Analytics of $T_{xy}T_{xy}(k=0)$ in $\lambda\phi^4$ theory

Andrei keeps asking me

"What is analytical structure of

$$\int d^4x e^{i\omega t}\langle T_{xy}(t,x)T_{xy}(0,0)\rangle = \langle TT\rangle(\omega,0)$$

in complex ω plane at weak coupling?"

I will try to address this in $\lambda\phi^4$ theory

Limited tools make it hard to find true analytical form,

just some limited information we will have to interpret.

What I will talk about

- What happens to a cut when you fit it with poles+zeros?
- How can you tell if it's just a cut, or a mix of poles and cuts?
- Kinetic theory review
- Analytics and poles/cuts in $\lambda\phi^4$

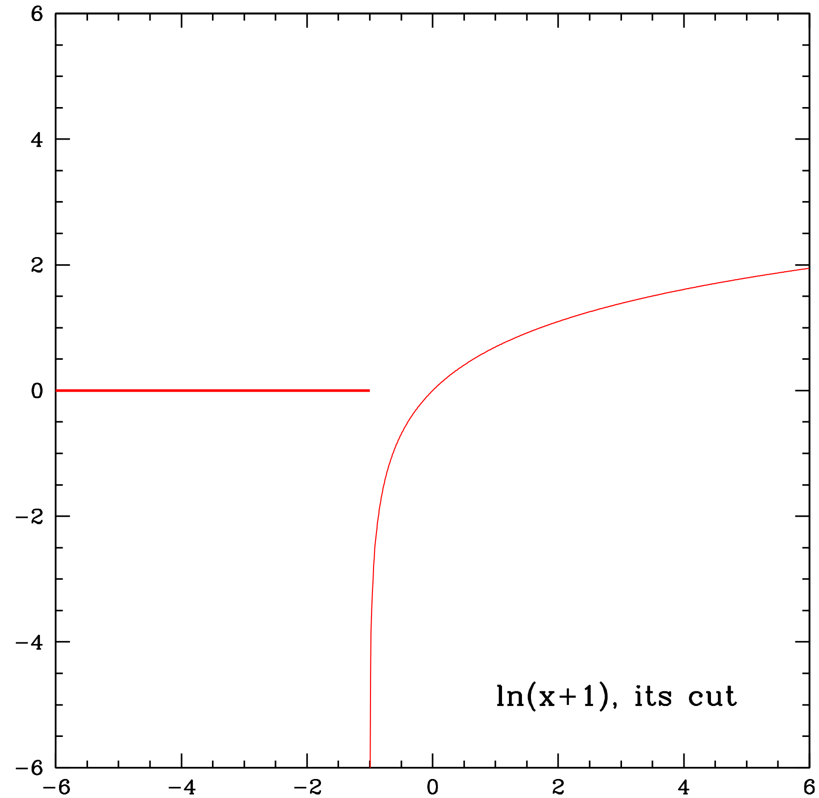
What does a cut look like

if I can only see poles + zeros?

Consider the function

$$f(x) = \ln(1 + x)$$

Cut from -1 to $-\infty$

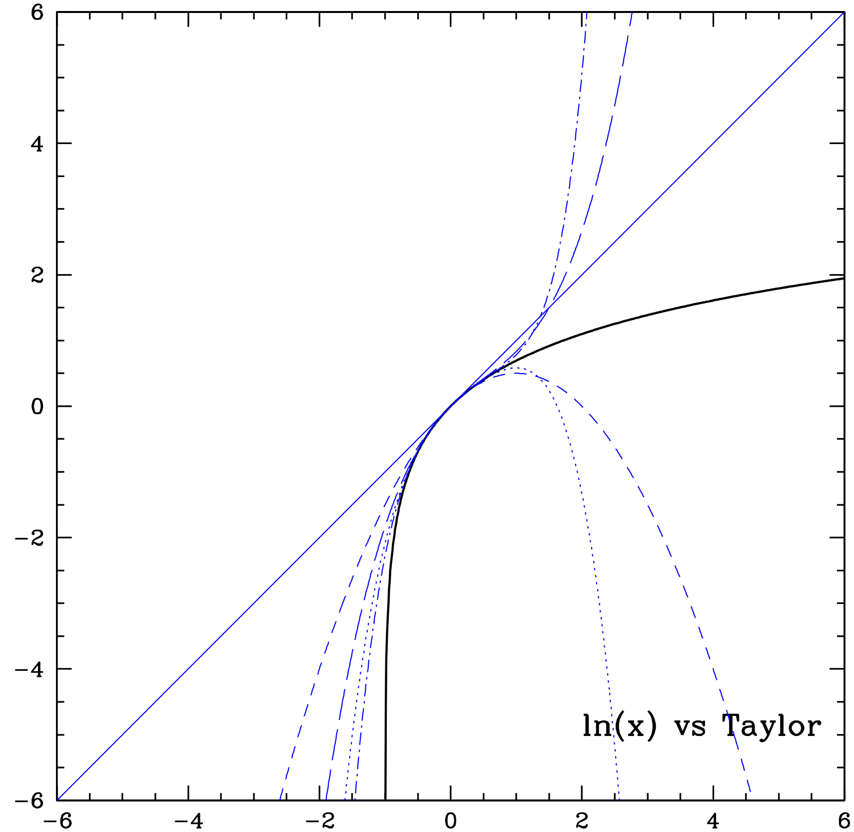


Suppose we only had information from the point $x = 0$.

Taylor:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

Does terrible job!



Why so bad?

Taylor is same as assuming function has n zeros and no poles.

Not good description of a cut!

Assume instead that function has 1 more zero than pole: Padé

$$P_{N,N-1}(x) = \frac{\sum_{n=1}^{N} d_n x^n}{1 + \sum_{n=1}^{N-1} c_n x^n}$$

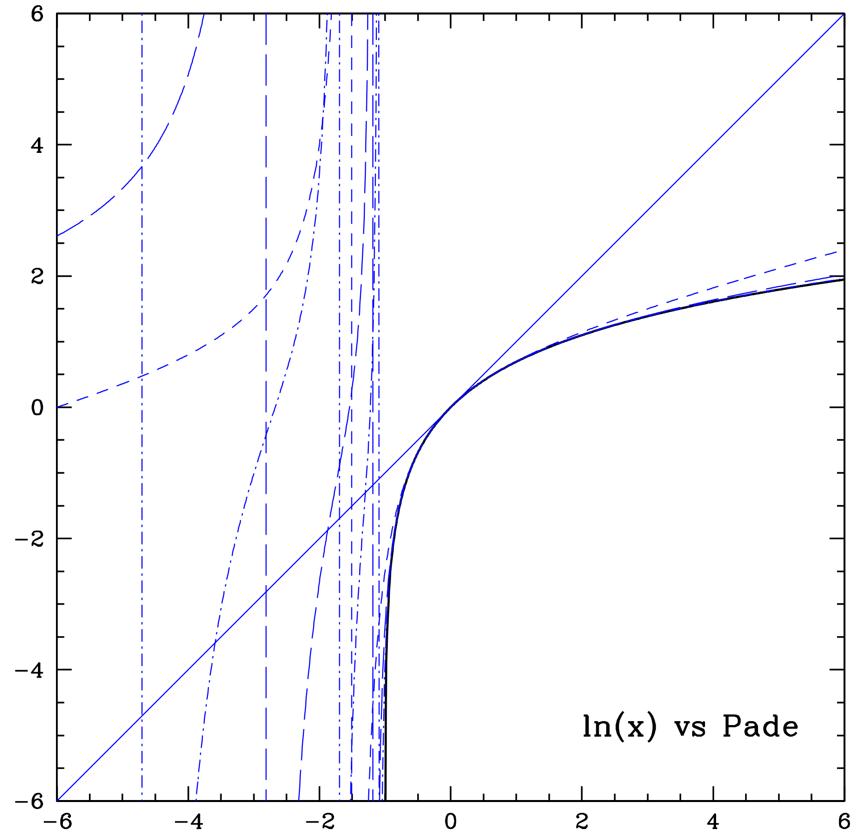
Taylor expand $P(x)$ to order $2N-1$

Choose unique d_n, c_n such that Taylor series of P and Taylor series of $\ln(1+x)$ agree through $2N-1$ terms

Does a far better job!

Padé Approximations of $\ln(1+x)$

Here are $(1,0)$, $(2,1)$, $(3,2)$, and $(4,3)$ Padé approximants of $\ln(1+x)$.



What is this mess at $x < -1$?

Padé is:

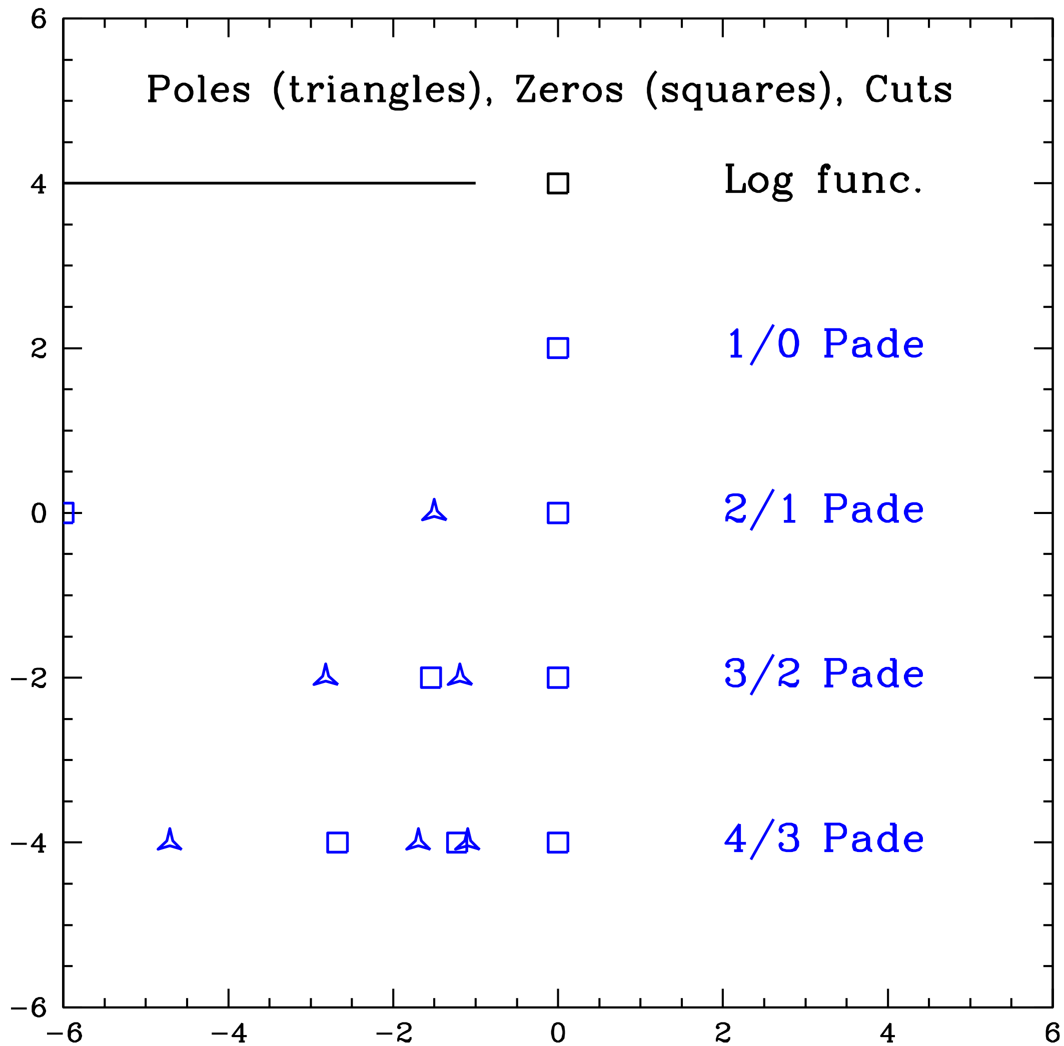
$$\frac{d_1 x + d_2 x^2 + \ldots}{1 + c_1 x + c_2 x^2 + \ldots} = \frac{A(x - z_1)(x - z_2)\ldots}{(x - p_1)(x - p_2)\ldots}$$

product of zeros and poles, at $z_1, \ldots$ and $p_1, \ldots$

Cut got replaced by series of zeros and poles.

Trying to describe a cut as a series of zeros and poles.

For last two, one zero is off edge of plot.



What if there is also a true pole?

Consider function

$$f(x) = \frac{\sqrt{(x+2)(x+3)/6}}{x+1}$$

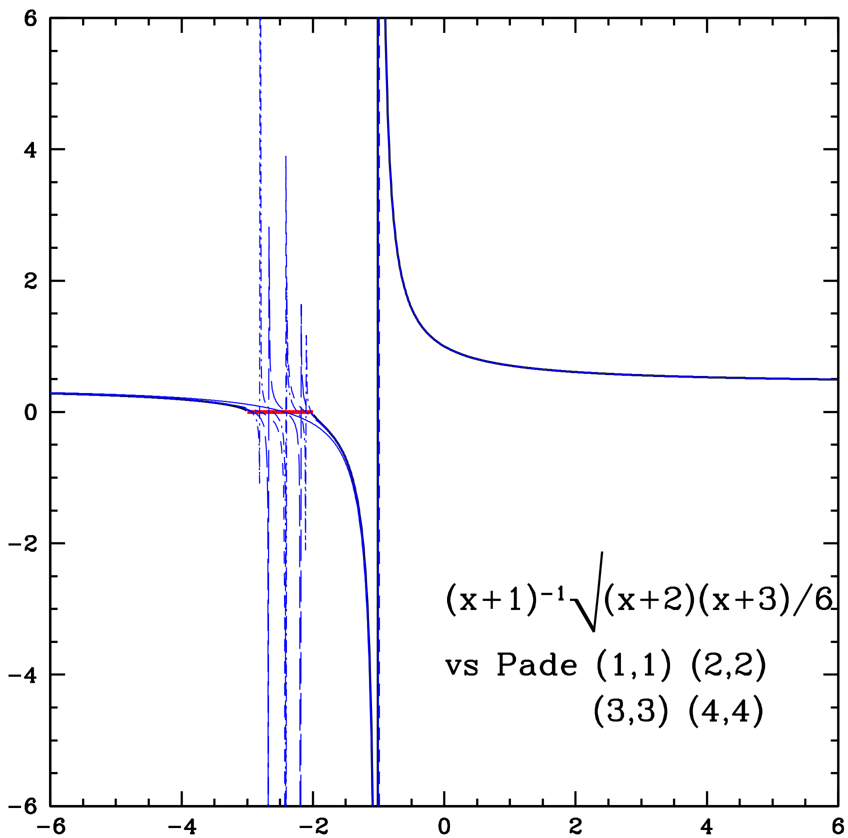
Pole at $x = -1$

Cut from $x = -2$ to $x = -3$

Fit it with an (N, N) Padé approximant

(Taylor series is, once again, crap)

Pole/zero fitting of a pole and cut

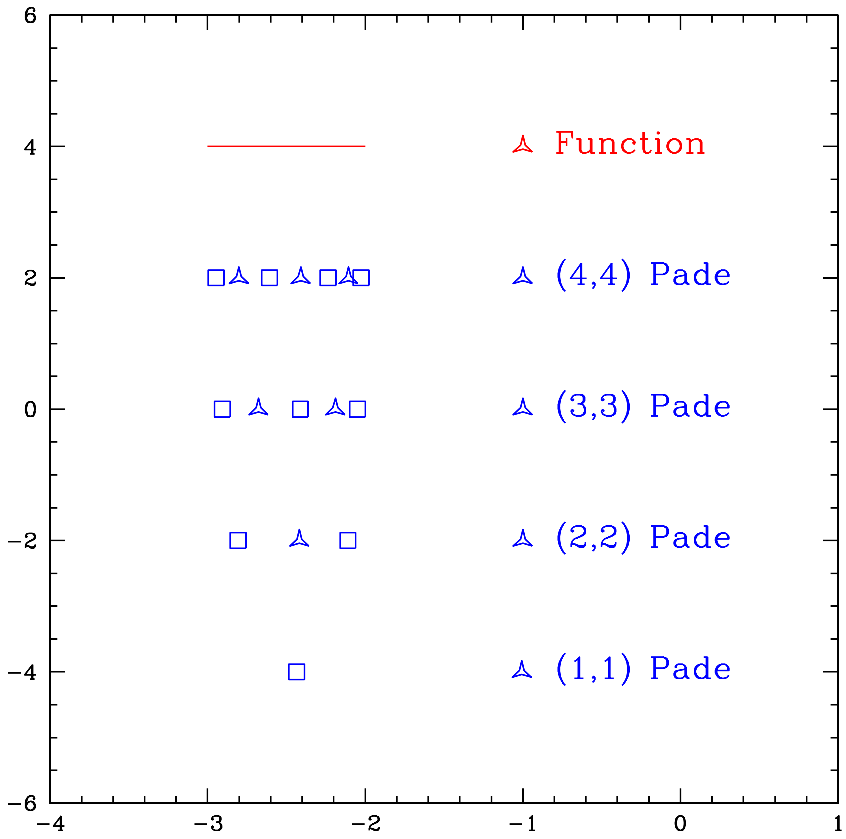


Even (1,1) Padé is great!

Pole treated as pole.

Cut $= N$ zeros, $N - 1$ poles

Pole/zero fitting of a pole and cut



Pole stays put as increase Padé size.
zeros/poles get tighter together.
Note: not evenly spaced

I can tell that there is an isolated pole in front of cut!

Back to TT correlation function

Shear viscosity determined by correlator

$$\eta = \frac{1}{6T} \lim_{\omega \to 0} \int d^3x dt e^{i\omega t} \langle T_{xy}(x,t) T_{xy}(0,0) \rangle$$

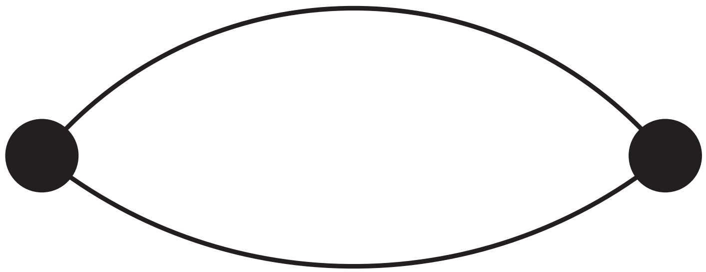
What is functional dependence on ω, keeping $\int d^3x$ (vanishing $\vec{k}$)?

Are there distinct poles? Purely imaginary, or real parts?

Or are there cuts? Where, what discontinuity?

Or both? What is nonanalyticity nearest the real axis?

Why we need resummations



Simplest diagram: 1 loop

Blobs are T_{xy} insertions

Propagators carry

4-momentum $\pm P^{\mu}$

Propagator

are "cut", eg,

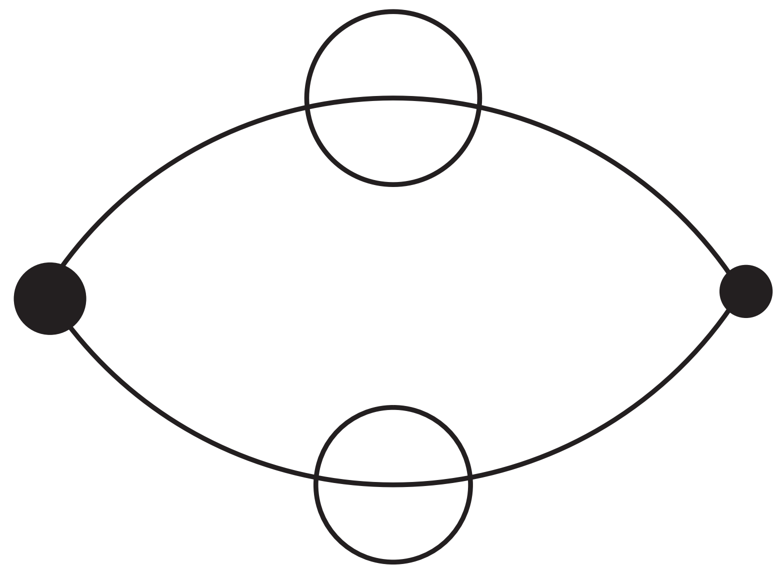
$$\Delta(p) = 2\pi[1+f(p)]\delta(p^2)$$

on-shell Delta function (at free level). Divergent:

$$\int d^4p 2\pi \; f(p)[1+f(p)]\,\delta(p^2)\,\delta(p^2)$$

Therefore you need

To get finite answer you **MUST** include scattering, width: on-shell δ becomes Lorentzian

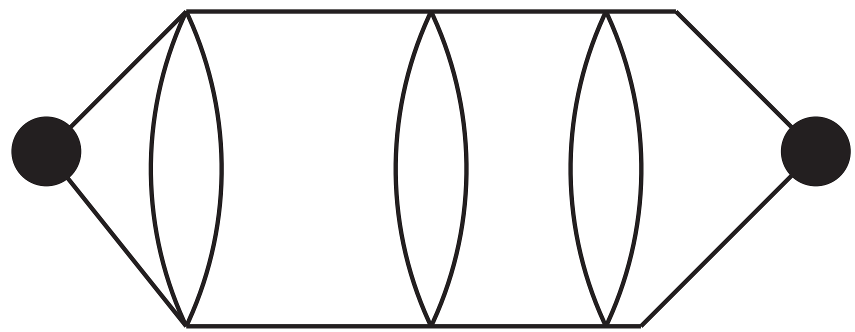


$$\int d^4p f[1+f]\Big(\delta(p^2)\Big)^2 \implies \int d^4p f[1+f]\left(\frac{\Gamma p^0}{(p^2)^2+\Gamma^2 p_0^2}\right)^2$$

Divergence becomes $T^5/\Gamma \sim T^4/\lambda^2$ (Γ is 2-loop, $\propto \lambda^2$)

Ladder resummation

Higher loops involve more powers of $1/\Gamma$. Compensate λ^2 loop "cost". Also restore stress-tensor conservation.



Each "rail" at different (matching pair of) momentum than last. Each rail $\propto \lambda^{-2}$, each "rung" $\propto \lambda^2$.

Neglecting these gets answer wrong by factor $\simeq 3$.

Effective kinetic theory

Effective theory resums these ladders.

Contribution of rung-pair described by

$$\delta f(k,t) = f_0(k)[1 + f_0(k)]\chi(k,t)$$

(f_0 Bose distribution) Evolves with time according to

$$\begin{aligned}
\partial_t \chi(k,t) &= S(k)\delta(t) - \mathcal{C}[\chi] \\
&= S(k)\delta(t) - \int d^3p\, \mathcal{C}_{p,k}\, \chi(p) \\
&= S(k)\delta(t) - \int d^3p \Big[\Gamma_k \delta^3(p-k) - \mathcal{C}_{k\to p} \Big] \chi(p)
\end{aligned}$$

First(loss), second(gain) term in [] from rails/rungs.

Connection to η

Correlator $\langle T_{xy}(t) T_{xy}(0) \rangle$ given by

$$T_{xy}(t) = \int d^3k \; \chi(k) S(k) f_0[1{+}f_0]$$

$\mathcal{C}$ is positive symmetric operator under this inner product

$$\langle \chi | \phi \rangle \equiv \int d^3k \; \chi(k) \phi(k) f_0[1{+}f_0]$$

In terms of inner product,

$$\partial_t | \chi \rangle = \delta(t) | S \rangle - \mathcal{C} | \chi \rangle$$

and

$$\eta = \frac{1}{6T} \int dt \langle S | \chi(t) \rangle = \frac{1}{3T} \langle S | \mathcal{C}^{-1} | S \rangle$$

Eigenspectrum of $\mathcal{C}$

Space of $|\chi\rangle$ is $\mathcal{L}^2$: ∞-dimensional.

Any positive symmetric operator has eigenspectrum

$$\mathcal{C} = \sum_i \lambda_i |\xi_i\rangle\langle\xi_i| + \int_D d\lambda' \lambda' |\xi(\lambda')\rangle\langle\xi(\lambda')|$$

discrete (pole) plus continuous (cut) spectrum, D the portion of $\Re^+$ which is cut.

Eigenvectors obey orthogonality

$$\langle\xi_i|\xi_j\rangle = \delta_{ij}\,, \quad \langle\xi_i|\xi(\lambda')\rangle = 0\,, \quad \langle\xi(\lambda')|\xi(\lambda'')\rangle = \delta(\lambda' - \lambda'')$$

Spectral decomposition solves Boltzmann equation:

$$|\chi(t)\rangle = \sum_i e^{-\lambda_i t}|\xi_i\rangle\langle\xi_i|S\rangle \; + \int_D d\lambda' e^{-\lambda' t}|\xi(\lambda')\rangle\langle\xi(\lambda')|S\rangle$$

Value of η is

$$3T\eta = \sum_i \lambda_i^{-1}\Big(\langle S|\xi_i\rangle\Big)^2 + \int_D d\lambda' \lambda'^{-1}\Big(\langle S|\xi(\lambda')\rangle\Big)^2$$

Retarded function has poles at $\omega = -i\lambda_i$, residue $\Big(\langle\xi_i|S\rangle\Big)^2$, and cuts along $-iD$ with discontinuity $\Big(\langle\xi(\lambda')|S\rangle\Big)^2$

If only I could find this decomposition explicitly.

Test function method

Work in finite-dimensional subspace spanned by test functions:

$$|\chi\rangle = \sum_{i=1}^{N} c_i |\phi_i\rangle$$

Test functions I will use:

$$\phi_{i,\mathrm{Yaffe}}(k) = \frac{k^{i+1} T^{M-i-2}}{(k+T)^{M-1}}\,, \qquad i = 1, \ldots, N\,, \quad N \geq M$$

Need to orthonormalize (easy). Large M: basis more complete everywhere. Large $N-M$: more complete UV. AMY used $N = M$ but we don't have to.

Test function method

Find “vector”

$$S_i = \langle S|\phi_i\rangle = \int d^3pS(p)\phi_i(p)f_0[1{+}f_0]$$

Find “matrix”

$$C_{ij} = \langle \phi_i|\mathcal{C}|\phi_j\rangle = \int d^3pd^3k\phi_i(p)\phi_j(k)\mathcal{C}_{k,p}f_0[1{+}f_0]$$

Eigenspectrum of C_{ij}: discrete spectrum as before

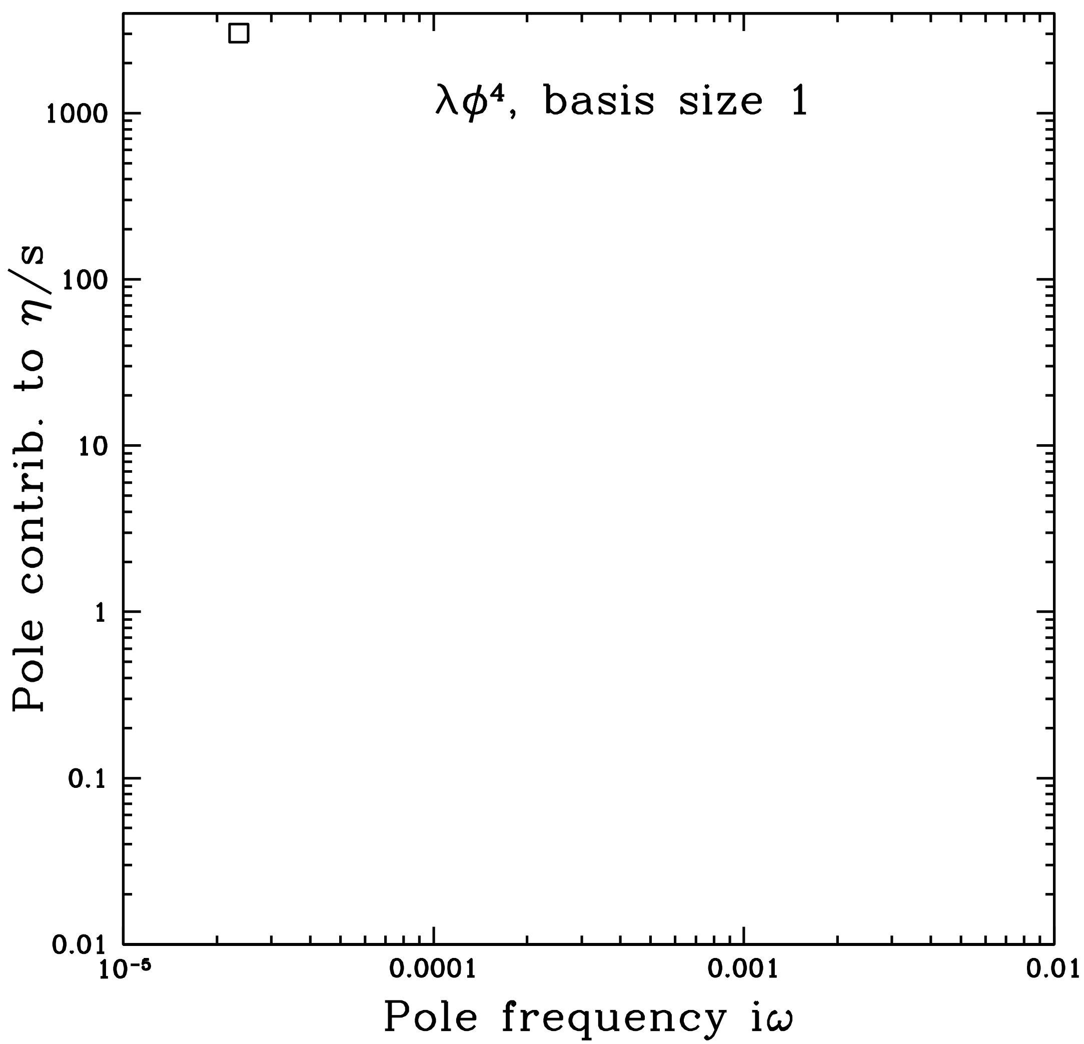
Test function method

Discontinuities purely on negative imaginary axis.
But this is from kinetic theory, not this approximate method.

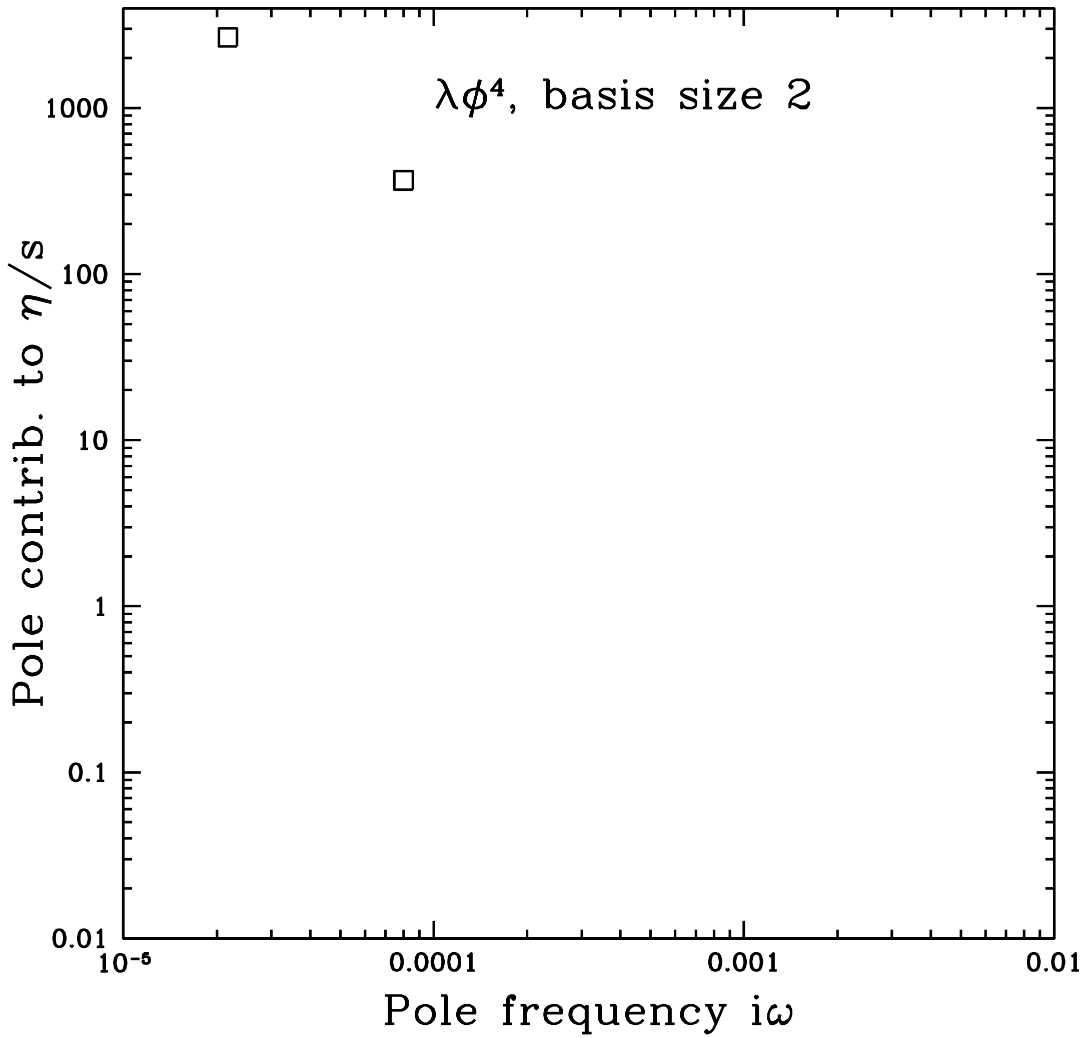
Method automatically "predicts" discrete spectrum of poles.
Like our Padé approximation – "forces" nonanalyticity structure through approximation scheme.

Try to tell if it's really poles or cuts by varying basis size, seeing whether poles stay put or "fill in" denser and denser.

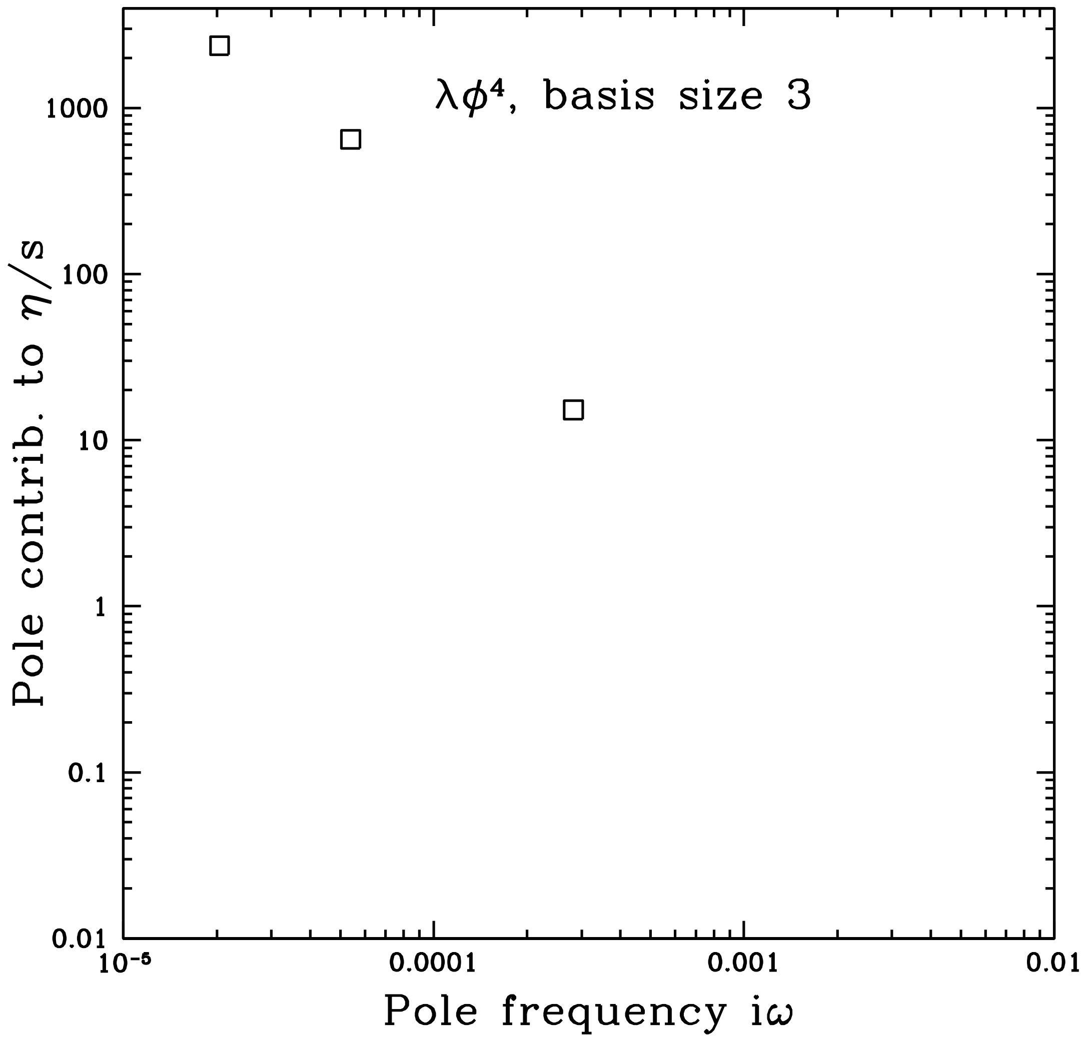
1 basis element



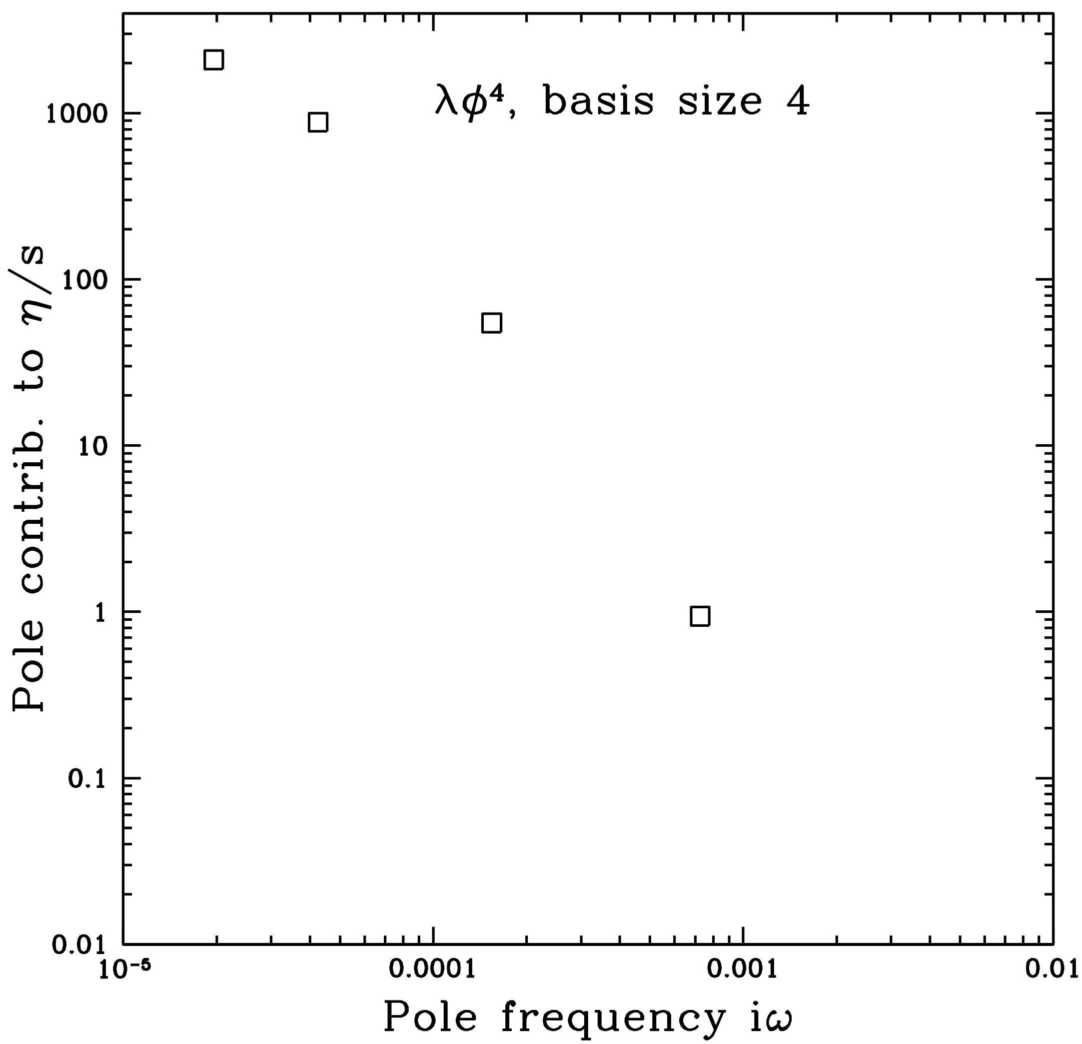
2 basis elements



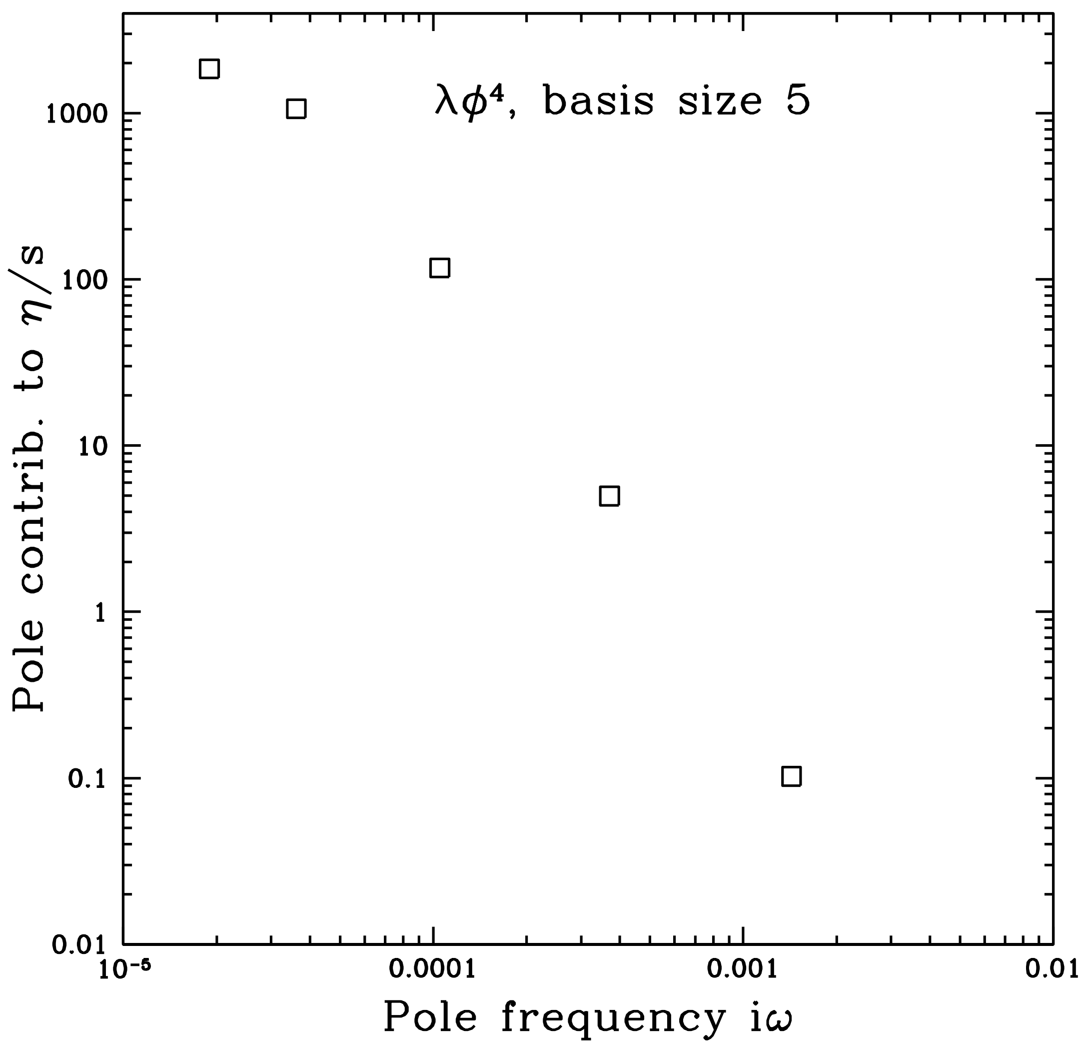
3 basis elements



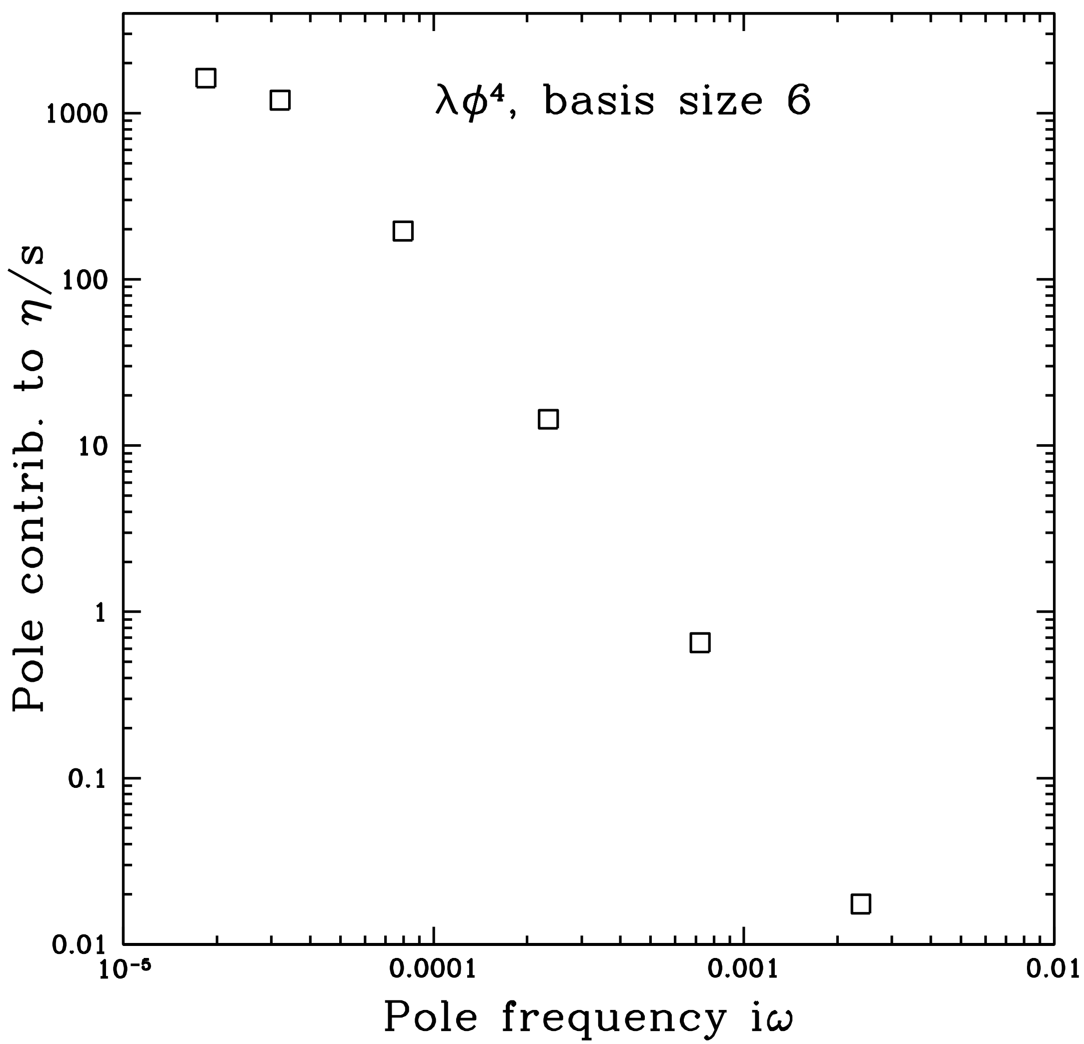
4 basis elements



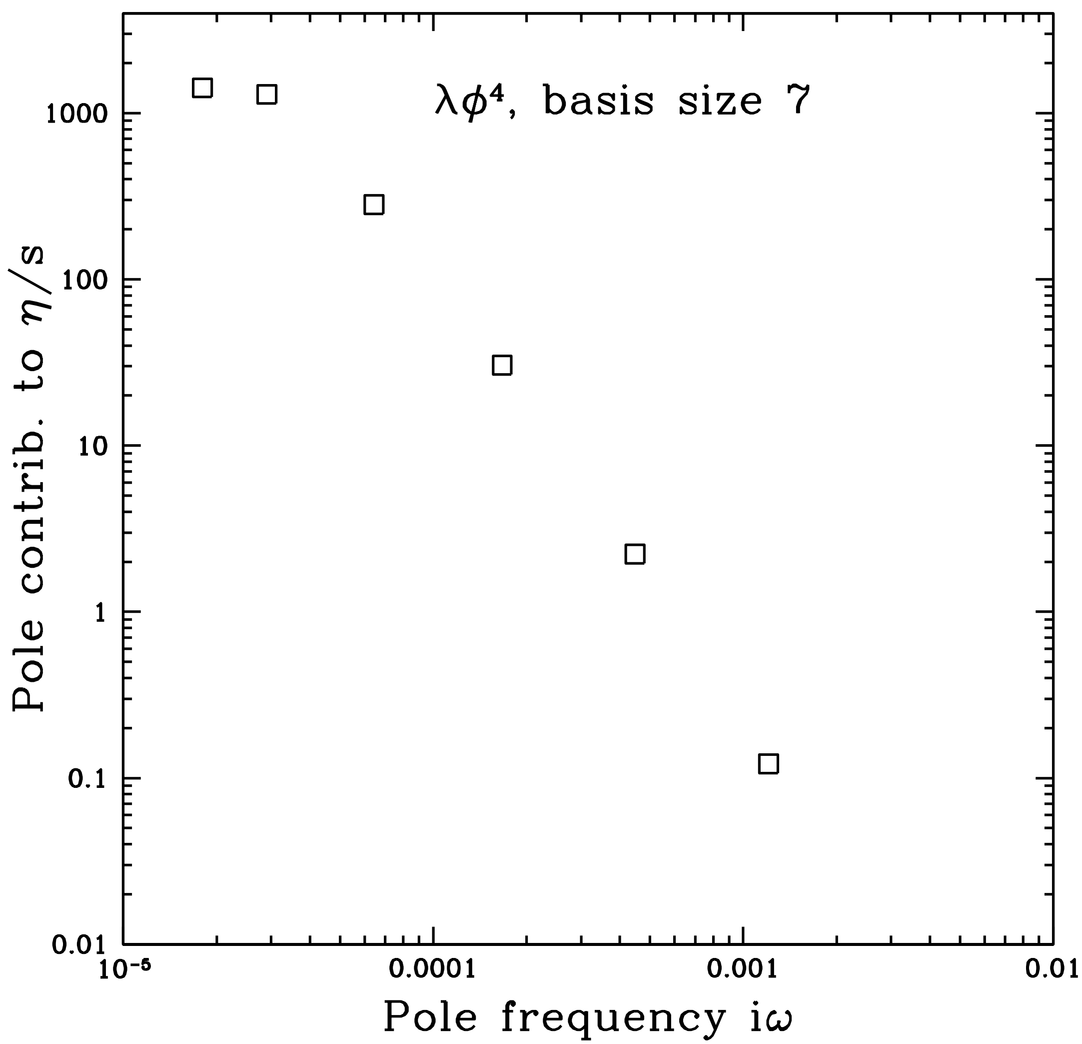
5 basis elements



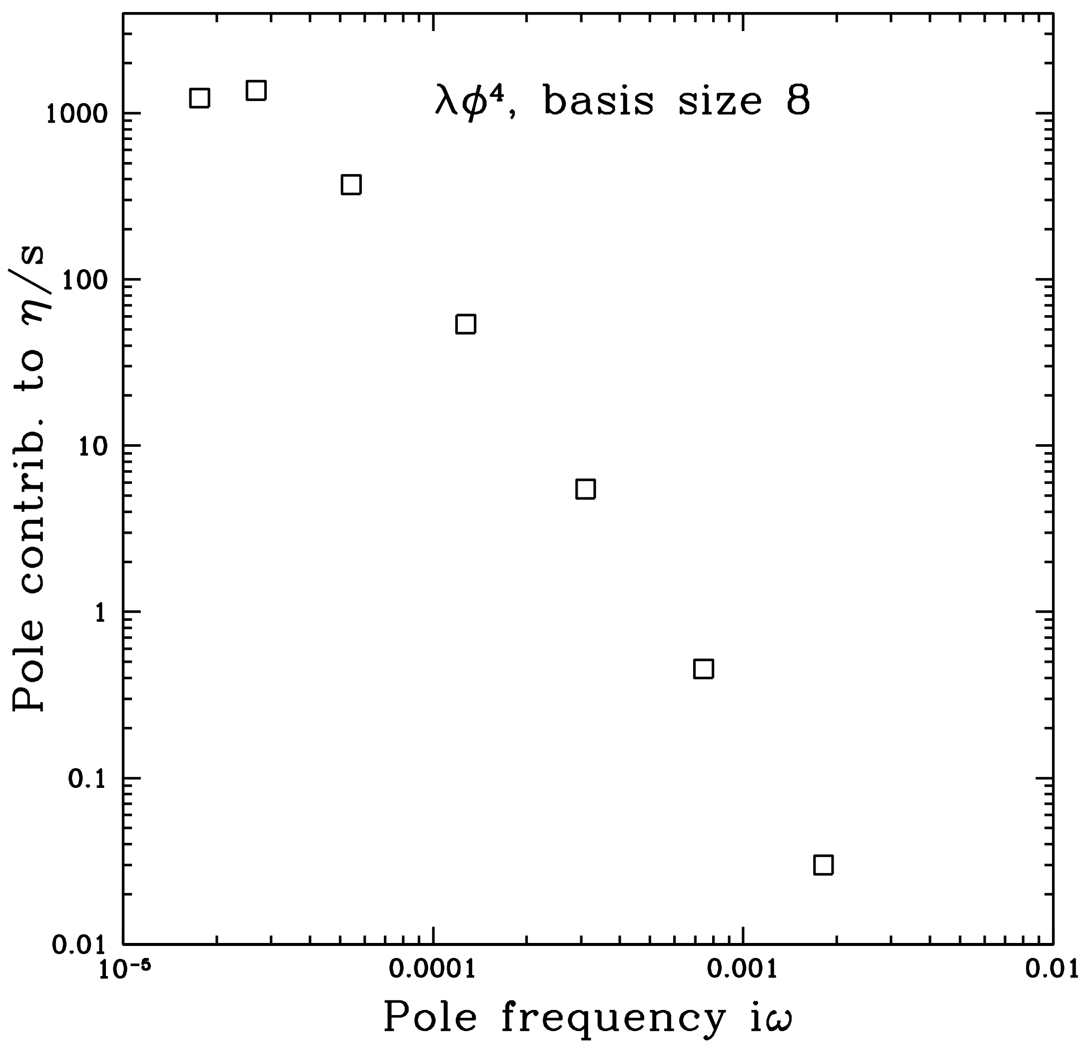
6 basis elements



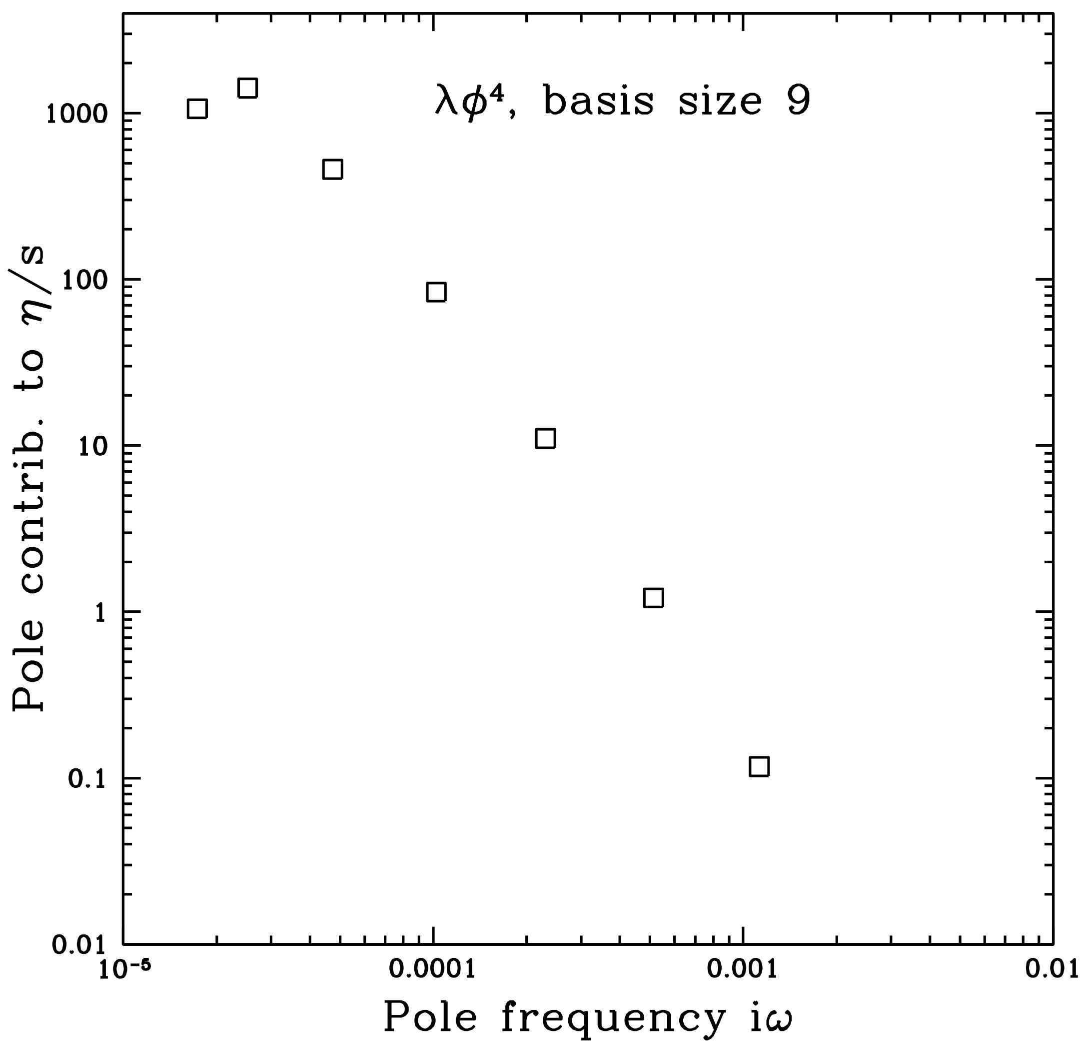
7 basis elements



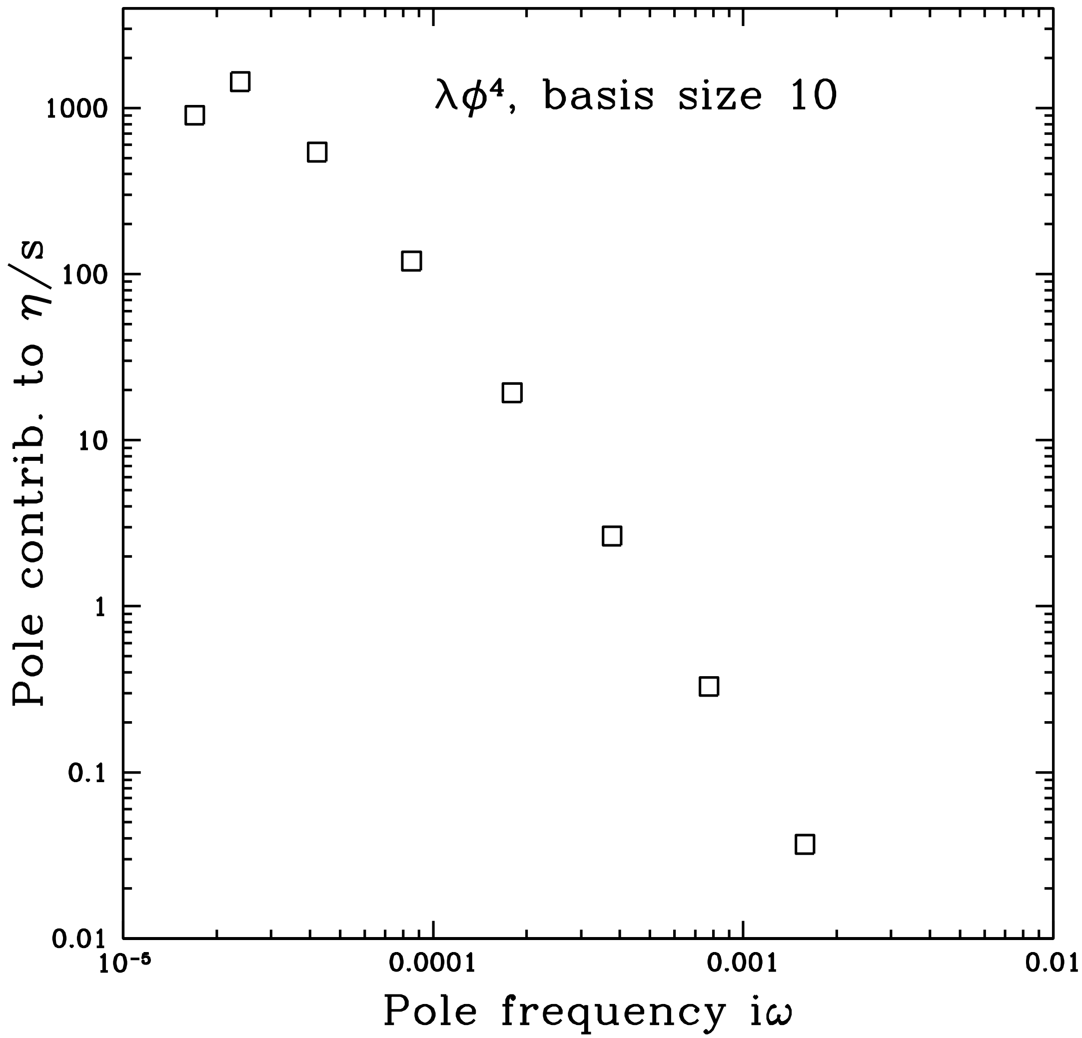
8 basis elements



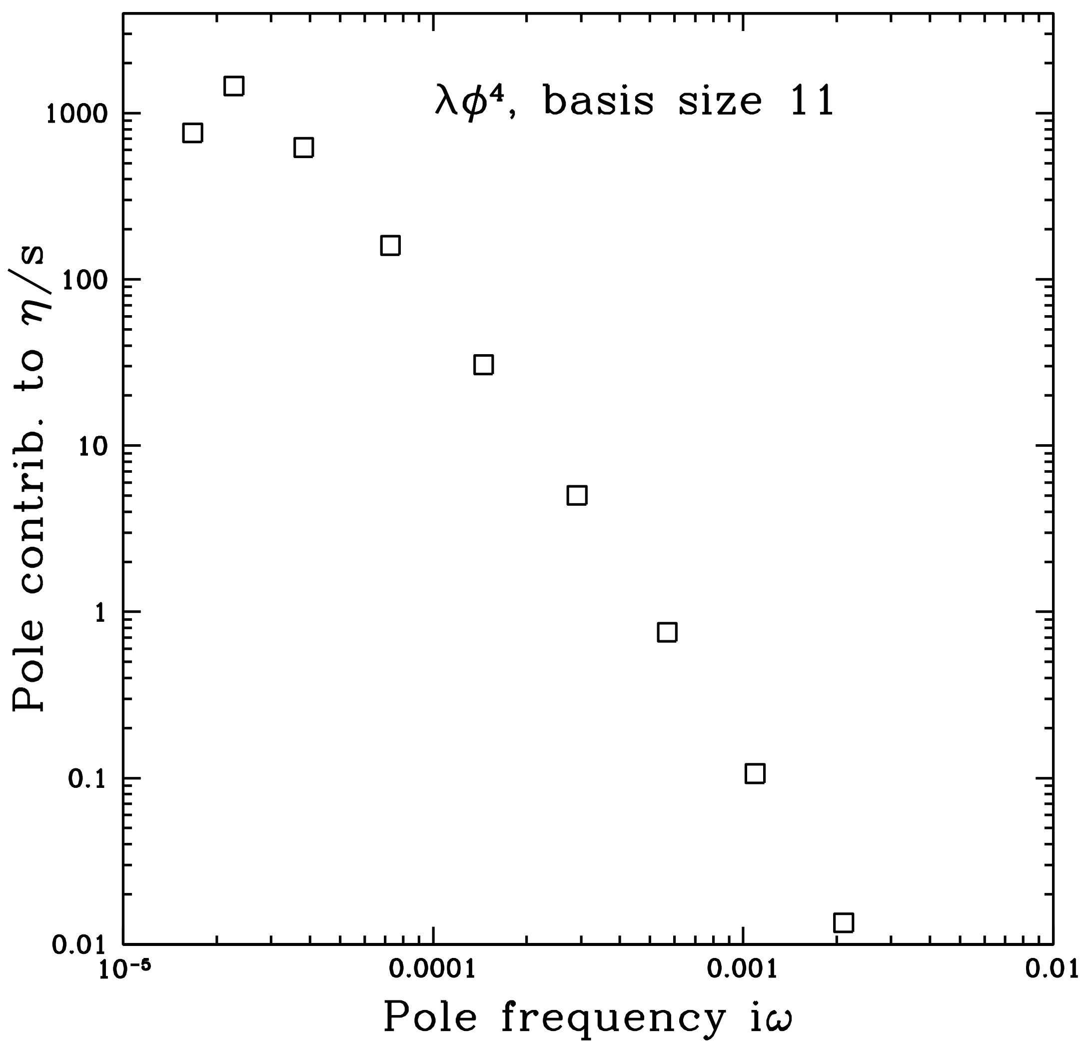
9 basis elements



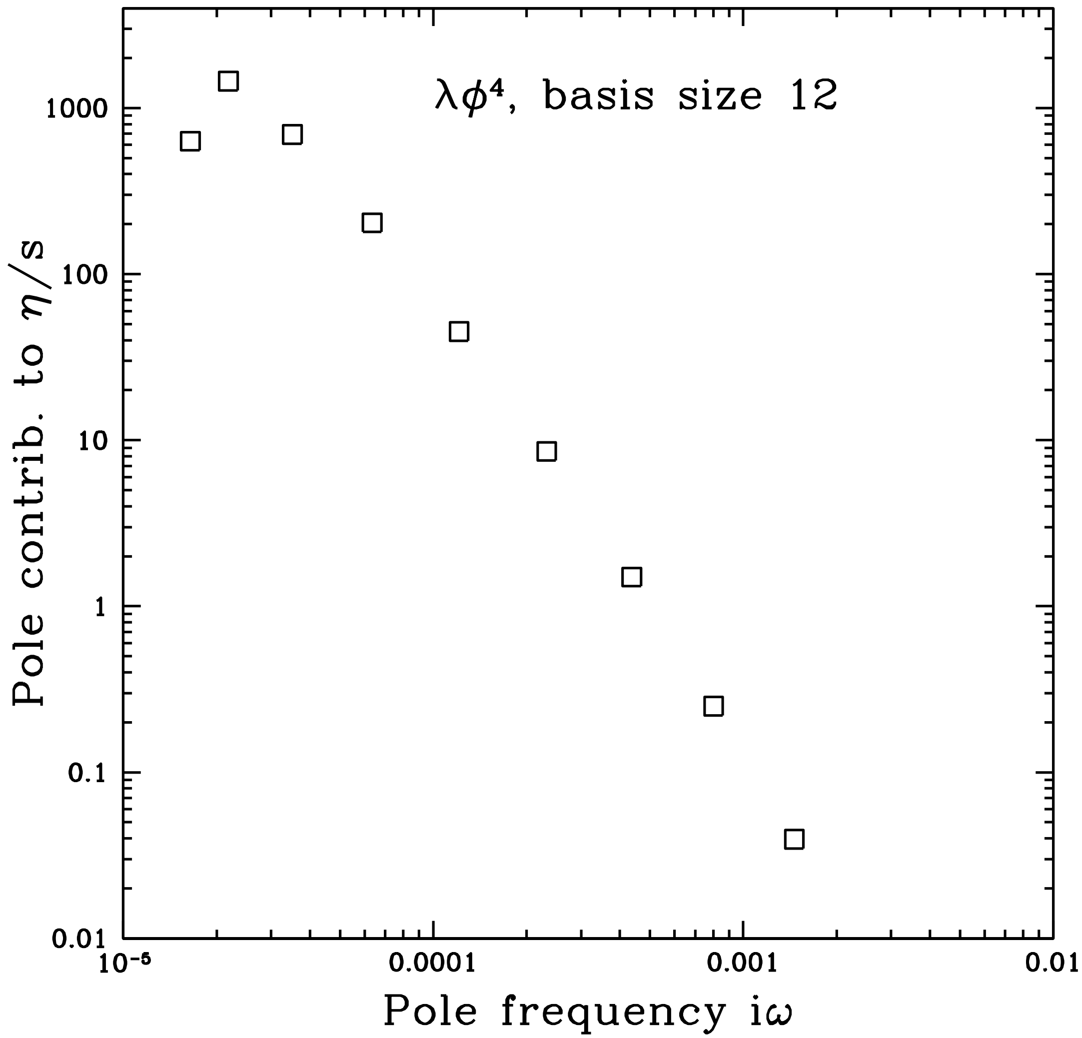
10 basis elements



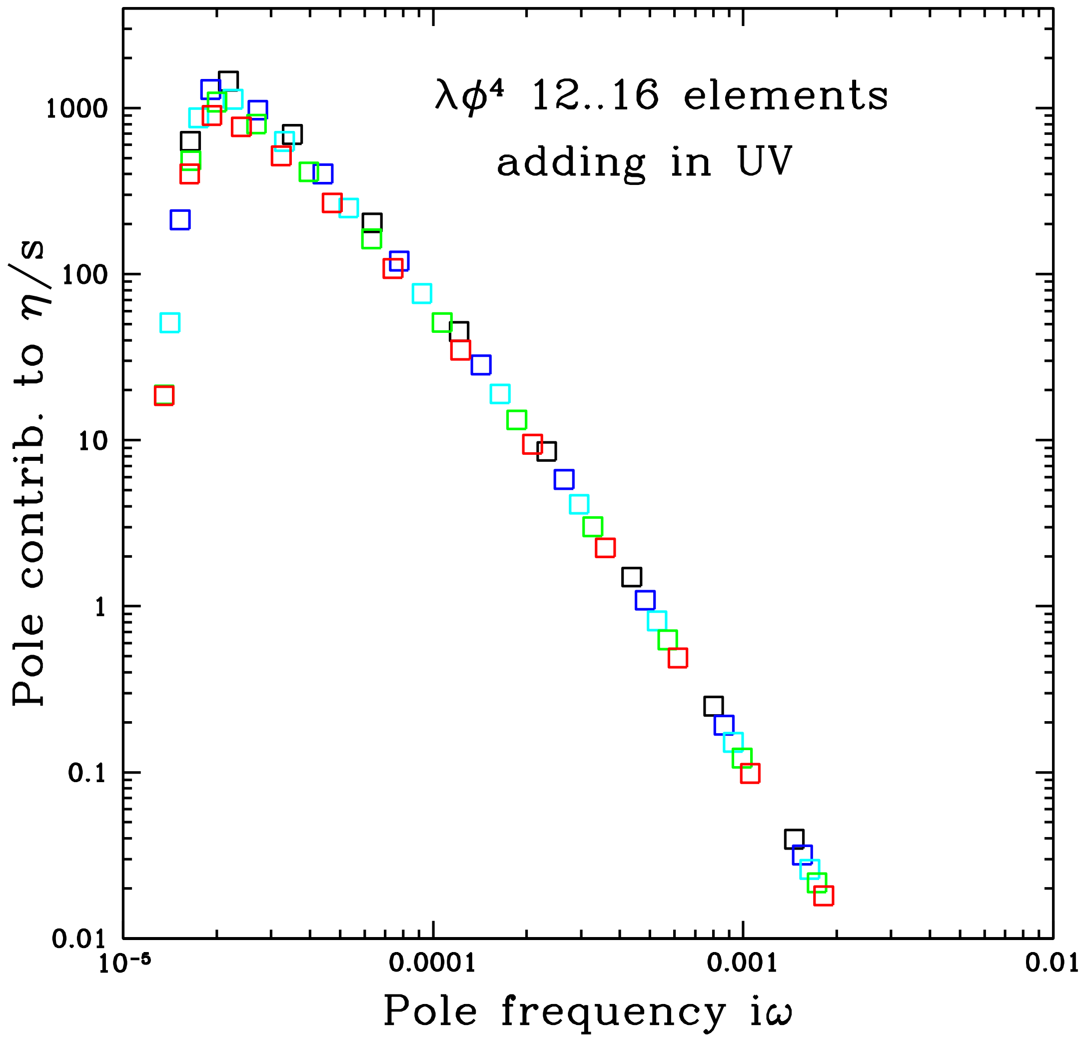
11 basis elements



12 basis elements



Expand UV power allowed



My interpretation

- Looks to me like a cut!
- Dominant contribution from one scale
- Cut discontinuity falls fast at smaller ω
- Discontinuity also falls fast at larger ω
- Large/small ω from small/large-k particles??

Conclusions

It looks to me like

- Considering $\langle T_{xy}T_{xy}(\omega \ll T, k=0)\rangle$
- $\lambda\phi^4$ theory at weak coupling has a cut at strictly imaginary ω
- Cut has a narrow region of large discontinuity
- Extends to larger ω with small discontinuity (forever? yes at small λ, cut off by thermal mass...)
- Extends to small ω with small discontinuity (all the way to $\omega = 0$? If so, exponentially small)