Energy-momentum tensor correlators in hot Yang-Mills theory

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Micro-workshop on analytic properties of thermal correlators
University of Oxford, 6.3.2017

Mikko Laine, Mikko Vepsäläinen, AV, 1008.3263, 1011.4439
Mikko Laine, AV, Yan Zhu, 1108.1259
York Schröder, Mikko Vepsäläinen, AV, Yan Zhu, 1109.6548
AV, Yan Zhu, 1212.3818, 1502.02556
Yan Zhu, Ongoing work

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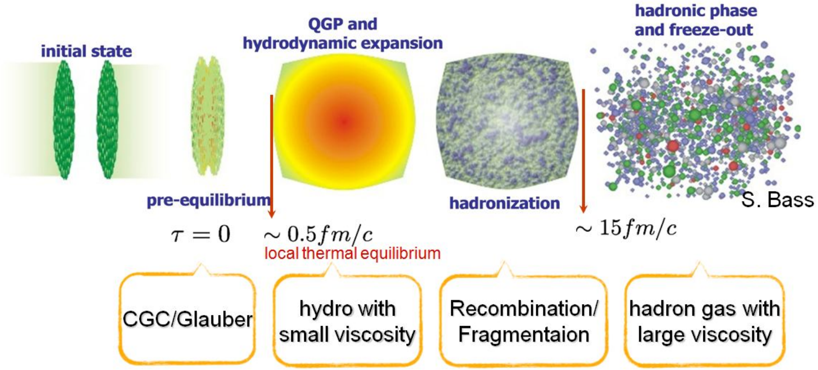
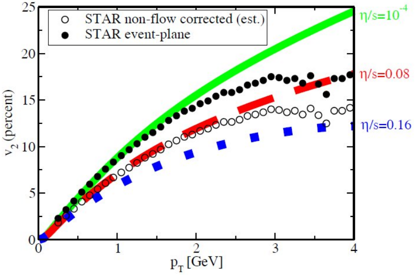
Background: Heavy ion collisions

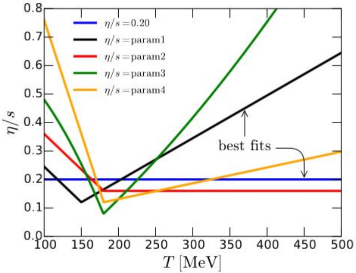


Expansion of thermalizing plasma surprisingly well described in terms of a low energy effective theory — hydrodynamics

- UV physics encoded in transport coefficients: η, ζ,..

Transport coefficients from data





Observation: Hydro results particularly sensitive to shear viscosity

- RHIC data indicated extremely low viscosity; recently attempts towards extracting $\eta(T)$ from RHIC+LHC data (Eskola et al.)

Related general question: Can the QGP be characterized as strongly/weakly coupled at RHIC/LHC?

Ultimate answers only from non-perturbative calculations in QCD

Why pQCD I: Transport coefficients from lattice

Kubo formulas: Transport coeffs. from IR limit of retarded Minkowski correlators — viscosities from those of energy momentum tensor $T_{\mu\nu}$:

$$\eta = \lim_{\omega\to 0}\frac{1}{\omega}\,\mathrm{Im}\,D^{\mathrm{R}}_{12,12}(\omega,\mathbf{k}=0) \equiv \lim_{\omega\to 0}\frac{\rho_{12,12}(\omega)}{\omega}$$

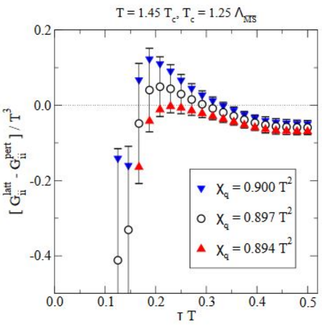
$$\zeta = \lim_{\omega\to 0}\frac{\pi}{9}\sum_{ij}\frac{1}{\omega}\,\mathrm{Im}\,D^{\mathrm{R}}_{ii,jj}(\omega,\mathbf{k}=0) \equiv \frac{\pi}{9}\sum_{ij}\lim_{\omega\to 0}\frac{\rho_{ii,jj}(\omega)}{\omega}$$

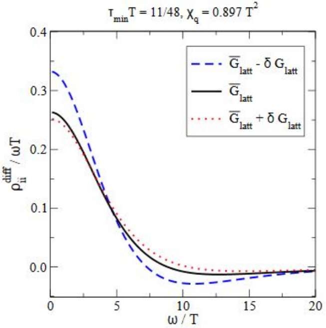
Problem: Lattice can only measure Euclidean correlators $\to$ Spectral density available only through inversion of

$$G(\tau) = \int_0^\infty \frac{\mathrm{d}\omega}{\pi}\rho(\omega)\frac{\cosh\frac{(\beta-2\tau)\omega}{2}}{\sinh\frac{\beta\omega}{2}}$$

$\therefore$ To extract IR limit of ρ, need to understand its behavior also at $\omega \gtrsim \pi T$ — perturbative input needed

Why pQCD I: Transport coefficients from lattice





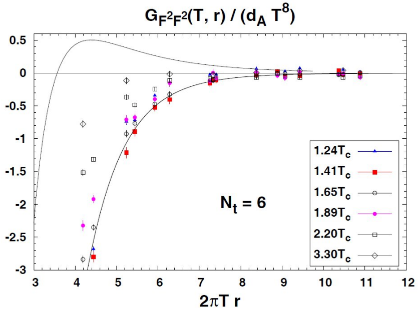
Analytic continuation from imaginary time correlator possible with precise lattice data and perturbative result

Successful example: nonperturbative flavor current spectral density and flavor diffusion coefficient [Burnier, Laine, 1201.1994]

Why pQCD II: Comparisons with lattice and AdS

Euclidean correlators provide direct information about medium ⇒ Comparisons between lattice QCD, pQCD and AdS/CFT valuable

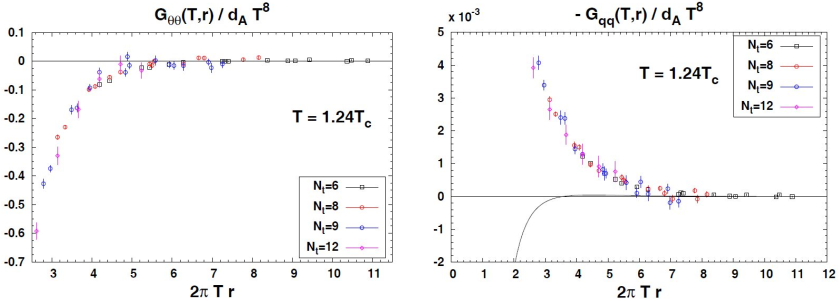
Iqbal, Meyer (0909.0582): Lattice data for spatial correlators of $\mathrm{Tr}\, F_{\mu\nu}^2$ in agreement with strongly coupled $\mathcal{N} = 4$ SYM, while leading order pQCD result completely off. How about NLO?



Why pQCD II: Comparisons with lattice and AdS

Euclidean correlators provide direct information about medium $\Rightarrow$ Comparisons between lattice QCD, pQCD and AdS/CFT valuable

Another curious result of Iqbal, Meyer (0909.0582): UV behavior of $\mathrm{Tr}\, F_{\mu\nu}^2$ and $-\mathrm{Tr}\, F_{\mu\nu}\widetilde{F}_{\mu\nu}$ correlators on the lattice completely different even though leading order OPEs identical



Challenge for perturbation theory

Goal: Perturbatively evaluate Euclidean and Minkowskian correlators of $T_{\mu\nu}$ in hot Yang-Mills theory to

1. Inspect Operator Product Expansions (OPEs) at finite temperature
2. Compare behavior of perturbative time-averaged spatial correlators to lattice QCD and AdS/CFT
3. Use spectral densities at zero wave vector to aid the determination of transport coefficients from lattice data

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Concretely: Specialize to scalar, pseudoscalar and shear operators

$$\theta \equiv c_\theta\, g_{\rm B}^2 F^a_{\mu\nu} F^a_{\mu\nu}\,, \quad \chi \equiv c_\chi\, g_{\rm B}^2 F^a_{\mu\nu} \widetilde{F}^a_{\mu\nu}\,, \quad \eta \equiv 2c_\eta T_{12} = -2c_\eta F^a_{1\mu} F^a_{2\mu}$$

and proceed from 1 to 3 working at NLO.

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When can perturbation theory be expected to work?

$$\bar{\Lambda}_{x,T} \simeq \sqrt{\left(\bar{\Lambda}_x\right)^2 + \left(\bar{\Lambda}_T\right)^2} \sim \sqrt{\frac{1}{x^2} + (2\pi T)^2}$$

At least, if either $x \ll 1/\Lambda_{\text{QCD}}$ ($\omega \gg \Lambda_{\text{QCD}}$) or $T \gg \Lambda_{\text{QCD}}$

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Correlation functions: generalities

Plenitude of different Minkowskian correlators:

$$\begin{aligned}
\Pi^{>}_{\alpha\beta}(\mathcal{K}) &\equiv \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \Big\langle \hat{\phi}_\alpha(\mathcal{X})\, \hat{\phi}^\dagger_\beta(0) \Big\rangle \,, \\
\Pi^{<}_{\alpha\beta}(\mathcal{K}) &\equiv \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \Big\langle \hat{\phi}^\dagger_\beta(0)\, \hat{\phi}_\alpha(\mathcal{X}) \Big\rangle \,, \\
\rho_{\alpha\beta}(\mathcal{K}) &\equiv \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \Big\langle \frac{1}{2} \Big[\hat{\phi}_\alpha(\mathcal{X}), \hat{\phi}^\dagger_\beta(0)\Big] \Big\rangle \,, \\
\Delta_{\alpha\beta}(\mathcal{K}) &\equiv \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \Big\langle \frac{1}{2} \Big\{\hat{\phi}_\alpha(\mathcal{X}), \hat{\phi}^\dagger_\beta(0)\Big\} \Big\rangle \,, \\
\Pi^{R}_{\alpha\beta}(\mathcal{K}) &\equiv i\int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \Big\langle \Big[\hat{\phi}_\alpha(\mathcal{X}), \hat{\phi}^\dagger_\beta(0)\Big]\theta(t) \Big\rangle \,, \\
\Pi^{A}_{\alpha\beta}(\mathcal{K}) &\equiv i\int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \Big\langle -\Big[\hat{\phi}_\alpha(\mathcal{X}), \hat{\phi}^\dagger_\beta(0)\Big]\theta(-t) \Big\rangle \,, \\
\Pi^{T}_{\alpha\beta}(\mathcal{K}) &\equiv \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \Big\langle \hat{\phi}_\alpha(\mathcal{X})\, \hat{\phi}^\dagger_\beta(0)\, \theta(t) + \hat{\phi}^\dagger_\beta(0)\, \hat{\phi}_\alpha(\mathcal{X})\, \theta(-t) \Big\rangle
\end{aligned}$$

One Euclidean correlator, computable on the lattice:

$$\Pi^{E}_{\alpha\beta}(K) \equiv \int_X e^{iK\cdot X} \Big\langle \hat{\phi}_\alpha(X)\, \hat{\phi}^\dagger_\beta(0) \Big\rangle$$

Correlation functions: generalities

However, in thermal equilibrium all correlators related through ρ:

$$
\begin{aligned}
\Pi^{<}_{\alpha\beta}(\mathcal{K}) &= 2n_{\rm B}(k^0)\rho_{\alpha\beta}(\mathcal{K})\,, \\
\Pi^{>}_{\alpha\beta}(\mathcal{K}) &= 2\frac{e^{\beta k^0}}{e^{\beta k^0}-1}\rho_{\alpha\beta}(\mathcal{K}) \;=\; 2[1+n_{\rm B}(k^0)]\,\rho_{\alpha\beta}(\mathcal{K})\,, \\
\Delta_{\alpha\beta}(\mathcal{K}) &= \frac{1}{2}\left[\Pi^{>}_{\alpha\beta}(\mathcal{K})+\Pi^{<}_{\alpha\beta}(\mathcal{K})\right] \;=\; \left[1+2n_{\rm B}(k^0)\right]\rho_{\alpha\beta}(\mathcal{K})\,. \\
\mathrm{Im}\,\Pi^{R}_{\alpha\beta}(\mathcal{K}) &= \rho_{\alpha\beta}(\mathcal{K})\,, \quad \mathrm{Im}\,\Pi^{A}_{\alpha\beta}(\mathcal{K}) = -\rho_{\alpha\beta}(\mathcal{K})\,, \\
\Pi^{T}_{\alpha\beta}(\mathcal{K}) &= -i\Pi^{R}_{\alpha\beta}(\mathcal{K})+\Pi^{<}_{\alpha\beta}(\mathcal{K})\,, \\
\Pi^{E}_{\alpha\beta}(K) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}k^0}{\pi}\frac{\rho_{\alpha\beta}(k^0,\mathbf{k})}{k^0-ik_n}
\end{aligned}
$$

Correlation functions: generalities

...and the spectral function can in turn be given in terms of the Euclidean correlator:

$$\rho_{\alpha\beta}(\mathcal{K}) \;=\; \mathrm{Im}\,\Pi^{E}_{\alpha\beta}(k_n \to -i[k^0 + i0^+], \mathbf{k}) \; .$$

$\therefore$ Analytic determination of Euclidean correlator, together with analytic continuation, enough to evaluate all Minkowskian Green's functions!

Surprising benefit: real-time quantities from the "simple" Feynman rules of the imaginary time formalism

Correlation functions: generalities

Simplest example: bosonic operator in free field theory

$$\begin{aligned}
\Pi^E(K) &= \frac{1}{k_n^2 + E_k^2}\,, \quad E_k = \sqrt{k^2+m^2}\,, \\
\Pi^R(\mathcal{K}) &= \frac{1}{-(k^0+i0^+)^2 + E_k^2} \\
&= -\mathbb{P}\left(\frac{1}{(k^0)^2 - E_k^2}\right) + \frac{i\pi}{2E_k}\Big[\delta(k^0 - E_k) - \delta(k^0 + E_k)\Big]\,, \\
\rho(\mathcal{K}) &= \frac{\pi}{2E_k}\Big[\delta(k^0 - E_k) - \delta(k^0 + E_k)\Big]\,, \\
\Pi^T(\mathcal{K}) &= \frac{i}{\mathcal{K}^2 - m^2 + i0^+} + 2\pi\,\delta(\mathcal{K}^2 - m^2)\,n_{\rm B}(|k^0|)
\end{aligned}$$

Setting up the two-loop calculation

Our program: work within finite-T SU(3) Yang-Mills theory

$$S_E = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^{3-2\epsilon}x \left\{ \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \right\},$$

write down diagrammatic expansions for Euclidean correlators of energy-momentum tensor components ($X \equiv (\tau, \mathbf{x})$)

$$G_\theta(X) \equiv \langle \theta(X)\theta(0)\rangle_\mathrm{c}\ , \quad G_\chi(X) \equiv \langle \chi(X)\chi(0)\rangle\ , \quad G_\eta(X) \equiv \langle \eta(X)\eta(0)\rangle_\mathrm{c},$$

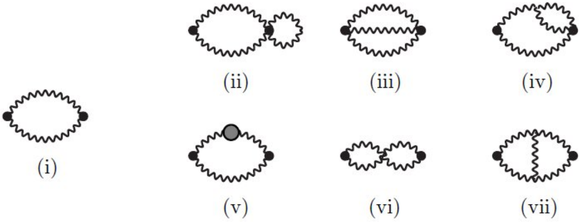
$$\widetilde{G}_\alpha(P) \equiv \int_X e^{-iP\cdot X} G_\alpha(X)\ , \quad \bar{G}_\alpha(x) \equiv \int_0^\beta d\tau\, G_\alpha(X)\ ,$$

$$\rho_\alpha(\omega) \equiv \mathrm{Im}\, \widetilde{G}_\alpha(p_0 = -i(\omega + i\epsilon), \mathbf{p} = 0),$$

and evaluate the necessary integrals up to NLO. For correct IR behavior, perform HTL resummation when needed.

Setting up the two-loop calculation

End up computing two-loop two-point diagrams in dimensional regularization, the black dots representing the operators:



NB: At $T \neq 0$, 4d integrals replaced by 3+1d sum-integrals

$$\int_Q \to \sum\!\!\!\!\!\!\!\int_Q \equiv T \sum_{q_0=2\pi nT} \int_q$$

Hard Thermal Loop resummation

For $\rho(\omega, \mathbf{p} = 0)$, three different energy regimes:

- $\omega \gtrsim \pi T$: Ordinary weak coupling expansion expected to converge, no resummations needed
- $g^2 T/\pi \ll \omega \ll \pi T$: Weak coupling expansion breaks down, but can be resummed using HTL effective theory
- $\omega \lesssim g^2 T/\pi$: All hell breaks loose

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- $\omega \lesssim g^2 T/\pi$: All hell breaks loose

For consistency, extend both bulk and shear results to $\omega \sim gT$ via HTL treatment:

$$\rho^{\text{QCD}}_{\text{resummed}} = \rho^{\text{QCD}}_{\text{resummed}} - \rho^{\text{HTL}}_{\text{resummed}} + \rho^{\text{HTL}}_{\text{resummed}} \approx \rho^{\text{QCD}}_{\text{naive}} - \rho^{\text{HTL}}_{\text{naive}} + \rho^{\text{HTL}}_{\text{resummed}}$$

In both cases, no HTL vertex functions necessary to match the IR behavior of the unresummed result.

Computational methods I: Identifying the masters

Step 1: Perform Wick contractions in Euclidean correlator and perform Lorentz algebra (typically with FORM)

Result: Expansion in terms of scalar 'masters'

$$\frac{\tilde{G}_\theta(P)}{4 d_A c_\theta^2 g_\mathrm{B}^4} = (D-2)\Big[-\mathcal{J}_\mathrm{a} + \frac{1}{2}\,\mathcal{J}_\mathrm{b}\Big]$$
$$+\quad g_\mathrm{B}^2 N_c \Big\{ 2(D-2)\Big[-(D-1)\mathcal{I}_\mathrm{a} + (D-4)\mathcal{I}_\mathrm{b}\Big] + (D-2)^2\Big[\mathcal{I}_\mathrm{c} - \mathcal{I}_\mathrm{d}\Big]$$
$$+ \frac{22-7D}{3}\mathcal{I}_\mathrm{f} - \frac{(D-4)^2}{2}\mathcal{I}_\mathrm{g} + (D-2)\Big[-3\mathcal{I}_\mathrm{e} + 3\mathcal{I}_\mathrm{h} + 2\mathcal{I}_\mathrm{i} - \mathcal{I}_\mathrm{j}\Big]\Big\}\,,$$

$$\mathcal{J}_\mathrm{a} \equiv \sum\!\!\!\!\!\!\!\int_Q \frac{P^2}{Q^2}\,, \quad \mathcal{J}_\mathrm{b} \equiv \sum\!\!\!\!\!\!\!\int_Q \frac{P^4}{Q^2(Q-P)^2}\,, \quad \mathcal{I}_\mathrm{a} \equiv \sum\!\!\!\!\!\!\!\int_{Q,R} \frac{1}{Q^2R^2}\,, \quad \mathcal{I}_\mathrm{b} \equiv \sum\!\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^2(R-P)^2}\,, \quad \cdots$$

$$\mathcal{I}_\mathrm{h} \equiv \sum\!\!\!\!\!\!\!\int_{Q,R} \frac{P^4}{Q^2R^2(Q-R)^2(R-P)^2}\,, \quad \mathcal{I}_\mathrm{i} \equiv \sum\!\!\!\!\!\!\!\int_{Q,R} \frac{(Q-P)^4}{Q^2R^2(Q-R)^2(R-P)^2}\,,$$

$$\mathcal{I}_\mathrm{i'} \equiv \sum\!\!\!\!\!\!\!\int_{Q,R} \frac{4(Q\cdot P)^2}{Q^2R^2(Q-R)^2(R-P)^2}\,, \quad \mathcal{I}_\mathrm{j} \equiv \sum\!\!\!\!\!\!\!\int_{Q,R} \frac{P^6}{Q^2R^2(Q-R)^2(Q-P)^2(R-P)^2}$$

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$$\mathcal{J}_{\mathrm{b}}^{0} \equiv \sum\!\!\!\!\!\!\int_{Q} \frac{P^4}{Q^2(Q-P)^2}\,, \tag{A.1}$$

$$\mathcal{J}_{\mathrm{b}}^{1} \equiv \sum\!\!\!\!\!\!\int_{Q} \frac{P^2}{Q^2(Q-P)^2} P_T(Q)\,, \tag{A.2}$$

$$\mathcal{J}_{\mathrm{b}}^{2} \equiv \sum\!\!\!\!\!\!\int_{Q} \frac{1}{Q^2(Q-P)^2} P_T(Q)^2\,, \tag{A.3}$$

$$\mathcal{I}_{\mathrm{b}}^{0} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^2(R-P)^2}\,, \tag{A.4}$$

$$\mathcal{I}_{\mathrm{b}}^{1} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{1}{Q^2R^2(R-P)^2} P_T(Q)\,, \tag{A.5}$$

$$\mathcal{I}_{\mathrm{b}}^{2} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{1}{Q^2R^2(R-P)^2} P_T(R)\,, \tag{A.6}$$

$$\mathcal{I}_{\mathrm{d}}^{0} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^4}{Q^2R^4(R-P)^2}\,, \tag{A.7}$$

$$\mathcal{I}_{\mathrm{d}}^{1} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^4(R-P)^2} P_T(Q)\,, \tag{A.8}$$

$$\mathcal{I}_{\mathrm{d}}^{2} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^4(R-P)^2} P_T(R)\,, \tag{A.9}$$

$$\mathcal{I}_{\mathrm{d}}^{3} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{1}{Q^2R^4(R-P)^2} P_T(R)^2\,, \tag{A.10}$$

$$\mathcal{I}_{\mathrm{f}}^{0} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2(Q-R)^2(R-P)^2}\,, \tag{A.11}$$

$$\mathcal{I}_{\mathrm{f}}^{1} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{1}{Q^2(Q-R)^2(R-P)^2} P_T(Q)\,, \tag{A.12}$$

$$\mathcal{I}_{\mathrm{h}}^{0} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^4}{Q^2R^2(Q-R)^2(R-P)^2}\,, \tag{A.13}$$

$$\mathcal{I}_{\mathrm{h}}^{1} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^2(Q-R)^2(R-P)^2} P_T(Q)\,, \tag{A.14}$$

$$\mathcal{I}_{\mathrm{h}}^{2} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^2(Q-R)^2(R-P)^2} P_T(R)\,, \tag{A.15}$$

$$\mathcal{I}_{\mathrm{h}}^{3} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^4}{Q^2R^4(Q-R)^2(R-P)^2} P_T(R)\,, \tag{A.16}$$

$$\mathcal{I}_{\mathrm{h}}^{4} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^4(Q-R)^2(R-P)^2} P_T(Q)^2\,, \tag{A.17}$$

$$\mathcal{I}_{\mathrm{h}}^{5} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^4(Q-R)^2(R-P)^2} P_T(R)^2\,, \tag{A.18}$$

$$\mathcal{I}_{\mathrm{h}}^{6} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^4(Q-R)^2(R-P)^2} P_T(Q)P_T(R)\,, \tag{A.19}$$

$$\mathcal{I}_{\mathrm{h}}^{7} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^4(Q-R)^2(R-P)^2} P_T(Q)P_T(Q-R)\,, \tag{A.20}$$

$$\mathcal{I}_{\mathrm{i}}^{0} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{(Q-P)^4}{Q^2R^2(Q-R)^2(R-P)^2}\,, \tag{A.21}$$

$$\mathcal{I}_{\mathrm{i}}^{1} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{(Q-P)^2}{Q^2R^2(Q-R)^2(R-P)^2} P_T(Q)\,, \tag{A.22}$$

$$\mathcal{I}_{\mathrm{i}}^{2} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2(Q-P)^2}{Q^2R^4(Q-R)^2(R-P)^2} P_T(Q)\,, \tag{A.23}$$

$$\mathcal{I}_{\mathrm{i}}^{3} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{(Q-P)^4}{Q^2R^4(Q-R)^2(R-P)^2} P_T(R)\,, \tag{A.24}$$

$$\mathcal{I}_{\mathrm{i}'} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{4(Q\cdot P)^2}{Q^2R^2(Q-R)^2(R-P)^2}\,, \tag{A.25}$$

$$\mathcal{I}_{\mathrm{j}}^{0} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^6}{Q^2R^2(Q-R)^2(Q-P)^2(R-P)^2}\,, \tag{A.26}$$

$$\mathcal{I}_{\mathrm{j}}^{1} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^4}{Q^2R^2(Q-R)^2(Q-P)^2(R-P)^2} P_T(Q)\,, \tag{A.27}$$

$$\mathcal{I}_{\mathrm{j}}^{2} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^4}{Q^2R^2(Q-R)^2(Q-P)^2(R-P)^2} P_T(Q-R)\,, \tag{A.28}$$

$$\mathcal{I}_{\mathrm{j}}^{3} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^2(Q-R)^2(Q-P)^2(R-P)^2} P_T(Q)^2\,, \tag{A.29}$$

$$\mathcal{I}_{\mathrm{j}}^{4} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^2(Q-R)^2(Q-P)^2(R-P)^2} P_T(Q-R)^2\,, \tag{A.30}$$

$$\mathcal{I}_{\mathrm{j}}^{5} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^2(Q-R)^2(Q-P)^2(R-P)^2} P_T(Q)P_T(R)\,, \tag{A.31}$$

$$\mathcal{I}_{\mathrm{j}}^{6} \equiv \sum\!\!\!\!\!\!\int_{Q,R} \frac{P^2}{Q^2R^2(Q-R)^2(Q-P)^2(R-P)^2} P_T(Q)P_T(Q-R)\,. \tag{A.32}$$

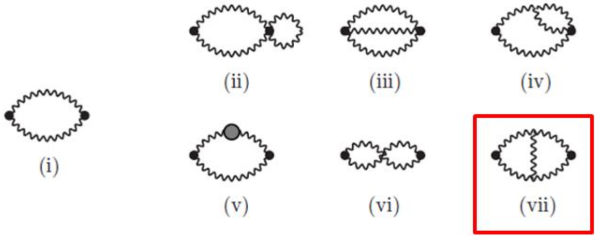
Computational methods II: Evaluating the masters

The optimal method for dealing with the master integrals depends on the particular problem:

- Operator Product Expansions
  - Explicit evaluation of Matsubara sums via ‘cutting’ methods
  - Expansion of the 3d integrals in powers of T^2/P^2
- Time-averaged spatial correlators
  - Set $p_0 = 0$ in mom. space correlator and perform 3d Fourier transf.
  - Calculation most conveniently handled directly in coordinate space, where the Matsubara sum trivializes
- Spectral functions: $\rho(\omega) \equiv \mathrm{Im}\, \widetilde{G}(p_0 = -i(\omega + i\epsilon), \mathbf{p} = 0)$
  - Matsubara sum via cutting rules, then take explicitly the imaginary part $\Rightarrow$ δ-function constraints for the 3-momenta
  - Most complicated part of the calculation: Dealing with the remaining spatial momentum integrals

Example: Integral *j* in the spectral density

Most complicated master integral in the bulk channel:



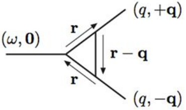
$$\mathcal{I}_{\rm j} \equiv \sum\!\!\!\!\!\!\!\int_{Q,R} \frac{P^6}{Q^2 R^2 (Q-R)^2 (Q-P)^2 (R-P)^2}$$

Example: Integral j in the spectral density

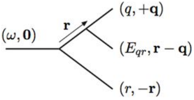
After performing the Matsubara sum and taking the imaginary part, $\mathcal{I}_j$ contribution written in terms of a two-fold phase space integral ($E_{qr} \equiv |\mathbf{q}-\mathbf{r}|$):

$$
\begin{aligned}
\rho_{\mathcal{I}_j}(\omega) = \int_{\mathbf{q},\mathbf{r}} \frac{\omega^6\pi}{4qrE_{qr}} \Bigg\{ & \frac{1}{8q^2}\Big[\delta(\omega-2q)-\delta(\omega+2q)\Big] \times \\
& \times \Bigg[\left(\frac{1}{(q+r-E_{qr})(q+r)} - \frac{1}{(q-r+E_{qr})(q-r)}\right)(1+2n_q)(n_{qr}-n_r) \\
& + \left(\frac{1}{(q+r+E_{qr})(q+r)} - \frac{1}{(q-r-E_{qr})(q-r)}\right)(1+2n_q)(1+n_{qr}+n_r)\Bigg] \\
& + \frac{1}{8r^2}\Big[\delta(\omega-2r)-\delta(\omega+2r)\Big] \times \\
& \times \Bigg[\left(\frac{1}{(q+r-E_{qr})(q+r)} - \frac{1}{(q-r-E_{qr})(q-r)}\right)(1+2n_r)(n_{qr}-n_q) \\
& + \left(\frac{1}{(q+r+E_{qr})(q+r)} - \frac{1}{(q-r+E_{qr})(q-r)}\right)(1+2n_r)(1+n_{qr}+n_q)\Bigg] \\
& + \Big[\delta(\omega-q-r-E_{qr})-\delta(\omega+q+r+E_{qr})\Big]\frac{(1+n_{qr})(1+n_q+n_r)+n_qn_r}{(q+r+E_{qr})^2(q-r+E_{qr})(q-r-E_{qr})} \\
& + \Big[\delta(\omega-q-r+E_{qr})-\delta(\omega+q+r-E_{qr})\Big]\frac{n_{qr}(1+n_q+n_r)-n_qn_r}{(q+r-E_{qr})^2(q-r+E_{qr})(q-r-E_{qr})} \\
& + \Big[\delta(\omega-q+r-E_{qr})-\delta(\omega+q-r+E_{qr})\Big]\frac{n_r(1+n_q+n_{qr})-n_qn_{qr}}{(q-r+E_{qr})^2(q+r+E_{qr})(q+r-E_{qr})} \\
& + \Big[\delta(\omega+q-r-E_{qr})-\delta(\omega-q+r+E_{qr})\Big]\frac{n_q(1+n_r+n_{qr})-n_rn_{qr}}{(q-r-E_{qr})^2(q+r+E_{qr})(q+r-E_{qr})}\Bigg\}.
\end{aligned}
$$

Factorized int./
Virtual correction



Phase space int./
Real correction



Example: Integral *j* in the spectral density

After quite some work: final result in terms of finite 1d and 2d integrals, amenable to numerical evaluation

$$
\begin{aligned}
&\frac{(4\pi)^3 \rho_{\mathcal{I}_\mathrm{j}}(\omega)}{\omega^4(1+2n_{\frac{\omega}{2}})} = \\
&\quad \int_0^{\frac{\omega}{4}} \mathrm{d}q\, n_q \left[\Big(\frac{1}{q-\frac{\omega}{2}} - \frac{1}{q}\Big) \ln\Big(1-\frac{2q}{\omega}\Big) - \frac{\frac{\omega}{2}}{q(q+\frac{\omega}{2})} \ln\Big(1+\frac{2q}{\omega}\Big) \right] \\
&+ \quad \int_{\frac{\omega}{4}}^{\frac{\omega}{2}} \mathrm{d}q\, n_q \left[\Big(\frac{2}{q-\frac{\omega}{2}} - \frac{1}{q}\Big) \ln\Big(1-\frac{2q}{\omega}\Big) - \frac{\frac{\omega}{2}}{q(q+\frac{\omega}{2})} \ln\Big(1+\frac{2q}{\omega}\Big) - \frac{1}{q-\frac{\omega}{2}} \ln\Big(\frac{2q}{\omega}\Big) \right] \\
&+ \quad \int_{\frac{\omega}{2}}^{\infty} \mathrm{d}q\, n_q \left[\Big(\frac{2}{q-\frac{\omega}{2}} - \frac{2}{q}\Big) \ln\Big(\frac{2q}{\omega}-1\Big) - \frac{\frac{\omega}{2}}{q(q+\frac{\omega}{2})} \ln\Big(1+\frac{2q}{\omega}\Big) + \Big(\frac{1}{q} - \frac{1}{q-\frac{\omega}{2}}\Big) \ln\Big(\frac{2q}{\omega}\Big) \right] \\
&+ \quad \int_0^{\frac{\omega}{2}} \mathrm{d}q \int_0^{\frac{\omega}{4}-|q-\frac{\omega}{4}|} \mathrm{d}r \left(-\frac{1}{qr}\right) \frac{n_{\frac{\omega}{2}-q}\, n_{q+r}(1+n_{\frac{\omega}{2}-r})}{n_r^2} \\
&+ \quad \int_{\frac{\omega}{2}}^{\infty} \mathrm{d}q \int_0^{q-\frac{\omega}{2}} \mathrm{d}r \left(-\frac{1}{qr}\right) \frac{n_{q-\frac{\omega}{2}}(1+n_{q-r})(n_q - n_{r+\frac{\omega}{2}})}{n_r n_{-\frac{\omega}{2}}} \\
&+ \quad \int_0^{\infty} \mathrm{d}q \int_0^{q} \mathrm{d}r \left(-\frac{1}{qr}\right) \frac{(1+n_{q+\frac{\omega}{2}}) n_{q+r} n_{r+\frac{\omega}{2}}}{n_r^2} \ .
\end{aligned}
$$

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Summary of NLO results

	OPEs	Coord. space	Spectral density
Scalar (θ)	[1]	[2]	[4]
Pseudoscalar (χ)	[1]	[2]	[4]
Shear (η)	[3]	–	[5,6]

[1] Mikko Laine, Mikko Vepsäläinen, AV, 1008.3263
[2] Mikko Laine, Mikko Vepsäläinen, AV, 1011.4439
[3] York Schröder, Mikko Vepsäläinen, AV, Yan Zhu, 1109.6548
[4] Mikko Laine, AV, Yan Zhu, 1108.1259
[5] Yan Zhu, AV, 1212.3818
[6] AV, Yan Zhu, 1502.02556

Note: Inclusion of fermions possible in all cases (ongoing work in shear channel by Yan Zhu).

Wilson coefficients for OPEs

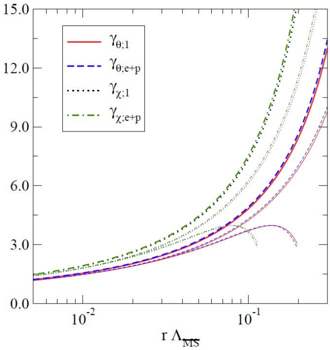
For $P \gg T$, perform large momentum expansion of Euclidean correlators to obtain $T \neq 0$ corrections to the OPEs

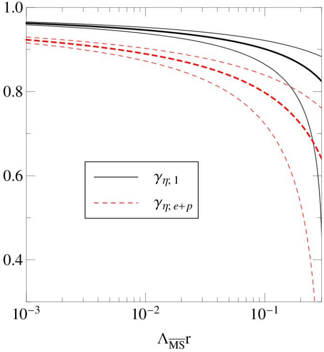
$$
\begin{aligned}
\frac{\Delta\widetilde{G}_\theta(P)}{4c_\theta^2 g^4} &= \frac{3}{P^2}\left(\frac{p^2}{3}-p_n^2\right)\left[1+\frac{g^2N_c}{(4\pi)^2}\left(\frac{22}{3}\ln\frac{\bar{\mu}^2}{P^2}+\frac{203}{18}\right)\right](e+p)(T) \\
&- \frac{2}{g^2b_0}\left[1+g^2b_0\ln\frac{\bar{\mu}^2}{\zeta_\theta P^2}\right](e-3p)(T)+\mathcal{O}\Big(g^4,\frac{1}{P^2}\Big) \\
\frac{\Delta\widetilde{G}_\chi(P)}{-16c_\chi^2 g^4} &= \frac{3}{P^2}\left(\frac{p^2}{3}-p_n^2\right)\left[1+\frac{g^2N_c}{(4\pi)^2}\left(\frac{22}{3}\ln\frac{\bar{\mu}^2}{P^2}+\frac{347}{18}\right)\right](e+p)(T) \\
&+ \frac{2}{g^2b_0}\left[1+g^2b_0\ln\frac{\bar{\mu}^2}{\zeta_\chi P^2}\right](e-3p)(T)+\mathcal{O}\Big(g^4,\frac{1}{P^2}\Big) \\
\frac{\Delta\widetilde{G}_\eta(P)}{4c_\eta^2} &= -\left\{1+\frac{3}{P^2}\left(\frac{p^2}{3}-p_n^2\right)-\frac{1}{3}\,\frac{g^2N_c}{(4\pi)^2}\left[22+\frac{41}{P^2}\left(\frac{p^2}{3}-p_n^2\right)\right]\right\}(e+p)(T) \\
&+ \frac{4}{3g^2b_0}\left[1-g^2b_0\ln\zeta_\eta\right](e-3p)(T)+\mathcal{O}\Big(g^4,\frac{1}{P^2}\Big)
\end{aligned}
$$

Note the appearance of Lorentz non-invariant operator $e+p$.

Wilson coefficients for OPEs

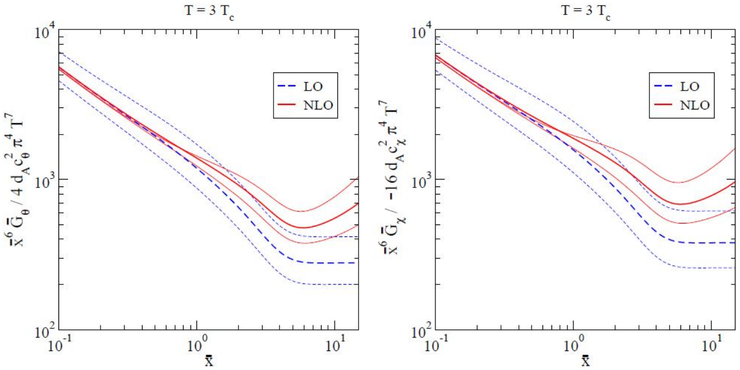
Behavior of Wilson coefficients in coordinate space:





Time averaged spatial correlators

To compare with lattice results for spatial correlators, and to assess validity of OPE, determine next time-averaged spatial correlators $\bar{G}_\theta(x) \equiv \int_0^\beta d\tau G_\theta(X)$, $\bar{G}_\chi(x) \equiv \int_0^\beta d\tau G_\chi(X)$:



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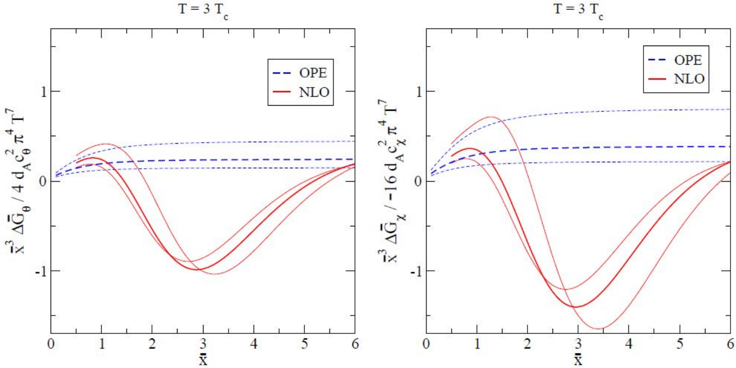

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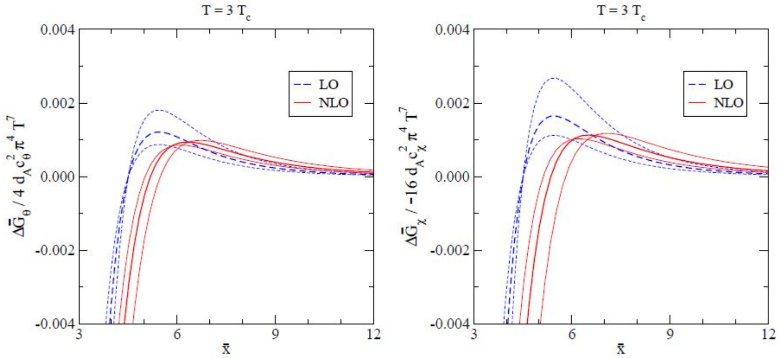
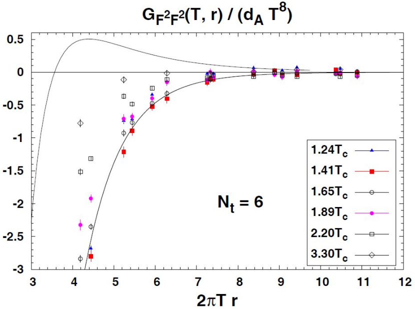
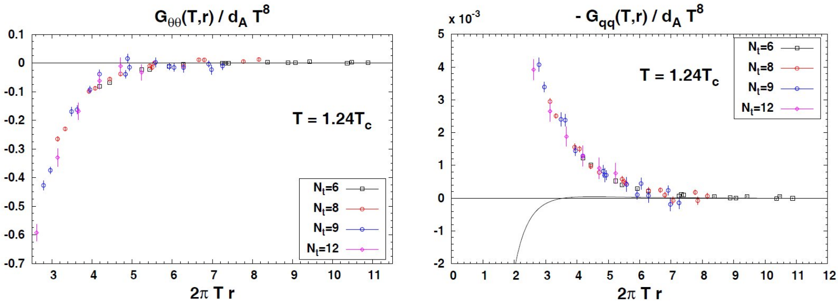


Time averaged spatial correlators



- Qualitatively, NLO results slightly closer to lattice than LO ones
- However: difference between θ and χ channels much more suppressed in perturbative results
- Caveat: lattice results for equal time, not time averaged correlator

Time averaged spatial correlators



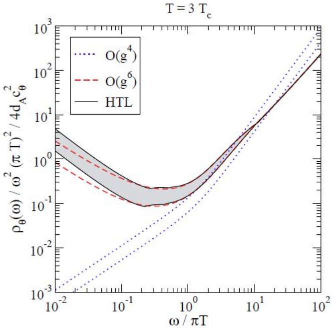
- Qualitatively, NLO results slightly closer to lattice than LO ones
- However: difference between θ and χ channels much more suppressed in perturbative results
- Caveat: lattice results for equal time, not time averaged correlator

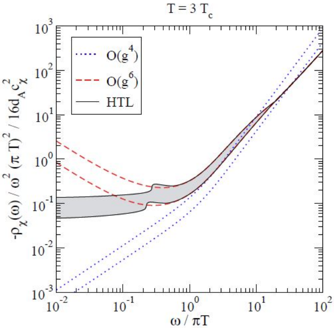
Spectral densities

Bulk channel spectral densities:

$$\frac{\rho_\theta(\omega)}{4d_A c_\theta^2} = \frac{\pi\omega^4}{(4\pi)^2}(1+2n_{\frac{\omega}{2}})\left\{g^4 + \frac{g^6 N_c}{(4\pi)^2}\left[\frac{22}{3}\ln\frac{\bar{\mu}^2}{\omega^2} + \frac{73}{3} + 8\,\phi_T(\omega)\right] + \frac{g^4 m_E^4}{\omega^4}\phi_\theta^{\rm HTL}(\omega)\right\} + \mathcal{O}(g^8)$$

$$\frac{-\rho_\chi(\omega)}{16d_A c_\chi^2} = \frac{\pi\omega^4}{(4\pi)^2}(1+2n_{\frac{\omega}{2}})\left\{g^4 + \frac{g^6 N_c}{(4\pi)^2}\left[\frac{22}{3}\ln\frac{\bar{\mu}^2}{\omega^2} + \frac{97}{3} + 8\,\phi_T(\omega)\right] + \frac{g^4 m_E^4}{\omega^4}\phi_\chi^{\rm HTL}(\omega)\right\} + \mathcal{O}(g^8)$$





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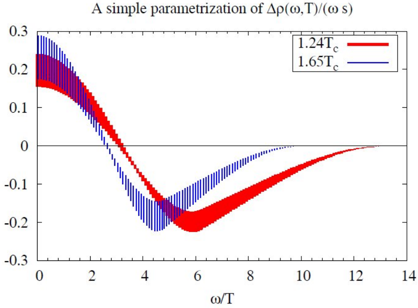
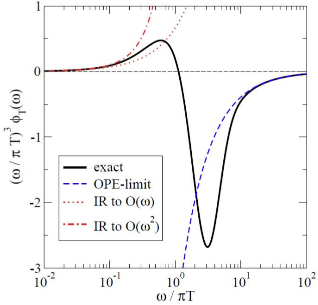
$$\frac{-\rho_\chi(\omega)}{16 d_A c_\chi^2} = \frac{\pi\omega^4}{(4\pi)^2}(1+2n_{\frac{\omega}{2}})\left\{g^4 + \frac{g^6 N_c}{(4\pi)^2}\left[\frac{22}{3}\ln\frac{\bar{\mu}^2}{\omega^2} + \frac{97}{3} + 8\,\phi_T(\omega)\right] + \frac{g^4 m_E^4}{\omega^4}\phi_\chi^{\mathrm{HTL}}(\omega)\right\} + \mathcal{O}(g^8)$$



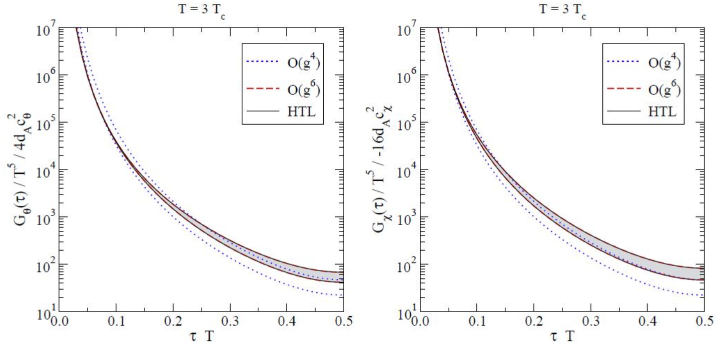


H. B. Meyer, 1002.3343

Spectral densities

In the imaginary time correlator, theoretical uncertainties (dependence on renormalization scale) considerably suppressed

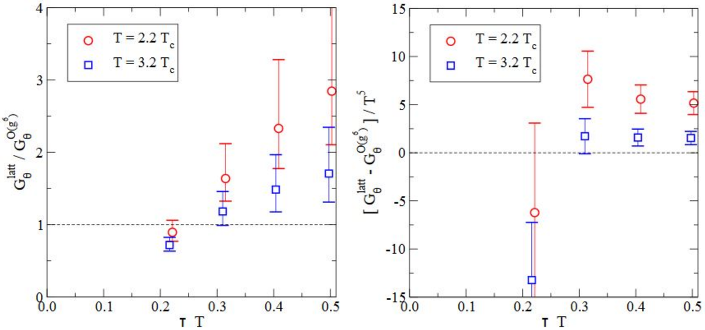
$$G(\tau) = \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \rho(\omega) \frac{\cosh \frac{(\beta - 2\tau)\omega}{2}}{\sinh \frac{\beta\omega}{2}}$$



Spectral densities

In the imaginary time correlator, impressive agreement with lattice results in the short distance limit; divergence absent in the difference

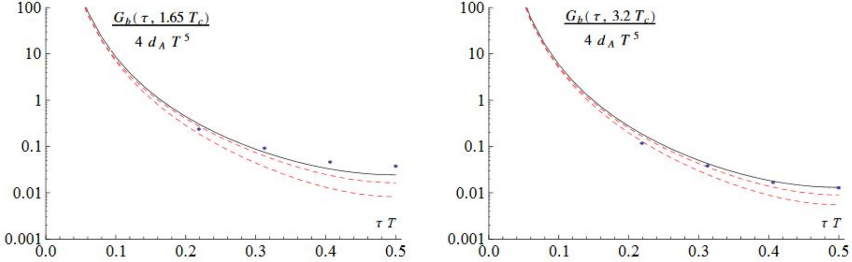
$$G(\tau) = \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \rho(\omega) \frac{\cosh \frac{(\beta - 2\tau)\omega}{2}}{\sinh \frac{\beta\omega}{2}}$$



Spectral densities

For the imaginary time correlator, comparison possible also to IHQCD

$$G(\tau) = \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \rho(\omega) \frac{\cosh \frac{(\beta - 2\tau)\omega}{2}}{\sinh \frac{\beta\omega}{2}}$$

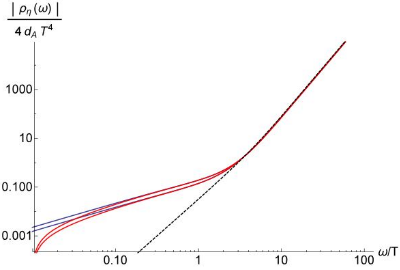


Spectral densities

In the shear channel, spectral density and imaginary time correlator converge wonderfully, but no lattice or AdS results to compare with at the moment:

$$\frac{\rho_\eta(\omega)}{4d_A} = \frac{\omega^4}{4\pi}(1+2n_{\frac{\omega}{2}})\left\{-\frac{1}{10}+\frac{g^2N_c}{(4\pi)^2}\left[\frac{2}{9}+\phi_\eta^T(\omega)\right]+\frac{m_E^4}{\omega^4}\phi_\eta^{\mathrm{HTL}}(\omega/T,m_E/T)\right\}+\mathcal{O}(g^8)\,,$$

$$G_\eta(\tau) = \int_0^\infty \frac{\mathrm{d}\omega}{\pi}\rho_\eta(\omega)\frac{\cosh\left[\left(\frac{\beta}{2}-\tau\right)\omega\right]}{\sinh\frac{\beta\omega}{2}}$$



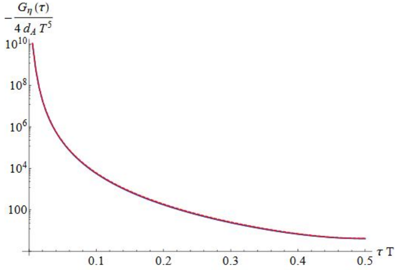


Table of contents

Conclusions

- Perturbative evaluation of energy-momentum tensor correlators in thermal QCD useful for disentangling properties of the QGP
  - Spectral densities needed to extract transport coefficients from lattice data for Euclidean correlators
  - Spatial correlators useful way to compare lattice, pQCD and holographic predictions
- NLO results derived for
  - OPEs in the bulk and shear channels
  - Time averaged spatial correlator in the bulk channel
  - Spectral density in the bulk and shear channels
- For further progress, accuracy of lattice results for Euclidean correlators the bottle neck