \dot{\mathbf{q}}, t)$ ,  $\dot{q}_i(t_0) = g_i(\mathbf{q}, \dot{\mathbf{q}}, t)$ , of their values at time  $t$ . Eliminating  $t$  from these  $2n$  equations leaves  $2n - 1$  constants of motion. There are few mechanical systems for which one can write down expressions for all  $2n - 1$  constants of motion, but we have already seen (e.g., motion of particle in central field) that finding

just  $n$  constants of motion is enough to understand the behaviour of a mechanical system with  $n$  degrees of freedom, at least qualitatively.

**Noether's theorem** states that for every continuous symmetry of the Lagrangian there is a corresponding conserved quantity. Suppose we apply a transformation to our mechanical system, which results in a small change in co-ordinates

$$\mathbf{q} \rightarrow \mathbf{q} + \epsilon \mathbf{K}(\mathbf{q}), \quad (1.55)$$

where  $\mathbf{K}(\mathbf{q})$  is a vector-valued function of  $\mathbf{q}$ . For example, we might move our favourite pendulum slightly to the left, or turn it anticlockwise a little. If the Lagrangian  $L(\mathbf{q}, \dot{\mathbf{q}}, t)$  is invariant under this transformation then there is a constant of motion

$$C(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \mathbf{K}. \quad (1.56)$$

**Proof:** Since the transformation (1.55) leaves  $L$  unchanged then, at  $\epsilon = 0$ ,

$$\begin{aligned} 0 &= \frac{dL}{d\epsilon} = \frac{\partial L}{\partial \mathbf{q}} \cdot \frac{\partial \mathbf{q}}{\partial \epsilon} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \frac{\partial \dot{\mathbf{q}}}{\partial \epsilon} \\ &= \frac{\partial L}{\partial \mathbf{q}} \cdot \mathbf{K} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{K}}. \end{aligned} \quad (1.57)$$

Using the EL equation to replace the  $\partial L/\partial \mathbf{q}$  factor,

$$0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \cdot \mathbf{K} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{K}} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \mathbf{K} \right). \quad (1.58)$$

**Example: homogeneity of space** The Lagrangian for a closed system of  $N$  particles,

$$L = \frac{1}{2} \sum_i m_i \dot{\mathbf{x}}_i^2 - \sum_{ij} V(|\mathbf{x}_i - \mathbf{x}_j|), \quad (1.59)$$

is invariant if we apply the translation  $\mathbf{x}_i \rightarrow \mathbf{x}_i + \epsilon \hat{\mathbf{n}}$  to all the particles' co-ordinates, for any choice of direction  $\hat{\mathbf{n}}$ . By Noether's theorem, this symmetry means that

$$\sum_i \left( \frac{\partial L}{\partial \dot{\mathbf{x}}_i} \right) \cdot \hat{\mathbf{n}} = \left( \sum_i m_i \dot{\mathbf{x}}_i \right) \cdot \hat{\mathbf{n}}, \quad (1.60)$$

is a constant of the motion. Since the relation holds for any  $\hat{\mathbf{n}}$ , we have that  $\sum_i m_i \dot{\mathbf{x}}_i$  is an invariant. Thus, translation invariance of  $L$  implies conservation of total linear momentum.

**Example: isotropy of space** Similarly, the Lagrangian (1.59) is invariant if we pick any direction  $\hat{\mathbf{n}}$  and carry out an infinitesimal rotation of the system about this axis:  $\mathbf{x}_i \rightarrow \mathbf{x}_i + \epsilon \hat{\mathbf{n}} \times \mathbf{x}_i$ . Noether tells us that there is a conserved quantity

$$\sum_i \left( \frac{\partial L}{\partial \dot{\mathbf{x}}_i} \right) \cdot (\hat{\mathbf{n}} \times \mathbf{x}_i) = \left( \sum_i \mathbf{x}_i \times m_i \dot{\mathbf{x}}_i \right) \cdot \hat{\mathbf{n}}. \quad (1.61)$$

In other words, rotational invariance of  $L$  leads to conservation of angular momentum.

**Example: particle in a uniform magnetic field** A particle moves in a uniform magnetic field  $\mathbf{B} = (0, 0, B) = B\mathbf{k}$ . From (1.24), the Lagrangian  $L = \frac{1}{2}m\dot{\mathbf{x}}^2 + Q(\dot{\mathbf{x}} \cdot \mathbf{A} - \phi)$ . Since  $\mathbf{E} = 0$ , we are free to choose  $\phi = 0$ . To find the constants of motion it proves easiest to consider two different choices for the vector potential  $\mathbf{A}$ , each of which lead to the same  $\mathbf{B}$  and therefore to the same equations of motion.

Our first choice is  $\mathbf{A} = (-By, 0, 0)$ . Then we get  $L = \frac{1}{2}m\dot{\mathbf{x}}^2 - Q\dot{\mathbf{x}}By$ . This is invariant under translations in either the  $i$  or  $k$  directions:  $\mathbf{x} \rightarrow \mathbf{x} + \epsilon i$ ,  $\mathbf{x} \rightarrow \mathbf{x} + \epsilon k$ . Therefore, two constants of the motion are

$$\frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot i = p_x = m\dot{x} - QB y \quad \text{and} \quad \frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \mathbf{k} = p_z = m\dot{z}. \quad (1.62)$$

Our second choice is  $\mathbf{A} = (0, Bx, 0)$ . Then  $L = \frac{1}{2}m\dot{\mathbf{x}}^2 - Q\dot{y}Bx$ , which is invariant under translations in either  $j$  or  $k$  directions, leading to the additional constant of motion

$$\frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot j = p_y = m\dot{y} + QBx. \quad (1.63)$$

The physical meaning of  $p_z$  is obvious. To understand  $p_x$  and  $p_y$ , consider

$$P \equiv p_x + ip_y = m(\dot{x} + i\dot{y}) + QB(ix - y) = m\dot{\xi} + iQB\xi, \quad (1.64)$$

where  $\xi \equiv x + iy$ . This is a first-order ODE for  $\xi$ . Multiplying by the integrating factor  $e^{i\omega t}$ , where the **Larmor frequency**  $\omega \equiv QB/m$ , the solution is

$$\xi(t) = \frac{P}{i\omega m} + Ke^{i\omega t}, \quad (1.65)$$

where  $K$  is a constant of integration. We now see that  $p_x$  and  $p_y$  (through  $P$ ) encode the  $x$  and  $y$  co-ordinates of the guiding centre around which the particle gyrates. The radius of gyration is given by the integration constant  $|K|$ , which sets the particle's energy.



## 2 Hamiltonian mechanics

### 2.1 Hamilton's equations

The Euler–Lagrange equation (1.8),

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} = 0, \quad (2.1)$$

when written out in component form becomes a set of  $n$  coupled second-order ODEs. Like any set of  $n$  coupled second-order ODEs, we can turn it into a set of  $2n$  first-order ODEs by introducing  $n$  additional variables. In this case, introduce  $\mathbf{p} \equiv \partial L / \partial \dot{\mathbf{q}}$  to obtain

$$\dot{\mathbf{p}} = \frac{\partial L}{\partial \mathbf{q}}, \quad \mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}, \quad (2.2)$$

the second of which is an awkward implicit equation for  $\dot{\mathbf{q}}$ .

We'd like to have a new function that somehow encodes the same information as  $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ , but with  $\dot{\mathbf{q}}$  replaced by  $\mathbf{p} \equiv \partial L / \partial \dot{\mathbf{q}}$ . Provided  $L(\mathbf{q}, \dot{\mathbf{q}}, t)$  is a convex function of the velocities  $\dot{\mathbf{q}}$ , then we can do just this by taking the Legendre transform (§0.3) of  $L(\mathbf{q}, \dot{\mathbf{q}}, t)$  with respect to  $\dot{\mathbf{q}}$ . This gives a new function, the **Hamiltonian**,

$$H(\mathbf{q}, \mathbf{p}, t) \equiv \mathbf{p} \cdot \dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}, t), \quad (2.3)$$

which is a function of the generalized co-ordinates  $\mathbf{q}$ , the conjugate momenta  $\mathbf{p}$  and time; we have to use the relation  $\mathbf{p} = \partial L / \partial \dot{\mathbf{q}}$  to express all  $\dot{\mathbf{q}}$  on the RHS in terms of  $\mathbf{p}$  and  $\mathbf{q}$  (and possibly  $t$ ).

To obtain the equations of motion in terms of this new function, take the total differential of each side of (2.3). The RHS gives

$$\begin{aligned} dH &= \dot{\mathbf{q}} \cdot d\mathbf{p} + \mathbf{p} \cdot d\dot{\mathbf{q}} - \left( \frac{\partial L}{\partial \mathbf{q}} \cdot d\mathbf{q} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot d\dot{\mathbf{q}} + \frac{\partial L}{\partial t} dt \right) \\ &= \dot{\mathbf{q}} \cdot d\mathbf{p} - \frac{\partial L}{\partial \mathbf{q}} \cdot d\mathbf{q} - \frac{\partial L}{\partial t} dt, \end{aligned} \quad (2.4)$$

using  $\mathbf{p} = \partial L / \partial \dot{\mathbf{q}}$  to cancel two of the terms on the first line. This must equal the total differential of the LHS,

$$dH = \frac{\partial H}{\partial \mathbf{q}} \cdot d\mathbf{q} + \frac{\partial H}{\partial \mathbf{p}} \cdot d\mathbf{p} + \frac{\partial H}{\partial t} dt. \quad (2.5)$$

Since (2.4) and (2.5) have to be equal for any choice of  $(d\mathbf{q}, d\mathbf{p}, dt)$ , it follows that

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}; \quad -\frac{\partial L}{\partial \mathbf{q}} = \frac{\partial H}{\partial \mathbf{q}}; \quad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}. \quad (2.6)$$

Using the relation  $\dot{\mathbf{p}} = \partial L / \partial \mathbf{q}$ , these become **Hamilton's equations**:

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}. \quad (2.7)$$

### 2.2 Why bother?

In Hamiltonian mechanics we can think of our mechanical system as a point  $(\mathbf{q}, \mathbf{p})$  moving in  $2n$ -dimensional **phase space** with velocity given by (2.7). Contrast this to Lagrangian mechanics in which there is no such simple geometrical interpretation of the corresponding paths through configuration space, even when the equations are written out in the coupled form (2.2).

In practice it usually turns out that Hamilton's equations are no easier to solve than the corresponding EL equations. The power of Hamiltonian mechanics comes from the ease with which one can use the explicit ODEs (2.7) to discover general properties of trajectories in phase space. This helps uncover much of the hidden structure that underpins classical mechanics. It turns out that very similar structures underlie quantum mechanics.

### 2.3 Examples

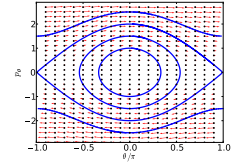
**Simple pendulum** The Lagrangian (1.9)  $L(\theta, \dot{\theta}) = \frac{1}{2} m l^2 \dot{\theta}^2 + m g l \cos \theta$ , so that  $p_\theta \equiv \partial L / \partial \dot{\theta} = m l^2 \dot{\theta}$ . Using the Legendre transform (2.3) to turn this  $L$  into a function of  $(\theta, p_\theta)$  gives

$$\begin{aligned} H(\theta, p_\theta) &= p_\theta \dot{\theta} - L \\ &= \frac{p_\theta^2}{2 m l^2} - m g l \cos \theta, \end{aligned} \quad (2.8)$$

in which we have used the expression for  $p_\theta$  to express all occurrences of  $\dot{\theta}$  in terms of  $p_\theta$ . Hamilton's equations (2.7) become

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m l^2}, \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -m g l \sin \theta. \quad (2.9)$$

Notice that there is essentially no difference between these and the corresponding EL equation. The Hamiltonian approach does, however, encourage us to think of the motion of the pendulum as taking place in a two-dimensional **phase plane**  $(\theta, p_\theta)$ , in which the velocity field is given by  $(\dot{\theta}, \dot{p}_\theta) = (\partial H / \partial p_\theta, -\partial H / \partial \theta)$  (red arrows on plot). The phase-space co-ordinates of the bob follow the **integral curves** of this velocity field (blue curves), so called because they are obtained by “integrating” (i.e., solving) Hamilton's equations to find the trajectory.



**Particle in a potential well** If the particle's potential energy  $V = V(x, t)$  then the Lagrangian  $L = \frac{1}{2} m \dot{x}^2 - V$ , giving  $\mathbf{p} = \partial L / \partial \dot{x} = m \dot{x}$ , the usual momentum familiar from Newtonian mechanics. The Legendre transform (2.3) of  $L$  is

$$\begin{aligned} H(x, \mathbf{p}, t) &= \mathbf{p} \cdot \dot{x} - L(x, \dot{x}, t) \\ &= \frac{\mathbf{p}^2}{2m} + V(x, t) \end{aligned} \quad (2.10)$$

Hamilton's equations (2.7) reduce to the very familiar

$$\dot{x} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}} = -\frac{\partial V}{\partial \mathbf{x}}. \quad (2.11)$$

This example serves as a useful reminder of which of Hamilton's equations has the minus sign.

**Particle in a central field** The Lagrangian (1.18)  $L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - V(r)$ , which gives momenta  $p_r = m \dot{r}$ ,  $p_\theta = m r^2 \dot{\theta}$  and  $p_\phi = m r^2 \sin^2 \theta \dot{\phi}$ . Applying the Legendre transform (2.3) to turn this  $L(r, \dot{r}, \dot{\theta}, \dot{\phi})$  into something that depends explicitly on  $(r, p_r, p_\theta, p_\phi)$  gives the Hamiltonian

$$\begin{aligned} H(r, p_r, p_\theta, p_\phi) &= p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - L(r, \dot{r}, \dot{\theta}, \dot{\phi}) \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2 \theta} + V(r). \end{aligned} \quad (2.12)$$

Hamilton's equations (2.7) become

$$\begin{aligned} \dot{r} &= \frac{p_r}{m}, & \dot{\theta} &= \frac{p_\theta}{mr^2}, & \dot{\phi} &= \frac{p_\phi}{mr^2 \sin^2 \theta}, \\ \dot{p}_r &= \frac{p_\theta^2}{r^3} + \frac{p_\phi^2}{r^3 \sin^2 \theta} - \frac{dV}{dr}, & \dot{p}_\theta &= \frac{p_\phi^2 \cos \theta}{mr^2 \sin^3 \theta}, & \dot{p}_\phi &= 0. \end{aligned} \quad (2.13)$$

Just as in the Lagrangian case (§1.3), if we orient our co-ordinate system so that the particle starts with  $\theta = \frac{\pi}{2}$  and  $\dot{\theta} = 0$  then Hamilton's equations tell us that  $p_\theta$  and  $\theta$  remain zero throughout the motion and  $p_\phi$  is a constant of motion. This is equivalent to motion in the simpler “effective” Hamiltonian  $H_{\text{eff}}(r, p_r) = p_r^2/2m + V_{\text{eff}}(r)$ , where the one-dimensional **effective potential**

$$V_{\text{eff}}(r) = \frac{p_\phi^2}{2mr^2} + V(r). \quad (2.14)$$

Notice that it is much easier to exploit the conservation of the momenta  $p_\theta$  and  $p_\phi$  in Hamiltonian mechanics than in Lagrangian mechanics.

**Motion of a particle referred to a rotating co-ordinate system** The Lagrangian (1.35)  $L = \frac{1}{2}m[\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r}]^2 - V(\mathbf{r}, t)$  from which  $\mathbf{p} \equiv \partial L/\partial \dot{\mathbf{r}} = m(\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r})$ . Using (2.3) to construct the corresponding Hamiltonian gives

$$\begin{aligned} H(\mathbf{r}, \mathbf{p}) &= \mathbf{p} \cdot \dot{\mathbf{r}} - L \\ &= m[\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r}] \cdot \dot{\mathbf{r}} - \frac{1}{2}m[\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r}]^2 + V \\ &= \frac{1}{2}m\dot{\mathbf{r}}^2 - \frac{1}{2}m(\boldsymbol{\Omega} \times \mathbf{r})^2 + V \\ &= \frac{\mathbf{p}^2}{2m} - \mathbf{p} \cdot (\boldsymbol{\Omega} \times \mathbf{r}) + V. \end{aligned} \quad (2.15)$$

**Exercise:** Write out Hamilton's equations for this system and show that they are equivalent to equation (1.38).

**Particle in an electromagnetic field** Similarly, for the Lagrangian (1.24),  $L = \frac{1}{2}m\dot{\mathbf{x}}^2 + Q(\dot{\mathbf{x}} \cdot \mathbf{A} - \phi)$ , the momenta  $\mathbf{p} = \partial L/\partial \dot{\mathbf{x}} = m\dot{\mathbf{x}} + Q\mathbf{A}$ . The Hamiltonian

$$\begin{aligned} H = \mathbf{p} \cdot \dot{\mathbf{x}} - L &= m\dot{\mathbf{x}}^2 + Q\mathbf{A} \cdot \dot{\mathbf{x}} - \left[ \frac{1}{2}m\dot{\mathbf{x}}^2 + Q(\dot{\mathbf{x}} \cdot \mathbf{A} - \phi) \right] \\ &= \frac{1}{2}m\dot{\mathbf{x}}^2 + Q\phi \\ &= \frac{(\mathbf{p} - Q\mathbf{A})^2}{2m} + Q\phi. \end{aligned} \quad (2.16)$$

**Exercise:** By writing the second of Hamilton's equations for this case in the form

$$\dot{p}_i = -\frac{1}{2m} \frac{\partial}{\partial x_i} \sum_j (p_j - QA_j)(p_j - QA_j) - Q \frac{\partial \phi}{\partial x_i} \quad (2.17)$$

show that  $\dot{\mathbf{p}} = Q\nabla(\dot{\mathbf{x}} \cdot \mathbf{A}) - Q\nabla\phi$  (remember  $\mathbf{x}$  and  $\mathbf{p}$  are independent co-ordinates in phase space). Hence show that Hamilton's equations reduce to the expected  $m\ddot{\mathbf{x}} = Q\dot{\mathbf{x}} \times (\nabla \times \mathbf{A}) - Q\nabla\phi$ .

## 2.4 General remarks

1. If a co-ordinate  $q_i$  doesn't appear in the Lagrangian, then, by construction, it doesn't appear in the Hamiltonian either. The corresponding momentum  $p_i$  is conserved because, from Hamilton's equations,

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0. \quad (2.18)$$

2. The rate of change of  $H$  along a trajectory  $(\mathbf{q}(t), \mathbf{p}(t))$ ,

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial t} + \dot{\mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{q}} + \dot{\mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{p}} \\ &= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{q}} - \frac{\partial H}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial H}{\partial t}, \end{aligned} \quad (2.19)$$

the second line following from the first on using Hamilton's equations (2.7). Thus  $H$  is conserved if it does not depend explicitly on time. (We have already seen this from the Beltrami identity in §0.1.)

3. If  $L = T - V$ , where  $T$  is a homogeneous quadratic form in the velocities  $\dot{\mathbf{q}}$ ,

$$T = \frac{1}{2} \sum_{ij} a_{ij}(\mathbf{q}, t) \dot{q}_i \dot{q}_j, \quad (2.20)$$

with  $a_{ij} = a_{ji}$  and  $V = V(\mathbf{q}, t)$ , then the Hamiltonian  $H = T + V$ . To see this, notice that

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k} = \frac{1}{2} \sum_{ij} (a_{ij} \delta_{ik} \dot{q}_j + a_{ij} \dot{q}_i \delta_{jk}) = \sum_i a_{ki} \dot{q}_i. \quad (2.21)$$

Constructing the Hamiltonian in the usual way, we have that

$$H = \mathbf{p} \cdot \dot{\mathbf{q}} - L = \sum_k \left[ \sum_i a_{ki} \dot{q}_i \right] \dot{q}_k - \left[ \frac{1}{2} \sum_{ij} a_{ij} \dot{q}_i \dot{q}_j - V \right] = T + V. \quad (2.22)$$

## 2.5 Liouville's theorem\*

The instantaneous state of a mechanical system is described by a point in phase space with co-ordinates  $(\mathbf{q}, \mathbf{p})$ . This point moves through phase space with a velocity  $(\dot{\mathbf{q}}, \dot{\mathbf{p}})$  given by Hamilton's equations (2.7). If we release an ensemble of systems with the same Hamiltonian but slightly different initial conditions, the phase points flow through phase space like a fluid.

Before tackling this particular flow, we first need a standard result: any flow in which the divergence of the velocity field is identically zero preserves volume (i.e., is incompressible). To show this, let us take a general velocity field

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \quad (2.23)$$

in an  $n$ -dimensional space and examine the relative motion of two nearby points  $\mathbf{x}_0(t)$  and  $\mathbf{x}_1(t)$ . Let  $\Delta(t) \equiv \mathbf{x}_1(t) - \mathbf{x}_0(t)$ . From a Taylor expansion of (2.23) we have that after time  $\delta t$ , dropping terms  $O(\delta t^2)$ ,

$$\Delta_j \rightarrow \Delta_j + \sum_k \frac{\partial f_j}{\partial x_k} \Delta_k \delta t = \left( \delta_{jk} + \frac{\partial f_j}{\partial x_k} \delta t \right) \Delta_k = J_{jk} \Delta_k, \quad (2.24)$$

where  $J_{jk} = \delta_{jk} + (\partial f_j / \partial x_k) \delta t$ . The change in volume effected by this transformation is given by  $\det J_{jk}$ .

**Exercise:** Show by direct expansion that  $\det(I + A\delta t) = 1 + (\text{tr } A)\delta t + O(\delta t^2)$  for any square matrix  $A$ .

Using the result of the exercise, the flow preserves volume if  $\sum_i (\partial f_i / \partial x_i) = 0$ , i.e., if the velocity field has zero divergence.

Now we return to phase space. Let us introduce the shorthand  $\mathbf{w} \equiv (\mathbf{q}, \mathbf{p}) = (q_1, \dots, q_n, p_1, \dots, p_n)$  for the co-ordinates of a point in  $2n$ -dimensional phase space. By Hamilton's equations, the “velocity field” in phase space is  $\dot{\mathbf{w}} = (\partial H / \partial \mathbf{p}, -\partial H / \partial \mathbf{q})$ . Its divergence

$$\text{div}(\dot{\mathbf{w}}(\mathbf{w}, t)) \equiv \frac{\partial}{\partial \mathbf{w}} \cdot (\dot{\mathbf{w}}(\mathbf{w}, t)) = \frac{\partial^2 H}{\partial \mathbf{q} \partial \mathbf{p}} - \frac{\partial^2 H}{\partial \mathbf{p} \partial \mathbf{q}} = 0. \quad (2.25)$$

So, the phase-space volume enclosed by the points representing our ensemble of systems is conserved. This is Liouville's theorem. Similarly we can introduce the phase-space mass density  $f(\mathbf{x}, \mathbf{v}, t)$ , which has to respect the (phase-space) continuity equation

$$\begin{aligned} 0 &= \frac{\partial f}{\partial t} + \operatorname{div}(f\dot{\mathbf{w}}) = \frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{q}} \cdot (f\dot{\mathbf{q}}) + \frac{\partial}{\partial \mathbf{p}} \cdot (f\dot{\mathbf{p}}) \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{\partial f}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} + f \underbrace{\left( \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{q}} + \frac{\partial \dot{\mathbf{p}}}{\partial \mathbf{p}} \right)}_{\operatorname{div} \dot{\mathbf{w}}=0} \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{q}}. \end{aligned} \quad (2.26)$$

This is known as Liouville's equation.

## 2.6 Poisson brackets

The **Poisson bracket**  $[A, B]$  of any two smooth phase-space functions  $A(\mathbf{q}, \mathbf{p}, t)$ ,  $B(\mathbf{q}, \mathbf{p}, t)$  is defined as

$$[A, B] \equiv \frac{\partial A}{\partial \mathbf{q}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{q}}. \quad (2.27)$$

It is straightforward to show that the Poisson bracket has the following properties (**verify them!**):

- (i)  $[A, B] = -[B, A]$  (antisymmetry);
- (ii)  $[\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C]$  for any real numbers  $\alpha, \beta$  (linearity);
- (iii)  $[AB, C] = [A, C]B + A[B, C]$  (chain rule);
- (iv)  $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$  (**Jacobi identity**).

Furthermore, phase-space co-ordinates satisfy the **canonical commutation relations** or **fundamental Poisson bracket relations**

$$[p_i, p_j] = 0, \quad [q_i, q_j] = 0 \quad \text{and} \quad [q_i, p_j] = \delta_{ij}. \quad (2.28)$$

This follows directly from the definition (2.27) and, remembering that  $(\mathbf{q}, \mathbf{p})$  are independent coordinates in phase space, the relations  $\partial p_i / \partial q_j = \partial q_i / \partial p_j = 0$  and  $\partial p_i / \partial p_j = \partial q_i / \partial q_j = \delta_{ij}$ .

In terms of Poisson brackets, Hamilton's equations become

$$\dot{q}_i = [q_i, H], \quad \dot{p}_i = [p_i, H]. \quad (2.29)$$

The rate of change of any function  $f(\mathbf{q}, \mathbf{p}, t)$  along a trajectory  $(\mathbf{q}(t), \mathbf{p}(t))$  is

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{\partial f}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{q}} \\ &= \frac{\partial f}{\partial t} + [f, H]. \end{aligned} \quad (2.30)$$

**Alternative notation** It is sometimes convenient to combine  $\mathbf{q}$  and  $\mathbf{p}$  into the single vector  $\mathbf{w} \equiv (\mathbf{q}, \mathbf{p}) = (q_1, \dots, q_n, p_1, \dots, p_n)$ . Then Hamilton's equations (2.7) can be written as

$$\dot{\mathbf{w}} = \mathbf{J} \cdot \frac{\partial H}{\partial \mathbf{w}}, \quad (2.31)$$

where the **symplectic matrix**

$$\mathbf{J} \equiv \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, \quad (2.32)$$

and  $0_n$  and  $I_n$  are the  $n \times n$  zero and identity matrices, respectively. The expression (2.27) for the Poisson bracket becomes

$$[A, B] = \left( \frac{\partial A}{\partial \mathbf{w}} \right)^T \cdot \mathbf{J} \cdot \frac{\partial B}{\partial \mathbf{w}} = \sum_{\alpha, \beta=1}^{2n} \frac{\partial A}{\partial w_\alpha} J_{\alpha\beta} \frac{\partial B}{\partial w_\beta}. \quad (2.33)$$

Because  $\partial w_i / \partial w_\alpha = \delta_{i\alpha}$  the canonical commutation relations (2.28) are simply

$$[w_i, w_j] = J_{ij}. \quad (2.34)$$

## 2.7 Symmetries and conservation laws

**Constants of motion** If a function  $F(\mathbf{q}, \mathbf{p})$  is a constant of the motion then, using (2.30),

$$0 = \dot{F} = \frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H] \quad \Rightarrow \quad [F, H] = 0. \quad (2.35)$$

The functions  $F$  and  $H$  are then said to (Poisson) commute. Conversely, if we can find a function  $F(\mathbf{q}, \mathbf{p})$  for which  $[F, H] = 0$ , then  $F$  is a constant of motion. Given two constants of motion,  $F(\mathbf{q}, \mathbf{p})$  and  $G(\mathbf{q}, \mathbf{p})$ , we have from the Jacobi identity that

$$[[F, G], H] + \underbrace{[[G, H], F]}_0 + \underbrace{[[H, F], G]}_0 = 0. \quad (2.36)$$

So,  $[F, G]$  is also a constant of motion. In some cases the new function  $[F, G]$  will turn out to be trivial (e.g., it might be zero or a straightforward function of the known invariants  $F$  and  $G$ ), but sometimes it will be a new, independent constant of motion.

**Example:** Let  $\mathbf{r} = (r_1, r_2, r_3)$  be Cartesian co-ordinates and  $\mathbf{p}$  the corresponding conjugate momentum. The angular momentum  $\mathbf{J} = \mathbf{r} \times \mathbf{p}$  has components

$$J_1 = r_2 p_3 - r_3 p_2, \quad J_2 = r_3 p_1 - r_1 p_3, \quad J_3 = r_1 p_2 - r_2 p_1. \quad (2.37)$$

The Poisson bracket of the first two,

$$\begin{aligned} [J_1, J_2] &= [r_2 p_3 - r_3 p_2, r_3 p_1 - r_1 p_3] \\ &= [r_2 p_3, r_3 p_1] - \underbrace{[r_2 p_3, r_1 p_3]}_0 - \underbrace{[r_3 p_2, r_3 p_1]}_0 + [r_3 p_2, r_1 p_3] \\ &= [r_2 p_3, r_3 p_1] + [r_3 p_2, r_1 p_3] \\ &= -r_2 p_1 + p_2 r_1 = J_3. \end{aligned} \quad (2.38)$$

If  $J_1$  and  $J_2$  are constants of motion, then so too is  $J_3$ . Or, more generally, the vector  $\mathbf{J}$  is conserved if any two of its components are.

**Exercise:** Show that  $[\mathbf{J}^2, J_i] = 0$ .

**Symmetries and conservation laws** In section §1.6 we saw that if a Lagrangian  $L$  is invariant under a small change in co-ordinates,  $\mathbf{q} \rightarrow \mathbf{q} + \epsilon \mathbf{K}(\mathbf{q})$ , then  $C = \mathbf{K} \cdot (\partial L / \partial \dot{\mathbf{q}})$  is a constant of motion. There is a corresponding relationship between symmetry and conservation laws in Hamiltonian mechanics.

We have seen how the Hamiltonian function defines a phase flow  $(\dot{q}, \dot{p}) = ([q, H], [p, H]) = (\partial H/\partial p, -\partial H/\partial q)$ . Similarly, *any* function  $G(q, p)$  defines a flow through phase space with

$$\begin{aligned}\frac{dq(\lambda)}{d\lambda} &= [q(\lambda), G], \\ \frac{dp(\lambda)}{d\lambda} &= [p(\lambda), G],\end{aligned}\quad (2.39)$$

in which the parameter  $\lambda$  takes the place of time. If we integrate the coupled ODEs (2.39) for a fixed interval in  $\lambda$ , we obtain a one-parameter mapping  $G_\lambda : (q(0), p(0)) \rightarrow (q(\lambda), p(\lambda))$  of phase space onto itself. The function  $G$  is the **generator** of this mapping. The solutions to (2.39) are the **integral curves** of  $G$ .

**Exercise:** Suppose that  $(x, p)$  are Cartesian co-ordinates in phase space. What mapping is generated by  $G = x_1$ ? By  $G = p_1$ ? By  $G = x_1 p_2 - x_2 p_1$ ?

**Exercise:** Show that the derivative of a function  $F(q, p)$  along the flow generated by  $G$  is

$$\frac{dF}{d\lambda} = \frac{\partial F}{\partial q} \cdot \frac{dq}{d\lambda} + \frac{\partial F}{\partial p} \cdot \frac{dp}{d\lambda} = [F, G]. \quad (2.40)$$

For small  $\lambda$ ,  $G_\lambda$  is the infinitesimal map

$$q \rightarrow q + \lambda \frac{\partial G}{\partial p}, \quad p \rightarrow p - \lambda \frac{\partial G}{\partial q}, \quad (2.41)$$

or  $\delta q = \lambda \partial G/\partial p$ ,  $\delta p = -\lambda \partial G/\partial q$ . If  $H$  is invariant under this map, then  $G$  is called a **symmetry** of the Hamiltonian. The condition for this is that

$$\begin{aligned}0 = \delta H &= \frac{\partial H}{\partial q} \cdot \delta q + \frac{\partial H}{\partial p} \cdot \delta p \\ &= \lambda \frac{\partial H}{\partial q} \cdot \frac{\partial G}{\partial p} - \lambda \frac{\partial H}{\partial p} \cdot \frac{\partial G}{\partial q} \\ &= \lambda [H, G].\end{aligned}\quad (2.42)$$

But  $\dot{G} = [G, H]$ , so if  $G$  is a symmetry then  $G$  is conserved. Conversely, for any constant of motion  $G(q, p)$  then there is a map (2.41) that leaves  $H$  invariant. (In contrast, in Lagrangian mechanics Noether's theorem says only that symmetry  $\Rightarrow$  constant; it does not show that constant  $\Rightarrow$  symmetry too.)

**Exercise:** What mapping does  $H$  generate? What condition do we need to impose on  $H$  for this to work?

**Exercise:** Show that the infinitesimal map (2.41) can be written as

$$\begin{aligned}w &\rightarrow w + \lambda J \left( \frac{\partial G}{\partial w} \right) \\ &= w - \lambda \left( \frac{\partial G}{\partial w} \right)^T J \\ &= \left[ I - \lambda \left( \frac{\partial G}{\partial w} \right)^T J \left( \frac{\partial}{\partial w} \right) \right] w,\end{aligned}\quad (2.43)$$

where  $w = \begin{pmatrix} q \\ p \end{pmatrix}$  and  $J$  is the symplectic matrix (2.32). By chaining many such mappings together, obtain a formal expression for the mapping  $G_\lambda$  for general  $\lambda$  using the definition  $\exp(X) = \lim_{n \rightarrow \infty} (I + X/n)^n$  for the exponential of a linear operator.

## 2.8 Poincaré integral invariants

The following two sections provide another way of looking at the structure of phase space in which time plays a more explicit role: you may have noticed that many of the results concerning Poisson brackets required us to restrict our attention to functions  $F(q, p)$  that do not depend explicitly on time.

**Some hydrodynamics: Stokes' lemma** Suppose we have a fluid in which points at position  $x$  move with velocity  $\dot{x} = \nabla \times u$ , for some  $u = u(x)$ . At time  $t_0$  we set up a closed loop  $\gamma_0$  of dye-releasing particles. At any later instant the particles will lie along another closed loop  $\gamma(t)$ . Between times  $t_0$  and  $t$  the particles will have traced out a surface  $S$ , the ends of which are the curves  $\gamma_0$  and  $\gamma(t)$ . The surface  $S$  is known as a **vortex tube**: individual particles within the loop move along integral curves of  $\nabla \times u$ , which are **vortex lines** of  $u$ . Because the particles always move parallel to  $\nabla \times u$  we have that  $\int_S (\nabla \times u) \cdot \hat{n} dS = 0$ , where  $\hat{n}$  is the outward-pointing unit normal to the vortex tube. Stokes' theorem tells us that

$$\oint_{\gamma_0} u \cdot dx - \oint_{\gamma(t)} u \cdot dx = \int_S (\nabla \times u) \cdot \hat{n} dS = 0, \quad (2.44)$$

where  $\gamma$  is any other loop that encircles the vortex tube.

**Extended phase space** Given a mechanical system with  $n$  degrees of freedom the corresponding phase space has  $2n$  dimensions. If we treat time  $t$  as an additional co-ordinate we obtain a  $(2n+1)$ -dimensional space, known as extended phase space.

**Exercise:** A one-dimensional simple harmonic oscillator has Hamiltonian  $H = \frac{1}{2}(p^2 + q^2)$ . Show that solutions to Hamilton's equations yield helical curves in extended phase space.

**Exercise:** Now suppose that  $H = \frac{1}{2}(p^2 + \omega^2 q^2)$ , where  $\omega = \omega(t)$ . How does the time dependence of the spring constant change the trajectories in extended phase space?

**Vortex tubes in 3d extended phase space** Consider a system with one degree of freedom so that extended phase space has three dimensions. We recycle our use of  $w$  from the previous section to label points in (extended) phase space and look at one-dimensional loops  $w(\gamma) = p(\gamma)\hat{p} + q(\gamma)\hat{q} + t(\gamma)\hat{t}$ , where  $(\hat{p}, \hat{q}, \hat{t})$  are the natural basis vectors to use for extended phase space. We assume that  $(\hat{p}, \hat{q}, \hat{t})$  (in that order) form a right-handed set.

We introduce a special vector  $u = p\hat{q} - H\hat{t}$ . The vortex lines of  $u$  are given by the solutions  $w(\tau)$  of

$$\left( \frac{dp}{d\tau}, \frac{dq}{d\tau}, \frac{dt}{d\tau} \right) = \frac{dw}{d\tau} = \nabla \times u = \begin{vmatrix} \hat{p} & \hat{q} & \hat{t} \\ \frac{\partial}{\partial p} & \frac{\partial}{\partial q} & \frac{\partial}{\partial t} \\ 0 & p & -H \end{vmatrix} = \left( -\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}, 1 \right), \quad (2.45)$$

using  $\partial p/\partial t = 0$  because  $(p, q, t)$  are independent co-ordinates of extended phase space. The vortex lines of the special vector  $u$  are the solutions to Hamilton's equations of motion!

Any one-dimensional loop  $\gamma_0$  we draw in extended phase space will form a vortex tube, the vortex lines of which are the integral curves of Hamilton's equations. The circulation about  $\gamma_0$  is

$$\oint_{\gamma_0} u \cdot dw = \oint_{\gamma_0} \left( p \frac{dq}{d\gamma} - H \frac{dt}{d\gamma} \right) d\gamma = \oint_{\gamma_0} (p dq - H dt). \quad (2.46)$$

Stokes' lemma tells us that

$$\oint_{\gamma_1} (p dq - H dt) = \oint_{\gamma_0} (p dq - H dt) \quad (2.47)$$

for any other loop  $\gamma_1$  that encircles the same vortex tube.

**Comments:**

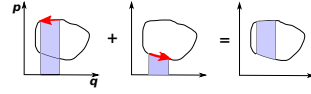
1. The "special" vector  $u = p\hat{q} - H\hat{t}$  is not unique: for any  $S(q, p, t)$  the vortex lines of  $u + \nabla S$  are the same as those of  $u$ ; we can add a total derivative  $dS$  to either integrand in (2.47).

2. If we choose  $\gamma_0$  and  $\gamma_1$  to lie on constant- $t$  slices,  $t = t_0$  and  $t = t_1$  respectively, of extended phase space, then  $dt = 0$  and (2.47) becomes

$$\oint_{t_0} p dq = \oint_{t_1} p dq \Rightarrow \int_{S(t_0)} dp dq = \int_{S(t_1)} dp dq. \quad (2.48)$$

That is, the area  $S(t)$  of the phase plane enclosed by a loop  $\gamma$  does not change as the loop evolves: we have rederived Liouville's theorem for the simplest case,  $n = 1$ , in a much more complicated way...

The figure on the right shows why  $\oint_{\gamma} p dq = \int_S dp dq$ , where  $S$  is the surface enclosed by the loop  $\gamma$ : both "terms" on the LHS of this graphical equation contribute to  $p dq$ , but  $dq$  is negative in the first term and positive in the second.



**General case** It turns out that this result can be generalized to systems with more than one degree of freedom. The "special" vector becomes  $\mathbf{u} = \sum_{i=1}^n p_i \hat{q}_i - H \hat{t}$ , the vortex lines of which again are precisely the integral curves of Hamilton's equations: every vortex line of  $\mathbf{u}$  is a solution to Hamilton's equations and every solution to Hamilton's equations is a vortex line of  $\mathbf{u}$ . A generalization of Stokes' lemma to  $(2n + 1)$ -dimensional space tells us that

$$\oint_{\gamma_0} (\mathbf{p} \cdot d\mathbf{q} - H dt) = \oint_{\gamma_1} (\mathbf{p} \cdot d\mathbf{q} - H dt) \quad (2.49)$$

for any one-dimensional loops  $\gamma_0$  and  $\gamma_1$  that encircle the same two-dimensional vortex tube. The quantity (2.49) is known as a **Poincaré (–Cartan) integral invariant**.

#### Comments

1. For later use we note that, for any function  $S(\mathbf{p}, \mathbf{q}, t)$ ,

$$\oint_{\gamma} \left( \frac{\partial S}{\partial \mathbf{p}} \cdot d\mathbf{p} + \frac{\partial S}{\partial \mathbf{q}} \cdot d\mathbf{q} + \frac{\partial S}{\partial t} dt \right) = 0. \quad (2.50)$$

This means we can add a total derivative of  $S$  to either, or both, sides of (2.49).

2. If we choose  $\gamma_0$  and  $\gamma_1$  to be constant- $t$  slices,  $t = t_0$  and  $t = t_1$ , of extended phase space then (2.49) becomes

$$\begin{aligned} \oint_{\gamma_0} \mathbf{p} \cdot d\mathbf{q} &= \oint_{\gamma_1} \mathbf{p} \cdot d\mathbf{q} \\ \Rightarrow \sum_i \oint_{\gamma_0} p_i dq_i &= \sum_i \oint_{\gamma_1} p_i dq_i \\ \Rightarrow \sum_i \int_{S_i(\gamma_0)} dp_i dq_i &= \sum_i \int_{S_i(\gamma_1)} dp_i dq_i, \end{aligned} \quad (2.51)$$

where  $S_i(\gamma)$  is the projection of the loop  $\gamma$  onto the  $(p_i, q_i)$  plane. If we set up a loop of particles at some time  $t_0$ , the sum of the projected areas of the loop  $\gamma$  onto the  $(p_i, q_i)$  planes is preserved as the loop evolves. Volume preservation (Liouville's theorem) can be viewed as a consequence of these more fundamental integral invariants.

## 2.9 Canonical maps

We have seen how easy it is to change variables  $\mathbf{q} \rightarrow \mathbf{Q}$  in the Lagrangian formulation of mechanics. The price we pay for the more interesting and powerful structure of phase space in Hamiltonian mechanics is that co-ordinate transformations are not so straightforward.

In this section we investigate how to change to new phase-space co-ordinates  $(\mathbf{Q}, \mathbf{P})$ ,

$$\begin{aligned} Q_i &= Q_i(\mathbf{q}, \mathbf{p}, t), \\ P_i &= P_i(\mathbf{q}, \mathbf{p}, t), \end{aligned} \quad (i = 1, \dots, n), \quad (2.52)$$

that preserve the Poincaré invariants of the previous section. First, some definitions: If for any loop  $\gamma$  in extended phase space we have that

$$\oint_{\gamma} (\mathbf{P} \cdot d\mathbf{Q} - K dt) = \oint_{\gamma} (\mathbf{p} \cdot d\mathbf{q} - H dt) \quad (2.53)$$

in which the function  $K(\mathbf{Q}, \mathbf{P}, t)$  is independent of the choice of  $\gamma$ , then the transformation (2.52) is called a **canonical map** (or a **canonical transformation**) and the new co-ordinates  $(\mathbf{P}, \mathbf{Q})$  are called **canonical co-ordinates**.<sup>†</sup>

Evolution under time is an example of a canonical map. To see this, suppose that we define  $(\mathbf{Q}, \mathbf{P})$  to be the values that  $(\mathbf{q}, \mathbf{p})$  will have one second in the future. Then (2.53) is clearly satisfied if we take  $K(\mathbf{Q}, \mathbf{P}, t) = H(\mathbf{Q}, \mathbf{P}, t + 1 \text{ sec})$ .

**Hamilton's equations in the new co-ords** The RHS of (2.53) is  $\oint_{\gamma} \mathbf{u} \cdot d\mathbf{w}$ , where  $\mathbf{w} = (\mathbf{p}(\gamma), \mathbf{q}(\gamma), t(\gamma))$  and  $\mathbf{u} = \sum_i p_i \hat{q}_i - H \hat{t}$ . We have already seen that the vortex lines for this  $\mathbf{u}$  are given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \quad (2.54)$$

Similarly, the LHS of (2.53) is  $\oint_{\gamma} \mathbf{U} \cdot d\mathbf{W}$ , where  $\mathbf{W} = (\mathbf{P}(\gamma), \mathbf{Q}(\gamma), t(\gamma))$  and  $\mathbf{U} = \sum_i P_i \hat{Q}_i - K \hat{t}$ . The vortex lines of  $\mathbf{U}$  are clearly given by

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}. \quad (2.55)$$

Both (2.54) and (2.55) describe the same vortex lines in the same extended phase space, but expressed in different co-ordinates. Therefore Hamilton's equations in the new co-ordinates are given by (2.55) with Hamiltonian  $K(\mathbf{Q}, \mathbf{P}, t)$ .

**Generating functions** Equation (2.53) can hold for all loops  $\gamma$  only if the integrands differ by a total derivative  $dS$  of any well-behaved function  $S(\mathbf{P}, \mathbf{Q}, t)$ :

$$\mathbf{P} \cdot d\mathbf{Q} - K dt + dS = \mathbf{p} \cdot d\mathbf{q} - H dt. \quad (2.56)$$

A powerful way of constructing canonical maps is by playing with the function  $S$ . Let us assume that we can express  $\mathbf{P} = \mathbf{P}(\mathbf{q}, \mathbf{Q}, t)$  so that we can eliminate  $\mathbf{P}$  from  $S$  to obtain  $S = F_1(\mathbf{q}, \mathbf{Q}, t)$ , a function of both the old and new co-ordinates and time, but not the momenta. Substituting this  $S = F_1$  into (2.56) and using the chain rule gives

$$\mathbf{P} \cdot d\mathbf{Q} - K dt + \frac{\partial F_1}{\partial \mathbf{q}} \cdot d\mathbf{q} + \frac{\partial F_1}{\partial \mathbf{Q}} \cdot d\mathbf{Q} + \frac{\partial F_1}{\partial t} dt = \mathbf{p} \cdot d\mathbf{q} - H dt. \quad (2.57)$$

As  $(d\mathbf{q}, d\mathbf{Q}, dt)$  can be varied independently (the equality above has to hold for any loop  $\gamma$ ) we must have

$$\mathbf{p} = \frac{\partial F_1}{\partial \mathbf{q}}, \quad \mathbf{P} = -\frac{\partial F_1}{\partial \mathbf{Q}}, \quad K = H + \frac{\partial F_1}{\partial t}. \quad (2.58)$$

<sup>†</sup> NB: Some books define a canonical map as one that preserves the form of Hamilton's equations. Our condition (2.53) is more stringent.

where  $H$  in the last equation is to be interpreted as the original  $H(\mathbf{q}, \mathbf{p}, t)$  substituting for  $\mathbf{q} = \mathbf{q}(\mathbf{Q}, \mathbf{P}, t)$ ,  $\mathbf{p} = \mathbf{p}(\mathbf{Q}, \mathbf{P}, t)$  to make it a function of  $(\mathbf{Q}, \mathbf{P}, t)$ . Thus the function  $F_1(\mathbf{q}, \mathbf{Q}, t)$  generates an implicit transformation from  $(\mathbf{q}, \mathbf{p}) \rightarrow (\mathbf{Q}, \mathbf{P})$ . By construction, it satisfies the condition (2.53) and therefore is canonical.

**Exercise:** What mapping is generated by  $F_1 = \mathbf{q} \cdot \mathbf{Q}$ ? Show that the Hamiltonian  $H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}(\mathbf{p}^2 + \omega^2 \mathbf{q}^2)$  is transformed to  $K(\mathbf{Q}, \mathbf{P}) = \frac{1}{2}(\mathbf{Q}^2 + \omega^2 \mathbf{P}^2)$ .

Unfortunately, generating functions of the form  $F_1(\mathbf{q}, \mathbf{Q}, t)$  are not suitable for constructing mappings close to the identity. So, instead of writing  $S = F_1(\mathbf{q}, \mathbf{Q}, t)$ , let us take

$$S = -\mathbf{P} \cdot \mathbf{Q} + F_2(\mathbf{q}, \mathbf{P}, t), \quad (2.59)$$

in which we treat  $Q$  as a function  $Q(\mathbf{q}, \mathbf{P}, t)$ . Substituting this into (2.56) and using the chain rule to expand  $dS$  gives:

$$\mathbf{P} \cdot d\mathbf{Q} - K dt - \mathbf{P} \cdot d\mathbf{Q} - \mathbf{Q} \cdot d\mathbf{P} + \frac{\partial F_2}{\partial \mathbf{q}} \cdot d\mathbf{q} + \frac{\partial F_2}{\partial \mathbf{P}} \cdot d\mathbf{P} + \frac{\partial F_2}{\partial t} dt = \mathbf{p} \cdot d\mathbf{q} - H dt. \quad (2.60)$$

One way of explaining the  $-\mathbf{P} \cdot \mathbf{Q}$  term that appears in (2.59) that it comes from taking the Legendre transform of  $F_1$ . An alternative, simpler approach is to note that (a) we are free to choose (almost) whatever we like for  $S$  and (b) including  $-\mathbf{P} \cdot \mathbf{Q}$  in  $S$  nicely cancels out the  $\mathbf{P} \cdot d\mathbf{Q}$  on the LHS of (2.56). As  $(d\mathbf{P}, d\mathbf{q}, dt)$  vary independently, we must have that

$$\mathbf{p} = \frac{\partial F_2}{\partial \mathbf{q}}, \quad \mathbf{Q} = \frac{\partial F_2}{\partial \mathbf{P}}, \quad K = H + \frac{\partial F_2}{\partial t}. \quad (2.61)$$

This is another implicit canonical mapping between  $(\mathbf{q}, \mathbf{p}, t)$  and  $(\mathbf{Q}, \mathbf{P}, t)$ .

**Exercise:** Show that  $F_2(\mathbf{q}, \mathbf{P}) = \mathbf{q} \cdot \mathbf{P}$  generates the identity map. What mapping does  $F_2(\mathbf{q}, \mathbf{P}) = \mathbf{q} \cdot \mathbf{P} + \epsilon \hat{n} \cdot \mathbf{q}$  produce? What about  $F_2 = \mathbf{q} \cdot \mathbf{P} + \epsilon \hat{n} \cdot \mathbf{P}$ ?

**Exercise:** Show that for small  $\lambda$  the generating function  $F_2(\mathbf{q}, \mathbf{P}) = \mathbf{q} \cdot \mathbf{P} + \lambda G(\mathbf{q}, \mathbf{P})$  produces the infinitesimal map (2.41). (Use the fact that  $\mathbf{P} \rightarrow \mathbf{p}$  as  $\lambda \rightarrow 0$ .)

**Is a given mapping canonical?** A simple way of testing whether a mapping is canonical is by examining  $\mathbf{P} \cdot d\mathbf{Q} - \mathbf{p} \cdot d\mathbf{q}$ . If that can be expressed as a total derivative  $dS(\mathbf{p}, \mathbf{q}, t)$  or  $dS(\mathbf{P}, \mathbf{Q}, t)$  then the mapping is canonical.

**Exercise:** Show that the mapping  $Q = -\log p$ ,  $P = pq$  is canonical. Find a function  $F_1(q, Q)$  that generates this mapping. Find another generating function of the form  $F_2(q, P)$ .

Another test is to return to the definition (2.53) of a canonical map and to check whether

$$\sum_i \int_{S_i(\gamma)} dP_i dQ_i = \sum_i \int_{s_i(\gamma)} dp_i dq_i \quad (2.62)$$

for all loops  $\gamma$ , where  $s_i(\gamma)$  and  $S_i(\gamma)$  are the projections of  $\gamma$  onto the  $(p_i, q_i)$  and  $(P_i, Q_i)$  planes. Let us look at the projections of the  $(p_k, q_k)$  planes onto all of the  $(Q_i, Q_j)$ ,  $(P_i, P_j)$  and  $(P_i, Q_j)$  planes. We have that

$$\begin{aligned} dQ_i dQ_j &= \sum_k \left( \frac{\partial Q_i}{\partial q_k} \frac{\partial Q_j}{\partial p_k} - \frac{\partial Q_j}{\partial q_k} \frac{\partial Q_i}{\partial p_k} \right) dp_k dq_k, \\ dP_i dQ_j &= \sum_k \left( \frac{\partial P_i}{\partial q_k} \frac{\partial Q_j}{\partial p_k} - \frac{\partial Q_j}{\partial q_k} \frac{\partial P_i}{\partial p_k} \right) dp_k dq_k, \\ dP_i dP_j &= \sum_k \left( \frac{\partial P_i}{\partial q_k} \frac{\partial P_j}{\partial p_k} - \frac{\partial P_j}{\partial q_k} \frac{\partial P_i}{\partial p_k} \right) dp_k dq_k, \end{aligned} \quad (2.63)$$

where the quantities in parentheses are the Jacobians of the transformation from  $(q_k, p_k)$  to the new  $(Q_i, Q_j)$  etc coordinates. The only way of making (2.62) hold for all choices of the loop  $\gamma$  is by requiring that

$$\begin{aligned} \sum_k \left( \frac{\partial Q_i}{\partial q_k} \frac{\partial Q_j}{\partial p_k} - \frac{\partial Q_j}{\partial q_k} \frac{\partial Q_i}{\partial p_k} \right) &= 0, \\ \sum_k \left( \frac{\partial P_i}{\partial q_k} \frac{\partial Q_j}{\partial p_k} - \frac{\partial Q_j}{\partial q_k} \frac{\partial P_i}{\partial p_k} \right) &= \delta_{ij}, \\ \sum_k \left( \frac{\partial Q_i}{\partial q_k} \frac{\partial Q_j}{\partial p_k} - \frac{\partial Q_j}{\partial q_k} \frac{\partial Q_i}{\partial p_k} \right) &= 0. \end{aligned} \quad (2.64)$$

That is, for a map to be canonical, the new coords  $(\mathbf{Q}, \mathbf{P})$  must themselves satisfy the canonical commutation relations (a.k.a. fundamental Poisson bracket relations)

$$[Q_i, Q_j] = [P_i, P_j] = 0, \quad [Q_i, P_j] = \delta_{ij}, \quad (2.65)$$

in which the Poisson brackets are understood to be evaluated with respect to the old  $(\mathbf{q}, \mathbf{p})$  coordinates, as in (2.64). Equation (2.65) is a necessary and sufficient condition for (2.62) to be true: a map  $(\mathbf{q}, \mathbf{p}) \rightarrow (\mathbf{Q}, \mathbf{P})$  is canonical if and only if the new coordinates  $(\mathbf{Q}, \mathbf{P})$  satisfy the canonical commutation relations (2.65).

**Invariance of Poisson brackets under canonical maps** We can use the condition (2.65) to show that *all* Poisson brackets are invariant under canonical maps. To simplify notation, we introduce

$$\mathbf{w} = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} \quad \text{and} \quad \mathbf{W} = \begin{pmatrix} \mathbf{Q} \\ \mathbf{P} \end{pmatrix}, \quad (2.66)$$

in terms of which the relations (2.65) become simply  $[W_i, W_j] = J_{ij}$ , where  $J_{ij}$  are the elements of the symplectic matrix (2.32). Then, using expression (2.33) for the Poisson bracket of the functions  $A(\mathbf{w}, t)$ ,  $B(\mathbf{w}, t)$ , we have that

$$\begin{aligned} [A, B]_{\mathbf{w}} &= \sum_{\alpha, \beta=1}^{2n} \frac{\partial A}{\partial w_\alpha} J_{\alpha\beta} \frac{\partial B}{\partial w_\beta} = \sum_{\alpha, \beta=1}^{2n} \left( \sum_{i=1}^{2n} \frac{\partial A}{\partial W_i} \frac{\partial W_i}{\partial w_\alpha} \right) J_{\alpha\beta} \left( \sum_{j=1}^{2n} \frac{\partial B}{\partial W_j} \frac{\partial W_j}{\partial w_\beta} \right) \\ &= \sum_{i,j=1}^{2n} \frac{\partial A}{\partial W_i} \left( \sum_{\alpha, \beta=1}^{2n} \frac{\partial W_i}{\partial w_\alpha} J_{\alpha\beta} \frac{\partial W_j}{\partial w_\beta} \right) \frac{\partial B}{\partial W_j} = \sum_{i,j=1}^{2n} \frac{\partial A}{\partial W_i} [W_i, W_j] \frac{\partial B}{\partial W_j} \\ &= \sum_{i,j=1}^{2n} \frac{\partial A}{\partial W_i} J_{ij} \frac{\partial B}{\partial W_j} = [A, B]_{\mathbf{W}}. \end{aligned} \quad (2.67)$$

So, canonical maps preserve *all* Poisson brackets.

**Exercise:** We have been cavalier about the choice of signs in the Jacobians in equation (2.63) above. Here is how to show that the signs in that expression are correct. Any pair of  $n$ -dimensional vectors  $(\mathbf{a}, \mathbf{b})$  defines a parallelogram in  $n$ -dimensional space. We define the *oriented area* of the projection of this parallelogram onto the  $(x_i, x_j)$  plane to be  $(dx_i \wedge dx_j)(\mathbf{a}, \mathbf{b}) = a_i b_j - a_j b_i$ . Show that  $(dx_i \wedge dx_j)(\mathbf{b}, \mathbf{a}) = -(dx_i \wedge dx_j)(\mathbf{a}, \mathbf{b}) = (dx_j \wedge dx_i)(\mathbf{a}, \mathbf{b})$ .

Given new coordinates  $\mathbf{X} = \mathbf{X}(\mathbf{x})$ , the projection of the  $(\mathbf{a}, \mathbf{b})$  parallelogram onto the  $(X_i, X_j)$  plane is

$$(dX_i \wedge dX_j)(\mathbf{a}, \mathbf{b}) = \left( \sum_k \frac{\partial X_i}{\partial x_k} dx_k \wedge \sum_l \frac{\partial X_j}{\partial x_l} dx_l \right) (\mathbf{a}, \mathbf{b}) = \sum_{kl} \frac{\partial X_i}{\partial x_k} \frac{\partial X_j}{\partial x_l} (dx_k \wedge dx_l)(\mathbf{a}, \mathbf{b}). \quad (2.68)$$

Hence show that the condition  $\sum_i (dP_i \wedge dQ_i)(\mathbf{a}, \mathbf{b}) = \sum_i (dp_i \wedge dq_i)(\mathbf{a}, \mathbf{b})$  for all  $(\mathbf{a}, \mathbf{b})$  implies (2.64). Now read §§12–16, 18–20, 32–48 of Arnol'd.

### 3 Linearisation and small oscillations (WIP)

A mechanical system is in equilibrium if all time derivatives vanish. In particular, if  $\mathbf{q} = \mathbf{q}_0$  is an equilibrium configuration, then we must have  $\dot{\mathbf{q}} = 0$  and, from the EL equation,  $\partial L/\partial \mathbf{q} = 0$  too. To study the behaviour of a system close to equilibrium, the usual first step is to linearize the equations of motion. This reduces the problem to modelling a coupled set of simple harmonic oscillators, making it easy to test whether the equilibrium is stable or unstable, to calculate the frequencies with which the system “rings” when knocked, and much more.

Expanding  $L(\mathbf{q}, \dot{\mathbf{q}})$  to second order as a Taylor series about  $(\mathbf{q}, \dot{\mathbf{q}}) = (\mathbf{q}_0, 0)$ ,

$$L(\mathbf{q} + \mathbf{h}, \dot{\mathbf{q}} + \dot{\mathbf{h}}) = L(\mathbf{q}_0, 0) + \sum_i h_i \left( \frac{\partial L}{\partial q_i} \right)_{(\mathbf{q}_0, 0)} + \sum_i \dot{h}_i \left( \frac{\partial L}{\partial \dot{q}_i} \right)_{(\mathbf{q}_0, 0)} + \frac{1}{2} \sum_{ij} \left[ h_i F_{ij} h_j + h_i C_{ij} \dot{h}_j + \dot{h}_i C_{ij}^T h_j + \dot{h}_i M_{ij} \dot{h}_j \right] + O(h^3), \quad (3.1)$$

where the constants  $F_{ij} = F_{ji} \equiv \partial^2 L/\partial q_i \partial q_j$ ,  $M_{ij} = M_{ji} \equiv \partial^2 L/\partial \dot{q}_i \partial \dot{q}_j$  and  $C_{ij} = C_{ji}^T \equiv \partial^2 L/\partial q_i \partial \dot{q}_j$ , all evaluated at  $(\mathbf{q}, \dot{\mathbf{q}}) = (\mathbf{q}_0, 0)$ . Remembering that  $\partial L/\partial q_i = 0$  at equilibrium, it is easy to see that none of the first three terms affect the equations of motion. The linearized EL equation for  $h_k$  is then

$$\frac{d}{dt} \left[ \frac{1}{2} \sum_i h_i C_{ik} + \frac{1}{2} \sum_j C_{kj}^T h_j + \sum_j M_{kj} \dot{h}_j \right] - \left[ \sum_j F_{kj} h_j + \frac{1}{2} \sum_j C_{kj} \dot{h}_j + \frac{1}{2} \sum_i \dot{h}_i C_{ik}^T \right] = 0 \quad (3.2)$$

$$\Rightarrow \sum_i \left[ M_{ki} \ddot{h}_i + (C_{ik} - C_{ki}) \dot{h}_i - F_{ki} h_i \right] = 0.$$

The solutions to this homogeneous linear equation are of the form  $\mathbf{h}(t) = \mathbf{Q} \exp(i\omega t)$ , with the vector  $\mathbf{Q}$  and  $\omega$  related through the eigenvalue equation

$$[\omega^2 M - i\omega \hat{C} + F] \mathbf{Q} = 0, \quad (3.3)$$

where  $\hat{C}_{ij} = (C_{ij} - C_{ji})$  is the antisymmetric part of  $C$ . Taking the determinant of this, the eigenfrequencies  $\omega$  are given by the roots of

$$\det(F - i\omega \hat{C} + \omega^2 M) = 0, \quad (3.4)$$

The system is (linearly) stable if all the eigenfrequencies are real.

For most problems

$$L = T - V = \frac{1}{2} \sum_{ij} a_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j - V(\mathbf{q}), \quad (3.5)$$

with some symmetric functions  $a_{ij}(\mathbf{q}) = a_{ji}(\mathbf{q})$  such that the kinetic energy is a positive definite quadratic form in the velocities. For typical cases it turns out that  $M_{ij} = a_{ij}(\mathbf{q}_0)$ ,  $F_{ij} = -\partial^2 V/\partial q_i \partial q_j$  and  $C_{ij} = 0$ . An exception is when there are velocity-dependent forces (e.g., motion in a rotating frame or in an electromagnetic field). We simply ignore such problems in the following and assume from now on that  $C_{ij} = 0$ .

It is easy to see that each of the eigenfrequencies  $\omega$  is either purely real or purely imaginary. Substituting  $h_i = Q_i \exp(i\omega t)$  into (3.2), multiplying by  $Q_k^*$ , summing over  $k$  and rearranging gives

$$\omega^2 = - \sum_{ki} F_{ki} Q_k^* Q_i / \sum_{ki} M_{ki} Q_k^* Q_i. \quad (3.6)$$

Each of the sums is real, because

$$\left( \sum_{ki} M_{ki} Q_k^* Q_i \right)^* = \sum_{ki} M_{ki} Q_k Q_i^* = \sum_{ki} M_{ik} Q_k Q_i^* = \sum_{ki} M_{ki} Q_k^* Q_i, \quad (3.7)$$

by the symmetry of  $M_{ki}$  (and  $F_{ki}$ ), and swapping labels  $(i, k)$  in the last step. Since the kinetic energy  $T = \frac{1}{2} M_{ij} \dot{h}_i \dot{h}_j$  is a positive definite quadratic form, it follows that all  $\omega^2 > 0$  (and therefore the system is stable) if  $\mathbf{q}_0$  is a local minimum of  $V$ .

**Normal co-ordinates** If the eigenfrequencies  $\omega_\alpha$  obtained by solving (3.4) are distinct, then the corresponding eigenvectors  $\mathbf{Q}_\alpha$  are orthogonal in the sense that

$$\mathbf{Q}_\beta^T M \mathbf{Q}_\alpha = 0, \quad \text{if } \omega_\alpha \neq \omega_\beta. \quad (3.8)$$

This follows on multiplying the  $(C = 0)$  eigenvalue equation (3.4)

$$(F + \omega_\alpha^2 M) \mathbf{Q}_\alpha = 0 \quad (3.9)$$

by another eigenvector  $\mathbf{Q}_\beta$  and then using the symmetry of  $F$  and  $M$  to show that  $(\omega_\beta - \omega_\alpha) \mathbf{Q}_\alpha^T M \mathbf{Q}_\beta = 0$ . The importance of this is that *any* small oscillation  $\mathbf{h}(t)$  that satisfies the linearized equation of motion (3.2) can be decomposed into a sum of normal modes,

$$\mathbf{h}(t) = \sum_\alpha a_\alpha \mathbf{Q}_\alpha \cos(\omega_\alpha t + \phi_\alpha), \quad (3.10)$$

where the amplitudes  $a_\beta$  and phases  $\phi_\beta$  can be found by premultiplying (3.10) by  $\mathbf{Q}_\beta^T M$  to obtain

$$\mathbf{Q}_\beta^T M \mathbf{h}(t) = a_\beta \cos(\omega_\beta t + \phi_\beta) \quad (3.11)$$

(assuming the  $\mathbf{Q}_i$  are normalized such that  $\mathbf{Q}_i^T M \mathbf{Q}_i = \delta_{i\beta}$ ). Thus for each  $\beta$ ,  $\mathbf{Q}_\beta^T M \mathbf{h}(t)$  is a combination of the original co-ordinates that oscillates sinusoidally at angular frequency  $\omega_\beta$ , regardless of how the system was set into motion. A combination of the co-ordinates that inevitably oscillates sinusoidally is called a **normal co-ordinate**.

**Exercise:** In terms of generalized co-ordinates  $(\theta_1, \theta_2)$ , a double pendulum has Lagrangian

$$L = ml^2 \dot{\theta}_1^2 + \frac{1}{2} ml^2 \dot{\theta}_2^2 + ml^2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 + 2mgl \cos \theta_1 + mgl \cos \theta_2. \quad (3.12)$$

Expanding about the equilibrium  $\theta_1 = \theta_2 = 0$  to second order, show that this may be written

$$L \simeq ml^2 \dot{\theta}_1^2 + \frac{1}{2} ml^2 \dot{\theta}_2^2 + ml^2 \dot{\theta}_1 \dot{\theta}_2 - mgl \theta_1^2 - \frac{1}{2} mgl \theta_2^2, \quad (3.13)$$

and that the EL equations in matrix form are

$$\ddot{\boldsymbol{\theta}} = -\frac{g}{l} \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \boldsymbol{\theta}, \quad (3.14)$$

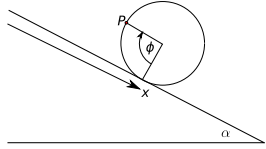
where  $\boldsymbol{\theta} = (\theta_1, \theta_2)^T$ . Find the normal modes. [Ans: eigenmodes  $(1, \sqrt{2})^T e^{i\omega_- t}$  and  $(1, -\sqrt{2})^T e^{i\omega_+ t}$ , with eigenfrequencies  $\omega_\pm^2 = (g/l)(2 \pm \sqrt{2})$ .]

## Attic\*

## A Rigid bodies\*

## A.1 Constraints\*

Sometimes it is convenient to have some redundancy among the co-ordinates. For example, a circular hoop of mass  $m$  rolls without slipping down a rigid wire inclined at an angle  $\alpha$  to the horizontal. The obvious co-ordinates to describe the system are the distance  $x$  of the hoop from its starting point and the angle  $\phi$  between the hoop's point of contact with the wire and a reference point  $P$  on its rim. But  $\dot{x} = R\dot{\phi}$  because the hoop rolls without slipping, so that



$$x = R\phi + \text{constant}, \quad (\text{A.1})$$

the constant depending on the initial conditions. So,  $x$  and  $\phi$  are not independent co-ordinates.

A **holonomic constraint** is a relation

$$g(\mathbf{q}; t) = 0, \quad (\text{A.2})$$

where  $g$  is a function of the  $n$  co-ordinates  $(q_1, \dots, q_n)$  and possibly time  $t$ . Equation (A.1) is an example of such a constraint. Following §0.2, the procedure for finding the equations of motion of a system with  $k$  independent holonomic constraints is to introduce a new generalized co-ordinate  $\lambda_i$  for each constraint  $g_i(\mathbf{q}; t) = 0$ . Treating these  $n + k$  co-ordinates as independent, consider motion in the  $(n + k)$ -dimensional augmented configuration space with Lagrangian

$$L'(\{\mathbf{q}, \boldsymbol{\lambda}\}, \dot{\mathbf{q}}, t) \equiv L(\mathbf{q}, \dot{\mathbf{q}}, t) + \sum_{i=1}^k \lambda_i g_i(\mathbf{q}, t). \quad (\text{A.3})$$

Writing down the  $\mathbf{q}$  and  $\lambda_i$  components of the EL equation for  $L'$ , the equations of motion are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} = \sum_{i=1}^k \lambda_i \frac{\partial g_i}{\partial \mathbf{q}}, \quad (\text{A.4})$$

$$g_1 = \dots = g_k = 0.$$

Each constraint  $g_i$  results in an additional generalized force on the RHS of the EL equation, the size of which is controlled by the co-ordinate  $\lambda_i$ .

**Exercise:** Consider a system with co-ordinates  $(r, \theta)$ , Lagrangian  $L = \frac{1}{2}mr^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr \cos \theta$  and constraint  $r - l = 0$ . Show that the constraint force has magnitude  $|mr\dot{\theta}^2 + mg \cos \theta|$ . More generally, show that if the co-ordinate  $q_1$  is held fixed when one solves the EL equations, then the magnitude of the corresponding (generalized) constraint force is given by  $|\partial L / \partial q_1|$ .

Returning to our example of a hoop on a wire, the hoop's kinetic energy  $T$  can be broken down into the energy due to translational motion of its centre of mass,  $\frac{1}{2}m\dot{x}^2$ , and the rotational energy about the centre of mass,  $\frac{1}{2}mR^2\dot{\phi}^2$ . The potential energy  $V = -mgx \sin \alpha$ . There is one constraint,

$$g(x, \phi) = x - R\phi = 0. \quad (\text{A.5})$$

The augmented Lagrangian

$$L' = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mR^2\dot{\phi}^2 + mgx \sin \alpha + \lambda(x - R\phi), \quad (\text{A.6})$$

for which the EL equations are

$$\begin{aligned} \frac{d}{dt} m\dot{x} - mg \sin \alpha &= \lambda, \\ \frac{d}{dt} mR^2\dot{\phi} &= -\lambda R, \\ \dot{x} - R\dot{\phi} &= 0. \end{aligned} \quad (\text{A.7})$$

Using the third equation to eliminate  $\dot{\phi}$  from the second, we have that  $\frac{d}{dt} m\dot{x} = -\lambda$ . Substituting this into the first equation and eliminating  $\lambda$ , gives

$$\ddot{x} = \frac{1}{2}g \sin \alpha. \quad (\text{A.8})$$

So the hoop rolls down the plane with only half the acceleration it would have in the frictionless case.

Not all constraints can be written as  $g(x, t) = 0$ . An example of a system with such a **non-holonomic constraint** is a hoop rolling without slipping down a plane instead of along a wire. Natural co-ordinates to use are the location  $(x, y)$  of the hoop's centre, the orientation  $\phi$  of a reference point on the rim of the hoop and another angle  $\theta$  giving the orientation of the hoop in the plane. The no-slip conditions mean that the hoop's velocity satisfies

$$\dot{x} = R\dot{\phi} \sin \theta, \quad \dot{y} = R\dot{\phi} \cos \theta, \quad (\text{A.9})$$

but these cannot be integrated to give an expression of the form  $g(x, y, \theta, \phi, t) = 0$ . To see this, think of rolling the hoop on closed circuits of different lengths around the plane, returning to the starting position  $(x, y)$  with the same  $\theta$ . The angle  $\phi$  at the end depends on the length of the circuit.

## A.2 Lagrangian mechanics of rigid bodies\*

A rigid body is a system of particles having masses  $m_i$  and positions  $\mathbf{x}_i$  satisfying constraints of the form  $|\mathbf{x}_i - \mathbf{x}_j| = r_{ij}$  for all pairs  $(i, j)$  of particles, where each  $r_{ij}$  is a constant.

**Exercise:** A rigid body moves in an external gravitational potential  $\Phi(\mathbf{x})$ . Show that extremizing the action integral (1.3) with Lagrangian  $L = T - V$ , where  $T = \frac{1}{2} \sum_i m_i \dot{\mathbf{x}}_i^2$  and  $V = \sum_i m_i \Phi(\mathbf{x}_i)$ , subject to the constraints that  $(\mathbf{x}_i - \mathbf{x}_j)^2 = r_{ij}^2$ , leads to the usual Newtonian equations of motion for a rigid body:

$$\frac{d}{dt} m_k \dot{\mathbf{x}}_k = -m_k \frac{\partial \Phi}{\partial \mathbf{x}_k} + \sum_{j \neq k} (\text{constraint forces}), \quad (\text{A.10})$$

where the constraint forces are of the form  $\lambda_{kj}(\mathbf{x}_k - \mathbf{x}_j)$  with  $\lambda_{kj} = \lambda_{jk}$ .

**Exercise:** Use (A.10) to show that if there are no external forces acting on the body, then its linear momentum  $\sum_i m_i \dot{\mathbf{x}}_i$  and angular momentum  $\sum_i \mathbf{x}_i \times m_i \dot{\mathbf{x}}_i$  are conserved. See §1.6 later for a more elegant way of obtaining this result.

The configuration space of a rigid body is six dimensional. In case this is not obvious, pick any three non-collinear points  $\mathbf{x}_1, \mathbf{x}_2$  and  $\mathbf{x}_3$  in the body. We need three numbers to specify  $\mathbf{x}_1$ , another two for  $\mathbf{x}_2$  (we already know  $r_{21}$ ) and a final one to fix  $\mathbf{x}_3$ . The positions  $\mathbf{x}_i$  of all the other points in the body are then completely determined by the constraints  $|\mathbf{x}_i - \mathbf{x}_1| = r_{i1}$ ,  $|\mathbf{x}_i - \mathbf{x}_2| = r_{i2}$  and  $|\mathbf{x}_i - \mathbf{x}_3| = r_{i3}$ , once we've chosen whether  $\mathbf{x}_4$  lies "above" or "below" the plane defined by  $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ .

Let us set up a co-ordinate system that moves with the body, its origin at  $\mathbf{x}_1$ , its first basis vector  $\hat{\mathbf{r}}_1 = (\mathbf{x}_2 - \mathbf{x}_1)/|\mathbf{x}_2 - \mathbf{x}_1|$ , the second,  $\hat{\mathbf{r}}_2$ , orthogonal to  $\hat{\mathbf{r}}_1$  but lying in the  $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$  plane and the third given by  $\hat{\mathbf{r}}_3 = \hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2$ . In this frame, the co-ordinates  $\mathbf{r}_i$  of particles in the body do not change with time,  $\dot{\mathbf{r}}_i = 0$ . In the inertial  $\mathbf{x}$  frame, the particles' co-ordinates

$$\mathbf{x}_i = \mathbf{R} + B\mathbf{r}_i, \quad (\text{A.11})$$

where  $\mathbf{R} = \mathbf{x}_1$  and the rotation matrix  $B$  is set by the orientation of  $(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3)$ . We use this  $\mathbf{R}$  and a set of three angles, known as Euler angles, that describe  $B$  as our six generalized co-ordinates for the body. The



particles' velocities

$$\begin{aligned}\dot{\mathbf{x}}_i &= \dot{\mathbf{R}} + \boldsymbol{\omega} \times (\mathbf{x}_i - \mathbf{R}) \\ &= \dot{\mathbf{R}} + B(\boldsymbol{\Omega} \times \mathbf{r}_i),\end{aligned}\quad (\text{A.12})$$

the first equality following from the definition (1.52) of the ( $\mathbf{x}$ -frame) angular velocity  $\boldsymbol{\omega}$ , the second from the definition of the ( $\mathbf{r}$ -frame) angular velocity  $\boldsymbol{\Omega} \equiv B^{-1}\boldsymbol{\omega}$  together with (A.11).

**Angular momentum of a rigid body rotating about a fixed point** Before defining the Euler angles, let us investigate the case of a rigid body rotating about a fixed point  $\mathbf{x} = \mathbf{r} = 0$ , so that  $\mathbf{R} = 0$ . Using the relations (A.11) and (A.12) above, the angular momentum in the  $\mathbf{x}$  frame

$$\mathbf{j} \equiv \sum_i \mathbf{x}_i \times m_i \dot{\mathbf{x}}_i = \sum_i B \mathbf{r}_i \times m_i B(\boldsymbol{\Omega} \times \mathbf{r}_i) = B \left[ \sum_i \mathbf{r}_i \times m_i (\boldsymbol{\Omega} \times \mathbf{r}_i) \right] = B \mathbf{J}, \quad (\text{A.13})$$

where the angular momentum vector in the  $\mathbf{r}$ -frame

$$\begin{aligned}\mathbf{J} &\equiv \sum_i \mathbf{r}_i \times m_i (\boldsymbol{\Omega} \times \mathbf{r}_i) \\ &\rightarrow \int \mathbf{r} \times \rho (\boldsymbol{\Omega} \times \mathbf{r}) d^3r = \int \rho [r^2 \boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \mathbf{r}) \mathbf{r}] d^3r,\end{aligned}\quad (\text{A.14})$$

and we have moved to the continuum limit to avoid a rash of indices in the following. In tensor notation this becomes

$$J_i = \sum_{j=1}^3 \mathcal{I}_{ij} \Omega_j, \quad \text{with } \mathcal{I}_{ij} \equiv \int d^3r \rho (r^2 \delta_{ij} - r_i r_j), \quad (\text{A.15})$$

where the Kronecker delta symbol  $\delta_{ij} = 1$  if  $i = j$  and is zero otherwise. Writing out the **inertia tensor**  $\mathcal{I}_{ij}$  explicitly,

$$\mathcal{I} = \int d^3r \rho(\mathbf{r}) \begin{pmatrix} Y^2 + Z^2 & -XY & -XZ \\ -XY & X^2 + Z^2 & -YZ \\ -XZ & -YZ & X^2 + Y^2 \end{pmatrix}, \quad (\text{A.16})$$

where  $\mathbf{r} = (X, Y, Z)$ . Since it is a real symmetric matrix, it has real eigenvalues  $I_i$  and eigenvectors  $\hat{\mathbf{b}}_i$ . If we orient our co-moving axes  $\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3$  so that  $\hat{\mathbf{r}}_i = \hat{\mathbf{b}}_i$  then  $\mathcal{I}$  becomes  $\text{diag}(I_1, I_2, I_3)$  and the angular momentum  $J_i = I_i \Omega_i$ . The  $\hat{\mathbf{b}}_i$  are known as the body's **principal** or **body axes** and the  $I_i$  its **principal moments of inertia**.

The following table shows  $\mathcal{I}_{ij}$  for some simple mass distributions, assuming that each has total mass  $M$  and that the mass is distributed uniformly. Unless stated otherwise,  $\mathcal{I}_{ij}$  is measured about the centre of mass.

	$\mathcal{I}_{ij}$
Rod length $a$ (about centre)	$\frac{1}{12} M a^2 \text{diag}(1, 1, 0)$
Rod length $a$ (about end)	$\frac{1}{3} M a^2 \text{diag}(1, 1, 0)$
Ring radius $a$	$\frac{1}{2} M a^2 \text{diag}(1, 1, 2)$
Disc radius $a$	$\frac{1}{4} M a^2 \text{diag}(1, 1, 2)$
Spherical shell radius $a$	$\frac{2}{3} M a^2 \text{diag}(1, 1, 1)$
Sphere radius $a$	$\frac{2}{5} M a^2 \text{diag}(1, 1, 1)$

**Exercise:** Verify these!

Note that the vectors  $\mathbf{x}$ ,  $\boldsymbol{\omega}$  and  $\mathbf{j}$  live in the  $\mathbf{x}$  frame, while  $\mathbf{r}$ ,  $\boldsymbol{\Omega}$  and  $\mathbf{J}$  live in the  $\mathbf{r}$  frame. Vectors in different frames can meet only through the intercession of the operator  $B$ . From (1.46), if we have a vector  $\mathbf{V}$  that lives in the  $\mathbf{r}$  frame, then the rate of change of the corresponding  $\mathbf{x}$ -frame vector,  $\mathbf{v} \equiv B\mathbf{V}$ , is given by

$$\frac{d\mathbf{v}}{dt} = B(\dot{\mathbf{V}} + \boldsymbol{\Omega} \times \mathbf{V}). \quad (\text{A.17})$$

We can immediately apply this to  $\mathbf{j} = B\mathbf{J}$  in the case of a *free* rigid body rotating about a fixed point (e.g., centre of mass of a freely falling body). Since there are no external torques  $\mathbf{j}$  is conserved. So,

$$\frac{d\mathbf{j}}{dt} = B(\dot{\mathbf{J}} + \boldsymbol{\Omega} \times \mathbf{J}) = 0 \quad \Rightarrow \quad \dot{\mathbf{J}} + \boldsymbol{\Omega} \times \mathbf{J} = 0, \quad (\text{A.18})$$

which is known as **Euler's equation**. In the principal-axis frame  $J_i = I_i \Omega_i$  and Euler's equation becomes

$$\begin{aligned}I_1 \frac{d\Omega_1}{dt} &= (I_2 - I_3) \Omega_2 \Omega_3, \\ I_2 \frac{d\Omega_2}{dt} &= (I_3 - I_1) \Omega_3 \Omega_1, \\ I_3 \frac{d\Omega_3}{dt} &= (I_1 - I_2) \Omega_1 \Omega_2.\end{aligned}\quad (\text{A.19})$$

The motion of rigid bodies is interesting because the angular momentum  $\mathbf{J}$  is not proportional to the angular velocity  $\boldsymbol{\Omega}$  (unless  $I_1 = I_2 = I_3$ ).

**Exercise:** (The "tennis racquet theorem") A rigid body rotates freely about its third principal axis, with  $\mathbf{J} = (0, 0, I_3 \Omega_3)$ . It is given a small perturbation, so that  $\Omega_1$  and  $\Omega_2$  are non-zero, but small. By substituting trial solutions of the form  $\Omega_1 = a_1 e^{kt}$  and  $\Omega_2 = a_2 e^{kt}$  into (A.19) and neglecting second-order terms such as  $\Omega_1 \Omega_2$ , show that the motion is stable ( $k^2 < 0$ ) if either  $I_3 > \max(I_1, I_2)$  or  $I_3 < \min(I_1, I_2)$ . Thus rotation about either the short or the long axis of a free rigid body is stable, but rotation about the intermediate axis is unstable.

**Exercise:** Consider a **free symmetric top** with  $I_1 = I_2 \neq I_3$ . Show that  $\Omega_3$  is a constant of motion and that the angular velocity  $\boldsymbol{\Omega}$  **precesses** around the  $\hat{\mathbf{r}}_3$  axis with frequency

$$\Omega_p = \frac{I_1 - I_3}{I_1} \Omega_3. \quad (\text{A.20})$$

Notice that the precession is retrograde ( $\Omega_p < 0$ ) if  $I_3 > I_1$  (i.e., if the body is oblate).

**Kinetic energy of a rigid body rotating about a fixed point** Using  $\dot{\mathbf{x}} = B(\boldsymbol{\Omega} \times \mathbf{r})$ , the kinetic energy  $T = \frac{1}{2} \sum_i m_i \dot{\mathbf{x}}_i^2$  becomes

$$\begin{aligned}T &= \frac{1}{2} \int \rho [B(\boldsymbol{\Omega} \times \mathbf{r})]^T \cdot [B(\boldsymbol{\Omega} \times \mathbf{r})] d^3r = \frac{1}{2} \int \rho (\boldsymbol{\Omega} \times \mathbf{r})^2 d^3r \\ &= \frac{1}{2} \int \rho [\Omega^2 r^2 - (\boldsymbol{\Omega} \cdot \mathbf{r})^2] d^3r.\end{aligned}\quad (\text{A.21})$$

We can rewrite the contents of the square brackets as

$$\Omega^2 r^2 - (\boldsymbol{\Omega} \cdot \mathbf{r})^2 = \sum_{ij} \Omega_i \Omega_j r^2 \delta_{ij} - \sum_i \Omega_i r_i \sum_j \Omega_j r_j = \sum_{ij} \Omega_i \Omega_j (r^2 \delta_{ij} - r_i r_j), \quad (\text{A.22})$$

so that

$$T = \frac{1}{2} \sum_{ij} \Omega_i \mathcal{I}_{ij} \Omega_j = \frac{1}{2} \boldsymbol{\Omega}^T \cdot \mathcal{I} \cdot \boldsymbol{\Omega}, \quad (\text{A.23})$$

where  $\mathcal{I}_{ij}$  are the components of the inertia tensor (A.15).

**Exercise:** Starting from (A.12), show that if we allow translational motion ( $\dot{\mathbf{R}} \neq 0$ ) and we take  $\mathbf{R}$  to be the position of the body's centre of mass, then the kinetic energy of the body can be split into two parts,

$$T = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \sum_{ij} \Omega_i \mathcal{I}_{ij} \Omega_j, \quad (\text{A.24})$$

where  $M$  is the total mass of the body. The first term is the kinetic energy of translational motion of the body's centre of mass, the second the rotational kinetic energy of the body about its centre of mass.

The rest of this section explains how to obtain a Lagrangian for a body moving in a uniform gravitational field about a fixed point (e.g., a top spinning about a point on a table). We first introduce Euler angles. These describe the rotation matrix  $B$  and serve as our generalized co-ordinates. Then we obtain the potential energy and kinetic energy in terms of these angles and the principal moments of inertia of the system.

**Euler angles** The standard method for parametrizing  $B$  is as follows. Start with the  $\hat{r}_1, \hat{r}_2, \hat{r}_3$  axes coincident with the inertial  $\hat{x}_1, \hat{x}_2, \hat{x}_3$  axes. Then apply the following sequence of rotations to the former:

1. Rotate through an angle  $\phi$  about the  $\hat{r}_3 = \hat{x}_3$  axis. Under this rotation  $\hat{r}_3 = \hat{x}_3$  is unchanged and  $\hat{r}_1$  (temporarily) picks out a new direction known as the "line of nodes". Label this direction  $\hat{r}_N$ .
2. Rotate through an angle  $\theta$  about the line of nodes (the temporary  $\hat{r}_1$  axis). The line of nodes remains fixed, and  $\hat{r}_3$  comes to its final position.
3. rotate through an angle  $\psi$  about the  $\hat{r}_3$  axis.

Writing out this sequence of operations explicitly, we have that

$$\begin{pmatrix} \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \end{pmatrix} = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix}, \quad (\text{A.25})$$

or  $\hat{r} = C\hat{x}$ , where  $C$  is the product of the three rotation matrices above. Since  $C$  is orthogonal, we have that  $\hat{x} = C^T\hat{r}$ , or  $\hat{x}_i = C_{ji}\hat{r}_j$ . So, a point  $P$  with co-ordinates  $r = r_i\hat{r}_i$  in the  $\hat{r}$  basis can be written as

$$\mathbf{a} = r_i\hat{r}_i = r_i C_{ij}\hat{x}_j. \quad (\text{A.26})$$

In other words, the point  $P$  has  $\mathbf{x}$  co-ordinates  $\mathbf{x} = B\mathbf{r}$ , where the (co-ordinate) rotation matrix  $B = C^T$ .

**Exercise:** It is perhaps not immediately obvious, so show that *any* rotation can be represented by some choice (sometimes not unique) of Euler angles  $(\phi, \theta, \psi)$ .

**Potential energy of an axisymmetric top** Assume that the top is rotationally symmetric about the  $\hat{r}_3$  axis. Then  $V$  does not change as  $\psi$  and  $\phi$  are varied, because in the definition of Euler angles the  $\psi$  and  $\phi$  rotations take place about this axis. Therefore  $V = mgl \cos \theta$ , where  $(r_1, r_2, r_3) = (0, 0, l)$  is the position of the top's centre of mass in the body frame.

**Kinetic energy in terms of Euler angles** Finally, we need to express the  $\Omega$  appearing in (A.23) in terms of  $(\phi, \theta, \psi)$  and their derivatives. Between times  $t$  and  $t + dt$  the orientation of the rigid body changes from  $(\phi, \theta, \psi)$  to  $(\phi + d\phi, \theta + d\theta, \psi + d\psi)$  and the  $\mathbf{x}$  co-ordinates of a point  $\mathbf{r} = \text{constant}$  on the body vary from  $\mathbf{x}$  to  $\mathbf{x} + d\mathbf{x}$ . Since  $\mathbf{x}(t) = B(t)\mathbf{r}$ , we have that (to first order in  $dt, d\phi$  etc)

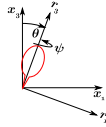
$$d\mathbf{x} = B(\phi + d\phi, \theta + d\theta, \psi + d\psi)B^{-1}(\phi, \theta, \psi)\mathbf{x} - \mathbf{x} = \boldsymbol{\omega} dt \times \mathbf{x}, \quad (\text{A.27})$$

the last equality following from the definition (1.52) of angular velocity. Taking the changes in each angle separately,

$$\begin{aligned} B(\phi + d\phi, \theta, \psi)B^{-1}(\phi, \theta, \psi)\mathbf{x} - \mathbf{x} &= \boldsymbol{\omega}_\phi dt \times \mathbf{x} \\ B(\phi, \theta + d\theta, \psi)B^{-1}(\phi, \theta, \psi)\mathbf{x} - \mathbf{x} &= \boldsymbol{\omega}_\theta dt \times \mathbf{x} \\ B(\phi, \theta, \psi + d\psi)B^{-1}(\phi, \theta, \psi)\mathbf{x} - \mathbf{x} &= \boldsymbol{\omega}_\psi dt \times \mathbf{x}, \end{aligned} \quad (\text{A.28})$$

and the total angular velocity  $\boldsymbol{\omega} = \boldsymbol{\omega}_\phi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\psi$  (again, to first order). Our job is to express  $\boldsymbol{\omega}_\phi, \boldsymbol{\omega}_\theta$  and  $\boldsymbol{\omega}_\psi$  in terms of either the  $\hat{x}$  or the  $\hat{r}$  basis. Let us do this first for the special case  $\phi = \psi = 0$ . The first of equations (A.28) is a rotation of  $d\phi$  about the  $\hat{x}_3$  axis, giving

$$\boldsymbol{\omega}_\phi = \dot{\phi}\hat{x}_3 = \dot{\phi}\sin\theta\hat{r}_2 + \dot{\phi}\cos\theta\hat{r}_3, \quad (\text{A.29})$$



because, solving (A.25) for  $\mathbf{x} = (0, 0, 1)^T$ ,  $\hat{x}_3 = \hat{r}_2 \sin \theta + \hat{r}_3 \cos \theta$  when  $\phi = \psi = 0$ . The second is a rotation of  $d\theta$  about the line of nodes  $\hat{r}_N (= \hat{r}_1$  when  $\psi = \phi = 0)$ . Therefore

$$\boldsymbol{\omega}_\theta = \dot{\theta}\hat{r}_1. \quad (\text{A.30})$$

The third is a rotation of  $d\psi$  about  $\hat{r}_3$ , so

$$\boldsymbol{\omega}_\psi = \dot{\psi}\hat{r}_3. \quad (\text{A.31})$$

Gathering these last three equations together,  $\boldsymbol{\omega} = \boldsymbol{\omega}_\phi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\psi = B(\Omega_1, \Omega_2, \Omega_3)^T$  with

$$\Omega_1 = \dot{\theta}, \quad \Omega_2 = \dot{\phi}\sin\theta, \quad \Omega_3 = \dot{\psi} + \dot{\phi}\cos\theta. \quad (\text{A.32})$$

The expression (A.23) for the kinetic energy of our axisymmetric ( $I_1 = I_2$ ) top becomes

$$T = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2. \quad (\text{A.33})$$

This is assuming that  $\phi = \psi = 0$ . But for an axisymmetric top the kinetic energy cannot depend on these cyclic co-ordinates and we can always choose the origin of reference for  $\phi$  and  $\psi$  to have  $\psi = \phi = 0$ . Thus the expression holds for all  $\psi, \phi$  and so the Lagrangian for an axisymmetric top moving about a fixed point in a uniform field is

$$L = T - V = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mgl\cos\theta. \quad (\text{A.34})$$

**Exercise:** (optional detour) It is also possible to obtain  $\Omega$  in terms of  $(\phi, \theta, \psi)$  without assuming that  $\phi = \psi = 0$ . Show that for general  $(\phi, \theta, \psi)$ ,

$$\begin{aligned} \boldsymbol{\Omega} &= \dot{\phi}\hat{x}_3 + \dot{\theta}\hat{r}_N + \dot{\psi}\hat{r}_3 \\ &= [\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi]\hat{r}_1 + [\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi]\hat{r}_2 + [\dot{\psi} + \dot{\phi}\cos\theta]\hat{r}_3, \end{aligned} \quad (\text{A.35})$$

using (A.25) to obtain  $\hat{x}_3$  and  $\hat{r}_N$  in terms of the  $\hat{r}_i$ . Substitute this into the expression (A.23) for the kinetic energy and show that it reduces to (A.33) when  $I_1 = I_2$ .

**Exercise: Free symmetric top revisited** We have already seen how the angular velocity  $\Omega$  of a free symmetric top precesses about  $\hat{r}_3$  at a rate  $\Omega_p = \Omega_3(I_1 - I_3)/I_1$  (A.20). Now let us look at it from an inertial frame. By substituting  $\psi = \Omega_p t$  into (A.35) or otherwise explain how one can identify  $\dot{\psi}$  with the precession rate  $\Omega_p$ . Use this together with  $\Omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$  from (A.35) to show that  $\dot{\phi} = I_3\Omega_3/I_1\cos\theta$  and hence that the wobble rate  $\dot{\phi}$  of a uniform disc ( $I_1 = I_2 = \frac{1}{2}I_3$ ) satisfies  $\dot{\phi} \simeq -2\dot{\psi}$  when  $\Omega_1, \Omega_2 \ll \Omega_3$ .

**Pinned axisymmetric top in uniform gravitational field** The generalized momenta

$$\begin{aligned} p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}I_1\sin^2\theta + \dot{\phi}I_3\cos^2\theta + \dot{\psi}I_3\cos\theta, \\ p_\psi &= \frac{\partial L}{\partial \dot{\psi}} = \dot{\psi}I_3 + \dot{\phi}I_3\cos\theta \end{aligned} \quad (\text{A.36})$$

in (A.34) are clearly constants of motion. So too is the energy  $E = T + V$ , which can be written as

$$E = \frac{1}{2}I_1\dot{\theta}^2 + \left[ \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3} + mgl\cos\theta \right]. \quad (\text{A.37})$$

This comes from using (A.36) to eliminate

$$\dot{\psi} = \frac{p_\psi}{I_3} - \dot{\phi}\cos\theta, \quad \dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}, \quad (\text{A.38})$$

from the expression (A.33) for  $T$  in favour of the constants  $p_\psi, p_\phi$ . Therefore the problem reduces to motion in the one-dimensional potential  $V_{\text{eff}}(\theta)$  given by the contents of the square bracket in (A.37).

Substituting  $u = \cos \theta$ , equation (A.37) becomes

$$E = \frac{1}{2} \frac{I_1 \dot{u}^2}{1-u^2} + \frac{(p_\phi - p_\psi u)^2}{2I_1(1-u^2)} + \frac{p_\psi^2}{2I_3} + mgl u, \quad (\text{A.39})$$

or, rearranging,

$$\dot{u}^2 = (\alpha - \beta u)(1-u^2) - (a - bu)^2 \equiv f(u) \quad (\text{A.40})$$

with constants

$$a = \frac{p_\phi}{I_1}, \quad b = \frac{p_\psi}{I_1}, \quad \alpha = \frac{1}{I_1} \left( 2E - \frac{p_\psi^2}{I_3} \right), \quad \beta = \frac{2mgl}{I_1} > 0. \quad (\text{A.41})$$

$f(u)$  is a cubic with  $f(u) \rightarrow \infty$  as  $u \rightarrow \infty$ . Since  $u = \cos \theta$ ,  $u$  must lie between -1 and 1. Looking at (A.41) we see that  $f(\pm 1) < 0$  unless  $a = b$ . Therefore  $f(u)$  has two roots  $u_1, u_2$  in the interval  $-1 < u < 1$ , between which  $\dot{u}^2 = f(u) > 0$ . This means that the inclination  $\theta$  **nutates** (nods) between two values  $\theta_1 = \cos^{-1} u_1$  and  $\theta_2 = \cos^{-1} u_2$ . Meanwhile the azimuthal angle  $\phi$  **precesses** at a rate

$$\dot{\phi} = \frac{a - bu}{1 - u^2}. \quad (\text{A.42})$$

If  $u = b/a$  lies between  $u_1$  and  $u_2$ ,  $\dot{\phi}$  changes sign meaning that the curve traced by the top on the  $(\theta, \phi)$  sphere has loops – see lectures!