Measuring Quantum Entanglement

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Max Born Lecture
University of Göttingen, December 2012
Quantum Entanglement is one of the most fascinating and counter-intuitive aspects of Quantum Mechanics.

Its existence was first recognised in early work of the pioneers of quantum mechanics\(^1\).

It is the basis of the celebrated Einstein-Podolsky-Rosen paper\(^2\) which argued that its predictions are incompatible with locality.

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\(^3\)Letter to Max Born, 4 December 1926
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Bell showed that EPR’s explanation, involving hidden variables, is inconsistent with the predictions of quantum mechanics – this was subsequently tested experimentally.

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3 Letter to Max Born, 4 December 1926

in this talk I am going to assume that conventional quantum mechanics (and the Copenhagen interpretation) holds and will address the questions:

- what is quantum entanglement and is there a universal measure of the amount of entanglement?
- how does this behave for systems with many degrees of freedom?
- how might it be measured experimentally?
A simple example

- a system consisting of two qubits (spin-$\frac{1}{2}$ particles) with a basis of states

$$ (|\uparrow\rangle_A, |\downarrow\rangle_A) \times (|\uparrow\rangle_B, |\downarrow\rangle_B) $$

- Alice observes qubit $A$, Bob observes qubit $B$

- the state

$$ |\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A |\uparrow\rangle_B + |\downarrow\rangle_A |\downarrow\rangle_B) $$

is entangled: before Alice measures $\sigma^z_A$, Bob can obtain either result $\sigma^z_B = \pm 1$, but after she makes the measurement the state in $B$ collapses and Bob can only get one result

- moreover this still holds if the subsystems $A$ and $B$ are far apart (the EPR paradox)
an entangled state is different from a classically correlated state, e.g. with density matrix

\[ \rho = \frac{1}{2} |\uparrow\rangle_A |\uparrow\rangle_B A \langle\uparrow|_B \langle\uparrow| + \frac{1}{2} |\downarrow\rangle_A |\downarrow\rangle_B A \langle\downarrow|_B \langle\downarrow| \]

in both cases

\[ \langle \sigma^z_A \sigma^z_B \rangle = 1 \]

but for the entangled state

\[ \langle \sigma^x_A \sigma^x_B \rangle = 1 \]

while it vanishes for the classically correlated state
Entanglement of pure states

- Is there a good way of characterising the degree of entanglement (of pure states)?
- **Schmidt decomposition theorem:** any state in $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written

$$|\Psi\rangle = \sum_j c_j |\psi_j\rangle_A |\psi_j\rangle_B$$  \hspace{1cm} (S)

where the states are orthonormal, $c_j > 0$, and $\sum_j c_j^2 = 1$

- The $c_j^2$ are the eigenvalues of the reduced density matrix

$$\rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi|$$

- If there is only one term in (S), $|\Psi\rangle$ is unentangled
- If all the $c_j$ are equal, $|\Psi\rangle$ is maximally entangled
a suitable measure is the entanglement entropy

\[ S_A = - \sum_j c_j^2 \log c_j^2 = -\text{Tr}_A \rho_A \log \rho_A = S_B \]

- it is zero for unentangled states and maximal when all the \( c_j \) are equal
- it is convex: \( S_{A_1 \cup A_2} \leq S_{A_1} + S_{A_2} \)
- it is basis-independent
it increases under Local Operations and Classical Communication
even for many-body systems it is often computable (analytically, or numerically by density matrix renormalization group methods or matrix product states)

but it is not the only such measure: eg the Rényi entropies

\[ S_A^{(n)} \propto - \log \text{Tr}_A \rho_A^n \]

are equally useful, and for different \( n \) give information about the whole entanglement spectrum of \( \rho_A \)
Entanglement Entropy in Extended Systems

Consider a system whose degrees of freedom are extended in space, e.g., a quantum magnet described by the Heisenberg model with Hamiltonian

\[ H = \sum_{r,r'} J(r - r') \vec{\sigma}(r) \cdot \vec{\sigma}(r') \]

The temperature is low enough so the system is in the ground state \( |0\rangle \) of \( H \)
suppose $A$ is a large but finite region of space: what is the degree of entanglement of the spins within $A$ with the reminder in $B$?

since $S_A = S_B$ it can’t be $\propto$ the volume of $A$ or $B$

in fact in almost all cases we have the area law:

$$S_A \sim C \times \text{Area of boundary}$$

where $D$ is the dimensionality of space. The constant $C$ is $\propto 1/\text{(lattice spacing)}^{D-1}$ and is non-universal in general.

entanglement occurs only near the boundary
One dimension

- When $D = 1$ something interesting happens: the constant is proportional to a logarithm

$$S_A \sim C \log(\text{correlation length } \xi)$$

- Now the constant $C$ is dimensionless and universal
- At a quantum critical point $\xi$ diverges and so does the entanglement entropy
Measuring Entanglement

- measuring entropy of many-body systems is conceptually difficult: even at finite temperature we do it by integrating the specific heat
- however the situation is better for the Rényi entropies

$$S_A^{(n)} \propto - \log \text{Tr} \rho_A^n = - \log \sum_j c_j^{2n}$$

- to simplify the discussion assume $n = 2$ and consider two independent identical copies of the whole system, so the composite system is in the state

$$|\Psi\rangle_1|\Psi\rangle_2 = \sum_{j_1} \sum_{j_2} c_{j_1} c_{j_2} |\psi_{j_1}\rangle_{A1}|\psi_{j_1}\rangle_{B1}|\psi_{j_2}\rangle_{A2}|\psi_{j_2}\rangle_{B2}$$
let $\mathcal{S}$ be a ‘swap’ operator which interchanges the states in $A_1$ with those in $A_2$ but leaves states in $B_1$ and $B_2$ the same:

$$\mathcal{S} |\psi_{j_1}\rangle_{A_1}|\psi_{j_1}\rangle_{B_1}|\psi_{j_2}\rangle_{A_2}|\psi_{j_2}\rangle_{B_2} = |\psi_{j_2}\rangle_{A_1}|\psi_{j_1}\rangle_{B_1}|\psi_{j_1}\rangle_{A_2}|\psi_{j_2}\rangle_{B_2}$$

then

$$\left(1 \langle \Psi |_2 \langle \Psi | \right) \mathcal{S} \left(|\Psi\rangle_1 |\Psi\rangle_2\right) = \sum_j c_j^4 = \text{Tr} \rho_A^2$$
on a system with local interactions, $\mathcal{S}$ can be implemented locally as a quantum switch: eg a 1D quantum spin chain:

\[
\begin{array}{c}
\text{initially the two decoupled chains have a hamiltonian } H = H_1 + H_2 \text{ with a ground state } |0\rangle = |\Psi\rangle_1 |\Psi\rangle_2 \\
\text{after the switch, the hamiltonian is } H' = \mathcal{S} H \mathcal{S}^{-1}, \text{ with a ground state } |0\rangle' = \mathcal{S} |0\rangle \text{ with the same energy} \\
\text{we need to measure the overlap} \\
M = \langle 0 | 0 \rangle' = \text{Tr} \rho_A^2
\end{array}
\]
two proposals for how to do this:

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- prepare the system in ground state $|0\rangle$ of $H$
- flip the switch so the new hamiltonian is $H'$
- the system finds itself in a higher energy state than $|0\rangle'$ and decays to this eg by emission of quasiparticles
- decay rate $\propto |M|^2$

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- introduce tunnelling between $|0\rangle$ and $|0\rangle'$, equivalent to adding a term $\propto \mathcal{S}$ to the hamiltonian
- the tunnelling amplitude is $\propto M$
- this can be detected by preparing in one state and observing Rabi oscillations

\[ 5 \text{JC, Phys. Rev. Lett. 106, 150404, 2011} \]
\[ 6 \text{Abanin DA and Demler E, Phys. Rev. Lett. 109, 020504, 2012} \]
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although the theory tells us a lot *die Alten* still have many secrets!