Functions of a Complex Variable (S1)
Lecture 10

- The argument principle
  - Winding number
  - Counting zeros and poles
  - Rouché theorem

- Applications to
  - Expansions in series of fractions
  - Infinite product expansions
The argument principle

Let $f$ be meromorphic inside and on a closed contour $C$, with no zeros or poles on $C$. Then

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = N - P = \frac{1}{2\pi} \Delta_C \arg f(z)$$

where $N =$ number of zeros of $f$ inside $C$,

$P =$ number of poles of $f$ inside $C$,

counted according to their multiplicity,

$\Delta_C \arg f(z) =$ change in the argument of $f$ over $C$. 
Part (a)

\[ z_k = \text{pole of order } n_k \implies \frac{f'(z)}{f(z)} = -\frac{1}{z - z_k} n_k + \phi(z) \text{ for } z \text{ near } z_k \]

\[ z_k = \text{zero of order } n_k \implies \frac{f'(z)}{f(z)} = \frac{1}{z - z_k} n_k + \phi(z) \text{ for } z \text{ near } z_k \]

Residue theorem \implies \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = \sum_{j_z=1}^{N_z} n_{j_z} - \sum_{j_p=1}^{N_p} n_{j_p} = N - P
Part (b)

Let $C$ be parameterized as $z = z(t)$ on $a \leq t \leq b$, with $z(a) = z(b)$. Then

$$
\frac{1}{2\pi i} \oint_{C} \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} \int_{a}^{b} \frac{f'(z(t))}{f(z(t))} \, z'(t) \, dt
$$

$$
= \frac{1}{2\pi i} \ln \left[ f(z(t)) \right]_{|a}^{b}
$$

$$
= \frac{1}{2\pi i} \ln \left[ |f(z(t))| \right]_{|a}^{b} + \frac{1}{2\pi i} \left( i [\text{arg} f(z(t))]_{|a}^{b} \right) = 0
$$

$$
= \frac{1}{2\pi} \Delta_{C} \arg f(z)
$$
Winding number

\[ w = f(z) = |f(z)| e^{i\theta} \]

\( \Delta_C \ \text{arg} w/(2\pi) \) gives the number of times the point \( w \) winds around the origin in the image curve \( C' \) when \( z \) moves around \( C \)

\[ \Rightarrow \text{“winding number” of } C' \text{ about the origin} \]

\[ \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} \oint_{C'} \frac{dw}{w} = \frac{1}{2\pi} \Delta_C \text{ arg} w \equiv J \]
The argument principle establishes a striking relationship between number of zeros — number of poles of $f$ in a domain $D$ and how $f$ maps the boundary $\partial D$ of the domain, namely, the number of times the image of $\partial D$ through $f$ winds around the origin.

$$N - P = J$$
EXAMPLES

(a) \( f(z) = z^2 - 1 \quad ; \quad C : |z - 1| = 1 \)

\[ w = f(z) = |z|^2 e^{2i\theta} - 1 \]
\[ \implies \Delta_C \arg w = 2\pi, \text{ i.e., } J = 1 \implies N - P = 1 \]
Indeed \( f \) has 1 zero (at \( z = 1 \)) and no poles inside \( C \).

(b) \( f(z) = \frac{z}{(z + 1)^2} \quad ; \quad C : |z| = 10 \)

\[ w = f(z) = \frac{|z| e^{i\theta}}{(|z| e^{i\theta} + 1)^2} \approx \frac{1}{|z|} e^{-i\theta} \]
\[ \implies \Delta_C \arg w = -2\pi, \text{ i.e., } J = -1 \implies N - P = -1 \]
Indeed \( f \) has 1 zero (at \( z = 0 \)) and 1 double pole (at \( z = -1 \)) inside \( C \).
EXAMPLE

• How many solutions does the equation $e^z - 2z = 0$ have inside the circle $|z| = 3$?

$$f(z) = e^z - 2z$$

▷ image of the circle winds around the origin *twice*  
▷ there are *no* poles

⇒ *two solutions*
A COROLLARY OF THE ARGUMENT PRINCIPLE: ROUCHÉ THEOREM

Let $F(z)$ and $G(z)$ be holomorphic on and inside a closed contour $C$. If $|F(z)| > |G(z) - F(z)|$ on $C$, then $F(z)$ and $G(z)$ have the same number of zeros inside $C$.

Let $w = \frac{G}{F}$; consider

$$\frac{1}{2\pi i} \oint_C \frac{w'(z)}{w(z)} \, dz .$$

$$|w(z) - 1| = \frac{|G - F|}{|F|} < 1 \text{ on } C .$$

Therefore the image of $C$ lies inside $|w - 1| < 1$

$$\implies \Delta_C \arg w = 0 \implies N = P \text{ for } w(z) .$$

Thus the number of zeros of $F(P)$ equals the number of zeros of $G(N)$. 


Rouché theorem may be used to

- locate solutions of equations in the complex plane
- arrive at results such as the fundamental theorem of algebra
  (alternative proof to that based on Liouville theorem)

and maximum modulus principle.
EXAMPLE

◊ Show that the polynomial $P(z) = z^5 + 14z + 2$ has 4 roots in the annulus $3/2 < |z| < 2$.

• Consider $C_2$ circle $|z| = 2$. Take $G = P(z)$, $F(z) = z^5$.

$|G - F| < |F|$ on $C_2 \implies P(z)$ has as many zeros inside $C_2$ as $F(z)$, which is 5.

• Next consider $C_1$ circle $|z| = 3/2$. Take $G = P(z)$, $F(z) = 14z$.

$|G - F| < |F|$ on $C_1 \implies P(z)$ has as many zeros inside $C_1$ as $F(z)$, which is 1.

Thus $5 - 1 = 4$ zeros of $P(z)$ lie between $C_2$ and $C_1$. 

AN EXTENDED VERSION OF THE ARGUMENT PRINCIPLE

If $f$ and $C$ satisfy the same hypotheses of the argument principle and $h(z)$ is holomorphic inside and on $C$, then

$$
\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \ h(z) \ dz = \sum_{j_z=1}^{N_z} n_{j_z} h(z_{j_z}) - \sum_{j_p=1}^{N_p} n_{j_p} h(z_{j_p})
$$
EXPANSIONS BASED ON POLES OF A FUNCTION

- Taylor and Laurent series provide power series expansions of a function \( f \).

- Other kinds of expansions can be useful based on poles of \( f \):

  \[ z_n, \quad n = 1, \ldots, \infty \quad \text{poles of function } f(z) \]

  \[
  i) \quad f(z) = \sum_{n \in \text{poles}} g_n(z, n)
  \]

  \[
  ii) \quad f(z) = \prod_{n \in \text{poles}} g_n(z, n)
  \]

▷ The (extended) argument principle may be used to obtain such expansions.
APPLICATION TO EXPANSION IN SERIES OF FRACTIONS

\[ I(\alpha) = \frac{I}{2\pi i} \oint_{\Gamma} \frac{\pi \cot \pi z}{\alpha^2 - z^2} \, dz \]

- Use extended argument principle with \( f(z) = \sin \pi z, \ h(z) = 1/(\alpha^2 - z^2), \) and \( z_k = k (k \in \mathbb{Z}) \) with \( n_k = 1. \)

So \( I(\alpha) = \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2}. \)

- Now note that for \( N \to \infty \) the integral on the square \( \to 0, \) and compute the integrals on the circles from the residues at \( \pm \alpha. \)

\[ \Rightarrow \frac{\pi \cot \pi \alpha}{\alpha} = \sum_{n=-\infty}^{\infty} \frac{1}{\alpha^2 - n^2} \]
\[ \pi \cot \pi z = z \sum_{n=-\infty}^{\infty} \frac{1}{z^2 - n^2} = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} = \frac{1}{z} + \frac{2z}{z^2 - 1} + \frac{2z}{z^2 - 4} + \ldots \]

- expansion of \( \pi \cot \pi z \) in a series of fractions based on the poles of the function

◊ Expansion of function \( f \), having poles \( z_j \), \( 0 < |z_1| \leq \ldots \leq |z_j| \leq \ldots \), with residues \( r_j \):

\[ f(z) = f(0) + \sum_j r_j \left( \frac{1}{z - z_j} + \frac{1}{z_j} \right) \]

- For \( f(z) = \cot z - 1/z \) this gives

\[ \cot z = \frac{1}{z} + \sum_{n=\pm 1, \pm 2, \ldots} \left( \frac{1}{z - n\pi} + \frac{1}{n\pi} \right) \]

i.e., the expansion at the top:

\[ \cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z - n\pi} + \frac{1}{z + n\pi} \right) = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2 \pi^2} \]
AN EXAMPLE OF INFINITE PRODUCT EXPANSION

\[
\frac{d}{dz} \ln \sin \pi z = \pi \cot \pi z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}
\]

\[\Rightarrow \ln \sin \pi z = \ln z + c_0 + \sum_{n=1}^{\infty} \left[ \ln(z^2 - n^2) + c_n \right] \]

with \( c_0 = \ln \pi \), \( c_n = -\ln(-n^2) \)

\[\Rightarrow \sin \pi z = \pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) \]

▷ infinite-product expansion of the function \( \sin \pi z \)