VII. Integral Transforms

- Fourier transform from an application of residue calculus
- Properties of Fourier transformation
- Applications to differential equations
- Laplace transform

Fourier transformation

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t}\ \widetilde{f}(\omega)\ d\omega \equiv \mathrm{FT}^{-1}\left[\widetilde{f}\right]$$

$$\widetilde{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t}\ f(t)\ dt \equiv \mathrm{FT}\left[f\right]$$

- We see next how to obtain this transformation starting from a complex integral representation of step functions.

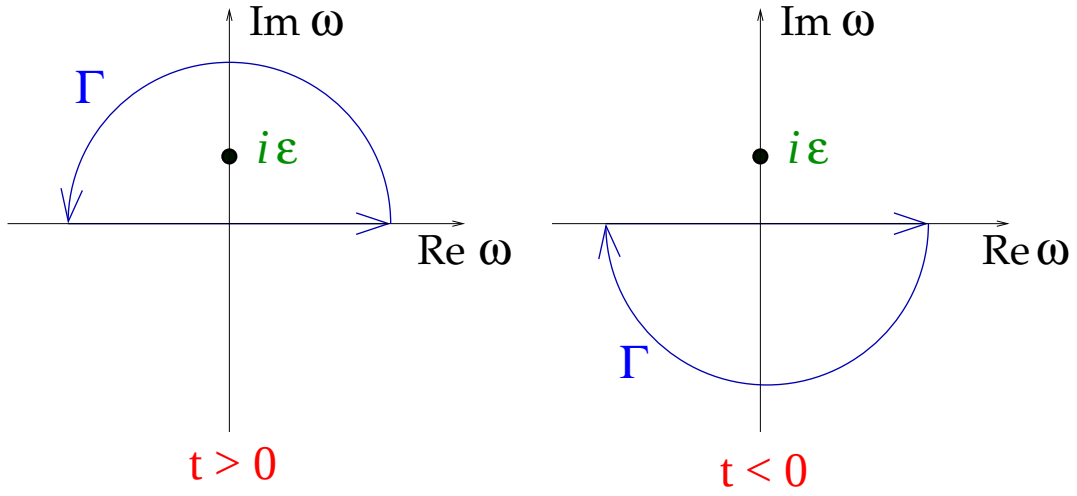
The step function $\Theta(t)$

- Consider $$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{i\omega t}\,\frac{1}{\omega - i\varepsilon}\,d\omega \qquad (\varepsilon \in \mathbb{R}^+)$$

- Evaluate this integral by complex contour integration:

for $t > 0$ Jordan lemma applies to semicircular arc in UHP $\Longrightarrow$ integral $=$ Res$_{i\varepsilon} = 1$;

for $t < 0$ Jordan lemma applies to semicircular arc in LHP $\Longrightarrow$ integral $= 0$.

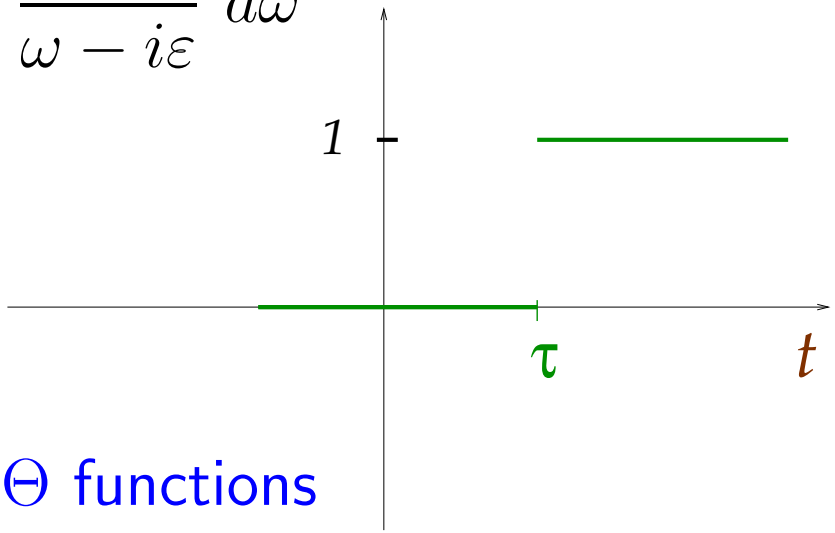


- Thus $$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{i\omega t}\,\frac{1}{\omega - i\varepsilon}\,d\omega = \Theta(t)$$

where $\Theta(t)$ is the step function: $\Theta(t) = 1$ for $t > 0$, $\Theta(t) = 0$ for $t < 0$.

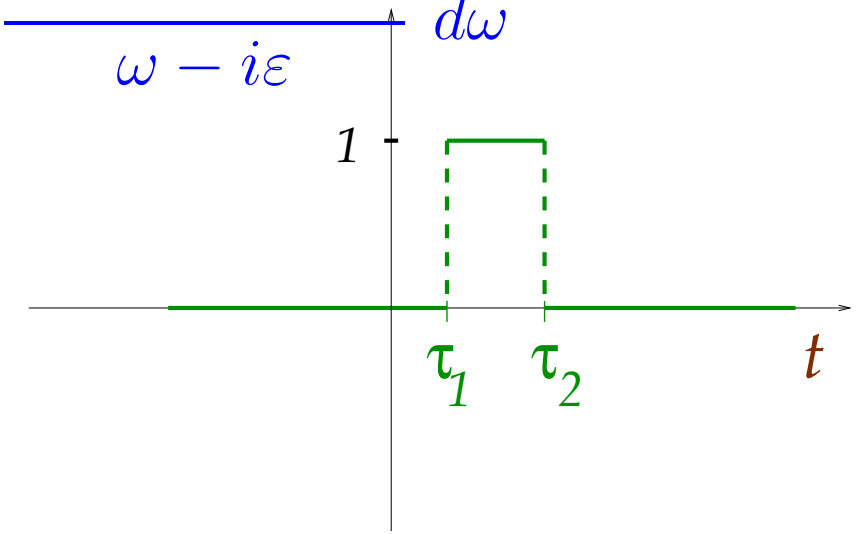
step function centred at $t = \tau$:

$$\Theta(t-\tau) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{i\omega(t-\tau)} \, \frac{1}{\omega - i\varepsilon} \, d\omega$$



"finite pulse": combination of Θ functions

$$\Theta(t-\tau_1) - \Theta(t-\tau_2) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{i\omega t} \, \frac{e^{-i\omega\tau_1} - e^{-i\omega\tau_2}}{\omega - i\varepsilon} \, d\omega$$

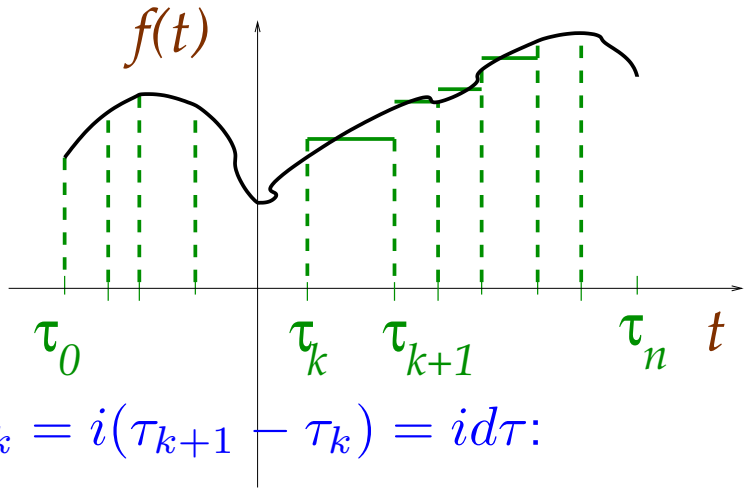


Construction of Fourier transform

- First construct approximation to $f(t)$ by sum of step functions

$$f(t) \simeq \sum_{k=0}^{n-1} f(\tau_k)\,[\Theta(t-\tau_k) - \Theta(t-\tau_{k+1})]$$

$$= \frac{1}{2\pi i} \sum_{k=0}^{n-1} f(\tau_k) \int_{-\infty}^{+\infty} e^{i\omega t}\, \frac{e^{-i\omega\tau_k} - e^{-i\omega\tau_{k+1}}}{\omega - i\varepsilon}\, d\omega$$



- Then let each $\tau_{k+1} - \tau_k \to 0$ and expand in $\Delta\tau_k = i(\tau_{k+1} - \tau_k) = id\tau$:

$$\Longrightarrow \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \underbrace{\int_{-\infty}^{+\infty} f(\tau)\, e^{-i\omega\tau}\, d\tau}_{\widetilde{f}(\omega)}\, d\omega$$

- Thus we obtain representation of f as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t}\ \widetilde{f}(\omega)\ d\omega$$

where $\widetilde{f}(\omega)$ is the *Fourier transform* defined by

$$\widetilde{f}(\omega) = \int_{-\infty}^{+\infty} f(t)\ e^{-i\omega t}\ dt$$

Properties of Fourier transformation

$$\mathrm{FT}\,[\alpha f + \beta g] = \alpha\ \mathrm{FT}\,[f] + \beta\ \mathrm{FT}\,[g]$$

$$\mathrm{FT}\,[f^{(n)}] = (i\omega)^n\ \widetilde{f}$$

$$i^n\ \widetilde{f}^{(n)} = \mathrm{FT}\,[t^n f]$$

$$\mathrm{FT}\,[f * g] = \widetilde{f}\ \widetilde{g}$$

$$\mathrm{FT}\,[f\,g] = \frac{1}{2\pi}\ \widetilde{f} * \widetilde{g}$$

$$\text{convolution product} \quad : \quad (f * g)(t) \stackrel{\mathrm{df}}{=} \int_{-\infty}^{+\infty} dy\ f(y)\ g(t-y)$$

Parseval identity

- Take particular case of convolution formula with $g(t) = \overline{f}(-t)$.

$$\Longrightarrow \quad \widetilde{g}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t}\ \overline{f}(-t)\ dt = \int_{-\infty}^{+\infty} e^{i\omega t}\ \overline{f}(t)\ dt = \overline{\widetilde{f}}(\omega)$$

$$\text{Then} \quad f * g = \mathrm{FT}^{-1}\left[\widetilde{f}\ \widetilde{g}\right] \quad \text{gives}$$

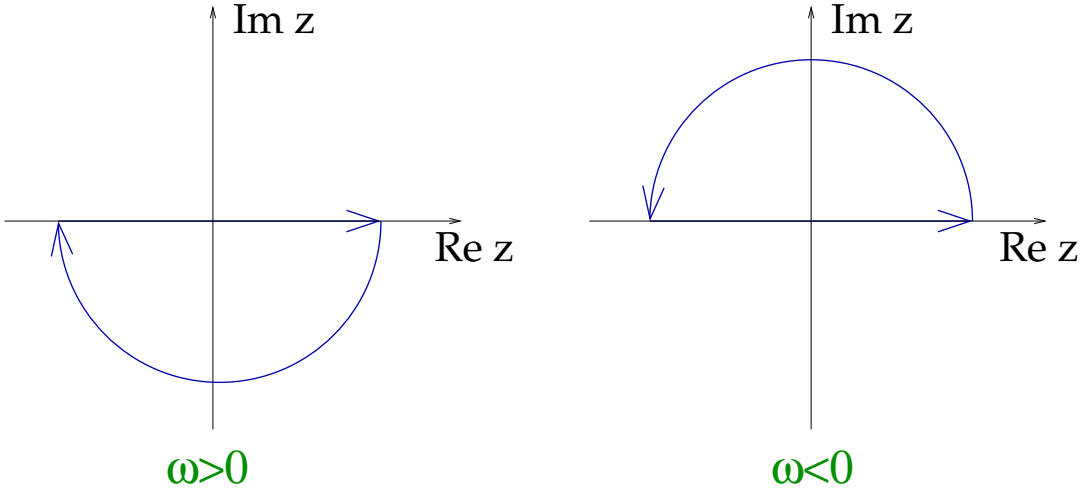
$$\int_{-\infty}^{+\infty} dy\ f(y)\ \overline{f}(y-t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega\ e^{i\omega t}\ \widetilde{f}(\omega)\ \overline{\widetilde{f}}(\omega)$$

$$t = 0 \quad \Rightarrow \quad \int_{-\infty}^{+\infty} dy\ |f(y)|^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega\ |\widetilde{f}(\omega)|^2$$

- Example: Compute the FT of $f(t) = 1/(1+t^2)$.

$$\widetilde{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t}\ \frac{1}{1+t^2}\ dt$$

This integral can be computed by complex contour integration methods, closing the contour as in the figure below depending on the sign of ω.



$$\omega > 0 \ : \quad \widetilde{f}(\omega) = -2\pi i\ \mathrm{Res}_{-i} = \pi e^{-\omega}$$

$$\omega < 0 \ : \quad \widetilde{f}(\omega) = 2\pi i\ \mathrm{Res}_{i} = \pi e^{\omega}$$

$$\text{Thus}\ \ \widetilde{f}(\omega) = \pi e^{-|\omega|}\ \ \text{for any}\ \ \omega\ \ .$$

Use of FT to solve differential equations

Example : $f''(t) - f(t) = e^{-t^2}$

Take FT of the equation:

$$-\omega^2 \widetilde{f} - \widetilde{f} = \text{FT}\,[e^{-t^2}] \quad \text{i.e.,} \quad \widetilde{f} = -\frac{1}{1+\omega^2}\,\text{FT}\,[e^{-t^2}]$$

$$\Longrightarrow \quad f(t) = -\,\text{FT}^{-1}\,[\frac{1}{1+\omega^2}] * (e^{-t^2})$$

$$= -\pi \int_{-\infty}^{+\infty} dy\; e^{-|y-t|-y^2}$$

Application of FT to boundary value problems

Example : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad , \quad y > 0$

$u(x,0) = G(x) \qquad (\text{Dirichlet boundary condition})$

- Taking FT in x at fixed y gives ODE for $\widetilde{u}(\omega, y)$:

$$-\omega^2 \widetilde{u} + \frac{\partial^2 \widetilde{u}}{\partial y^2} = 0 \ , \quad \widetilde{u}(\omega,0) = \widetilde{G}(\omega) \quad \Longrightarrow \quad \widetilde{u}(\omega,y) = \widetilde{G}(\omega) e^{-|\omega| y}$$

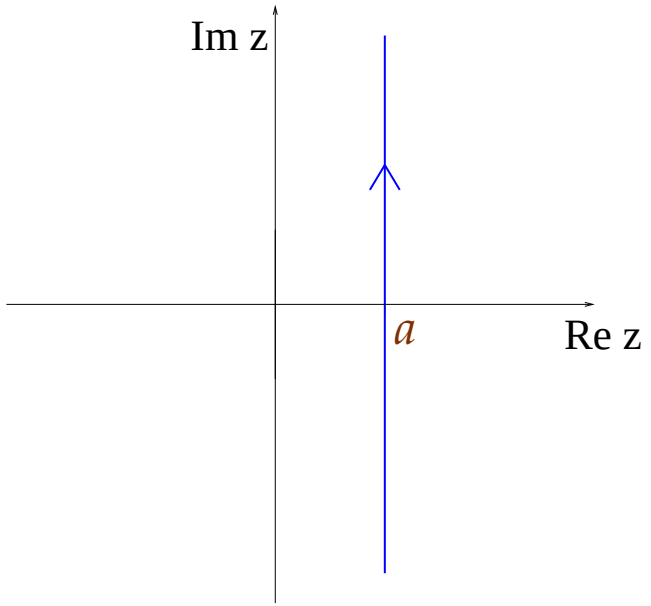
$$\text{Thus} \ \ u(x,y) = G * \underbrace{\text{FT}^{-1}\left[e^{-|\omega| y}\right]}_{[\pi y(1+x^2/y^2)]^{-1}}$$

- Evaluating the convolution product gives the Poisson integral formula:

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\xi \ \frac{y}{(\xi - x)^2 + y^2} \ G(\xi)$$

Laplace transformation

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{zt}\ \widetilde{f}(z)\ dz \equiv \mathrm{LT}^{-1}\left[\widetilde{f}\right]$$

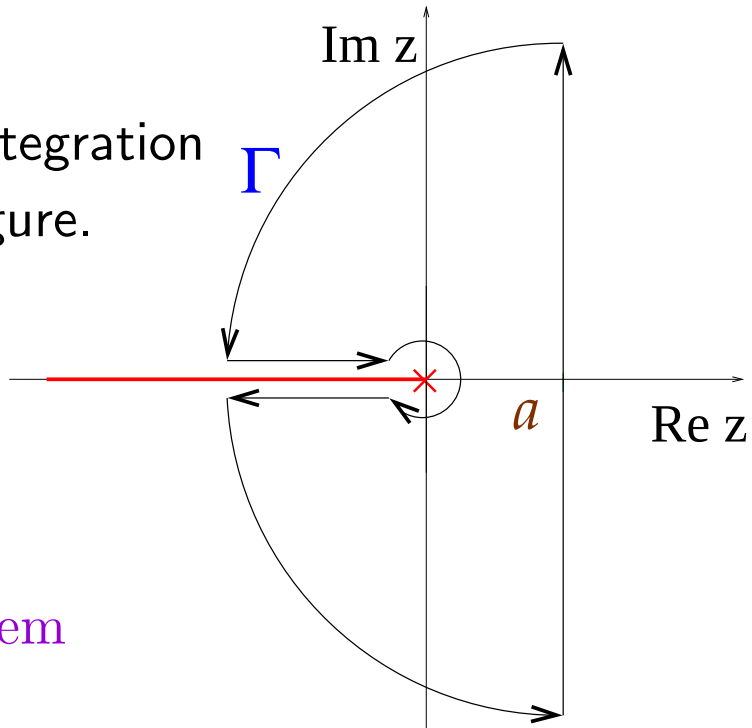


$$\widetilde{f}(z) = \int_{0}^{+\infty} e^{-zt}\ f(t)\ dt \equiv \mathrm{LT}\left[f\right]$$

◊ Example: Compute the LT^{-1} of $F(z) = 1/\sqrt{z}$.

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{zt} \, \frac{1}{\sqrt{z}} \, dz \qquad (a > 0)$$

This integral can be computed by complex integration methods, closing the contour as in the figure.



- branch cut along $\mathbb{R}^-$
- $\oint_\Gamma e^{zt} \, \frac{1}{\sqrt{z}} \, dz = 0$ by Cauchy theorem
- integrals on arc of large circle and small circle go to 0 in the asymptotic limits
- integrals above the branch cut and below the branch cut give

$$f(t) = \frac{1}{\sqrt{\pi t}}$$