1 Loops and renormalization

Tree-level processes are fully specified by the Lagrangian. Loop corrections to tree-level processes arise at higher orders of perturbation theory. For instance, the graph in Fig. 1 is an example of a loop correction to fermion pair production.

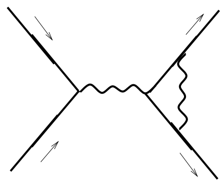

Figure 1: *Loop correction to fermion pair production.*

The part of the theory that deals with loop effects is renormalization. This note gives an introduction to the idea of renormalization and its physical implications.

1.1 Basic ideas and examples

In this section we introduce general principles of renormalization and describe two specific examples, the renormalization of the electric charge and the electron's anomalous magnetic moment. In the next section we extend the discussion by introducing the renormalization group.

1.1.1 General principles

While the Lagrangian specifies interaction processes at tree level, the method of renormalization is required to treat processes including loops.

A symptom that renormalization is required is that Feynman graphs with loops may give rise to integrals containing divergences from high-momentum regions. Renormalization allows one to give meaning to the occurrence of these ultraviolet divergences.

Ultraviolet power counting provides the basic approach to renormalization. For a Feynman graph involving a loop integral of the form

$$\int d^4k \, \frac{N(k)}{M(k)} \quad , \tag{1}$$

consider the superficial degree of divergence defined as

$$D = (powers\ \ of\ \ k\ \ in\ \ N\ +\ 4)\quad -\quad (powers\ \ of\ \ k\ \ in\ \ M)\ \ . \tag{2}$$

If $D \geq 0$, the integral is ultraviolet divergent. A first way of characterizing a theory as "renormalizable" is that the number of ultraviolet divergent amplitudes is finite. This is the case for instance with QED. There are 3 ultraviolet divergent amplitudes in QED, depicted in Fig. 2. QCD has a few more, due to the more complex structure of the strong interactions, but still a finite number. In a renormalizable theory there can of course be infinitely many Feynman graphs that are ultraviolet divergent, but they are so because they contain one of the few primitively divergent amplitudes as a subgraph.

(a)

(b)

(c)

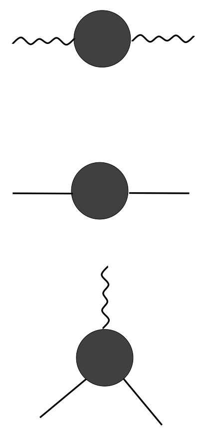

Figure 2: *Ultraviolet divergent amplitudes in QED: (a) photon self-energy; (b) electron self-energy; (a) electron-photon vertex.*

The main point about renormalizability is that it implies that all the ultraviolet divergences can be absorbed, according to a well-prescribed procedure specified below, into rescalings of the parameters and wave functions in the theory. For a given quantity ϕ, the rescaling is of the form

$$\phi \to \phi_0 = Z\ \phi\ \ , \tag{3}$$

where ϕ_0 and ϕ are respectively the unrenormalized and renormalized quantities, and Z is a calculable constant, into which the divergence can be absorbed. Here Z is the renormalization constant, possibly divergent but unobservable. Once the rescalings are done and the predictions of the theory are expressed in terms of renormalized quantities, all physical observables are finite and free of divergences.

This leads to a characterization of the renormalization program which we can formulate as a sequence of steps as follows.

- Compute the divergent amplitudes, by prescribing a "regularization method". Examples of regularization methods are a cut-off Λ on the ultraviolet integration region, where the result diverges as we let $\Lambda \to \infty$, or, as we will see in explicit calculations later, dimensional regularization.
- Assign parameter and wave-function rescalings to eliminate divergences. In the case of QED, these involve the electromagnetic potential A, the electron wave function ψ and mass m, and the coupling e. Using traditional notation for the QED renormalization constants Z_i, the rescalings can be written as

$$\begin{aligned} A \to A_0 &= \sqrt{Z_3}\, A \;, \\ \psi \to \psi_0 &= \sqrt{Z_2}\, \psi \;, \\ m \to m_0 &= \frac{Z_m}{Z_2}\, m \;, \\ e \to e_0 &= \frac{Z_1}{Z_2 \sqrt{Z_3}}\, e \;. \end{aligned} \tag{4}$$

  Here Z_3 and Z_2 are the respectively the renormalization constants for the photon and electron wave function, Z_1 is the vertex renormalization constant and Z_m is the electron mass renormalization constant.
- Once the rescalings are done, all physical observables are calculable, i.e., unambiguously defined in terms of renormalized quantities, and free of divergences.

Theories for which this program succeeds in giving finite predictions for physical quantities are renormalizable theories. Non-renormalizable theories are theories in which one cannot absorb all divergences in a finite number of Z: for instance, as we go to higher orders new divergences appear and an infinite number of Z is needed.

The above program, while it appears quite abstract at first, gives in fact testable, measurable effects. In the next few subsections we see specific examples of this.

A further, general point is that gauge invariance places strong constraints on renormalization, implying relations among the divergent amplitudes of the theory, and thus among the renormalization constants. Here is an example for the case of QED. Gauge invariance establishes the following relation between the electron-photon vertex Γ_μ dotted into the photon momentum q^μ and the electron propagators S,

$$q^\mu \Gamma_\mu = S^{-1}(p+q) - S^{-1}(p) \;. \tag{5}$$

Eq. (5), pictured in Fig. 3, is referred to as the Ward identity and is valid to all orders. Using the renormalization constants Z_1 and Z_2 defined by the rescalings in Eq. (4),

$$\Gamma^\mu = \frac{1}{Z_1}\, \gamma^\mu + \dots \;, \qquad S(p) = \frac{Z_2}{\not p - m} + \dots \;, \tag{6}$$

we have

$$\frac{1}{Z_1}\, \not{q} = \frac{1}{Z_2}\, [(\not{p} + \not{q} - m) - (\not{p} - m)] \quad . \tag{7}$$

Thus in the abelian case

$$Z_1 = Z_2 \qquad \text{(QED)} \quad . \tag{8}$$

As a result, the rescaling relation in Eq. (4) defining the renormalized coupling in QED becomes

$$e^2 = Z_3 e_0^2 \quad . \tag{9}$$

That is, the renormalization of the electric charge is entirely determined by the renormalization constant Z_3, associated with the photon wave function, and does not depend on any other quantity related to the electron.

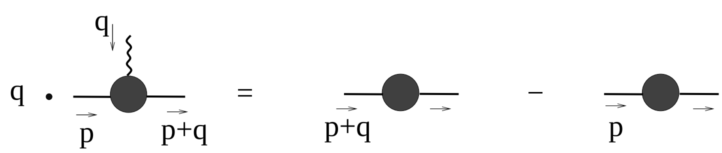

Figure 3: *Relation between electron-photon vertex and electron propagators.*

In the non-abelian case the relation (8) does not apply. However, it is still valid that non-abelian gauge invariance sets constraints on renormalization, leading to other, more complex relations among the renormalization constants. We will see examples of this in Sec. 1.2.3.

In the rest of this section we describe specific calculations of renormalization at one loop.

1.1.2 The gauge boson self-energy

Let us consider the gauge boson self-energy. This is one of the divergent amplitudes shown in Fig. 2. The Feynman graphs contributing to the self-energy through one loop are given in Fig. 4 for the photon and gluon cases. In the photon case one has the fermion loop graph only, while in the gluon case one has in addition gluon loop and ghost loop graphs.

Because of the relations (8),(9), in the QED case the calculation of the gauge boson self-energy is all that is needed to determine the renormalization of the coupling. So the result of this subsection will be used in Subsec. 1.1.3 to discuss the renormalized electric charge.

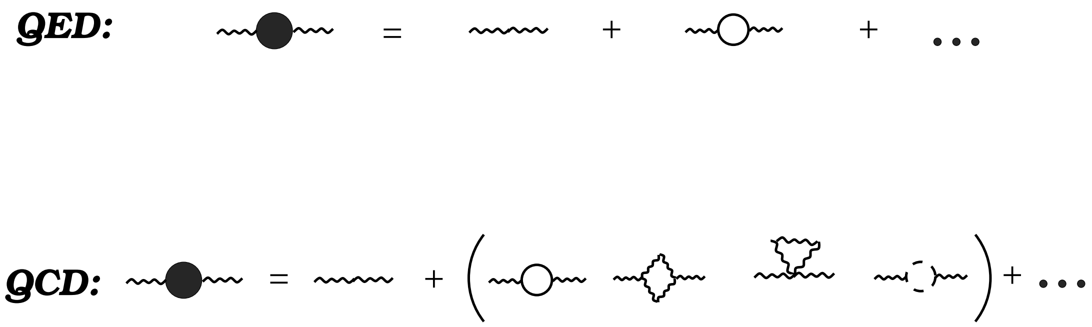

Figure 4: *(top) Photon and (bottom) gluon self-energy through one loop.*

We now compute the fermion loop graph in Fig. 5. As shown in Fig. 4, in the QED case the fermion loop is all that contributes to the self-energy, while in the QCD case this gives one of the required contributions.

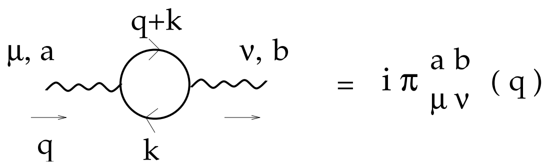

Figure 5: *Fermion loop contribution to the gauge boson self-energy.*

The graph in Fig. 5 is given by

$$i\pi^{ab}_{\mu\nu}(q) = -g^2 \ \mathrm{Tr}(T^a T^b) \int \frac{d^4k}{(2\pi)^4} \ \frac{\mathrm{Tr}[\gamma_\mu(\not{k} + \not{q} + m)\gamma_\nu(\not{k} + m)]}{(k^2 - m^2 + i0^+)((k+q)^2 - m^2 + i0^+)} \quad . \tag{10}$$

This expression is written in general for the non-abelian case. In this case the color-charge factor equals

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2} \ \delta^{ab} \quad . \tag{11}$$

The QED case is obtained from Eq. (10) by taking

$$\begin{aligned} g^2 \longrightarrow e^2 \quad &= \quad 4\pi\alpha \ , \\ \mathrm{Tr}(T^a T^b) \quad &\longrightarrow \quad 1 \ . \end{aligned} \tag{12}$$

The integral in Eq. (10) is ultraviolet divergent. By superficial power counting in the loop momentum k, the divergence is quadratic. Gauge invariance however requires that $\pi_{\mu\nu}$ be proportional to the transverse projector $g_{\mu\nu}q^2 - q_\mu q_\nu$, that is,

$$\pi_{\mu\nu} = \left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) \Pi(q^2) \quad . \tag{13}$$

This reduces the degree of divergence by two powers of momentum. As a result, the divergence in Eq. (10) is not quadratic but logarithmic.

We need a regularization method to calculate the integral (10) and parameterize the divergence. We take the method of dimensional regularization. This consists of continuing the integral from 4 to $d = 4 - 2\varepsilon$ dimensions by introducing the dimensionful mass-scale parameter μ so that

$$g^2 \ \frac{d^4k}{(2\pi)^4} \longrightarrow g^2(\mu^2)^\varepsilon \ \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \quad . \tag{14}$$

In dimensional regularization a logarithmic divergence d^4k/k^4 appears as a pole at $\varepsilon = 0$ (i.e., $d = 4$). We thus identify ultraviolet divergences in the integral (10) by identifying poles in $1/\varepsilon$.

By carrying out the calculation in dimensional regularization, the result for $\pi_{\mu\nu}$ is

$$\begin{aligned} \pi^{ab}_{\mu\nu}(q) &= -\left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) \mathrm{Tr}(T^aT^b) \ \frac{g^2}{4\pi^2} \ \Gamma(\varepsilon) \int_0^1 dx \ \left(\frac{4\pi\mu^2}{m^2 - x(1-x)q^2}\right)^\varepsilon \ 2x(1-x) \\ &\equiv \left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) \Pi(q^2) \quad . \end{aligned} \tag{15}$$

We can interpret the different factors in this result. As mentioned above, the first factor on the right hand side, consistent with the gauge-invariance requirement (13), implies that the gauge boson self-energy is purely transverse,

$$\left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) q^\mu = \left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) q^\nu = 0 \quad . \tag{16}$$

Owing to the transversality of the self-energy, loop corrections do not give mass to gauge bosons in QED and QCD. The factor $\mathrm{Tr}(T^aT^b)$ in Eq. (15) is the non-abelian charge factor, which just reduces to 1 in the QED case according to Eq. (12). Next, $g^2/(4\pi^2)$ is the coupling factor, which becomes $e^2/(4\pi^2) = \alpha/\pi$ in the QED case (12). The Euler gamma function $\Gamma(\varepsilon)$ contains the logarithmic divergence, i.e., the pole at $\varepsilon = 0$ ($d = 4$) in dimensional regularization:

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - C_E + \mathcal{O}(\varepsilon) \qquad , \qquad C_E \simeq .5772 \quad . \tag{17}$$

The first factor in the integrand of Eq. (15) results from the regularization method, depending on the ratio between the dimensional-regularization scale μ^2 and a linear combination of the physical mass scales m^2 and q^2. The last factor in the integrand, $2x(1-x)$, depends on the details of the calculated Feynman graph.

We can extract the ultraviolet divergent part of the self-energy by computing the integral in Eq. (15) at $q^2 = 0$. Higher q^2 powers in the expansion of $\Pi(q^2)$ give finite contributions. We have

$$\begin{aligned}\Pi(0) &= -\mathrm{Tr}(T^a T^b)\, \frac{g^2}{4\pi^2}\, \Gamma(\varepsilon) \int_0^1 dx\, \left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon} 2x(1-x) \\ &\simeq -\mathrm{Tr}(T^a T^b)\, \frac{g^2}{4\pi}\, \frac{1}{3\pi}\, \frac{1}{\varepsilon} + \dots \quad , \end{aligned} \tag{18}$$

where in the last line we have used the expansion (17) of the gamma function and computed the integral in dx. Specializing to the QED case according to Eq. (12) gives

$$\Pi(0) \simeq -\frac{\alpha}{3\pi}\, \frac{1}{\varepsilon} + \dots \qquad \text{(QED)} \quad . \tag{19}$$

We will next use the results in Eqs. (15),(19) to discuss the renormalization of the electromagnetic coupling.

1.1.3 Renormalization of the electromagnetic coupling

Suppose we consider a physical process occurring via photon exchange, and ask what the effect is of the renormalization on the photon propagator. Fig. 6 illustrates this effect by multiple insertions of the photon self-energy,

$$D_0 \to D = D_0 + D_0 \pi D_0 + D_0 \pi D_0 \pi D_0 + \dots \quad , \tag{20}$$

where D_0 is the photon propagator and π is the photon self-energy computed in Eq. (15). We can sum the series in Eq. (20) by applying repeatedly the transverse projector in π and using that longitudinal contributions vanish by gauge invariance, and we get

$$D_0 \to D = D_0\, \frac{1}{1 + \Pi(q^2)} \quad . \tag{21}$$

Then the effect of renormalization in the photon exchange process amounts to

$$\frac{e_0^2}{q^2} \longrightarrow \frac{e_0^2}{q^2}\, \frac{1}{1 - \Pi(q^2)} \quad , \tag{22}$$

where q is the photon momentum.

Let us now rewrite the denominator on the right hand side of Eq. (22) by separating the divergent part and the finite part in Π. According to the discussion around Eq. (18), this can be achieved by

$$1 - \Pi(q^2) = [1 - \Pi(0)] \left[1 - \left(\Pi(q^2) - \Pi(0)\right)\right] + \mathcal{O}(\alpha^2) \quad . \tag{23}$$

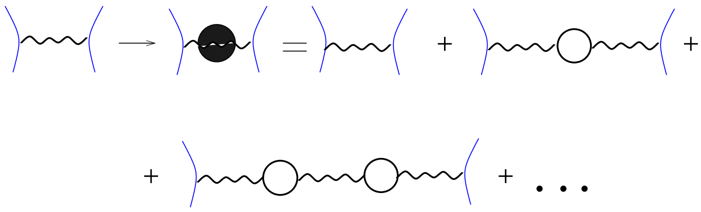

Figure 6: *Effect of renormalization in a photon exchange process.*

Therefore Eq. (22) gives

$$\begin{aligned} \frac{e_0^2}{q^2} &\longrightarrow \frac{e_0^2}{q^2}\frac{1}{1-\Pi(q^2)} \\ &\simeq \frac{1}{q^2}\underbrace{\frac{e_0^2}{1-\Pi(0)}}_{e^2\equiv Z_3 e_0^2}\ \underbrace{\frac{1}{1-[\Pi(q^2)-\Pi(0)]}}_{q^2-\text{dependence}} \quad . \end{aligned} \tag{24}$$

In the last line of Eq. (24) we have underlined two distinct effects in the result we obtain from renormalization. The first is that the strength of the coupling is modified to

$$\frac{e_0^2}{1-\Pi(0)} \equiv e^2 \quad , \tag{25}$$

from which, by comparison with Eq. (9), we identify the renormalization constant Z_3:

$$\begin{aligned} Z_3 &\simeq 1+\Pi(0) \\ &= 1-\frac{\alpha}{3\pi}\frac{1}{\varepsilon}+\ldots \quad , \end{aligned} \tag{26}$$

where in the last line we have used the explicit result for $\Pi(0)$ in Eq. (19). The coupling e in Eq. (25) is the physical coupling, that is, the renormalized coupling. This is obtained from the unrenormalized one, e_0, via a divergent, but unobservable, rescaling, according to the general procedure outlined below Eq. (3).

The second effect in Eq. (24) is that the coupling acquires a dependence on the momentum transfer q^2, controlled by the finite part of the self-energy, $\Pi(q^2)-\Pi(0)$. This dependence is free of divergences and observable. The q^2-dependence of the electromagnetic coupling is a new physical effect due to loop corrections. Using the explicit expression for Π in Eq. (15), we obtain that for low q^2

$$\Pi(q^2)-\Pi(0) \to 0 \qquad \text{for} \quad q^2\to 0 \quad , \tag{27}$$

and for high q^2

$$\Pi(q^2) - \Pi(0) \simeq \frac{\alpha}{3\pi} \ln \frac{q^2}{m^2} \qquad \text{for} \quad q^2 \gg m^2 \ . \tag{28}$$

Thus e^2 in Eq. (25) is the value of the coupling at $q^2 = 0$; the coupling increases as q^2 increases. Substituting Eqs. (25),(28) into Eq. (24) and rewriting it in terms of the fine structure, we have for large momenta

$$\alpha(q^2) = \frac{\alpha}{1 - (\alpha/(3\pi)) \ln(q^2/m^2)} \ . \tag{29}$$

The q^2-dependence of the coupling is referred to as running coupling. We will discuss this topic further in Sec. 1.2.

The result for the electromagnetic coupling that we have just found can be viewed as summing a series of perturbative large logarithms for $q^2 \gg m^2$. By expanding Eq. (29) in powers of α, we have

$$\begin{aligned} \alpha(q^2) &= \frac{\alpha}{1 - (\alpha/(3\pi)) \ln(q^2/m^2)} \\ &= \alpha \left(1 + \frac{\alpha}{3\pi} \ln \frac{q^2}{m^2} + \ldots + \frac{\alpha^n}{(3\pi)^n} \ln^n \frac{q^2}{m^2} + \ldots \right) \ . \end{aligned} \tag{30}$$

This is the simplest example of a conceptual framework referred to as resummation in QED and QCD. The point is that if the result (24) for the physical process is expressed in terms of an expansion in powers of α, as in Eq. (30), perturbative coefficients to higher orders are affected by large logarithmic corrections. On the other hand, one obtains a well-behaved perturbation series, without large higher-order coefficients, if the result is expressed in terms of the effective charge $\alpha(q^2)$.

1.1.4 Vertex correction and anomalous magnetic moment

In this section we study the one-loop vertex correction of Fig. 7. In particular we compute its contribution to the electron's magnetic moment,

$$\boldsymbol{\mu} = g \, \frac{e}{2m} \, \boldsymbol{S} \ , \tag{31}$$

where $\boldsymbol{S}$ is the spin operator and g is the gyromagnetic ratio. This computation gives

$$g = g_{\rm Dirac} + \frac{\alpha}{\pi} + \mathcal{O}(\alpha^2) \ , \quad g_{\rm Dirac} = 2 \ , \tag{32}$$

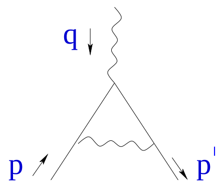

Figure 7: *One-loop vertex correction in QED.*

where $g_{\rm Dirac} = 2$ is the prediction from the Dirac equation and α/π is the correction from the graph in Fig. 7. Higher order corrections arise from multi-loop graphs. The deviations from the Dirac value are referred to as the electron's anomalous magnetic moment.

Let us consider first the Dirac equation coupled to electromagnetism,

$$(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi = 0 \quad , \tag{33}$$

and write the magnetic interaction term explicitly. We can recast Eq. (33) in two-component notation, including the electromagnetic coupling, as

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \boldsymbol{\sigma} \cdot (\mathbf{p} - e\boldsymbol{A}) \\ \boldsymbol{\sigma} \cdot (\mathbf{p} - e\boldsymbol{A}) & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} \quad . \tag{34}$$

Now substitute the bottom equation in (34) into the top equation, use the Pauli σ matrix relation

$$\boldsymbol{\sigma} \cdot \mathbf{a}\, \boldsymbol{\sigma} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + i\, \boldsymbol{\sigma} \cdot \mathbf{a} \wedge \mathbf{b} \quad , \tag{35}$$

and take the nonrelativistic limit $E \simeq m$, in which $\phi \ll \chi$. We then obtain that the action of the hamiltonian on the spinor ψ can be written as

$$H\psi \simeq \left((\mathbf{p} - e\boldsymbol{A})^2 - \frac{e}{2m}\, \mathbf{B} \cdot 2\boldsymbol{S} \right) \psi \quad , \tag{36}$$

where $\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A}$ is the magnetic field and $\boldsymbol{S}$ is the spin operator given in terms of the σ matrices,

$$\boldsymbol{S} = \frac{1}{2}\, \boldsymbol{\Sigma} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \quad . \tag{37}$$

We recognize that the second term in the right hand side of Eq. (36) is the magnetic interaction

$$-\boldsymbol{\mu} \cdot \mathbf{B} \quad , \quad \text{with } \boldsymbol{\mu} = \frac{e}{2m}\, 2\boldsymbol{S} \quad . \tag{38}$$

That is, the Dirac equation prediction for the gyromagnetic ratio g in Eq. (31) is

$$g_{\rm Dirac} = 2 \quad . \tag{39}$$

Let us consider now the vertex function $\Gamma^\nu(p, p')$ represented at one loop in Fig. 7. We can determine the general structure of the vertex function based on relativistic invariance and gauge invariance. Because $\Gamma^\nu(p, p')$ transforms like a Lorentz vector, we can write it as a linear combination of γ^ν, p^ν, p'^ν, or equivalently

$$\Gamma^\nu(p, p') = A\ \gamma^\nu + B(p + p')^\nu + C(p - p')^\nu \quad , \tag{40}$$

where A, B and C are scalar functions of q^2 only ($q = p' - p$).

Gauge invariance requires

$$q_\nu \Gamma^\nu = 0 \quad . \tag{41}$$

By dotting q_ν into Eq. (40), the term in B gives zero, and the term in A gives zero once it is sandwiched between $\bar{u}(p')$ and $u(p)$. Thus $C = 0$. We can further show that the following identity holds,

$$\bar{u}(p')\gamma^\nu u(p) = \frac{1}{2m}\ \bar{u}(p')(p + p')^\nu u(p) + \frac{i}{m}\ \bar{u}(p')\ \Sigma^{\nu\rho}\ q_\rho\ u(p) \quad , \tag{42}$$

where $\Sigma^{\nu\rho} \equiv (i/4)\ [\gamma^\nu, \gamma^\rho]$. This implies that the term in $(p + p')^\nu$ in Eq. (40) can be traded for a linear combination of a term in γ^ν and a term in $\Sigma^{\nu\rho} q_\rho$. Therefore the vertex function can be decomposed in general as

$$\Gamma^\nu(p, p') = F_1(q^2)\ \gamma^\nu + \frac{i}{m}\ F_2(q^2)\ \Sigma^{\nu\rho}\ q_\rho \ , \tag{43}$$

where the scalar functions $F_1(q^2)$ and $F_2(q^2)$ are the electron's electric and magnetic form factors. At tree level, $\Gamma^\nu = \gamma^\nu$, thus $F_1 = 1$ and $F_2 = 0$. In general, F_1 and F_2 receive radiative corrections from loop graphs and are related to the electron's charge and magnetic moment,

$$F_1(0) = Q \quad , \tag{44}$$

$$F_2(0) = \frac{g - 2}{2} \quad , \tag{45}$$

where Q is the electron's charge in units of e and g is the electron's magnetic moment in units of $(e/(2m))S$ where S is the electron spin. $F_1(0)$ is 1 to all orders, that is, radiative corrections to F_1 vanish at $q^2 = 0$. We next compute the correction to $F_2(0)$ at one loop.

To this end, consider the one-loop graph in Fig. 7. This is given by

$$\bar{u}(p') ie\Gamma^\nu u(p) = e^3 \int \frac{d^4k}{(2\pi)^4}\ \frac{\bar{u}(p')\ \gamma^\lambda\ (\not{k} + \not{q} + m)\ \gamma^\nu\ (\not{k} + m)\ \gamma_\lambda\ u(p)}{[(k + q)^2 - m^2 + i\varepsilon]\ [k^2 - m^2 + i\varepsilon]\ [(p - k)^2 + i\varepsilon]} \ . \tag{46}$$

The integral in Eq. (46) can be handled starting with the following Feynman's parameterization of the three denominators in the integrand,

$$\begin{aligned}
&\frac{1}{[(k+q)^2 - m^2 + i\varepsilon]\ [k^2 - m^2 + i\varepsilon]\ [(p-k)^2 + i\varepsilon]} \\
&= \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3\ \frac{2\ \delta(x_1+x_2+x_3-1)}{[x_1\ ((k+q)^2 - m^2) + x_2\ (k^2 - m^2) + x_3\ (p-k)^2 + i\varepsilon]^3} \\
&= \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3\ \frac{2\ \delta(x_1+x_2+x_3-1)}{(\widetilde{k}^2 - K + i\varepsilon)^3} \quad ,
\end{aligned} \tag{47}$$

where in the last line we have set $\widetilde{k} = k + x_1 q - x_3 p$, $K = m^2(1-x_3)^2 - q^2 x_1 x_2$.

Next change integration variable $k \to \widetilde{k}$ in Eq. (46), and note that the numerator in the integrand can be rewritten according to

$$\begin{aligned}
&\gamma^\lambda\ (\not{k} + \not{q} + m)\ \gamma^\nu\ (\not{k} + m)\ \gamma_\lambda \qquad\qquad (48) \\
&= \gamma^\nu[\widetilde{k}^2 - 2q^2(1-x_1)(1-x_2) + 2m^2(4x_3 - 1 - x_3^2)] - 4mi\Sigma^{\nu\rho} q_\rho x_3(1-x_3)\ .
\end{aligned}$$

Then Eq. (46) can be recast in the form

$$\begin{aligned}
\bar{u}(p')\Gamma^\nu u(p) &= -ie^2 \bar{u}(p') \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3\ 2\ \delta(x_1+x_2+x_3-1) \\
&\times \int \frac{d^4\widetilde{k}}{(2\pi)^4} \left[\gamma^\nu\ \frac{\widetilde{k}^2 - 2q^2(1-x_1)(1-x_2) - 2m^2(1 - 4x_3 + x_3^2)}{[\widetilde{k}^2 - K]^3}\right. \\
&+ \left. \frac{i}{m}\ \Sigma^{\nu\rho} q_\rho\ \frac{-4m^2 x_3(1-x_3)}{[\widetilde{k}^2 - K]^3}\right] u(p) \quad .
\end{aligned} \tag{49}$$

Comparing Eq. (49) with the general decomposition in Eq. (43), we see that the two terms in the second and third line of Eq. (49) give one-loop integral representations for, respectively, the form factors $F_1(q^2)$ and $F_2(q^2)$. Let us concentrate on the calculation of F_2:

$$\begin{aligned}
F_2(q^2) &= -ie^2 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3\ 2\ \delta(x_1+x_2+x_3-1) \\
&\times \int \frac{d^4\widetilde{k}}{(2\pi)^4}\ \frac{-4m^2 x_3(1-x_3)}{[\widetilde{k}^2 - K]^3} \quad .
\end{aligned} \tag{50}$$

While the integral for F_1 in Eq. (49) has divergences that need regularization, the integral (50) for F_2 is finite. Let us compute the result for $q^2 = 0$.

The integration over the four-momentum $\widetilde{k}$ in Eq. (50) can be done by using the transformation of variables $\widetilde{k}^0 \to -e^{i\pi/2}\widetilde{k}^0$ in the integral over the time component of the momentum. This yields the result

$$\int \frac{d^4\widetilde{k}}{(2\pi)^4} \frac{1}{[\widetilde{k}^2 - K]^3} = -\frac{i}{32\pi^2 K} \quad . \tag{51}$$

Then we have ($e^2 = 4\pi\alpha$)

$$\begin{aligned} F_2(0) &= \frac{\alpha}{\pi} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \; \delta(x_1 + x_2 + x_3 - 1) \; \frac{m^2 x_3 (1 - x_3)}{(1 - x_3)^2 m^2} \\ &= \frac{\alpha}{\pi} \int_0^1 dx_3 \int_0^{1-x_3} dx_2 \; \frac{x_3}{1 - x_3} \\ &= \frac{\alpha}{\pi} \int_0^1 dx_3 (1 - x_3) \; \frac{x_3}{1 - x_3} = \frac{\alpha}{2\pi} \quad . \end{aligned} \tag{52}$$

We thus obtain that the one-loop contribution to the electron's anomalous magnetic moment $g - 2 = 2F_2(0)$ is given by

$$g - 2 = 2F_2(0) = \frac{\alpha}{\pi} \; . \tag{53}$$

1.2 Renormalization group

We next discuss renormalization from the standpoint of the renormalization group. We have seen that renormalization introduces dependence on a renormalization scale μ in loop calculations. As the value of μ is arbitrary, physics must be invariant under changes in this scale. This invariance is expressed in a precise manner by the renormalization group. We will see that by studying the dependence on the renormalization scale μ we gain insight into the asymptotic behavior of the theory at short distances.

1.2.1 Renormalization scale dependence and evolution equations

In this section we illustrate how the relation between renormalized and unrenormalized quantities, applied to a given physical quantity G, can be used to to study the dependence on the renormalization scale μ and to obtain renormalization group evolution equations.

Renormalizability implies that the divergent dependence in the unrenormalized quantity G_0 can be factored out in the renormalization constant Z, provided we re-express renormalized G in terms of the renormalized coupling and renormalization scale μ,

$$G_0(p_i, \alpha_0) = Z G(p_i, \alpha, \mu) \quad . \tag{54}$$

Here p_i is the set of physical momenta on which G depends, α is the renormalized coupling and α_0 is the unrenormalized coupling. Because the left hand side in Eq. (54) does not depend on μ,

$$\frac{d}{d\ln\mu^2}\, G_0 = 0 \quad , \tag{55}$$

we have

$$\frac{d}{d\ln\mu^2}\,(ZG) = 0 \quad \Longrightarrow \quad \frac{\partial G}{\partial\ln\mu^2} + \frac{\partial G}{\partial\alpha}\frac{\partial\alpha}{\partial\ln\mu^2} + \frac{\partial\ln Z}{\partial\ln\mu^2}\, G = 0 \quad . \tag{56}$$

By defining

$$\beta(\alpha) = \frac{\partial\alpha}{\partial\ln\mu^2} \quad , \tag{57}$$

$$\gamma(\alpha) = \frac{\partial\ln Z}{\partial\ln\mu^2} \quad , \tag{58}$$

we can rewrite Eq. (56) as

$$\left[\frac{\partial}{\partial\ln\mu^2} + \beta(\alpha)\,\frac{\partial}{\partial\alpha} + \gamma(\alpha)\right] G(p_i,\alpha,\mu) = 0 \quad , \tag{59}$$

where $\beta(\alpha)$ and $\gamma(\alpha)$ are calculable functions of α.

Suppose we measure G at a physical mass-scale Q. Let us rescale by Q the arguments in G and set

$$G(p_i,\alpha,\mu) = F(x_i,t,\alpha) \quad , \tag{60}$$

where

$$x_i = \frac{p_i}{Q} \quad , \qquad t = \ln\frac{Q^2}{\mu^2} \quad . \tag{61}$$

In this notation Eq. (59) can be written as

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha)\,\frac{\partial}{\partial\alpha} + \gamma(\alpha)\right] F(t,\alpha) = 0 \quad , \tag{62}$$

where from now on we will not write explicitly the dependence on the rescaled physical momenta x_i in F.

Eq. (62) is the renormalization group evolution equation, which we can solve with boundary condition $F(0,\alpha)$ at $t = 0$, i.e., $\mu = Q$. To do this, we first write the solution for the case $\gamma = 0$ and then generalize this solution to any γ.

For $\gamma = 0$ we have

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha)\,\frac{\partial}{\partial\alpha}\right] F(t,\alpha) = 0 \qquad (\gamma = 0) \quad . \tag{63}$$

Now observe that if we construct $\alpha(t)$ such that

$$t = \int_{\alpha}^{\alpha(t)} \frac{d\alpha'}{\beta(\alpha')} \quad , \tag{64}$$

then any F of the form

$$F(t, \alpha) = F(0, \alpha(t)) \tag{65}$$

satisfies the equation and the boundary condition.

Eq. (64) defines $\alpha(t)$ as an implicit function. To verify that Eq. (65) is solution, note first that the boundary condition at $t = 0$ is

$$t = 0 \quad , \quad \alpha(0) = \alpha \quad \Longrightarrow \quad F = F(0, \alpha) \quad . \tag{66}$$

Next evaluate the derivative of Eq. (64) with respect to t,

$$1 = \frac{1}{\beta(\alpha(t))} \frac{\partial \alpha(t)}{\partial t} \quad , \tag{67}$$

and with respect to α,

$$0 = \frac{1}{\beta(\alpha(t))} \frac{\partial \alpha(t)}{\partial \alpha} - \frac{1}{\beta(\alpha)} \quad . \tag{68}$$

Then the differential operator in Eq. (63) applied to $F(0, \alpha(t))$ gives

$$\begin{aligned} & \left[-\frac{\partial}{\partial t} + \beta(\alpha) \frac{\partial}{\partial \alpha} \right] F(0, \alpha(t)) \\ & = - \frac{\partial F}{\partial \alpha(t)} \left[\underbrace{\frac{\partial \alpha(t)}{\partial t}}_{\beta(\alpha(t))} - \beta(\alpha) \underbrace{\frac{\partial \alpha(t)}{\partial \alpha}}_{\beta(\alpha(t))/\beta(\alpha)} \right] = 0 \quad , \end{aligned} \tag{69}$$

where in the last line we have used Eqs. (67),(68).

In the general case $\gamma \neq 0$, the solution to Eq. (62) is obtained from the $\gamma = 0$ answer (65) by multiplication by the exponential of a γ integral, as follows

$$\begin{aligned} F(t, \alpha) &= F(0, \alpha(t)) \exp \left[\int_{\alpha}^{\alpha(t)} d\alpha' \, \frac{\gamma(\alpha')}{\beta(\alpha')} \right] \\ &= F(0, \alpha(t)) \exp \left[\int_{0}^{t} dt' \, \gamma(\alpha(t')) \right] \quad . \end{aligned} \tag{70}$$

In the second line in Eq. (70) we have made the integration variable transformation using Eq. (64). We can verify that Eq. (70) is solution by a method similar to that employed above for the case $\gamma = 0$.

Eq. (70) indicates that once ultraviolet divergences are removed through renormalization, all effects of varying the scale in F from μ to Q can be taken into account by i) replacing α by $\alpha(t)$, and ii) including the t-dependence given by the exponential factor in γ. The latter factor breaks scaling in t, modifying the "engineering" dimensions of F by γ-dependent terms. For this reason γ is referred to as anomalous dimension. By expanding the exponential factor in powers of the coupling, we see that this factor sums terms of the type $(\alpha t)^n$ to all orders in perturbation theory. Eq. (70) thus provides a second example, besides that seen in Eq. (30) for the electric charge, of perturbative resummation of logarithmic corrections to all orders in the coupling, giving rise to an improved perturbation expansion, in which coefficients of higher order are free of large logarithms.

In QCD the e^+e^- annihilation cross section $\sigma(e^+e^- \to \text{hadrons})$ is an example of the $\gamma = 0$ case in Eq. (65), while deep-inelastic scattering structure functions are an example of the $\gamma \neq 0$ case in Eq. (70).

1.2.2 RG interpretation of the photon self-energy

Let us revisit the analysis of the photon self-energy in Sec. 1 from the viewpoint of the renormalization group. The divergent part of the renormalization constant Z_3 computed in Eq. (26) determines the QED β function at one loop.

According to Eq. (57), the variation of the coupling α with the energy scale μ is governed by the β function, calculable as a function of α. In dimensional regularization, from

$$\alpha \left(\mu^2\right)^{\varepsilon} = Z_3 \; \alpha_0 \quad , \tag{71}$$

by using Eq. (26) we have

$$\begin{aligned} \frac{\partial \alpha}{\partial \ln \mu^2} &= -\varepsilon \left(1 - \frac{\alpha}{3\pi}\frac{1}{\varepsilon}\right) \alpha_0 \left(\mu^2\right)^{-\varepsilon} \\ &= \frac{1}{3\pi}\,\alpha^2 \quad . \end{aligned} \tag{72}$$

The leading term of the QED β function at small coupling is given by (Fig. 8),

$$\begin{aligned} \beta(\alpha) &= b\alpha^2 + \mathcal{O}(\alpha^3) \quad , \\ b &= \frac{1}{3\pi} \quad . \end{aligned} \tag{73}$$

Inserting the result (73) into Eq. (57) gives the differential equation

$$\frac{\partial \alpha}{\partial \ln \mu^2} = b\alpha^2 \quad . \tag{74}$$

This can be solved by

$$\frac{d\alpha}{\alpha^2} = b\,\frac{d\mu^2}{\mu^2} \quad \Longrightarrow \quad -\frac{1}{\alpha(q^2)} + \frac{1}{\alpha} = b\ln\frac{q^2}{q_0^2} \quad , \tag{75}$$

which gives

$$\alpha(q^2) = \frac{\alpha}{1 - b\alpha\ln(q^2/q_0^2)} \quad , \qquad b = 1/(3\pi) \quad , \tag{76}$$

that is, the result (29) derived directly in Subsec. 1.1.3.

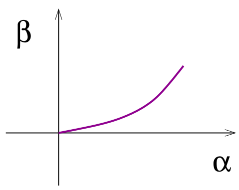

Figure 8: *Small-coupling approximation to the β function in QED.*

To sum up, we have found from the analysis of electric charge renormalization in Sec. 1.1.3 and in this section that as a result of loop graphs the electromagnetic coupling is energy-dependent. We can regard this result as illustrating the breaking of scale invariance as an effect of the quantum corrections taken into account by renormalization. We start at tree level with a coupling that is scale invariant. Then we include loops. This implies introducing an unphysical mass scale, such as the renormalization scale μ, to treat quantum fluctuations at short distances, or high momenta. At the end of the calculation in the renormalized theory, the unphysical mass scale disappears from physical quantities. But an observable, physical effect from including loop corrections remains in the fact that scale invariance is broken. The physical coupling depends on the energy scale at which we probe the interaction. The renormalization group provides the appropriate framework to describe this phenomenon, in which the rescalings (4) of the couplings and wave functions, necessary to compensate variations in the arbitrary renormalization scale, are governed by universal functions, respectively the β and γ functions (57),(58) of the theory.

1.2.3 QCD β function at one loop

We now extend the discussion to the case of renormalization in QCD at one loop, and determine the one-loop β function.

In the QCD case we assign rescaling relations analogous to those in Eq. (4) for the abelian theory. For wave function and mass renormalization we set

$$\begin{aligned} A \to A_0 &= \sqrt{Z_3}\; A \ , \\ \psi \to \psi_0 &= \sqrt{Z_2}\; \psi \ , \\ c \to c_0 &= \sqrt{\widetilde{Z}_3}\; c \ , \\ m \to m_0 &= \frac{Z_m}{Z_2}\; m \ . \end{aligned} \tag{77}$$

where, in addition to the renormalization constants of the abelian case, we introduce $\widetilde{Z}_3$ for ghost renormalization. For renormalization of quark-gluon, ghost-gluon and gluon self-coupling vertices we set

$$\begin{aligned} Z_2\sqrt{Z_3}g_0 &= Z_1\; g \ , \\ \widetilde{Z}_3\sqrt{Z_3}g_0 &= \widetilde{Z}_1\; g \ , \\ Z_3^{3/2}g_0 &= Z_{1,3}\; g \ , \\ Z_3^2 g_0^2 &= Z_{1,4}\; g \ . \end{aligned} \tag{78}$$

Non-abelian gauge invariance requires that the vertices have equal couplings. This implies relations among the different Z in Eq. (78), as follows

$$\frac{\widetilde{Z}_1}{\widetilde{Z}_3} = \frac{Z_1}{Z_2} = \frac{Z_{1,3}}{Z_3} = \sqrt{\frac{Z_{1,4}}{Z_3}} \ . \tag{79}$$

In the non-abelian theory, unlike QED, in general one has $Z_1 \neq Z_2$. The relations in Eq. (79) can be seen as non-abelian generalizations of the QED result $Z_1 = Z_2$ given in Eq. (8).

We can define the renormalized coupling from the quark-gluon vertex. The analogue of Eq. (71) for the QCD case is

$$\alpha_s \left(\mu^2\right)^{\varepsilon} = \frac{Z_2^2}{Z_1^2}\; Z_3\; \alpha_{s0} \ . \tag{80}$$

Each of the renormalization constants Z_i has a perturbation series expansion, with the coefficients of the expansion being ultraviolet divergent. In dimensional regularization the ultraviolet divergences appear as poles at $\varepsilon = 0$, so that the Z_i have the form

$$Z_i = 1 + \alpha_s\, \frac{1}{\varepsilon}\; c_i + \text{ finite } \ , \tag{81}$$

where the coefficients c_i of the divergent terms are to be calculated. By using Eqs. (80) and (81), the β function is given by

$$\begin{aligned} \beta(\alpha_s) &= \frac{\partial \alpha_s}{\partial \ln \mu^2} \\ &= -\varepsilon \alpha_B \left(\mu^2\right)^{-\varepsilon} [1 - 2(Z_1 - 1) + 2(Z_2 - 1) + (Z_3 - 1)] \\ &= 2\alpha_s^2 (c_1 - c_2 - \frac{1}{2}\, c_3) \ . \end{aligned} \tag{82}$$

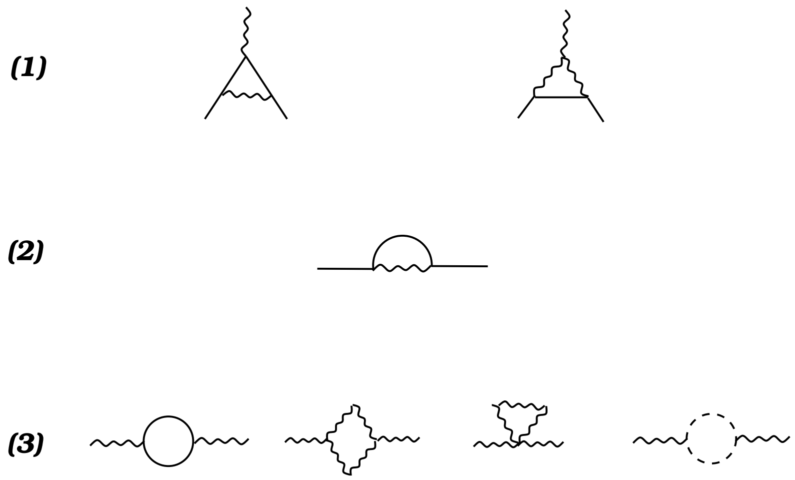

Figure 9: *One-loop corrections to (1) quark-gluon vertex; (2) quark self-energy; (3) gluon self-energy.*

The Feynman graphs contributing to c_1, c_2 and c_3 are the one-loop graphs for, respectively, the quark-gluon vertex renormalization, quark self-energy renormalization, and gluon self-energy renormalization, and they are shown in Fig. 9. The calculation of these graphs proceeds similarly to the calculation done in Sec. 1.1.2 for the fermion loop contribution. By computing these graphs, working in Feynman gauge $\xi = 1$, we obtain the results for the renormalization constants Z_i,

$$Z_1 = 1 - \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} \left(C_F + C_A\right) \ , \tag{83}$$

$$Z_2 = 1 - \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon}\, C_F \ , \tag{84}$$

$$Z_3 = 1 + \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} \left(\frac{5}{3}\, C_A - \frac{4}{3}\, N_f T_F\right) \ , \tag{85}$$

where N_f is the number of quark flavors and the color charge factors are

$$C_A = N = 3 \ , \quad C_F = \frac{N^2-1}{2N} = \frac{4}{3} \ , \quad T_F = \frac{1}{2} \ . \tag{86}$$

Note from the expression for Z_3 that the second term in the bracket in Eq. (85) is the term computed in Sec. 1.1.2 from the fermion loop graph, which, in the abelian limit $N_f T_F \to 1$, gives the QED contribution $-\alpha/(3\pi\varepsilon)$ of Eq. (26).

From Eqs. (83)-(85) we read the coefficients c_i to be put into Eq. (82) to determine the β function. We obtain

$$\begin{aligned} \beta(\alpha_s) &= 2\alpha_s^2(c_1 - c_2 - \frac{1}{2}\, c_3) = 2\, \frac{\alpha_s^2}{4\pi} \left(-C_F - C_A + C_F - \frac{1}{2}\, \frac{5}{3}\, C_A + \frac{1}{2}\, \frac{4}{3}\, N_f T_F \right) \\ &= \frac{\alpha_s^2}{4\pi} \left(-\frac{11}{3}\, C_A + \frac{4}{3}\, N_f T_F \right) = -\frac{\alpha_s^2}{12\pi}\, (11N - 2N_f) \ . \end{aligned} \tag{87}$$

Eq. (87) shows that for $N_f < 11N/2$ the β function in the non-abelian case has negative sign at small coupling (Fig. 10),

$$\beta(\alpha_s) = -\beta_0 \alpha_s^2 + \mathcal{O}(\alpha_s^3) \ , \tag{88}$$

where

$$\beta_0 = \frac{1}{12\pi}(11N - 2N_f) \ . \tag{89}$$

This behavior of the β function is opposite to the behavior of the β function in QED, Eq. (73) (Fig. 8).

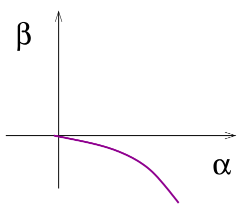

Figure 10: *Small-coupling approximation to the β function in QCD.*

The behavior of the β function in Eqs. (88),(89) implies that QCD is asymptotically free, i.e., weakly coupled at short distances. By inserting Eq. (88) into the renormalization group evolution equation,

$$\frac{\partial \alpha_s}{\partial \ln \mu^2} = \beta(\alpha_s) \simeq -\beta_0 \alpha_s^2 \quad , \tag{90}$$

and solving Eq. (90), we obtain

$$\alpha_s(q^2) = \frac{\alpha_s(\mu^2)}{1 + \beta_0 \ \alpha_s(\mu^2) \ln q^2/\mu^2} \ , \tag{91}$$

where β_0 is given in Eq. (89). Eq. (91) expresses the q^2-dependence of the QCD running coupling at one loop. The QCD coupling decreases logarithmically as the momentum scale q^2 increases. This property is the basis for the perturbative calculability of scattering processes due to strong interactions at large momentum transfers.

1.2.4 The QCD scale Λ

From Eq. (91) we also see that QCD becomes strongly coupled in the infrared, low-momentum region. This behavior is opposite to that in QED. In the QED case, taking $q_0 \sim m$ in Eq. (76), with m the electron mass, we have strong coupling in the ultraviolet region for

$$q^2 \sim m^2 e^{3\pi/\alpha} \ , \tag{92}$$

corresponding to enormously high energies.

In the QCD case, calling Λ the mass scale at which the denominator in Eq. (91) vanishes, we have

$$1 + \beta_0 \ \alpha_s(\mu^2) \ln \frac{\Lambda^2}{\mu^2} = 0 \quad \Longrightarrow \quad \Lambda^2 = \mu^2 e^{-1/(\beta_0 \alpha_s(\mu^2))} \ . \tag{93}$$

In QED and QCD we thus get the different pictures in Fig. 11 for the scale, referred to as the Landau pole, at which the coupling becomes strong.

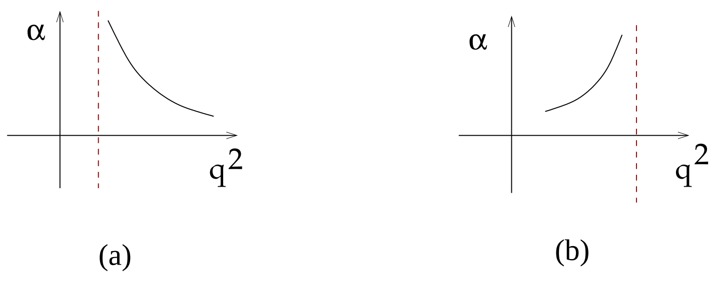

Figure 11: *Landau pole pictures in (a) QCD and (b) QED.*

The scale Λ in Eq. (93) is renormalization-group invariant, i.e., it is independent of μ. Under transformations

$$\begin{aligned} \mu^2 &\longrightarrow \mu'^2 = \mu^2 e^t \ , \\ \alpha_s(\mu^2) &\longrightarrow \alpha_s(\mu'^2) = \frac{\alpha_s(\mu^2)}{1+\beta_0\alpha_s(\mu^2)t} \ , \end{aligned} \tag{94}$$

we have

$$\begin{aligned} \Lambda^2 &\longrightarrow \mu'^2 e^{-1/(\beta_0\alpha_s(\mu'^2))} \ , \\ &= \mu^2 e^t e^{-(1+\beta_0\alpha_s(\mu^2)t)/(\beta_0\alpha_s(\mu^2))} = \mu^2 e^t e^{-1/(\beta_0\alpha_s(\mu^2))} e^{-t} = \Lambda^2 \ . \end{aligned} \tag{95}$$

The scale Λ is a physical mass scale of the theory of strong interaction. Its measured value is about 200 MeV.

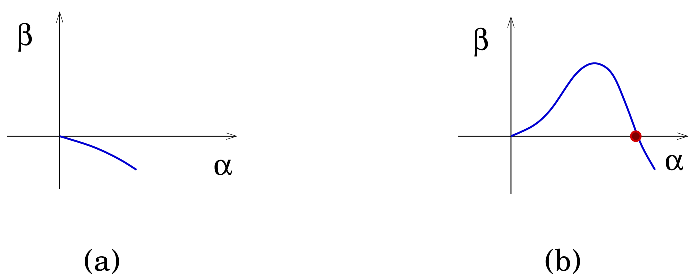

Figure 12: *(a) Trivial and (b) nontrivial ultraviolet fixed points of the β function.*

The running coupling (91) can be equivalently expressed in terms of Λ,

$$\begin{aligned} \alpha_s(q^2) &= \frac{\alpha_s(\mu^2)}{1+\beta_0\ \alpha_s(\mu^2)\,[\ln(q^2/\Lambda^2) - 1/(\beta_0\ \alpha_s(\mu^2))]} \\ &= \frac{1}{\beta_0 \ln(q^2/\Lambda^2)} \ . \end{aligned} \tag{96}$$

The rewriting (96) of Eq. (91) makes it manifest that the running coupling α_s does not depend on the choice of the renormalization scale μ.

Remark. The zero of the QCD β function at the origin, sketched in Fig. 12a, is responsible for the theory being weakly coupled at short distances. This behavior is referred to as a trivial ultraviolet fixed point. A behavior such as that in Fig. 12b (nontrivial ultraviolet fixed point), leading to strong coupling at short distances, is in principle possible but not realized in nature as far as we know. This is the reason why renormalization can be understood perturbatively and Feynman graphs provide a very effective method to investigate physical theories of fundamental interactions.