Constraining light-cone spectral functions with the lattice[1]

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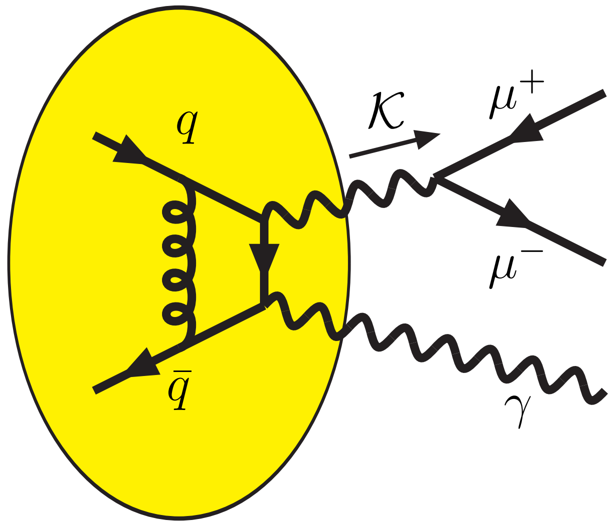
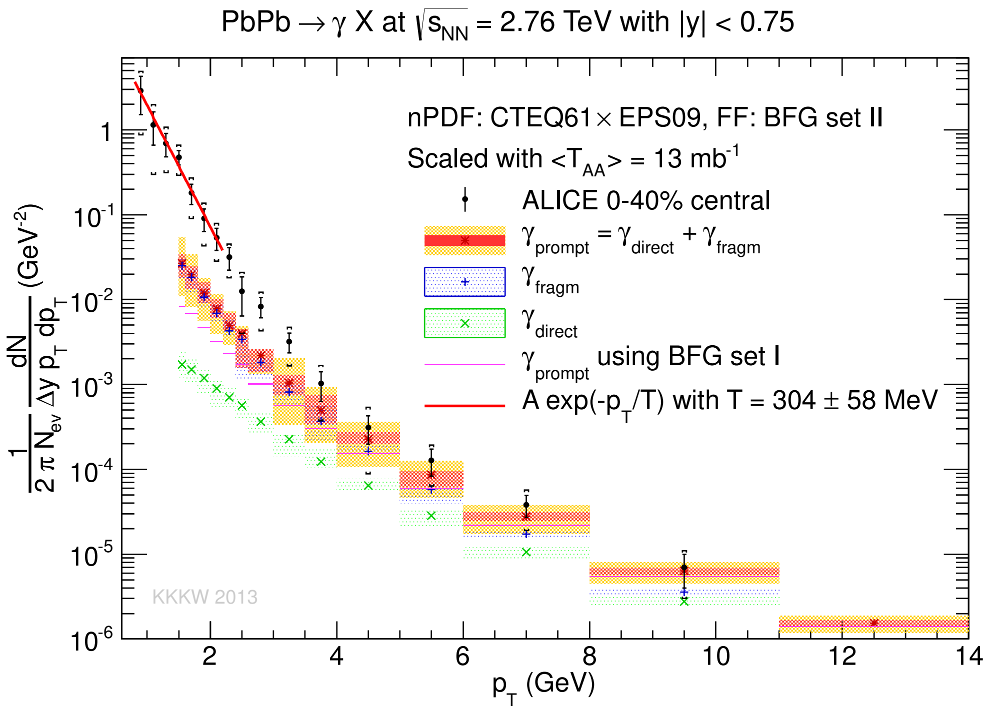
Motivation

Production rate of photons from the quark-gluon plasma:

$$\frac{\mathrm{d}n_\gamma}{\mathrm{d}t\,\mathrm{d}\ln k} = \frac{k^2}{2\pi^2}\int_{\mathcal{X}} e^{ik(t-z)}\langle\, J^1_{\mathsf{em}}(0)\, J^1_{\mathsf{em}}(\mathcal{X})\,\rangle + \mathcal{O}(\alpha^2_{\mathsf{em}}) \;.$$



ALICE/LHC data:[2]



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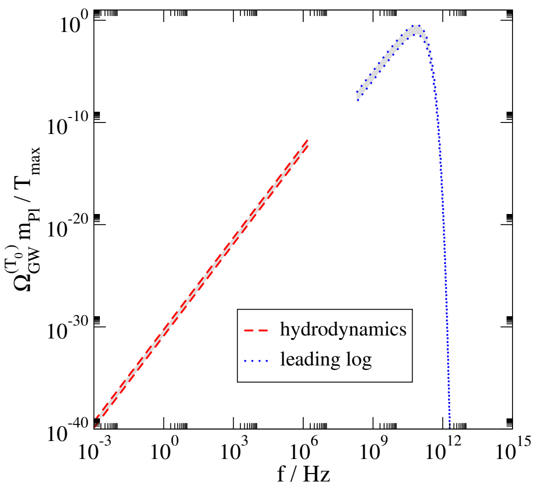
[2] M. Klasen, C. Klein-Bösing, F. König and J.P. Wessels, *How robust is a thermal photon interpretation of the ALICE low-p_T data?*, 1307.7034.

A similar production rate applies for gravitational waves:[3]

$$\frac{\mathrm{d}\rho_{\mathrm{GW}}}{\mathrm{d}t\,\mathrm{d}\ln k} = \frac{8k^3}{\pi m_{\mathrm{Pl}}^2}\int_{\mathcal{X}} e^{ik(t-z)}\langle\, T_{12}(0)\, T_{12}(\mathcal{X})\,\rangle \;.$$

This constrains the highest temperature after Big Bang.



$$\Omega_{\mathrm{GW}}\frac{m_{\mathrm{Pl}}}{T_{\mathrm{max}}} = y\;,$$

$$\Omega_{\mathrm{GW}} = y \times \frac{T_{\mathrm{max}}}{m_{\mathrm{Pl}}}\;.$$

[3] J. Ghiglieri, ML, *Gravitational wave background from Standard Model physics: Qualitative features*, 1504.02569.

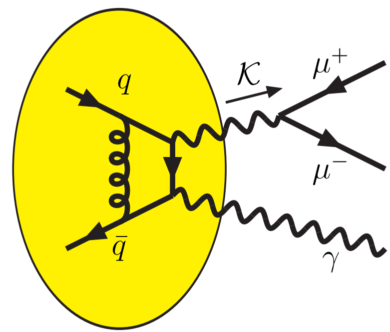
Basic definitions

It is convenient to rewrite the photon production rate as

$$\frac{\mathrm{d}n_\gamma}{\mathrm{d}t\,\mathrm{d}^3\mathbf{k}} = \frac{2\alpha_{\text{em}}\chi_{\text{q}}}{3\pi^2}\, n_{\text{B}}(k) D_{\text{eff}}(k) \; + \; \mathcal{O}(\alpha_{\text{em}}^2) \; .$$

Here $n_{\text{B}}(k) \equiv 1/(e^{\beta k} - 1)$ is the Bose distribution, and $\chi_{\text{q}} \sim T^2$ is a quark-number susceptibility which is easy to measure with lattice QCD / compute with pQCD.

The relation applies to all orders in α_{s}.



The strong interactions are hidden in $D_{\text{eff}}(k)$

The "effective diffusion coefficient" is defined as

$$D_{\text{eff}}(k) \equiv \begin{cases} \dfrac{\rho_{\text{V}}(k, \mathbf{k})}{2\chi_{\text{q}} k} & , \quad k > 0 \\ \lim\limits_{\omega \to 0^+} \dfrac{\rho_{\text{V}}(\omega, \mathbf{0})}{3\chi_{\text{q}}\omega} & , \quad k = 0 \end{cases} .$$

Hydrodynamics shows that $\lim_{k\to 0} D_{\text{eff}}(k) = D$ (cf. later).

Vector spectral function:

$$\rho_{\text{V}}(\omega, \mathbf{k}) \equiv \int_{\mathcal{X}} e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \langle \frac{1}{2}[V^{\mu}(t, \mathbf{x}) \, , \, V_{\mu}(0)] \rangle_{\text{c}} \, ,$$

$$V^{\mu} \equiv \bar{\psi}\gamma^{\mu}\psi \, , \quad \eta = (-+++) \, .$$

General structure of ρ_{V}

(i) Hydrodynamic regime

For small k the general theory of statistical fluctuations applies,[4] and permits for a "hydrodynamic" prediction:[5]

$$\frac{\rho_{\rm V}(\omega,\mathbf{k})}{\omega} = \Big(\frac{\omega^2 - k^2}{\omega^2 + D^2 k^4} + 2\Big)\chi_{\rm q} D \; .$$

Here $D \equiv \lim_{k\to 0} D_{\rm eff}(k)$ is the diffusion coefficient, and $\chi_{\rm q}$ is the quark number susceptibility, parametrizing the constant correlator $\int_{\mathbf{x}} \langle V^0(\tau,\mathbf{x}) V^0(0,\mathbf{0})\rangle = \chi_{\rm q} T$.

Note that $\rho_{\rm V}$ can be negative in the space-like domain $\omega < k$.

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[4] Cf. e.g. E.M. Lifshitz and L.P. Pitaevskii, *Statistical Physics, Part 2*, §88-89.

[5] Cf. e.g. J. Hong and D. Teaney, *Spectral densities for hot QCD plasmas in a leading log approximation*, 1003.0699.

(i) pQCD

Leading order (LO) at $M \equiv \sqrt{\omega^2 - k^2} \neq 0$:[6]

$$\rho_{\rm V}(\omega, \mathbf{k}) = \frac{N_{\rm c} T M^2}{2\pi k} \Big\{ \ln \Big[\frac{\cosh(\frac{\omega+k}{4T})}{\cosh(\frac{\omega-k}{4T})} \Big] - \frac{\omega\, \theta(k-\omega)}{2T} \Big\} \; .$$

Leading-log order (LL) at $M = 0$:[7]

$$\rho_{\rm V}(k, \mathbf{k}) = \frac{\alpha_{\rm s} N_{\rm c} C_{\rm F} T^2}{4} \ln\Big(\frac{1}{\alpha_{\rm s}}\Big) [1 - 2 n_{\mathsf{F}}(k)] + \mathcal{O}(\alpha_{\rm s} T^2) \; .$$

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[6] e.g. G. Aarts and J.M. Martínez Resco, *Continuum and lattice meson spectral functions at nonzero momentum and high temperature*, hep-lat/0507004.

[7] J.I. Kapusta, P. Lichard and D. Seibert, *High-energy photons from quark-gluon plasma versus hot hadronic gas*, PRD 44 (1991) 2774; R. Baier, H. Nakkagawa, A. Niégawa and K. Redlich, *Production rate of hard thermal photons and screening of quark mass singularity*, ZPC 53 (1992) 433.

Current status

LO at $M = 0$.[8] (only numerical result)

NLO at $M = 0$.[9] (only numerical result)

NLO at $M \sim gT$.[10] (only numerical result)

NLO at $M \sim \pi T$.[11] (only numerical result)

N^{4}LO at $M \gg \pi T$.[12] (analytic result)

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[8] P.B. Arnold, G.D. Moore and L.G. Yaffe, *Photon emission from ultrarelativistic plasmas*, hep-ph/0109064; *Photon emission from quark gluon plasma: Complete leading order results*, hep-ph/0111107.

[9] J. Ghiglieri *et al*, *Next-to-leading order thermal photon production in a weakly coupled quark-gluon plasma*, 1302.5970.

[10] J. Ghiglieri and G.D. Moore, *Low Mass Thermal Dilepton Production at NLO in a Weakly Coupled Quark-Gluon Plasma*, 1410.4203.

[11] ML, *NLO thermal dilepton rate at non-zero momentum*, 1310.0164.

[12] S. Caron-Huot, *Asymptotics of thermal spectral functions*, 0903.3958; P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Order α_s^4 QCD Corrections to Z and τ Decays*, 0801.1821.

(iii) AdS/CFT[13]

In the IR the hydrodynamic prediction is reproduced, with the specific values $D = 1/(2\pi T)$ and $\chi_{\mathrm{q}} = N_{\mathrm{c}}^2 T^2/8$.

One can also ask when hydrodynamics applies: the spectral function is close to hydrodynamics for $k \lesssim 0.5/D$, and becomes negative at the smallest ω for $k \lesssim 1.07/D$.

[13] G. Policastro, D.T. Son and A.O. Starinets, *From AdS / CFT correspondence to hydrodynamics*, hep-th/0205052; S. Caron-Huot *et al*, *Photon and dilepton production in supersymmetric Yang-Mills plasma*, hep-th/0607237.

General comment on the real world

$N_\mathrm{f} = 0$:

$m_{0_{++}} \gg 1$ GeV $\Rightarrow$ need to heat the system "a lot".

Concretely, $T_\mathrm{c}/\Lambda_{\overline{\mathrm{MS}}} \simeq 1.24 \Rightarrow \alpha_\mathrm{s}(2\pi T_\mathrm{c}) =$ "small".

$N_\mathrm{f} = 3$:

$m_\pi \ll 1$ GeV $\Rightarrow$ don't need to heat a lot.

Concretely, $T_\mathrm{c}/\Lambda_{\overline{\mathrm{MS}}} \simeq 0.45 \Rightarrow \alpha_\mathrm{s}(2\pi T_\mathrm{c}) =$ "large".

So at least for the unquenched case, pQCD is not sufficient.

Non-perturbative approach: idea 1/2

What can we do with Euclidean lattice?

$$G_{\rm V}(\tau,\mathbf{k}) = \int_0^\infty \frac{{\rm d}\omega}{\pi}\,\rho_{\rm V}(\omega,\mathbf{k})\,\frac{\cosh[\omega(\frac{\beta}{2}-\tau)]}{\sinh[\frac{\omega\beta}{2}]}\;, \quad \beta \equiv \frac{1}{T}\;.$$

In principle inversion **is** possible by the Cuniberti method,[14] if the perturbative UV tail ($\tau \ll \beta$, $\omega \gg \pi T$) is first subtracted.[15]

In practice there is a "sign problem" in the inversion $\Rightarrow$ fragile unless very high statistical precision available.[16]

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[14] G. Cuniberti, E. De Micheli and G.A. Viano, *Reconstructing the thermal Green functions at real times from those at imaginary times*, cond-mat/0109175; F. Ferrari, *The Analytic Renormalization Group*, 1602.07355.

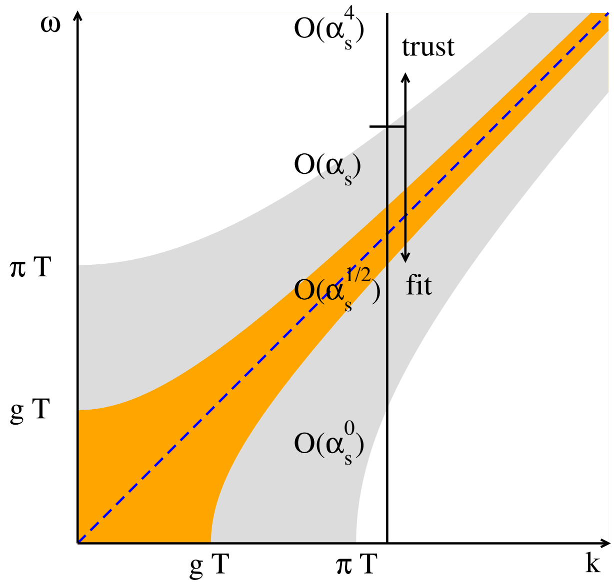
[15] Y. Burnier, ML, *Towards flavour diffusion coefficient and electrical conductivity without ultraviolet contamination*, 1201.1994.

[16] Y. Burnier, ML, L. Mether, *A Test on analytic continuation of thermal imaginary-time data*, 1101.5534.

Here: down-to-earth approach

Trust UV from pQCD, fit an interpolating function in the IR.[17]
Only a few coefficients can be fitted, so a "good" basis is needed.

($g \equiv \sqrt{4\pi\alpha_s}$):



[17] J. Ghiglieri, O. Kaczmarek, ML, F. Meyer, *Lattice constraints on the thermal photon rate*, 1604.07544.

Polynomial interpolation (assuming analyticity, $V \to \infty$)

Pick a point above which pQCD should apply, for instance

$$\omega_0 \simeq \sqrt{k^2 + (\pi T)^2} \ ,$$

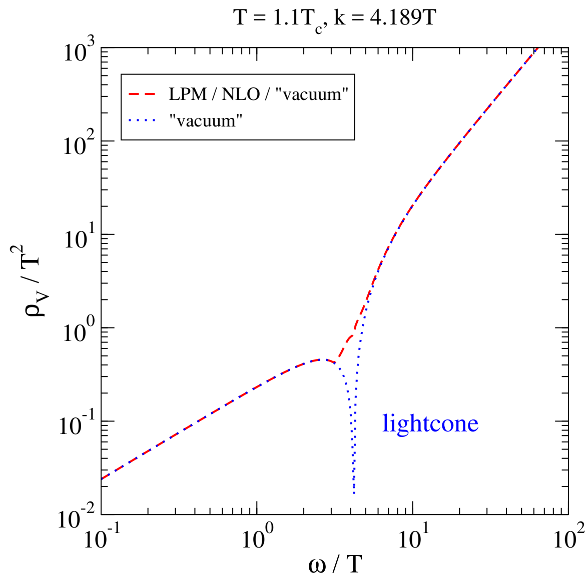
and use that to fix two coefficients:

$$\rho_{\rm V}(\omega_0, \mathbf{k}) \ \equiv \ \beta \ , \quad \partial_\omega \rho_{\rm V}(\omega_0, \mathbf{k}) \ \equiv \ \gamma \ .$$

Then the most general polynomial odd in ω takes the form

$$\rho_{\text{fit}} \equiv \frac{\beta\, \omega^3}{2\omega_0^3}\Big(5 - \frac{3\omega^2}{\omega_0^2}\Big) - \frac{\gamma\, \omega^3}{2\omega_0^2}\Big(1 - \frac{\omega^2}{\omega_0^2}\Big) + \sum_{n \geq 0}^{n_{\text{max}}} \frac{\delta_n \omega^{1+2n}}{\omega_0^{1+2n}}\Big(1 - \frac{\omega^2}{\omega_0^2}\Big)^2 \ .$$

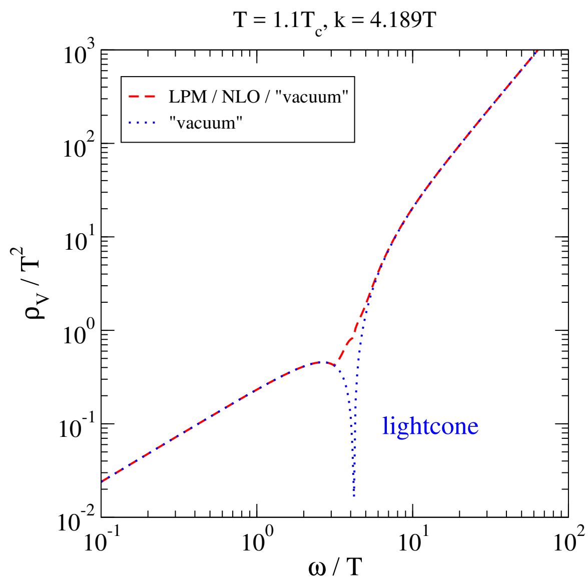
How does the pQCD result look like? (“vacuum”≡LO+...)[18]



[18] $3T < \omega < 10T$ from J. Ghiglieri and G.D. Moore, *Low Mass Thermal Dilepton Production at NLO in a Weakly Coupled Quark-Gluon Plasma*, 1410.4203 ; $\omega \gtrsim 10T$ from I. Ghisoiu and ML, *Interpolation of hard and soft dilepton rates*, 1407.7955 ; $\omega \gg 10T$ from ML, *NLO thermal dilepton rate at non-zero momentum*, 1310.0164. The best available perturbative data, both for $N_{\rm f} = 0$ and $N_{\rm f} = 3$, can be found at J. Ghiglieri and ML, web page `http://www.laine.itp.unibe.ch/dilepton-lattice/`

Missing ingredient

Below the light cone, ρ_V is only known at LO. More information there could be an "inexpensive" way to constrain the fit (for now the whole IR domain $\omega \le \omega_0$ is fitted).



Lattice details

Imaginary-time observable:

$$G_{\mathrm{V}}(\tau,\mathbf{k}) \equiv \int_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}} \langle V^i(\tau,\mathbf{x})\, V^i(0) - V^0(\tau,\mathbf{x})\, V^0(0)\rangle_{\mathsf{c}} \;.$$

Consider the full G_{V} rather than G^{ii} because this is relevant for dileptons and because much more is known within pQCD.

Momenta are chosen along the lattice axes. With periodic boundary conditions this requires

$$k = 2\pi n T \times \frac{N_\tau}{N_{\mathrm{s}}} \;,$$

where N_{s}, N_τ are the spatial and temporal lattice extents.

Ensemble

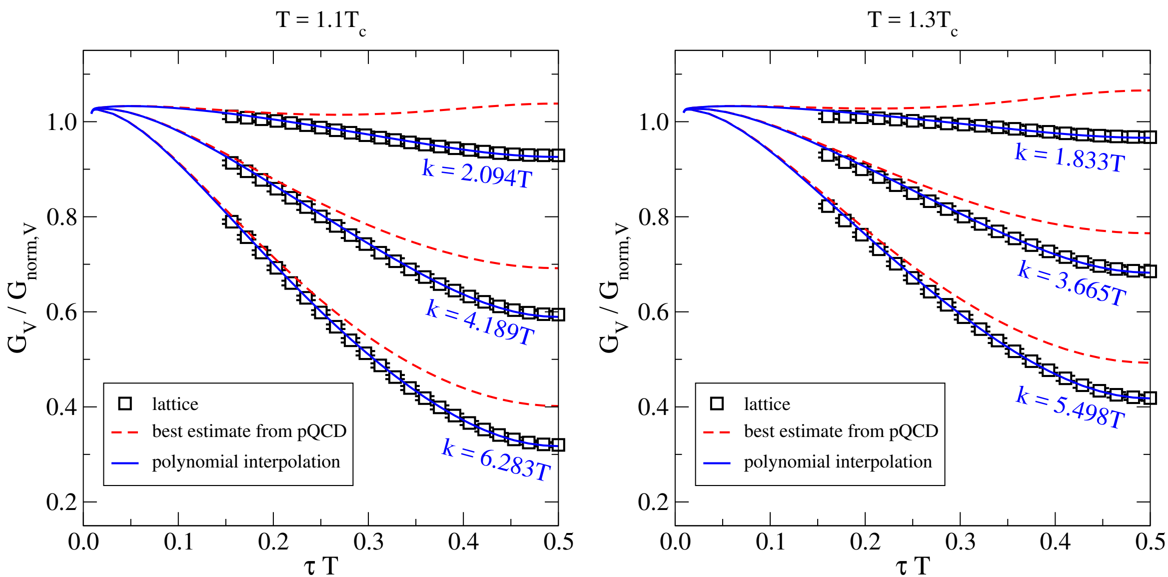
β_0	$N_s^3 \times N_\tau$	confs	$T/T_c\|_{t_0}$	k/T
7.192	$96^3 \times 32$	314	1.12	2.094,4.189,6.283
7.544	$144^3 \times 48$	358	1.14	
7.793	$192^3 \times 64$	242	1.15	
7.192	$96^3 \times 28$	232	1.28	1.833,3.665,5.498
7.544	$144^3 \times 42$	417	1.31	
7.793	$192^3 \times 56$	273	1.31	

With such large β_0 we are frozen to the trivial topological sector,[19] but do not expect this to affect the results dramatically.

[19] S. Schaefer *et al.* [ALPHA Collaboration], *Critical slowing down and error analysis in lattice QCD simulations*, 1009.5228.

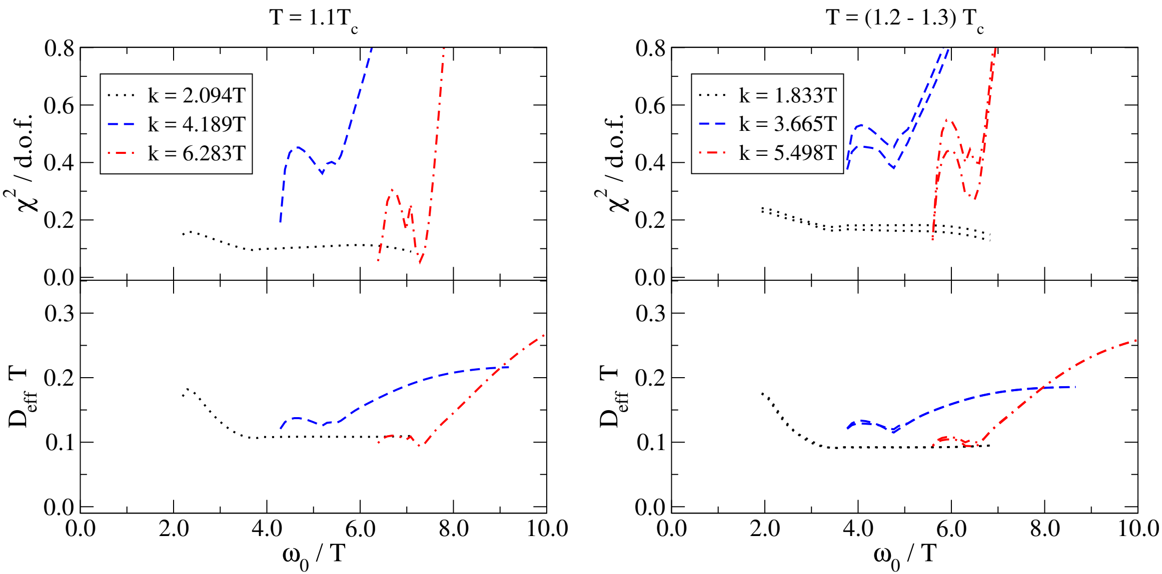
Numerical results

Imaginary-time correlators after continuum extrapolation



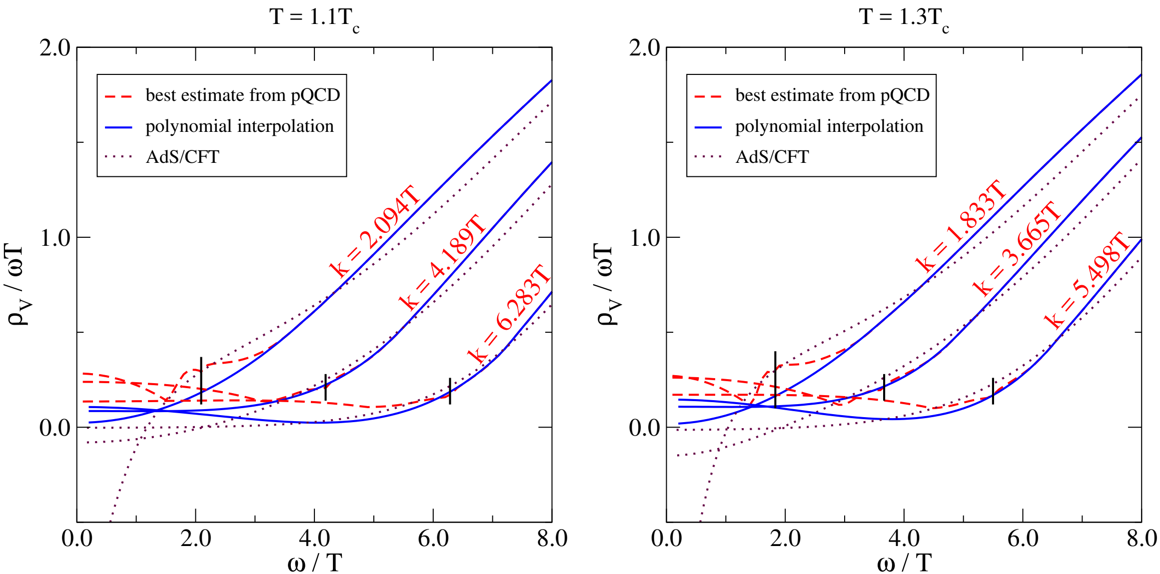
$$\frac{G_{\text{norm, V}}}{6T^3} \equiv \pi(1-2\tau T)\frac{1+\cos^2(2\pi\tau T)}{\sin^3(2\pi\tau T)} + \frac{2\cos(2\pi\tau T)}{\sin^2(2\pi\tau T)} .$$

One-parameter fits (δ_0) as a function of ω_0



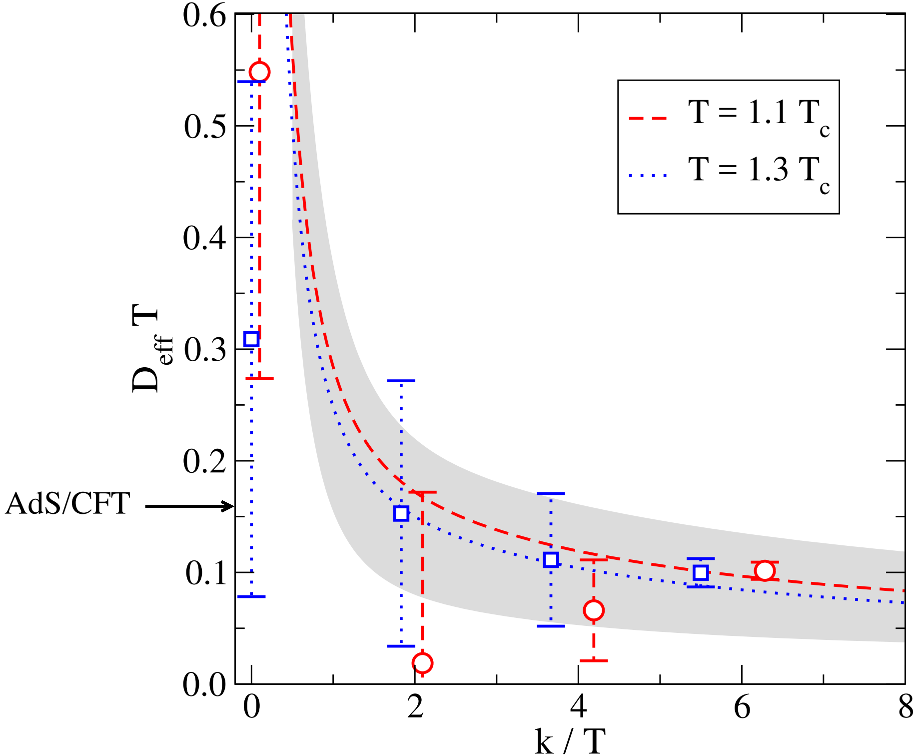
Final results are from two-parameter fits (δ_0, δ_1) to a full bootstrap ensemble for the continuum-extrapolated correlator.

One-parameter best fits for the spectral function



There is indeed a clear reduction in the spacelike domain.

Value at the photon point[20]



[20] J. Ghiglieri, O. Kaczmarek, ML, F. Meyer, *Lattice constraints on the thermal photon rate*, 1604.07544.

What did we learn?

(i) Lattice side

If had:

(a) continuum-extrapolated lattice data

(b) bootstrap ensemble for error estimation

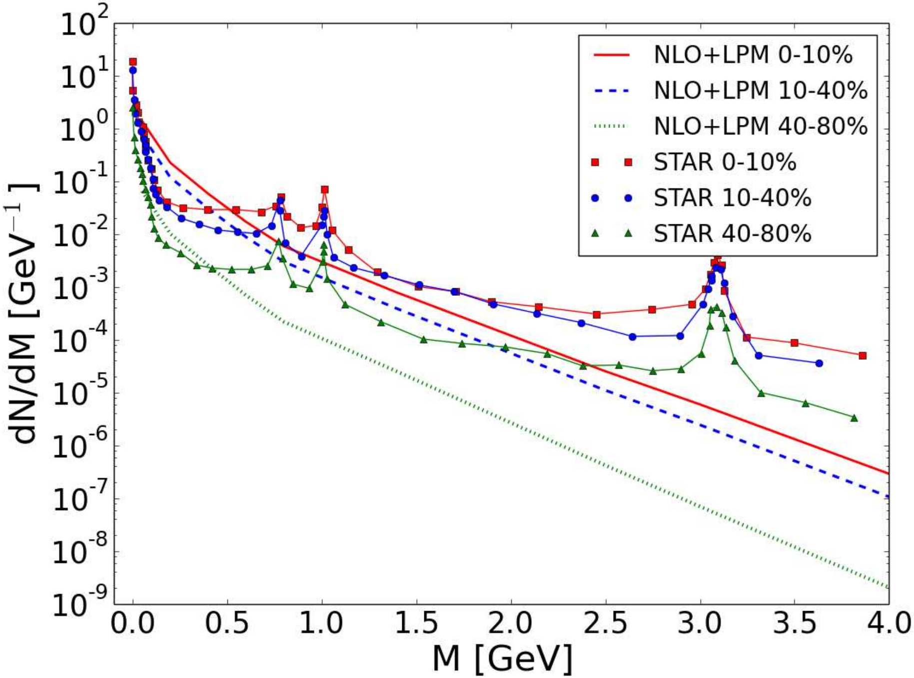
(c) high-order pQCD predictions for UV

(d) a well-motivated functional basis for IR

(e) a somewhat increased statistical precision

then results might be brought under reasonable control.

(ii) Physics: reduction could agree with phenomenology!?[21]



[21] Y. Burnier and C. Gastaldi, *Contribution of next-to-leading order and Landau-Pomeranchuk-Migdal corrections to thermal dilepton emission in heavy-ion collisions*, 1508.06978.

Non-perturbative approach: idea 2/2

Something to learn from "dynamic" analytic continuation?

$$G^{(\omega_n)}_{\mu\nu}(z) \equiv \int_{\mathbf{x}} \int_0^{\frac{1}{T}} \mathrm{d}\tau \, e^{i\omega_n \tau} \langle V_\mu(\tau, \mathbf{x}, z) V_\nu(0) \rangle \ .$$

Normally: fix momentum through $\int_{-\infty}^{\infty} \mathrm{d}z \, e^{ikz} G^{(\omega_n)}_{\mu\nu}(z)$, attempt analytic continuation into $\rho_{\mu\nu} = \mathrm{Im}(\ldots)_{\omega_n \to -i[\omega + i0^+]}$.

Now: fix ω_n, measure correlators in z. Asymptotics is fixed by a screening mass $M^{(\omega_n)}_{\mu\nu}$ and by an "amplitude" $A^{(\omega_n)}_{\mu\nu}$:

$$G^{(\omega_n)}_{\mu\nu}(z) \stackrel{z \gg 1/T}{=} A^{(\omega_n)}_{\mu\nu} e^{-|z| M^{(\omega_n)}_{\mu\nu}} \ .$$

The claim is that A and M probe real-time light-cone physics!

Pieces of evidence to support the claim.

(i) Up to NLO, the light-cone scattering rate ("transverse collision kernel") is related to a Euclidean static potential,[22] which can be estimated from lattice measurements in the z-direction.[23]

[22] S. Caron-Huot, *$O(g)$ plasma effects in jet quenching*, 0811.1603.

[23] M. Panero, K. Rummukainen and A. Schäfer, *Lattice Study of the Jet Quenching Parameter*, 1307.5850; M. D'Onofrio, A. Kurkela and G.D. Moore, *Renormalization of Null Wilson Lines in EQCD*, 1401.7951.

(ii) Concretely, the NLO determination of M and A is affected by the same "potential" as LPM resummation for jet quenching.[24]

$$G^{(\omega_n)}_{\mu\nu}(z) \overset{\mu=\nu}{=} \int_0^\infty \frac{\mathrm{d}\omega}{\pi}\, e^{-\omega|z|}\, \tilde{\rho}^{(\omega_n)}_{\mu\nu}(\omega) \; .$$

$$\left(\omega_n - \omega + \frac{m_\infty^2 - \nabla_\perp^2}{2M_\mathrm{r}} + V^+ - i0^+ \right) g(\omega;\mathbf{y}) = \delta^{(2)}(\mathbf{y}) \; ,$$

$$\frac{1}{M_\mathrm{r}} \equiv \frac{1}{p_n} + \frac{1}{\omega_n - p_n} \; , \quad 0 < p_n < \omega_n \; ,$$

$$\tilde{\rho}^{(\omega_n > 0)}_{00}(\omega) = - \sum_{0<p_n<\omega_n} 2N_\mathrm{c}T \lim_{\mathbf{y}\to\mathbf{0}} \mathrm{Im}\, g(\omega;\mathbf{y}) \; .$$

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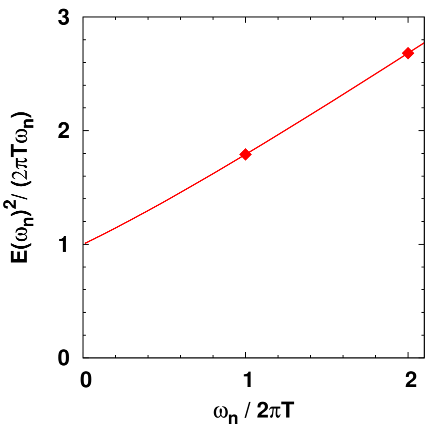
[24] B.B. Brandt, A. Francis, ML, H.B. Meyer, *A relation between screening masses and real-time rates*, 1404.2404.

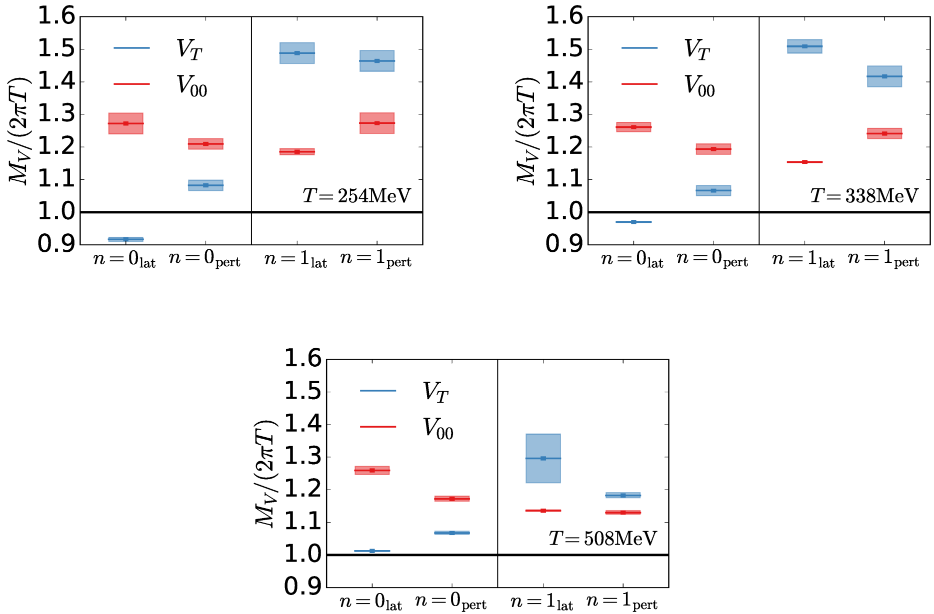
(iii) Within AdS/CFT, Harvey Meyer realized that screening masses indeed extrapolate to the diffusion coefficient:[25]

$$\rho_{00}(\omega, \mathbf{k}) = \mathrm{Im}\, \frac{k^2 \chi_{\mathrm{q}} D}{-i\omega + Dk^2} \overset{\omega \to i\omega_n}{\Rightarrow} \lim_{\omega_n \to 0} \frac{(M_{00}^{(\omega_n)})^2}{\omega_n} = \frac{1}{D} \, .$$



[25] B.B. Brandt, A. Francis, ML and H.B. Meyer, *Vector screening masses in the quark-gluon plasma and their physical significance*, 1408.5917; P.K. Kovtun and A.O. Starinets, *Quasinormal modes and holography*, hep-th/0506184; R.C. Brower *et al*, *Discrete spectrum of the graviton in the AdS(5) black hole background*, hep-th/9908196.

In any case, lattice simulations are “easy” for M and A:[26]



[26] B.B. Brandt, A. Francis, H.B. Meyer, A. Steinberg and K. Zapp, *Static and non-static vector screening masses*, 1611.09689.

Conclusions

Summary: current best "lattice-boosted" photon rate

Large distances $k < 2T \rightarrow$ strong interactions $\rightarrow$ less thermodynamic fluctuations $\rightarrow$ less currents $\rightarrow$ less photons.

The onset of the hydrodynamic regime can be empirically monitored through the k-dependence of $D_{\text{eff}}(k)$.

In principle the results can be implemented in hydrodynamic codes and compared with experimental data for the photon rate.

"Modest" improvements needed to get systematics under control.

Outlook: what could be done better?

More knowledge (NLO?) about the vector spectral function in the spacelike domain could constrain the fit.

Could the idea of "dynamic analytic continuation" be promoted beyond NLO & CFT into a truly non-perturbative tool?

Could shear viscosity be estimated from $k > 0$, by employing one of the two approaches discussed here ("fit" / "dynamic")?