Wave propagation in an inhomogeneous plasma

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1. Introduction

In the cold plasma waves notes, we treated the simplest case possible: a homogeneous, steady state plasma. In these notes we consider waves in a steady state, slightly inhomogeneous plasma, i.e. the background density and magnetic field depend on position but not on time. We assume that the characteristic length of variation of the background density and magnetic field,

$$L \sim |\nabla \ln n_s|^{-1} \sim B|\nabla \mathbf{B}|^{-1}, \tag{1.1}$$

is much longer than the wavelength of the waves,

$$kL \gg 1. \tag{1.2}$$

We use the WKB approximation to solve this problem.

2. WKB approximation for cold plasma waves

In the cold plasma waves notes, we assumed that the electromagnetic fields of the wave were of the form

$$\begin{aligned} \delta\mathbf{E} &= \tilde{\mathbf{E}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \mathrm{i}\omega t), \\ \delta\mathbf{B} &= \tilde{\mathbf{B}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \mathrm{i}\omega t). \end{aligned} \tag{2.1}$$

In an inhomogeneous, steady state plasma, we cannot assume that the dependence on $\mathbf{r}$ is simply $\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})$. Instead of the form in (2.1), we use

$$\begin{aligned} \delta\mathbf{E} &= \tilde{\mathbf{E}}(\mathbf{r}) \exp(\mathrm{i}S(\mathbf{r}) - \mathrm{i}\omega t), \\ \delta\mathbf{B} &= \tilde{\mathbf{B}}(\mathbf{r}) \exp(\mathrm{i}S(\mathbf{r}) - \mathrm{i}\omega t). \end{aligned} \tag{2.2}$$

We assume that the <u>eikonal function</u> S and the coefficients $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ have characteristic lengths of the order of L,

$$\frac{|\nabla S|}{S} \sim \frac{|\nabla \tilde{\mathbf{E}}|}{|\tilde{\mathbf{E}}|} \sim \frac{|\nabla \tilde{\mathbf{B}}|}{|\tilde{\mathbf{B}}|} \sim \frac{1}{L}, \tag{2.3}$$

but the function $S(\mathbf{r})$ is large,

$$S(\mathbf{r}) \sim kL \gg 1, \tag{2.4}$$

giving

$$|\nabla S| \sim k \gg \frac{1}{L}. \tag{2.5}$$

We have not generalized the time dependence in equation (2.2) because we are considering a plasma background independent of time. The frequency of the wave is set by the antenna that launches it.

Using equation (2.2), we can repeat the derivation in the notes on cold plasma waves by replacing $\mathrm{i}\mathbf{k}\tilde{\mathbf{E}}$ by $\mathrm{i}\nabla S\,\tilde{\mathbf{E}} + \nabla\tilde{\mathbf{E}}$,

$$\mathrm{i}\mathbf{k}\tilde{\mathbf{E}} \rightarrow \mathrm{i}\nabla S\,\tilde{\mathbf{E}} + \nabla\tilde{\mathbf{E}}. \tag{2.6}$$

We find

$$\frac{c^2}{\omega^2}(\mathbf{k} - \mathrm{i}\nabla) \times [(\mathbf{k} - \mathrm{i}\nabla) \times \tilde{\mathbf{E}}] + \boldsymbol{\epsilon} \cdot \tilde{\mathbf{E}} = 0, \tag{2.7}$$

where we have defined the local wave vector as

$$\mathbf{k} = \nabla S. \tag{2.8}$$

The terms with $-\mathrm{i}\nabla$ are small in $(kL)^{-1} \ll 1$, and the electric field $\tilde{\mathbf{E}}$ can be expanded in $(kL)^{-1} \ll 1$,

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 + \underbrace{\tilde{\mathbf{E}}_1}_{\sim (kL)^{-1}\tilde{\mathbf{E}}_0 \ll \tilde{\mathbf{E}}_0} + \ldots \tag{2.9}$$

To lowest order in $(kL)^{-1} \ll 1$, equation (2.7) becomes

$$\left[\frac{c^2k^2}{\omega^2}(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon}\right] \cdot \tilde{\mathbf{E}}_0 = 0, \tag{2.10}$$

which is the cold plasma dispersion relation. As we have seen in the notes on the cold plasma dispersion relation, equation (2.10) gives k and the direction of $\tilde{\mathbf{E}}_0$ if the direction of the wave vector, $\hat{\mathbf{k}}$, is known. Thus, we write $\tilde{\mathbf{E}}_0$ as an amplitude $A(\mathbf{r})$ and a polarization $\mathbf{e}(\mathbf{r})$,

$$\tilde{\mathbf{E}}_0(\mathbf{r}) = A(\mathbf{r})\mathbf{e}(\mathbf{r}). \tag{2.11}$$

The polarization $\mathbf{e}$ is the only piece of $\tilde{\mathbf{E}}_0$ that is determined by the dispersion relation,

$$\left[\frac{c^2k^2}{\omega^2}(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon}\right] \cdot \mathbf{e} = 0, \tag{2.12}$$

In general, there are two solutions for k: the fast and the slow wave, with wavenumbers k_f and k_s, and polarizations $\mathbf{e}_f$ and $\mathbf{e}_s$. For example, for propagation perpendicular to the magnetic field line, $\hat{\mathbf{k}} \cdot \hat{\mathbf{b}} = 0$, we have the ordinary and extraordinary mode solutions with wavenumbers

$$k_O = \frac{\omega}{c}\sqrt{\epsilon_\parallel} = \frac{\sqrt{\omega^2 - \omega_{pe}^2}}{c} \tag{2.13}$$

and

$$k_X = \frac{\omega}{c}\sqrt{\epsilon_\perp - \frac{g^2}{\epsilon_\perp}} = \frac{\omega}{c}\sqrt{\frac{(\omega^2 - \omega_L^2)(\omega^2 - \omega_R^2)}{(\omega^2 - \omega_{UH}^2)(\omega^2 - \omega_{LH}^2)}}, \tag{2.14}$$

and polarizations

$$\mathbf{e}_O = \hat{\mathbf{b}} \tag{2.15}$$

and

$$\mathbf{e}_X = \mathrm{i}g\hat{\mathbf{k}} + \epsilon_\perp \hat{\mathbf{k}} \times \hat{\mathbf{b}}. \tag{2.16}$$

Since equation (2.7) is linear, the total solution is a linear combination of the waves

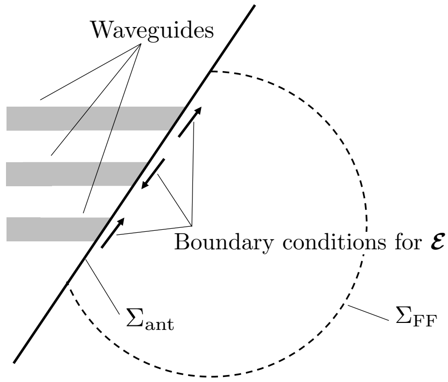

FIGURE 1. Sketch of the boundary conditions by an antenna.

that are solutions to equation (2.12). As an example, we can can consider two waves with the same frequency,

$$\delta\mathbf{E} = A_a(\mathbf{r})\mathbf{e}_a(\mathbf{r})\exp(\mathrm{i}S_a(\mathbf{r}) - \mathrm{i}\omega t) + A_b(\mathbf{r})\mathbf{e}_b(\mathbf{r})\exp(\mathrm{i}S_b(\mathbf{r}) - \mathrm{i}\omega t). \tag{2.17}$$

This example is useful because in general, an antenna that emits waves at a given frequency will launch two different waves. However, note that for some frequencies there might be only one or even no wave that propagates; for example, the ordinary mode does not propagate for $\omega < \omega_{pe}$. We need to determine the functions $A_a(\mathbf{r})$, $\mathbf{e}_a(\mathbf{r})$, $S_a(\mathbf{r})$, $A_b(\mathbf{r})$, $\mathbf{e}_b(\mathbf{r})$ and $S_b(\mathbf{r})$. We first discuss how these functions are determined at certain locations by the antenna, and we then calculate them at every point.

We consider the schematic antenna in figure 1. We know the boundary conditions for $\delta\mathbf{E}$ and $\delta\mathbf{B}$ at the surface of the antenna, $\Sigma_{\rm ant}$. In the near-field region (within a few wavelengths of the antenna), the electromagnetic field is not of the form (2.2). Instead, we use the more general assumption

$$\begin{aligned} \delta\mathbf{E} &= \boldsymbol{\mathcal{E}}(\mathbf{r})\exp(-\mathrm{i}\omega t), \\ \delta\mathbf{B} &= \boldsymbol{\mathcal{B}}(\mathbf{r})\exp(-\mathrm{i}\omega t). \end{aligned} \tag{2.18}$$

Repeating the derivation in the notes on cold plasma waves with this form for the fields, we obtain the equation

$$-\frac{c^2}{\omega^2}\nabla\times(\nabla\times\boldsymbol{\mathcal{E}}) + \boldsymbol{\epsilon}\cdot\boldsymbol{\mathcal{E}} = 0. \tag{2.19}$$

Since we are assuming $kL \gg 1$, the tensor $\boldsymbol{\epsilon}$ can be assumed to be constant in the near-field region, and we can solve equation (2.19) with the boundary conditions at the antenna surface, $\Sigma_{\rm ant}$, and outgoing-wave boundary conditions on a surface $\Sigma_{\rm FF}$ that is several wavelengths away from the antenna. Here the subscript FF stands for far-field. If we locate the surface $\Sigma_{\rm FF}$ sufficiently far from the antenna, the solution will be sufficiently close to the solution (2.17) that we can determine $\hat{\mathbf{k}}_a = \nabla S_a/|\nabla S_a|$, $\hat{\mathbf{k}}_b = \nabla S_b/|\nabla S_b|$, A_a and A_b from the value of $\boldsymbol{\mathcal{E}}$ at $\Sigma_{\rm FF}$. Note that $k_a = |\nabla S_a|$, $k_b = |\nabla S_b|$, $\mathbf{e}_a$ and $\mathbf{e}_b$ are determined by the plasma dispersion relation (2.12) once $\hat{\mathbf{k}}_a$ and $\hat{\mathbf{k}}_b$ are known.

From here on, we consider only a pure wave with a given polarization $\mathbf{e}$,

$$\delta\mathbf{E} = A(\mathbf{r})\mathbf{e}(\mathbf{r})\exp(\mathrm{i}S(\mathbf{r}) - \mathrm{i}\omega t). \tag{2.20}$$

Solutions with two or more waves can be constructed by summing over all the waves.

3. Ray tracing

To calculate the eikonal function $S(\mathbf{r})$ in (2.20), we need to solve equations such as (2.13) or (2.14), derived from (2.12). These equations are of the form

$$D(\nabla S, \omega, \mathbf{r}) \equiv \mathbf{e}^* \cdot \left[\frac{c^2 k^2}{\omega^2} (\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon} \right] \cdot \mathbf{e} = 0, \tag{3.1}$$

where $D(\mathbf{k}, \omega, \mathbf{r})$ is the local dispersion relation for the polarization $\mathbf{e}$. By solving for ω, we can write this equation as

$$\omega = \omega(\nabla S, \mathbf{r}). \tag{3.2}$$

To obtain information about the direction of ∇S, we take a gradient of (3.1) holding the frequency ω fixed,

$$\nabla\nabla S \cdot \nabla_{\mathbf{k}} D + \nabla D = 0. \tag{3.3}$$

Using the definition of $\mathbf{k}$ in (2.8), we obtain $\nabla\nabla S = \nabla \mathbf{k}$, leading to

$$\nabla_{\mathbf{k}} D \cdot \nabla \mathbf{k} = -\nabla D. \tag{3.4}$$

Given that $\nabla\nabla S$ is a symmetric tensor, we could have written this equation as $\nabla\mathbf{k} \cdot \nabla_{\mathbf{k}} D = -\nabla D$, but this latter form is not useful.

Equation (3.4) can be integrated following the characteristics, the lines parallel to the vector $\nabla_{\mathbf{k}} D$. These lines are called rays, and for this reason, integrating equation (3.4) (or some other version of this equation) is known as ray tracing. To make physical sense of (3.4), we use the group velocity

$$\mathbf{v}_g = \nabla_{\mathbf{k}} \omega = -\frac{\nabla_{\mathbf{k}} D}{\partial D / \partial \omega}, \tag{3.5}$$

where the first definition of $\mathbf{v}_g$ assumes that we have written the dispersion relation in the form (3.2), whereas the second definition is based on a dispersion relation of the form (3.1). Using the definition of $\mathbf{v}_g$ in (3.5), equation (3.4) becomes

$$\mathbf{v}_g \cdot \nabla \mathbf{k} = \frac{\nabla D}{\partial D / \partial \omega} = -\nabla \omega. \tag{3.6}$$

Thus, the rays follow the group velocity $\mathbf{v}_g$, and $\mathbf{k}$ can be found along the ray by integrating (3.6). Once $\mathbf{k}$ is known, we can integrate $\nabla S = \mathbf{k}$ to obtain S.

As an example of ray tracing, we consider the configuration shown in figure 2: a plasma with electron density $n_e(x)$ and uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. The density gradient is such that $\partial n_e / \partial x > 0$. We launch an ordinary wave ($\mathbf{e} = \hat{\mathbf{b}}$), which according to (2.13) satisfies the dispersion relation

$$\omega^2 = \omega_{pe}^2 + k^2 c^2. \tag{3.7}$$

The group velocity of this wave is

$$\mathbf{v}_g = \frac{\mathbf{k} c^2}{\omega}. \tag{3.8}$$

As a result, the ray tracing equation (3.6) gives

$$\frac{c^2}{\omega} \mathbf{k} \cdot \nabla \mathbf{k} = -\frac{\omega_{pe}}{\omega} \nabla \omega_{pe}. \tag{3.9}$$

Since $\partial n_e / \partial x > 0$ and $\partial n_e / \partial y = 0$, we find

$$\mathbf{k} \cdot \nabla k_x = -\frac{\omega_{pe}}{c^2} \frac{\partial \omega_{pe}}{\partial x} < 0 \tag{3.10}$$

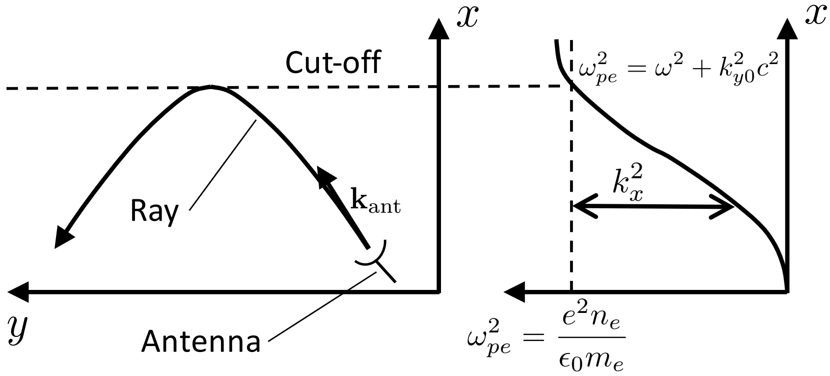

FIGURE 2. Ray for an ordinary wave in a plasma with electron density $n_e(x)$ and uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. The unit vector $\hat{\mathbf{z}}$ is pointing out of the paper.

and

$$\mathbf{k} \cdot \nabla k_y = 0. \tag{3.11}$$

Thus k_y is constant. Due to the symmetry of the system, we expect k_x to depend only on x, giving

$$k_x \frac{\partial k_x}{\partial x} = -\frac{\omega_{pe}}{c^2} \frac{\partial \omega_{pe}}{\partial x} < 0. \tag{3.12}$$

The solution to this equation is

$$k_x(x) = \pm \sqrt{k_x^2(x_{\text{ant}}) + \frac{\omega_{pe}^2(x_{\text{ant}}) - \omega_{pe}^2(x)}{c^2}} = \pm \frac{\sqrt{\omega^2 - k_y^2 c^2 - \omega_{pe}^2(x)}}{c}, \tag{3.13}$$

where $x = x_{\text{ant}}$ is the location of the antenna. Note that (3.13) could have been deduced from (3.7) and the fact that k_y is a constant. Using (3.8) and (3.13), we find the equation for the x component of the group velocity,

$$v_{gx}(x) = \pm c \sqrt{\frac{v_{gx}^2(x_{\text{ant}})}{c^2} + \frac{\omega_{pe}^2(x_{\text{ant}}) - \omega_{pe}^2(x)}{\omega^2}}. \tag{3.14}$$

The y component of the group velocity is constant, $v_{gy}(x) = v_{gy}(x_{\text{ant}})$. These two components of the group velocity give the ray shown in figure 2. Note that for a sufficiently large density increase, there exists a position x for which $v_{gx}(x) = 0$ and the ray reflects.

4. Equation for the amplitude A

Once S is known everywhere, the polarization of the wave is known because it is deduced from (2.12), but we still need to find the spatial dependence of the magnitude of $\tilde{\mathbf{E}}_0$, $A(\mathbf{r})$ in (2.20). To determine A we need to expand to first order in $(kL)^{-1} \ll 1$. The first order terms of equation (2.7) are

$$-\frac{\mathrm{i}c^2}{\omega^2} \nabla \times (\mathbf{k} \times \tilde{\mathbf{E}}_0) - \frac{\mathrm{i}c^2}{\omega^2} \mathbf{k} \times (\nabla \times \tilde{\mathbf{E}}_0) + \left[\frac{c^2 k^2}{\omega^2} (\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbf{I}) + \boldsymbol{\epsilon} \right] \cdot \tilde{\mathbf{E}}_1 = 0. \tag{4.1}$$

To eliminate the term that contains $\tilde{\mathbf{E}}_1$, we pre-multiply this equation by the complex conjugate of the polarization tensor $\mathbf{e}$ (recall that $\boldsymbol{\epsilon}$ is Hermitian),

$$\mathbf{e}^* \cdot [\nabla \times (A\mathbf{k} \times \mathbf{e})] + \mathbf{e}^* \cdot \{\mathbf{k} \times [\nabla \times (A\mathbf{e})]\} = 0. \tag{4.2}$$

Using $\nabla \times (A\mathbf{k} \times \mathbf{e}) = \nabla A \times (\mathbf{k} \times \mathbf{e}) + A\nabla \times (\mathbf{k} \times \mathbf{e})$ and $\mathbf{k} \times [\nabla \times (A\mathbf{e})] = \mathbf{k} \times (\nabla A \times \mathbf{e}) + A\mathbf{k} \times (\nabla \times \mathbf{e})$, we find

$$[\mathbf{e}^* \times (\mathbf{k} \times \mathbf{e}) + (\mathbf{e}^* \times \mathbf{k}) \times \mathbf{e}] \cdot \nabla \ln A = \mathbf{e}^* \cdot [\nabla \times (\mathbf{k} \times \mathbf{e})] + (\mathbf{e}^* \times \mathbf{k}) \cdot (\nabla \times \mathbf{e}). \quad (4.3)$$

Here it is important to point out that the dispersion relation (2.12) determines $\mathbf{e}$ up to a complex multiplier, that is, given a polarization $\mathbf{e}$ (a solution to equation (2.12)), we can find another valid polarization $\underline{\mathbf{e}}$ simply by multiplying it by any complex function $T(\mathbf{r}) = |T|(\mathbf{r}) \exp(\mathrm{i}\tau(\mathbf{r}))$, $\underline{\mathbf{e}} = T\mathbf{e}$. We should find that the complex function $A(\mathbf{r})$ "corrects" for any choices that we make when determining $\mathbf{e}$. Thus, if A is the amplitude that corresponds to $\mathbf{e}$ and $\underline{A}$ the amplitude that corresponds to $\underline{\mathbf{e}}$, we expect $A\mathbf{e} = \underline{A}\,\underline{\mathbf{e}}$, giving $A = T\underline{A}$. This property is satisfied by equation (4.3) by construction.

To solve equation (4.3), we use the polar form of A, $A = |A| \exp(\mathrm{i}\alpha)$. Then, $\nabla \ln A = \nabla \ln |A| + \mathrm{i}\nabla\alpha$, and we can get independent equations for $|A|$ and α by splitting equation (4.3) into its real and imaginary parts. Using the fact that $\mathbf{e}^* \times (\mathbf{k} \times \mathbf{e}) + (\mathbf{e}^* \times \mathbf{k}) \times \mathbf{e}$ is a real vector, we find that the real part of equation (4.3) is

$$[\mathbf{e}^* \times (\mathbf{k} \times \mathbf{e}) + (\mathbf{e}^* \times \mathbf{k}) \times \mathbf{e}] \cdot \nabla \ln |A| = \mathrm{Re}\Big(\mathbf{e}^* \cdot [\nabla \times (\mathbf{k} \times \mathbf{e})] + (\mathbf{e}^* \times \mathbf{k}) \cdot (\nabla \times \mathbf{e})\Big). \quad (4.4)$$

and the imaginary part is

$$[\mathbf{e}^* \times (\mathbf{k} \times \mathbf{e}) + (\mathbf{e}^* \times \mathbf{k}) \times \mathbf{e}] \cdot \nabla \alpha = \mathrm{Im}\Big(\mathbf{e}^* \cdot [\nabla \times (\mathbf{k} \times \mathbf{e})] + (\mathbf{e}^* \times \mathbf{k}) \cdot (\nabla \times \mathbf{e})\Big). \quad (4.5)$$

We proceed to simplify these two equations.

4.1. *Equation for $|A|$: energy conservation*

Using the relations

$$\mathrm{Re}\Big((\mathbf{e}^* \times \mathbf{k}) \cdot (\nabla \times \mathbf{e})\Big) = \mathrm{Re}\Big([(\mathbf{e} \times \mathbf{k}) \cdot (\nabla \times \mathbf{e}^*)]^*\Big) = \mathrm{Re}\Big((\mathbf{e} \times \mathbf{k}) \cdot (\nabla \times \mathbf{e}^*)\Big) \quad (4.6)$$

and

$$\mathbf{e}^* \cdot [\nabla \times (\mathbf{k} \times \mathbf{e})] + (\mathbf{e} \times \mathbf{k}) \cdot (\nabla \times \mathbf{e}^*) = -\nabla \cdot [\mathbf{e}^* \times (\mathbf{k} \times \mathbf{e})], \quad (4.7)$$

we can rewrite equation (4.4) as

$$\begin{aligned} [\mathbf{e}^* \times (\mathbf{k} \times \mathbf{e}) + (\mathbf{e}^* \times \mathbf{k}) \times \mathbf{e}] \cdot \nabla \ln |A| &= -\nabla \cdot \Big[\mathrm{Re}\Big(\mathbf{e}^* \times (\mathbf{k} \times \mathbf{e})\Big)\Big] \\ &= -\frac{1}{2}\nabla \cdot [\mathbf{e}^* \times (\mathbf{k} \times \mathbf{e}) + (\mathbf{e}^* \times \mathbf{k}) \times \mathbf{e}]. \quad (4.8) \end{aligned}$$

Multiplying this equation by $2|A|^2$, we find the useful expression

$$\nabla \cdot \Big\{[\mathbf{e}^* \times (\mathbf{k} \times \mathbf{e}) + (\mathbf{e}^* \times \mathbf{k}) \times \mathbf{e}]|A|^2\Big\} = 0. \quad (4.9)$$

Equation (4.9) can be expressed in a more physical form that can be deduced from the dispersion relation (3.1) multiplied by ω^2,

$$\begin{aligned} \omega^2(\mathbf{k}, \mathbf{r}) D(\mathbf{k}, \omega(\mathbf{k}, \mathbf{r}), \mathbf{r}) \equiv{}& c^2 \mathbf{e}^*(\mathbf{k}, \mathbf{r}) \cdot [\mathbf{k} \times (\mathbf{k} \times \mathbf{e}(\mathbf{k}, \mathbf{r}))] \\ &+ \omega^2(\mathbf{k}, \mathbf{r}) \mathbf{e}^*(\mathbf{k}, \mathbf{r}) \cdot \boldsymbol{\epsilon}(\mathbf{k}, \omega(\mathbf{k}, \mathbf{r}), \mathbf{r}) \cdot \mathbf{e}(\mathbf{k}, \mathbf{r}) \equiv 0. \quad (4.10) \end{aligned}$$

Differentiating this expression with respect to $\mathbf{k}$ holding $\mathbf{r}$ fixed, we find

$$-c^2[\mathbf{e}^* \times (\mathbf{k} \times \mathbf{e}) + (\mathbf{e}^* \times \mathbf{k}) \times \mathbf{e}] + \nabla_{\mathbf{k}}\omega\, \mathbf{e}^* \cdot \frac{\partial}{\partial \omega}(\omega^2 \boldsymbol{\epsilon}) \cdot \mathbf{e} = 0, \quad (4.11)$$

where the derivatives with respect to $\mathbf{k}$ of $\mathbf{e}$ and $\mathbf{e}^*$ do not appear because $c^2[\mathbf{k} \times (\mathbf{k} \times$

$\mathbf{e})] + \omega^2 \boldsymbol{\epsilon} \cdot \mathbf{e} = 0 = c^2[(\mathbf{e}^* \times \mathbf{k}) \times \mathbf{k}] + \omega^2 \mathbf{e}^* \cdot \boldsymbol{\epsilon}$. Using equation (4.11), we can rewrite (4.9) as

$$\nabla \cdot \left[\mathbf{e}^* \cdot \frac{\partial}{\partial \omega}(\omega^2 \boldsymbol{\epsilon}) \cdot \mathbf{e} |A|^2 \mathbf{v}_g \right] = 0, \tag{4.12}$$

where we have used the definition of the group velocity $\mathbf{v}_g = \nabla_{\mathbf{k}} \omega$.

Result (4.12) is related to energy conservation. Poynting's electromagnetic energy conservation equation is

$$\frac{\partial}{\partial t} \left(\frac{\epsilon_0 |\mathbf{E}|^2}{2} + \frac{|\mathbf{B}|^2}{2\mu_0} \right) + \nabla \cdot \left(\frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \right) = -\mathbf{J} \cdot \mathbf{E}. \tag{4.13}$$

We perturb this equation using the electric field, magnetic field and current density of a single wave,

$$\begin{aligned} \frac{\partial}{\partial t} \left(\epsilon_0 \mathbf{E} \cdot \delta \mathbf{E} + \frac{\mathbf{B} \cdot \delta \mathbf{B}}{\mu_0} + \frac{\epsilon_0 |\delta \mathbf{E}|^2}{2} + \frac{|\delta \mathbf{B}|^2}{2\mu_0} \right) + \nabla \cdot \Bigg(\frac{1}{\mu_0} \delta \mathbf{E} \times \mathbf{B} + \frac{1}{\mu_0} \mathbf{E} \times \delta \mathbf{B} \\ + \frac{1}{\mu_0} \delta \mathbf{E} \times \delta \mathbf{B} \Bigg) = -\delta \mathbf{J} \cdot \mathbf{E} - \mathbf{J} \cdot \delta \mathbf{E} - \delta \mathbf{J} \cdot \delta \mathbf{E}. \end{aligned} \tag{4.14}$$

Here

$$\begin{aligned} \delta \mathbf{E} &= \mathrm{Re}(\tilde{\mathbf{E}}(\mathbf{r}) \exp(\mathrm{i}S(\mathbf{r}) - \mathrm{i}\omega t)), \\ \delta \mathbf{B} &= \mathrm{Re} \left(\frac{\mathbf{k}(\mathbf{r}) \times \tilde{\mathbf{E}}(\mathbf{r})}{\omega} \exp(\mathrm{i}S(\mathbf{r}) - \mathrm{i}\omega t) \right), \\ \delta \mathbf{J} &= \mathrm{Re} \left(\boldsymbol{\sigma}(\mathbf{r}) \cdot \tilde{\mathbf{E}}(\mathbf{r}) \exp(\mathrm{i}S(\mathbf{r}) - \mathrm{i}\omega t) \right), \end{aligned} \tag{4.15}$$

and $\boldsymbol{\sigma}$ is the conductivity tensor. Note that we only consider the real part of the WKB form. Averaging equation (4.14) over a period of the wave, $\langle \ldots \rangle_t = (\omega/2\pi) \int_t^{t+2\pi/\omega} (\ldots) \, \mathrm{d}t'$, we find

$$\nabla \cdot \left(\frac{1}{\mu_0} \langle \delta \mathbf{E} \times \delta \mathbf{B} \rangle_t \right) = -\langle \delta \mathbf{J} \cdot \delta \mathbf{E} \rangle_t. \tag{4.16}$$

Using (4.15), we find

$$\frac{1}{\mu_0} \langle \delta \mathbf{E} \times \delta \mathbf{B} \rangle_t = \frac{\epsilon_0 c^2}{4\omega} [\tilde{\mathbf{E}}^* \times (\mathbf{k} \times \tilde{\mathbf{E}}) + \tilde{\mathbf{E}} \times (\mathbf{k} \times \tilde{\mathbf{E}}^*)] \tag{4.17}$$

and

$$\langle \delta \mathbf{J} \cdot \delta \mathbf{E} \rangle_t = \frac{1}{4} \tilde{\mathbf{E}}^* \cdot (\boldsymbol{\sigma} + \boldsymbol{\sigma}^\dagger) \cdot \tilde{\mathbf{E}}. \tag{4.18}$$

Since the cold plasma wave conductivity tensor is anti-Hermitian, $\boldsymbol{\sigma}^\dagger = -\boldsymbol{\sigma}$, the term $\langle \delta \mathbf{J} \cdot \delta \mathbf{E} \rangle_t$ vanishes, $\langle \delta \mathbf{J} \cdot \delta \mathbf{E} \rangle_t = 0$. Then, equation (4.16) becomes

$$\nabla \cdot \left(\frac{1}{\mu_0} \langle \delta \mathbf{E} \times \delta \mathbf{B} \rangle_t \right) = 0. \tag{4.19}$$

This equation gives A. Indeed, using $\tilde{\mathbf{E}} \simeq \tilde{\mathbf{E}}_0 = A\mathbf{e}$ in (4.17), we find that the time averaged Poynting vector becomes

$$\frac{1}{\mu_0} \langle \delta \mathbf{E} \times \delta \mathbf{B} \rangle_t = \frac{\epsilon_0 c^2 |A|^2}{4\omega} [\mathbf{e}^* \times (\mathbf{k} \times \mathbf{e}) + (\mathbf{e}^* \times \mathbf{k}) \times \mathbf{e}]. \tag{4.20}$$

Note that equations (4.19) and (4.20) are the same as equation (4.9) but for a few constants. Due to its relation to energy conservation, it is common to see equation (4.12) written as

$$\nabla \cdot \left(\frac{1}{\mu_0} \langle \delta \mathbf{E} \times \delta \mathbf{B} \rangle_t \right) = \nabla \cdot (W_{\text{wave}} \mathbf{v}_g) = 0, \tag{4.21}$$

where the energy density of the wave is

$$W_{\text{wave}} = \frac{\epsilon_0 |A|^2}{4\omega} \mathbf{e}^* \cdot \frac{\partial}{\partial \omega} (\omega^2 \boldsymbol{\epsilon}) \cdot \mathbf{e}. \tag{4.22}$$

4.2. *Equation for α*

Using equation (4.11), we rewrite equation (4.5) as

$$\mathbf{v}_g \cdot \nabla \alpha = \left[\frac{1}{c^2} \mathbf{e}^* \cdot \frac{\partial}{\partial \omega} (\omega^2 \boldsymbol{\epsilon}) \cdot \mathbf{e} \right]^{-1} \text{Im}\Big(\mathbf{e}^* \cdot [\nabla \times (\mathbf{k} \times \mathbf{e})] + (\mathbf{e}^* \times \mathbf{k}) \cdot (\nabla \times \mathbf{e}) \Big). \tag{4.23}$$

The phase α corrects for the phase that we have chosen for the polarization vector $\mathbf{e}$. As explained after equation (4.3), if we had chosen $\underline{\mathbf{e}} = \mathbf{e} \exp(-\mathrm{i}\alpha)$, we would have found a phase $\underline{\alpha} = 0$. In other words, it is always possible to choose the phase of $\mathbf{e}$ such that $\text{Im}(\mathbf{e}^* \cdot [\nabla \times (\mathbf{k} \times \mathbf{e})] + (\mathbf{e}^* \times \mathbf{k}) \cdot (\nabla \times \mathbf{e})) = 0$. With this choice, the phase α is constant.

Finally, it is often useful to rewrite equation (4.23) in a different form. Using $\mathbf{e}^* \cdot [\nabla \times (\mathbf{k} \times \mathbf{e})] = \mathbf{e} \cdot \nabla \mathbf{k} \cdot \mathbf{e}^* + (\mathbf{e}^* \cdot \mathbf{k})(\nabla \cdot \mathbf{e}) - |\mathbf{e}|^2 \nabla \cdot \mathbf{k} - \mathbf{k} \cdot \nabla \mathbf{e} \cdot \mathbf{e}^*$ and $(\mathbf{e}^* \times \mathbf{k}) \cdot (\nabla \times \mathbf{e}) = \mathbf{e}^* \cdot \nabla \mathbf{e} \cdot \mathbf{k} - \mathbf{k} \cdot \nabla \mathbf{e} \cdot \mathbf{e}^*$, we can rewrite equation (4.23) as

$$\mathbf{v}_g \cdot \nabla \alpha = \left[\frac{1}{c^2} \mathbf{e}^* \cdot \frac{\partial}{\partial \omega} (\omega^2 \boldsymbol{\epsilon}) \cdot \mathbf{e} \right]^{-1} \text{Im}\Big(\mathbf{e} \cdot \nabla \mathbf{k} \cdot \mathbf{e}^* + (\mathbf{e}^* \cdot \mathbf{k})(\nabla \cdot \mathbf{e}) + \mathbf{e}^* \cdot \nabla \mathbf{e} \cdot \mathbf{k} - 2\mathbf{k} \cdot \nabla \mathbf{e} \cdot \mathbf{e}^* \Big). \tag{4.24}$$

Note that $\mathbf{k} = \nabla S$, and hence $\text{Im}(\mathbf{e} \cdot \nabla \mathbf{k} \cdot \mathbf{e}^*) = \text{Im}(\mathbf{e} \cdot \nabla \nabla S \cdot \mathbf{e}^*) = 0$ because $\nabla \nabla S$ is symmetric. Thus, equation (4.24) becomes

$$\mathbf{v}_g \cdot \nabla \alpha = \left[\frac{1}{c^2} \mathbf{e}^* \cdot \frac{\partial}{\partial \omega} (\omega^2 \boldsymbol{\epsilon}) \cdot \mathbf{e} \right]^{-1} \text{Im}\Big((\mathbf{e}^* \cdot \mathbf{k})(\nabla \cdot \mathbf{e}) + \mathbf{e}^* \cdot \nabla \mathbf{e} \cdot \mathbf{k} - 2\mathbf{k} \cdot \nabla \mathbf{e} \cdot \mathbf{e}^* \Big). \tag{4.25}$$

The phase α only depends on the gradient of the polarization $\mathbf{e}$.

4.3. *Example*

To see how equations (4.21) and (4.25) determine the amplitude A, we consider the simple case in figure 2. For the ordinary mode, we can choose $\mathbf{e} = \hat{\mathbf{b}}$. Since the magnetic field is constant, $\nabla \mathbf{e} = 0$ and equation (4.25) becomes $\mathbf{v}_g \cdot \nabla \alpha = 0$ for our choice of $\mathbf{e}$. Then, $\alpha = \alpha(x_{\text{ant}})$ everywhere. Equation (4.21) gives

$$W_{\text{wave}} = \frac{\epsilon_0 |A|^2}{4\omega} \hat{\mathbf{b}} \cdot \frac{\partial}{\partial \omega} \left(\omega^2 \boldsymbol{\epsilon} \right) \cdot \hat{\mathbf{b}} = \frac{\epsilon_0 |A|^2}{4\omega} \frac{\partial}{\partial \omega} \left(\omega^2 \epsilon_\parallel \right) = \frac{\epsilon_0 |A|^2}{2}, \tag{4.26}$$

and then equation (4.21) becomes

$$\frac{\partial}{\partial x} \left(\frac{\epsilon_0 c^2}{2\omega} |A|^2 k_x \right) = 0. \tag{4.27}$$

The final result for $A = |A| \exp(\mathrm{i}\alpha)$ is

$$A(x) = A(x_{\text{ant}}) \sqrt{\frac{k_x(x_{\text{ant}})}{|k_x(x)|}} = A(x_{\text{ant}}) \left(\frac{k_x^2(x_{\text{ant}}) c^2}{k_x^2(x_{\text{ant}}) c^2 + \omega_{pe}^2(x_{\text{ant}}) - \omega_{pe}^2(x)} \right)^{1/4}, \tag{4.28}$$

where we have used the result in (3.13). Note that the amplitude of the wave diverges when k_x vanishes. The reason for this divergence is that v_{gx} vanishes, and as a result

$W_{\rm wave}$ must go to infinity to make $v_{gx}W_{\rm wave}$ finite and equal to the energy flux coming from the antenna. To avoid this divergence, we need to treat carefully the region where A becomes large, as we do in the next section.

5. Cutoffs and resonances

There will be regions in the plasma where the solution to the local dispersion relation in (2.12) gives $k^2 < 0$, or where the square of one of the components of $\mathbf{k}$ is negative (for example, $k_x^2 < 0$). A plasma wave cannot propagate into this region. At the surfaces limiting these forbidden regions, k^2 or the square of one of the components of $\mathbf{k}$, k_i^2, may have vanished (this is the case for $\omega = \omega_{pe}$ in (2.13), or $\omega = \omega_L, \omega_R$ in (2.14)), or it may have diverged (for example, $\omega = \omega_{UH}, \omega_{LH}$ in (2.14)). These limiting surfaces are cutoffs and resonances.

5.1. *Cutoffs*

When $k^2 = 0$ or $k_i^2 = 0$, the plasma wave reflects. This result is intuitively shown in figure 2. To show that the wave reflects, and to calculate the phase between the incoming wave and the outgoing wave, the region near the cutoff must be treated carefully. In this region, $k_i^2 \to 0$, and the assumption $k_i L \gg 1$ is not satisfied. The characteristic length l of the region in which the assumption $k_i L \gg 1$ is not valid is determined by $k_i l \sim 1$. Since the cold plasma wave dispersion relation gives $k_i^2 = F(\mathbf{r})$, near the cut-off where $k_i^2 = 0$, $k_i^2 \sim l|\nabla F|$. Thus, $k_i l \sim 1$ leads to $l^{3/2}\sqrt{|\nabla F|} \sim 1$, and we obtain

$$l \sim \frac{1}{|\nabla F|^{1/3}} \sim \frac{L}{(kL)^{2/3}} \ll L, \tag{5.1}$$

where we have used the order of magnitude estimate $|\nabla F| \sim k^2/L$. At a distance l from the cutoff, the behavior of the wave is described by the Airy equation. To illustrate the procedure needed to resolve the cut-off, we consider the case shown in figure 2.

The cutoff location x_c in figure 2 is determined by the equation

$$\omega_{pe}^2(x_c) = \omega^2 - k_y^2 c^2. \tag{5.2}$$

At a distance l from the cutoff, the WKB form that we have assumed, given in (2.2), fails, as demonstrated by the fact that the amplitude diverges at the cutoff. Thus, we use the more general solution

$$\delta\mathbf{E} = E_c(x)\hat{\mathbf{b}}\exp(\mathrm{i}k_y y - \mathrm{i}\omega t). \tag{5.3}$$

Note that we have already included the direction of the polarization that is of interest to us. Repeating the derivation in the notes on cold plasma waves with this form for the electric field, we obtain

$$\frac{\mathrm{d}^2 E_c}{\mathrm{d}x^2} + \left(\frac{\omega^2 - \omega_{pe}^2(x)}{c^2} - k_y^2\right) E_c = 0. \tag{5.4}$$

Since we are only interested in a region $x - x_c \sim l \ll L$, we can Taylor expand $\omega_{pe}^2(x)$ around x_c, $\omega_{pe}^2(x) \simeq \omega_{pe}^2(x_c)[1 + (\mathrm{d}\ln n_e/\mathrm{d}x)(x - x_c)]$. Using this Taylor expansion and equation (5.2), equation (5.4) becomes

$$\frac{\mathrm{d}^2 E_c}{\mathrm{d}x^2} - \frac{\omega_{pe}^2(x_c)}{c^2}\frac{\mathrm{d}\ln n_e}{\mathrm{d}x}(x - x_c)E_c = 0. \tag{5.5}$$

To solve this equation, we use the normalized spatial variable

$$z = \frac{x - x_c}{l}, \tag{5.6}$$

where the characteristic length l for this problem is

$$l = \left(\frac{\omega_{pe}^2(x_c)}{c^2} \frac{\mathrm{d} \ln n_e}{\mathrm{d}x} \right)^{-1/3}. \tag{5.7}$$

Note the connection with equation (5.1). Using the normalized variable z, equation (5.5) becomes the Airy equation,

$$\frac{\mathrm{d}^2 E_c}{\mathrm{d}z^2} - z E_c = 0. \tag{5.8}$$

We need to impose boundary conditions for E_c at large $|z| \sim |x - x_c|/l$. For negative $x - x_c$ and $|x - x_c| \gg l$, the solution should contain the incoming wave

$$A(x)\hat{\mathbf{b}} \exp\left(\mathrm{i} \int_{x_{\mathrm{ant}}}^{x} |k_x(x')| \, \mathrm{d}x' + \mathrm{i}k_y y - \mathrm{i}\omega t \right), \tag{5.9}$$

with $k_x(x)$ given in equation (3.13) and A given in equation (4.28). We also expect a reflected or outgoing wave that according to the ray tracing equations should be of the form

$$R(x)\hat{\mathbf{b}} \exp\left(-\mathrm{i} \int_{x_{\mathrm{ant}}}^{x} |k_x(x')| \, \mathrm{d}x' + \mathrm{i}k_y y - \mathrm{i}\omega t \right), \tag{5.10}$$

where k_x also satisfies equation (3.13), and R satisfies an equation similar to (4.28),

$$R(x) = R(x_{\mathrm{ant}}) \sqrt{\frac{k_x(x_{\mathrm{ant}})}{|k_x(x)|}}. \tag{5.11}$$

Since we are interested in $|x - x_c| \ll L$, we can Taylor expand k_x around x_c to find

$$k_x(x) \simeq \pm \sqrt{-\frac{\omega_{pe}^2(x_c)}{c^2} \frac{\mathrm{d} \ln n_e}{\mathrm{d}x} (x - x_c)} = \pm \frac{(-z)^{1/2}}{l} \tag{5.12}$$

and hence $\int_{x_{\mathrm{ant}}}^{x} |k_x(x')| \, \mathrm{d}x' = \int_{x_{\mathrm{ant}}}^{x_c} |k_x(x')| \, \mathrm{d}x' - \int_{x}^{x_c} |k_x(x')| \, \mathrm{d}x' \simeq \int_{x_{\mathrm{ant}}}^{x_c} |k_x(x')| \, \mathrm{d}x' - (2/3)(-z)^{3/2}$. Then, the solution in the region $l \ll x_c - x \ll L$ is

$$E_c(z) \simeq \frac{\sqrt{k_x(x_{\mathrm{ant}}) l}}{(-z)^{1/4}} \left[A(x_{\mathrm{ant}}) \exp\left(\mathrm{i} \int_{x_{\mathrm{ant}}}^{x_c} |k_x(x')| \, \mathrm{d}x' - \frac{2\mathrm{i}}{3} (-z)^{3/2} \right) \right.$$
$$\left. + R(x_{\mathrm{ant}}) \exp\left(-\mathrm{i} \int_{x_{\mathrm{ant}}}^{x_c} |k_x(x')| \, \mathrm{d}x' + \frac{2\mathrm{i}}{3} (-z)^{3/2} \right) \right]. \tag{5.13}$$

This solution is consistent with the solutions of equation (5.8) at $z \to -\infty$. We also need to impose a boundary condition for positive $x - x_c$ or, equivalently, for positive z. The solutions of equation (5.8) at $z \to \infty$ are

$$E_c(z) \propto \frac{1}{z^{1/4}} \exp\left(\pm \frac{2}{3} z^{3/2} \right). \tag{5.14}$$

One of the solutions diverges, and we do not expect that solution to be physical, so we choose the other one by requiring that $E_c(z) \to 0$ for $z \to +\infty$.

Note that the boundary conditions imply that E_c vanishes for $z \to \pm\infty$. Thus, to solve

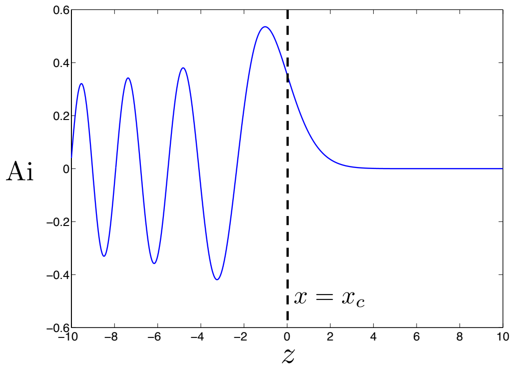

FIGURE 3. Graph of the Airy function. The location of the cut-off is indicated as a dashed line.

equation (5.8), we can Fourier transform E_c, $\tilde{E}_c(s) = \int_{-\infty}^{\infty} E_c(z)\exp(-\mathrm{i}sz)\,\mathrm{d}z$. Fourier transforming equation (5.8), we obtain

$$-s^2\tilde{E}_c - \mathrm{i}\frac{\mathrm{d}\tilde{E}_c}{\mathrm{d}s} = 0. \tag{5.15}$$

The solution to this equation is $\tilde{E}_c = K\exp(\mathrm{i}s^3/3)$, where K is a constant of integration. Using the inverse of the Fourier transform, we obtain the solution

$$E_c(z) = K\,\mathrm{Ai}(z), \tag{5.16}$$

where

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(\frac{\mathrm{i}s^3}{3} + \mathrm{i}sz\right)\mathrm{d}s \tag{5.17}$$

is the Airy function (see figure 3 for a graph of the Airy function). This solution decays exponentially for $z \to +\infty$, as expected. For negative and large z, the behavior of $\mathrm{Ai}(z)$ can be calculated using the method of stationary phase (see Appendix A),

$$\mathrm{Ai}(z) \simeq \frac{1}{2\sqrt{\pi}(-z)^{1/4}}\left[\exp\left(-\frac{2\mathrm{i}}{3}(-z)^{3/2} + \frac{\mathrm{i}\pi}{4}\right) + \exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2} - \frac{\mathrm{i}\pi}{4}\right)\right]. \tag{5.18}$$

Using this approximation, we can match the solution (5.16) with the boundary condition (5.13), finding

$$\begin{aligned}\frac{K\exp(\mathrm{i}\pi/4)}{2\sqrt{\pi}} &= \sqrt{k_x(x_{\mathrm{ant}})l}A(x_{\mathrm{ant}})\exp\left(\mathrm{i}\int_{x_{\mathrm{ant}}}^{x_c}|k_x(x')|\,\mathrm{d}x'\right),\\ \frac{K\exp(-\mathrm{i}\pi/4)}{2\sqrt{\pi}} &= \sqrt{k_x(x_{\mathrm{ant}})l}R(x_{\mathrm{ant}})\exp\left(-\mathrm{i}\int_{x_{\mathrm{ant}}}^{x_c}|k_x(x')|\,\mathrm{d}x'\right).\end{aligned} \tag{5.19}$$

Thus, the cutoff region solution that does not diverge for $x > x_c$ imposes

$$R(x_{\mathrm{ant}}) = A(x_{\mathrm{ant}})\exp\left(2\mathrm{i}\int_{x_{\mathrm{ant}}}^{x_c}|k_x(x')|\,\mathrm{d}x' - \frac{\mathrm{i}\pi}{2}\right). \tag{5.20}$$

All the energy is reflected at the cutoff ($A(x)$ and $R(x)$ have the same magnitude), and the reflected wave has an added phase of $-\pi/2$.

5.2. *Resonance*

When $k^2 \to \infty$, the wave resonates with the plasma, and usually the wave is absorbed. In the region of the plasma where k^2 is large, the cold plasma condition $kv_t/\omega \ll 1$ is not satisfied, and we have to start considering thermal effects. These effects will in general lead to resonances and absorption, but in some cases they can give rise to instabilities.

REFERENCES

Bender, C.M. & Orszag, S.A. 1999 *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*. Springer.

Appendix A. Airy function for large negative argument

For negative z and $|z| \gg 1$, the integral in (5.17) is dominated by the values of s around the maxima and minima of the phase $s^3/3 + sz$, that is, by the values of s around $\sqrt{-z}$ and $-\sqrt{-z}$. To obtain the integral, we follow the stationary phase method (Bender & Orszag 1999). Taylor expanding the exponent of the integrand of (5.17), we find that

$$\exp\left(\frac{\mathrm{i}s^3}{3} + \mathrm{i}sz\right) \simeq \exp\left(-\frac{2\mathrm{i}}{3}(-z)^{3/2} + \mathrm{i}\sqrt{-z}(s - \sqrt{-z})^2\right) \tag{A 1}$$

around $s = \sqrt{-z}$ and

$$\exp\left(\frac{\mathrm{i}s^3}{3} + \mathrm{i}sz\right) \simeq \exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2} - \mathrm{i}\sqrt{-z}(s + \sqrt{-z})^2\right) \tag{A 2}$$

around $s = -\sqrt{-z}$. For $|s - \sqrt{-z}| \gg 1/(-z)^{1/4}$ and $|s + \sqrt{-z}| \gg 1/(-z)^{1/4}$, the integrand $\exp(\mathrm{i}s^3/3 + \mathrm{i}sz)$ is highly oscillatory and it does not contribute much to the integral in (5.17), as we will show below.

For $-z \gg 1$, we write

$$\begin{aligned}
\mathrm{Ai}(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(\frac{\mathrm{i}s^3}{3} + \mathrm{i}sz\right) \mathrm{d}s \\
&\simeq \frac{1}{2\pi}\int_{-\sqrt{-z}-A/(-z)^{1/4}}^{-\sqrt{-z}+A/(-z)^{1/4}} \exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2} - \mathrm{i}\sqrt{-z}(s + \sqrt{-z})^2\right) \mathrm{d}s \\
&+ \frac{1}{2\pi}\int_{\sqrt{-z}-A/(-z)^{1/4}}^{\sqrt{-z}+A/(-z)^{1/4}} \exp\left(-\frac{2\mathrm{i}}{3}(-z)^{3/2} + \mathrm{i}\sqrt{-z}(s - \sqrt{-z})^2\right) \mathrm{d}s \\
&+ \frac{1}{2\pi}\int_{\mathrm{rest}} \exp\left(\frac{\mathrm{i}s^3}{3} + \mathrm{i}sz\right) \mathrm{d}s,
\end{aligned} \tag{A 3}$$

where A is a large positive number that satisfies

$$1 \ll A \ll (-z)^{3/4}. \tag{A 4}$$

We have assumed that $A \ll (-z)^{3/4}$ to ensure that the Taylor expansions (A 1) and (A 2) are valid. We will show that the exact value of A is not important. The last integral in (A 3) (the integral over the "rest") is the integral over what is left after subtracting the intervals $[-\sqrt{-z} - A/(-z)^{1/4}, -\sqrt{-z} + A/(-z)^{1/4}]$ and $[\sqrt{-z} - A/(-z)^{1/4}, \sqrt{-z} + A/(-z)^{1/4}]$,

$$\begin{aligned}
\int_{\mathrm{rest}} \exp\left(\frac{\mathrm{i}s^3}{3} + \mathrm{i}sz\right) \mathrm{d}s &= \int_{-\infty}^{-\sqrt{-z}-A/(-z)^{1/4}} \exp\left(\frac{\mathrm{i}s^3}{3} + \mathrm{i}sz\right) \mathrm{d}s \\
&+ \int_{-\sqrt{-z}+A/(-z)^{1/4}}^{\sqrt{-z}-A/(-z)^{1/4}} \exp\left(\frac{\mathrm{i}s^3}{3} + \mathrm{i}sz\right) \mathrm{d}s + \int_{\sqrt{-z}+A/(-z)^{1/4}}^{\infty} \exp\left(\frac{\mathrm{i}s^3}{3} + \mathrm{i}sz\right) \mathrm{d}s.
\end{aligned} \tag{A 5}$$

We will show at the end of this appendix that these integrals are negligible.

To calculate the first two integrals in equation (A 3), we use the complex plane. The first integral in (A 3) is equal to the integrals over the paths shown in figure 4(a): C (the straight line through $s = -\sqrt{-z}$ at a $-\pi/4$ angle with respect to the real axis), $C_{-\infty}$

and C_∞ (the two circumference sectors at large $|s+\sqrt{-z}|$). Thus,

$$\frac{1}{2\pi}\int_{-\sqrt{-z}-A/(-z)^{1/4}}^{-\sqrt{-z}+A/(-z)^{1/4}} \exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2}-\mathrm{i}\sqrt{-z}(s+\sqrt{-z})^2\right)\,\mathrm{d}s = \frac{1}{2\pi}\int_C \exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2}-\mathrm{i}\sqrt{-z}(s+\sqrt{-z})^2\right)\,\mathrm{d}s + \frac{1}{2\pi}\int_{C_{-\infty}} \exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2}-\mathrm{i}\sqrt{-z}(s+\sqrt{-z})^2\right)\,\mathrm{d}s + \frac{1}{2\pi}\int_{C_\infty} \exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2}-\mathrm{i}\sqrt{-z}(s+\sqrt{-z})^2\right)\,\mathrm{d}s. \tag{A 6}$$

The integral over C dominates. We calculate this integral using the new integration variable $s=-\sqrt{-z}+t\exp(-\mathrm{i}\pi/4)/(-z)^{1/4}$,

$$\frac{1}{2\pi}\int_C \exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2}-\mathrm{i}\sqrt{-z}(s+\sqrt{-z})^2\right)\,\mathrm{d}s = \frac{1}{2\pi(-z)^{1/4}}\exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2}-\frac{\mathrm{i}\pi}{4}\right)\int_{-A}^{A}\exp(-t^2)\,\mathrm{d}t. \tag{A 7}$$

Since we have chosen $A\gg 1$, we find $\int_{-A}^{A}\exp(-t^2)\,\mathrm{d}t \simeq \int_{-\infty}^{\infty}\exp(-t^2)\,\mathrm{d}t=\sqrt{\pi}$, leading to

$$\frac{1}{2\pi}\int_C \exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2}-\mathrm{i}\sqrt{-z}(s+\sqrt{-z})^2\right)\,\mathrm{d}s \simeq \frac{1}{2\sqrt{\pi}(-z)^{1/4}}\exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2}-\frac{\mathrm{i}\pi}{4}\right). \tag{A 8}$$

We proceed to show that the integrals over $C_{-\infty}$ and C_∞ are negligible. For the integral over $C_{-\infty}$ we use $s=-\sqrt{-z}+[A/(-z)^{1/4}]\exp(\mathrm{i}(\pi-\theta))$,

$$\frac{1}{2\pi}\int_{C_{-\infty}} \exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2}-\mathrm{i}\sqrt{-z}(s+\sqrt{-z})^2\right)\,\mathrm{d}s = -\frac{\mathrm{i}A}{2\pi(-z)^{1/4}}\exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2}\right)\int_0^{\pi/4}\exp\left(-\mathrm{i}A^2\exp(-2\mathrm{i}\theta)+\mathrm{i}(\pi-\theta)\right)\,\mathrm{d}\theta. \tag{A 9}$$

Using that in the interval $0<\theta<\pi/4$,

$$\left|\exp\left(-\mathrm{i}A^2\exp(-2\mathrm{i}\theta)+\mathrm{i}(\pi-\theta)\right)\right| = \exp\left(-A^2\sin 2\theta\right) \leqslant \exp\left(-\frac{4A^2}{\pi}\theta\right), \tag{A 10}$$

we find

$$\frac{1}{2\pi}\left|\int_{C_{-\infty}} \exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2}-\mathrm{i}\sqrt{-z}(s+\sqrt{-z})^2\right)\,\mathrm{d}s\right| \leqslant \frac{A}{2\pi(-z)^{1/4}}\int_0^{\pi/4}\exp\left(-\frac{4A^2}{\pi}\theta\right)\,\mathrm{d}\theta = O\left(\frac{1}{A(-z)^{1/4}}\right) \ll \frac{1}{(-z)^{1/4}}. \tag{A 11}$$

Thus, the integral over the path $C_{-\infty}$ is negligible compared to (A 8). Using a similar method, we can show that the integral over C_∞ is negligible as well, leaving

$$\frac{1}{2\pi}\int_{-\sqrt{-z}-A/(-z)^{1/4}}^{-\sqrt{-z}+A/(-z)^{1/4}} \exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2}-\mathrm{i}\sqrt{-z}(s+\sqrt{-z})^2\right)\,\mathrm{d}s \simeq \frac{1}{2\sqrt{\pi}(-z)^{1/4}}\exp\left(\frac{2\mathrm{i}}{3}(-z)^{3/2}-\frac{\mathrm{i}\pi}{4}\right). \tag{A 12}$$

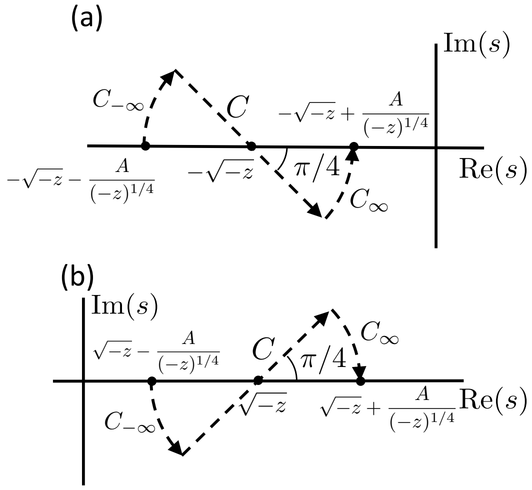

FIGURE 4. Contours in the complex plane used to take the integrals in equation (A 3).

We can calculate the second integral in (A 3) using the path shown in figure 4(b). Following the procedure that we used to obtain (A 12), we find

$$\frac{1}{2\pi}\int_{\sqrt{-z}-A/(-z)^{1/4}}^{\sqrt{-z}+A/(-z)^{1/4}} \exp\left(-\frac{2\mathrm{i}}{3}(-z)^{3/2}+\mathrm{i}\sqrt{-z}(s-\sqrt{-z})^2\right)\,\mathrm{d}s$$
$$\simeq \frac{1}{2\sqrt{\pi}(-z)^{1/4}}\exp\left(-\frac{2\mathrm{i}}{3}(-z)^{3/2}+\frac{\mathrm{i}\pi}{4}\right). \qquad \text{(A 13)}$$

Adding the integrals in (A 12) and (A 13), we find equation (5.18).

We finish by arguing that the integrals in (A 5) are negligible. We can prove it by integrating by parts. We show the procedure for the first integral in the right side of (A 5). Integrating by parts this integral, we find

$$\int_{-\infty}^{-\sqrt{-z}-A/(-z)^{1/4}} \exp\left(\frac{\mathrm{i}s^3}{3}+\mathrm{i}sz\right)\,\mathrm{d}s = -\left[\frac{\mathrm{i}\exp(\mathrm{i}s^3/3+\mathrm{i}sz)}{s^2+z}\right]_{-\infty}^{-\sqrt{-z}-A/(-z)^{1/4}}$$
$$+\mathrm{i}\int_{-\infty}^{-\sqrt{-z}-A/(-z)^{1/4}} \exp\left(\frac{\mathrm{i}s^3}{3}+\mathrm{i}sz\right)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+z}\right)\,\mathrm{d}s. \qquad \text{(A 14)}$$

In the first term of this equation, the limit $s=-\sqrt{-z}-A/(-z)^{1/4}$ dominates, giving

$$-\left[\frac{\mathrm{i}\exp(\mathrm{i}s^3/3+\mathrm{i}sz)}{s^2+z}\right]_{-\infty}^{-\sqrt{-z}-A/(-z)^{1/4}} = O\left(\frac{1}{A(-z)^{1/4}}\right). \qquad \text{(A 15)}$$

The second integral in the right side of (A 14) can be bounded. The integrand of the second integral in the right side of (A 14) diverges as $1/\sqrt{-z}(s+\sqrt{-z})^2$ for s near $-\sqrt{-z}$. Thus, we will find its maximum value in this region. Taking this into consideration, in the interval $(-\infty,-\sqrt{-z}-A/(-z)^{1/4}]$, there is a constant $K\sim 1$ such that

$$\left|\exp\left(\frac{\mathrm{i}s^3}{3}+\mathrm{i}sz\right)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+z}\right)\right| \leqslant \frac{K}{\sqrt{-z}(s+\sqrt{-z})^2}, \qquad \text{(A 16)}$$

leading to

$$\left| \mathrm{i} \int_{-\infty}^{-\sqrt{-z}-A/(-z)^{1/4}} \exp\left(\frac{\mathrm{i}s^3}{3} + \mathrm{i}sz\right) \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+z}\right) \mathrm{d}s \right|$$
$$\leqslant \int_{-\infty}^{-\sqrt{-z}-A/(-z)^{1/4}} \frac{K}{\sqrt{-z}(s+\sqrt{-z})^2}\, \mathrm{d}s = O\left(\frac{1}{A(-z)^{1/4}}\right). \tag{A 17}$$

This bound is not very accurate, and it can be made better by integrating by parts again. However, to prove that the integral is negligible, this bound is sufficient. Estimates (A 15) and (A 17) give

$$\int_{-\infty}^{-\sqrt{-z}-A/(-z)^{1/4}} \exp\left(\frac{\mathrm{i}s^3}{3} + \mathrm{i}sz\right) \mathrm{d}s = O\left(\frac{1}{A(-z)^{1/4}}\right) \ll \frac{1}{(-z)^{1/4}}. \tag{A 18}$$

Thus, this integral is much smaller than the main contribution (5.18). All the integrals in (A 5) are of the same order, that is, the integrals in (A 5) are negligible,

$$\int_{\mathrm{rest}} \exp\left(\frac{\mathrm{i}s^3}{3} + \mathrm{i}sz\right) \mathrm{d}s = O\left(\frac{1}{A(-z)^{1/4}}\right) \ll \frac{1}{(-z)^{1/4}}. \tag{A 19}$$