Functions of a Complex Variable (S1)

VI. RESIDUE CALCULUS

▷ Definition: residue of a function f at point z_0

▷ Residue theorem

▷ Relationship between complex integration and power series expansion

▷ Techniques and applications of complex contour integration

RESIDUE CALCULUS

- Complex differentiation, complex integration and power series expansions provide three approaches to the theory of holomorphic functions.

- Cauchy integral formulas can be seen as providing the relationship between the first two.

- Residues serve to formulate the relationship between complex integration and power series expansions.

DEFINITION OF RESIDUE

◇ Let f be holomorphic everywhere within and on a closed curve C except possibly at a point z_0 in the interior of C where f may have an isolated singularity.

♣ Define residue of f at z_0:

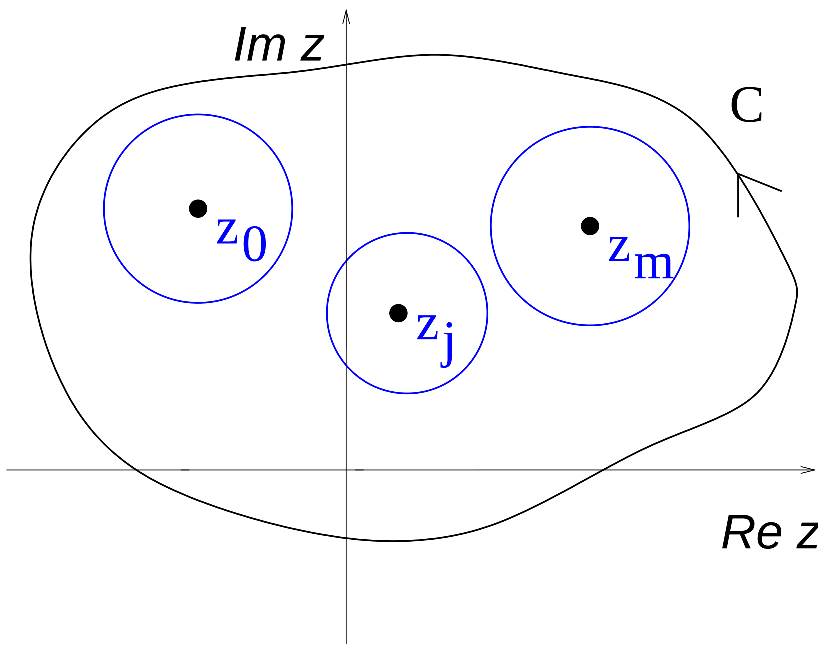
$$\mathrm{Res}_{z_0} f = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

If z_0 is a non-singular point, $\mathrm{Res}_{z_0} f = 0$. Otherwise, $\mathrm{Res}_{z_0} f$ may be $\neq 0$.

RESIDUE THEOREM

◊ Let C be closed path within and on which f is holomorphic except for m isolated singularities. Then

$$\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}_{z_j} f$$



▷ reformulation of Cauchy theorem via arguments similar to those used for deformation theorem

Relationship with Laurent expansion

Consider Laurent series expansion of f about z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \ , \quad \text{where} \quad c_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi$$

$\triangleright$ $\mathrm{Res}_{z_0} f$ is nothing but the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion of f about z_0

$$c_{-1} = \frac{1}{2\pi i} \oint_\Gamma d\xi \ f(\xi) = \mathrm{Res}_{z_0} f$$

- Laurent expansion thus provides a general method to compute residues.

EXAMPLES

- Compute the residue at the singularity of the function

$$f(z) = \frac{1-z}{(1-2z)^2}$$

$z = 1/2$ pole of order 2

$$\frac{1-z}{(1-2z)^2} = \frac{1}{8}\frac{1}{(z-1/2)^2} \underbrace{-\frac{1}{4}}_{residue} \frac{1}{z-1/2} \Rightarrow \text{Res}_{z=1/2} f = -\frac{1}{4}$$

- Compute the residue at the singularity of the function

$$f(z) = e^{1/z^2}$$

$z = 0$ essential singularity

$$e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z^4} + \ldots \Rightarrow \text{Res}_{z=0} f = 0$$

Calculation of residues in the case of poles

▷ If z_0 is pole of order n for f, then

$$f(z) = \frac{h(z)}{(z-z_0)^n} \ , \quad h \ \text{holomorphic and} \ h(z_0) \neq 0$$

Substituting this into the definition of residue gives

$$\begin{aligned}
\text{Res}_{z_0} f &= \frac{1}{2\pi i} \oint_C \frac{h(z)}{(z-z_0)^n} \, dz \\
&= \frac{1}{(n-1)!} \left[\frac{d^{n-1} h(z)}{dz^{n-1}} \right]_{z=z_0} \quad \text{by Cauchy formula} \\
&= \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} [(z-z_0)^n f(z)]
\end{aligned}$$

$$\text{Ex.}(\textit{previous slide}) \ \ f(z) = \frac{1-z}{(1-2z)^2}$$

$$\text{Res}_{z=1/2} f = \lim_{z \to 1/2} \frac{d}{dz} [(z-1/2)^2 f(z)] = \lim_{z \to 1/2} \frac{d}{dz} [\frac{1-z}{4}] = -\frac{1}{4}$$

METHODS TO CALCULATE RESIDUES

♠ General method: from Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \ , \quad \text{where} \quad c_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi$$

$$\begin{aligned} \mathrm{Res}_{z_0} f &= c_{-1} \\ &= \frac{1}{2\pi i} \oint_\Gamma d\xi \ f(\xi) \end{aligned}$$

♠ Method for z_0 pole of order n:

$$\mathrm{Res}_{z_0} f = \lim_{z \longrightarrow z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)]$$

$$\text{For} \ \ n = 1 : \quad \mathrm{Res}_{z_0} f = \lim_{z \longrightarrow z_0} [(z - z_0) f(z)]$$

$$\mathcal{I} = \oint_C \frac{\sin z}{z^6}\ dz$$

where C is the circle of centre $z = 0$ and radius 1

$z = 0$ pole of order 5

$$\frac{\sin z}{z^6} = \frac{1}{z^6}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\right) = \frac{1}{z^5} - \frac{1}{6}\frac{1}{z^3} + \underbrace{\frac{1}{120}}_{\text{residue}}\ \frac{1}{z} + \text{analytic part}$$

$$\Longrightarrow\ \mathcal{I} = \oint_C \frac{\sin z}{z^6}\ dz = 2\pi i \text{Res}_{z=0}(\text{Integrand}) = \frac{i\pi}{60}$$

$$\text{Note} \quad : \quad \oint_C \frac{\cos z}{z^6}\ dz = 0$$

$z = 0$ pole of order 6 with zero residue

CALCULATION OF CONTOUR INTEGRALS BY RESIDUE THEOREM

Let C be the circle of centre $z = 0$ and radius 3.

$$\oint_C dz \ \frac{5z-2}{z(z-1)} = 2\pi i \left[\text{Res}_{z=0}(\text{Integrand}) + \text{Res}_{z=1}(\text{Integrand})\right]$$

$$= 2\pi i(2+3) = 10\pi i$$

$$\oint_C dz \ e^{-1/z} = 2\pi i \ \text{Res}_{z=0}(\text{Integrand})$$

$$= 2\pi i(-1) = -2\pi i$$

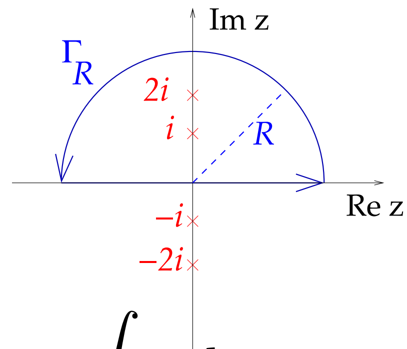
$$\oint_C dz \ \frac{1}{z^2+1} \ e^{\pi z/4} = 2\pi i \left[\text{Res}_{z=+i}(\text{Integrand}) + \text{Res}_{z=-i}(\text{Integrand})\right]$$

$$= 2\pi i \left[\frac{e^{i\pi/4}}{2i} + \frac{e^{-i\pi/4}}{-2i}\right] = 2\pi i \sin\frac{\pi}{4} = i\pi\sqrt{2}$$

EVALUATION OF REAL INTEGRALS BY COMPLEX CONTOUR INTEGRATION METHODS

$$I = \int_{-\infty}^{\infty} dx \, \frac{x^2}{(x^2+1)(x^2+4)}$$

$$\oint_{\Gamma_R} \frac{z^2}{(z^2+1)(z^2+4)} \, dz$$



$$= \int_{-R}^{R} dx \, \frac{x^2}{(x^2+1)(x^2+4)} + \int_{S_R} dz \, \frac{z^2}{(z^2+1)(z^2+4)}$$

- By residue theorem $\oint_{\Gamma_R} = 2\pi i [\mathrm{Res}_{z=+i} f + \mathrm{Res}_{z=+2i} f] = 2\pi i (-\frac{1}{6i} + \frac{1}{3i}) = \frac{\pi}{3}$.

- By Jordan lemma $\int_{S_R} \to 0$ for $R \to \infty$ because $|z f(z)| \to 0$.

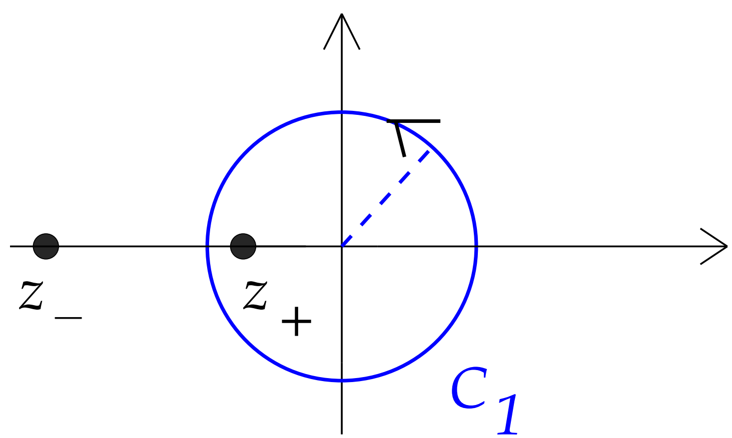
Thus $I = \pi/3$.

$$I = \int_0^{2\pi} d\theta \, \frac{1}{2 + \cos\theta}$$

$$z = e^{i\theta} \quad ; \quad dz = iz d\theta \quad ; \quad \cos\theta = \frac{z + 1/z}{2}$$

So $I = \oint_{C_1} dz \, \frac{1}{iz} \, \frac{1}{2 + (z + 1/z)/2} = \oint_{C_1} dz \, \frac{2}{i} \, \frac{1}{z^2 + 4z + 1}$

$$z_{\pm} = -2 \pm \sqrt{3}$$



- By residue theorem $I = 2\pi i \, [\mathrm{Res}_{z=z_+} f] = \frac{2\pi}{\sqrt{3}}$.

$$I_1 = \int_0^{2\pi} d\theta \, e^{-\cos\theta} \; \cos(\theta + \sin\theta) \; , \quad I_2 = \int_0^{2\pi} d\theta \, e^{-\cos\theta} \; \sin(\theta + \sin\theta)$$

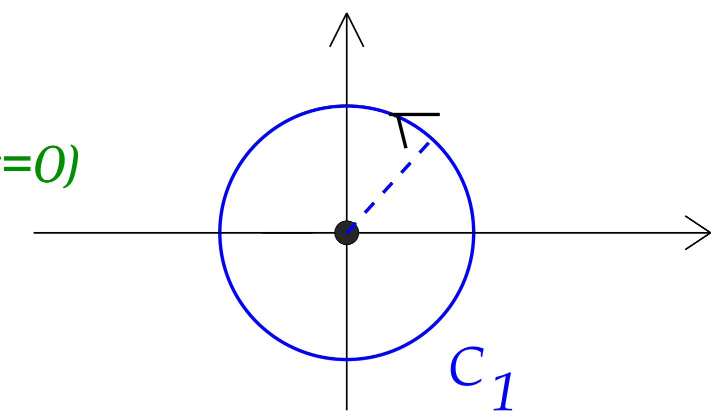
$$z = e^{i\theta} \;\; ; \quad dz = i e^{i\theta} d\theta \;\; ; \quad e^{-1/z} = e^{-e^{-i\theta}} = e^{-\cos\theta + i\sin\theta}$$

$$\text{So} \; \oint_{C_1} dz \; e^{-1/z} = i \int_0^{2\pi} d\theta \; e^{-\cos\theta} \; e^{i(\theta+\sin\theta)} = i(I_1 + i I_2)$$

- *By residue theorem*

$$\oint_{C_1} dz \, e^{-1/z} = 2\pi \; i \; Res(z=0)$$

$$= -2\pi \; i$$

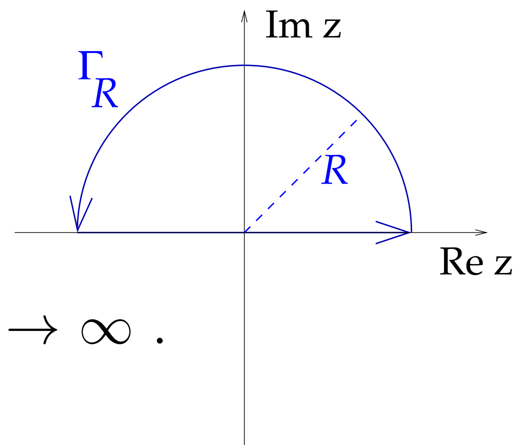


$$\text{Thus} \quad -2\pi i = i(I_1 + i I_2) \;\; \Rightarrow \quad I_1 = -2\pi \quad , \quad I_2 = 0 \quad .$$

TECHNIQUES OF CONTOUR INTEGRATION:
Choice of integrand in the complex z plane

Example : Consider $I = \int_0^\infty dx \, \frac{\cos x}{x^2+1}$.

$\int_{S_R} dz \, \frac{\cos z}{z^2+1}$ does not vanish on semicircle S_R for $R \to \infty$.



Take instead $f(z) = \frac{e^{iz}}{z^2+1}$.

- By Jordan lemma $\int_{S_R} \frac{e^{iz}}{z^2+1} \, dz \to 0$ for $R \to \infty$.

- By residue theorem $\oint_{\Gamma_R} \frac{e^{iz}}{z^2+1} \, dz = 2\pi i \, \mathrm{Res}_{z=i} f = 2\pi i \, \frac{e^{-1}}{2i} = \frac{\pi}{e}$.

Thus $\int_{-\infty}^{+\infty} dx \, \frac{\cos x}{x^2+1} + i \int_{-\infty}^{+\infty} dx \, \frac{\sin x}{x^2+1} = \frac{\pi}{e} \implies I = \frac{\pi}{2e}$.

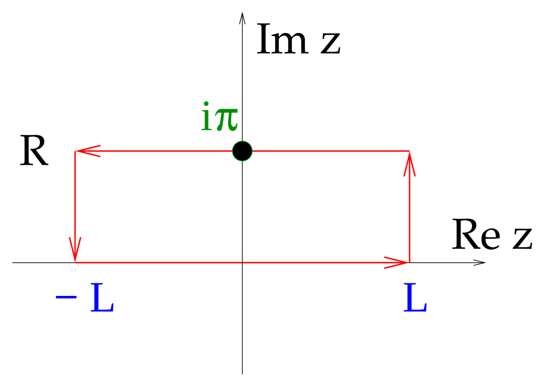
TECHNIQUES OF CONTOUR INTEGRATION:
Choice of contour in the complex z plane

$$\text{Example : Consider } I = \int_{-\infty}^{\infty} dx \, \frac{e^{x/2}}{\cosh x} \; .$$

- $f(z) = \dfrac{e^{z/2}}{\cosh z}$ has infinitely many poles, $z = i(\pi/2 + n\pi)$, $n \in \mathbb{Z}$.

Choose contour so as to enclose only a finite number of poles:

- Rectangular contour R encircles one only, $z = i\pi/2$, for any L .



By residue theorem $\displaystyle \oint_{R} dz \, \frac{e^{z/2}}{\cosh z} = 2\pi i \ \text{Res}_{z=i\pi/2}[f] = 2\pi e^{i\pi/4}$

Let $L \to \infty$. Integrals along vertical sides vanish because

$$|\cosh(L+iy)| = |e^{L+iy}+e^{-L-iy}|/2 \geq ||e^{L+iy}|-|e^{-L-iy}||/2 = (e^{L}-e^{-L})/2 \geq e^{L}/4 \ ,$$

and so, by Darboux inequality,

$$\left| \int_0^{\pi} i \ dy \ \frac{e^{(L+iy)/2}}{\cosh{(L+iy)}} \right| \leq \int_0^{\pi} dy \ \frac{e^{L/2}}{e^{L}/4} = 4\pi \ e^{-L/2} \to 0 \ \text{ for } \ L \to \infty \ .$$

Similarly for the other vertical side.

Because $\cosh(x+i\pi) = -\cosh x$, integrals along horizontal sides are related by

$$\int_L^{-L} dx \ \frac{e^{(x+i\pi)/2}}{\cosh{(x+i\pi)}} = e^{i\pi/2} \int_{-L}^{L} dx \ \frac{e^{x/2}}{\cosh x} \ .$$

$$\text{Taking } \ L \to \infty \ \Longrightarrow \ I = \int_{-\infty}^{+\infty} \frac{e^{x/2}}{\cosh x} \ dx = \frac{2\pi \ e^{i\pi/4}}{1+e^{i\pi/2}} = \frac{\pi}{\cos(\pi/4)} = \pi\sqrt{2} \ .$$