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The RG in statistical mechanics and in particle theory

These notes describe the relation between the RG as discussed in the lectures and the book and its formulation in field theory. Some familiarity with the ideas of renormalisation in field theory is assumed.

We have argued that, in high enough dimensionality d, the Ising model is in the same universality class as a continuous spin model with reduced Hamiltonian

$$\mathcal{H} = -\sum_{r,r'} J(r-r')S(r)S(r') + g_0 \sum_r (S(r)^2 - 1)^2 \tag{1}$$

In particle theory this would be identified as the euclidean action for a lattice field theory, in $d-1$ space dimensions and imaginary time. It goes over, in the naive continuum limit, to

$$\mathcal{H} = \int (\tfrac{1}{2}(\nabla S)^2 + \tfrac{1}{2}m_0^2 S^2 + \tfrac{1}{4}\lambda_0 S^4) d^d r \tag{2}$$

where $m_0^2 \propto t a^2$ and $\lambda_0 \propto g_0 a^{4-d}$. However, we have to remember that the lattice spacing a is also hidden in the UV cut-off $|k| < \Lambda \sim a^{-1}$.

In the language of field theory, the parameters m_0^2 and λ_0 are called the bare parameters, and $S(r)$ is called the bare field. It is important to understand that in statistical mechanics and condensed matter applications, these are the *physical* quantities. We might expect the action (2) to describe their behaviour on large distances scales, but not at short distances, where we have butchered the theory too badly to be able to see the lattice structure.

When $\lambda = 0$ the phase transition happens at $m_0^2 = 0$. When we turn on the coupling this is no longer true. Instead it happens when the renormalized mass m_R^2 (which may be defined, for example, as the position $q^2 = -m_R^2$ of the nearest pole to the origin in the Fourier transform $G_0(q^2)$ of the 2-point correlation function) vanishes. This happens at some value $m_{0c}^2 = \Lambda^2 f(g_0)$ where f is not universal (e.g. it depends on the precise way the cut-off is enforced.)

For simplicity we now assume that m_0^2 is tuned so that $m_R^2 = 0$. The theory then depends only on λ_0 and Λ (equivalently the dimensionless g_0 and Λ.) We implement the RG by changing $a \to a(1+\delta\ell)$ (equivalently $\Lambda \to \Lambda(1-\delta\ell)$), and 'keep the long-distance physics the same' by suitably compensating by changing g_0. This gives the RG equation $dg_0/d\ell = -\beta_0(g_0)$ where

$$\beta_0(g_0) \equiv \Lambda \frac{\partial}{\partial\Lambda} g_0 \bigg|_{\text{long-distance physics}} \tag{3}$$

The actual form of the beta-function will depend on what we use to specify the 'long-distance physics.' In the continuum real-space perturbative RG discussed in the lectures and in the book, as well as the Wilson RG, this means the form of the partition function. However, it is not at all clear how much information we really need to specify the long-distance physics - if we choose too little information, or the wrong information, we'll get nonsense, while if we try to use too much, we might get inconsistencies.

These questions are fortunately answered by renormalisation theory, as practiced by particle theorists. In particle theory, there is no lattice. In fact we want to be able to define the theory in the limit when the cut-off $\Lambda \to \infty$.[1]

The **statement of renormalisability** of this theory near four dimensions states (in the massless case) that:

- we can define renormalised fields $S_R = Z^{1/2}S$, and the renormalised coupling λ_R as the value of the one-particle irreducible vertex function $\Gamma_R^{(4)}$ (which is derived from the four-point function $\langle S_R S_R S_R S_R\rangle$ by some elementary steps) at a fixed but arbitrary momentum scale $q_j \sim \mu$, so that

- *all* the correlation functions $\langle S_R(r_1)\ldots S_R(r_N)\rangle$, when expressed in terms of λ_R, have a finite limit as $\Lambda \to \infty$.

For particle physics, this is what we need: it says that if we measure one number λ_R, all other measurable quantities are in principle calculable. For condensed matter, it tells us exactly what it means to 'keep the physics the same': it means holding λ_R constant.

Notice, however, that we had to introduce the arbitrary scale μ in order to define the renormalised theory. In the massless theory we can't choose $\mu = 0$ because of IR divergences. However, we can use this arbitrariness to obtain RG equations in the following way. The Fourier transform $G_0(q)$ of the 2-point function $\langle SS\rangle$ is related to its renormalised version by

$$G_R(q, g_R, \mu) = Z(\mu/\Lambda, g_R)G_0(q, g_0, \Lambda) \tag{4}$$

where we have been careful to exhibit the dependences: G_R being a quantity in the renormalised theory, depends on μ and also λ_R. It is useful to trade this in for dependence on μ and the dimensionless renormalised coupling $g_R = \lambda_R\mu^{4-d}$. G depends on the bare quantities, and Z, being dimensionless, can depend only on μ/Λ and g_R (or equivalently g_0).

From this we get the usual version of the Callan-Symanzik equation found in particle physics books: we observe that G_0 does know anything about μ, so we can write

$$\mu\frac{\partial}{\partial\mu}Z(\mu/\Lambda, g_R)^{-1}G_R(q, g_R, \mu)\bigg|_{\lambda_0} = 0 \tag{5}$$

which expands to

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_R(g_R)\frac{\partial}{\partial g} - \gamma(g_R)\right)G_R(q, g_R, \mu) = 0 \tag{6}$$

In particular, we see the appearance of the beta-function

$$\beta_R(g_R) = \mu\frac{\partial}{\partial\mu}g_R\bigg|_{\lambda_0} \tag{7}$$

[1] In the old days this was taken literally. Nowadays we accept that there will always be some cut-off at sufficiently short distances where the theory breaks down: that is, all field theories are only effective ones over a certain range of distances or energies. But if we want such theories to be much use their predictions should be insensitive to the actual value of Λ, which amounts to saying that the limit should make sense.

and $\gamma_R(g_R) = \mu(\partial/\partial\mu)\ln Z|_{\lambda_0}$.
The C-S equation by itself has no physics: it needs to combined with naive dimensional analysis, which tells us that

$$\left(q\frac{\partial}{\partial q} + \mu\frac{\partial}{\partial \mu} + 2\right) G_R(q, g_R, \mu) = 0 \tag{8}$$

Subtracting the two, we relate the dependence on q to the dependence on g_R: probing the UV behaviour $q \to \infty$ is equivalent to taking g_R towards a UV stable zero of $\beta_R(g_R)$, with negative slope, while the IR behaviour is given by the IR stable zero.
However, notice that we can also derive another equation, by observing that G_R knows nothing about Λ (at least in the large Λ limit.) This gives

$$\Lambda\frac{\partial}{\partial\Lambda} Z(\mu/\Lambda, g_0)G(q, g_0, \Lambda)\Big|_{\lambda_R} = 0 \tag{9}$$

which expands to

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta_0(g_0)\frac{\partial}{\partial g_0} - \gamma_0(g_0)\right) G(q, g_0, \Lambda) = 0 \tag{10}$$

This is much more in the spirit of the RG we discussed earlier: we keep the 'physics' (i.e. λ_R) the same, and see how we need to vary g_0 as Λ is changed. The beta-function is now

$$\beta_0(g_0) = \Lambda\frac{\partial}{\partial\Lambda} g_0\Big|_{\lambda_R} \tag{11}$$

while $\gamma_0(g_0) = -\Lambda(\partial/\partial\Lambda)\ln Z|_{\lambda_R}$.
(β_R, γ_R) and (β_0, γ_0) are in general different functions (although to one loop they are the same). Nevertheless they are related, and one can show that if β_R has a zero at g_R^*, then β_0 also has a zero at some g_0^*, and that $\beta_R'(g_R^*) = \beta_0'(g_0^*)$ and $\gamma_R(g_R^*) = \gamma_0(g_0^*)$, that is, the values of the critical exponents are the same in both schemes.
In statistical mechanics we are usually interested only in the IR stable zeros of the beta function. The UV behaviour of the field theory does not describe the short-distance behaviour of the lattice.
You may also have seen other renormalisation schemes in field theory: minimal substraction, and renormalisation in the massive theory at $q = 0$. While these are calculationally less cumbersome, especially to higher loops, than using a UV cut-off Λ, they can obscure the physics.