Hydrodynamic fluctuations, long time tails, and critical dynamics

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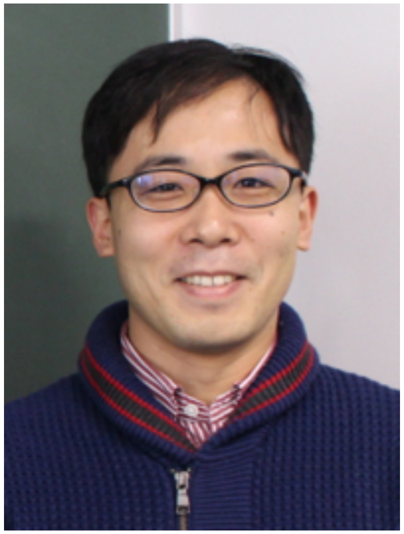
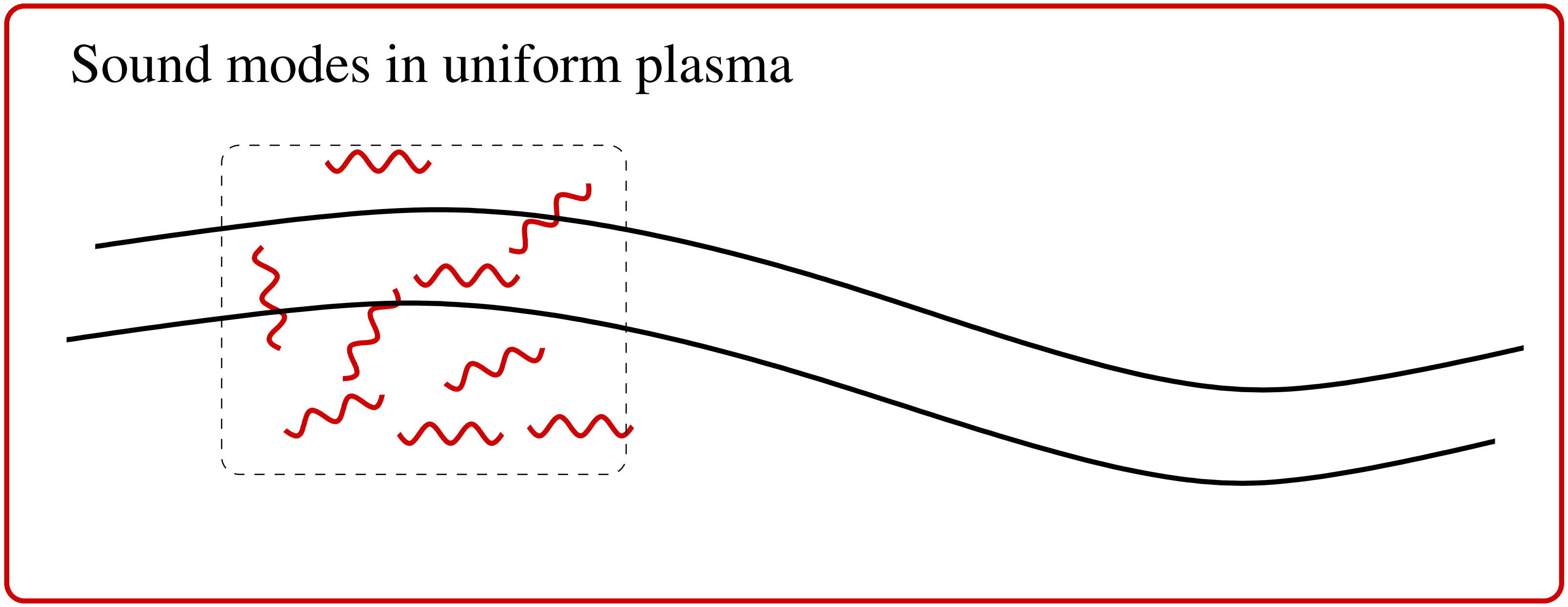
- Yukinao Akamatsu, Aleksas Mazeliauskas, DT, arXiv:1606.07742
- Yukinao Akamatsu, Aleksas Mazeliauskas, DT; in progress
- Y. Akamatsu, DT, Fanglida Yan, Yi Yin; in progress



Thermal fluctuations:



These hard sound modes are part of the bath, giving to the pressure and shear viscosity

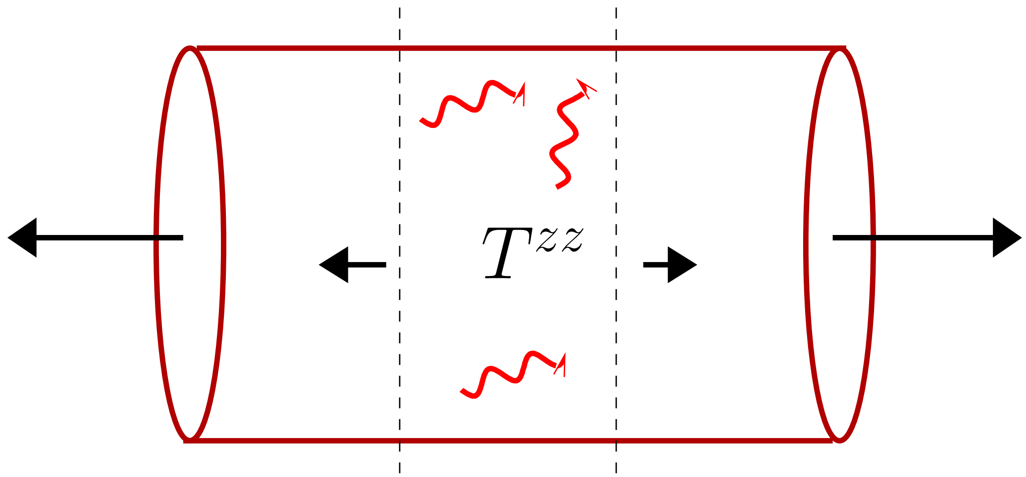
$$N^{\rm eq}_{ee}(\boldsymbol{k},t) \equiv \underbrace{\langle e^*(\boldsymbol{k},t) e(\boldsymbol{k},t)\rangle}_{\text{energy-density flucts}} = (e+p)T/c_s^2$$

$$N^{\rm eq}_{gg}(\boldsymbol{k},t) \equiv \underbrace{\left\langle g^{*i}(\boldsymbol{k},t) g^j(\boldsymbol{k},t)\right\rangle}_{\text{momentum, } g^i \equiv T^{0i}} = (e+p)T\delta^{ij}$$

In an expanding system these correlators will be driven out of equilibrium.

This changes the evolution of the slow modes.

A Bjorken expansion



1. The system has an expansion rate of $\partial_\mu u^\mu = 1/\tau$

2. The hydrodynamic expansion parameter is

$$\epsilon \equiv \frac{\gamma_\eta}{\tau} \ll 1 \qquad \gamma_\eta \equiv \frac{\eta}{e+p}$$

and corrections to hydrodynamics are organized in powers of ϵ

$$T^{zz} = p\Big[1 + \underbrace{\mathcal{O}(\epsilon)}_{\text{1st order}} + \underbrace{\mathcal{O}(\epsilon^2)}_{\text{2nd order}} + \ldots\Big]$$

High k modes are brought to equilibrium by the dissipation and noise

The transition regime:

- There is a wave number where the damping rate competes with the expansion

$$\underbrace{\gamma_\eta k^2}_{\text{damping rate}} \sim \underbrace{\frac{1}{\tau}}_{\text{expansion rate}}$$

and thus the transition happens for: $\gamma_\eta \equiv \eta/(e+p)$

$$k \sim k_* \equiv \frac{1}{\sqrt{\gamma_\eta \tau}}$$

need $k \gg k_*$ to reach equilibrium!

- This is an intermediate scale $k_* \equiv 1/(\tau\sqrt{\epsilon})$, $\epsilon \equiv \eta/(e+p)\tau$

$$\frac{1}{\tau} \ll k_* \ll \frac{1}{\ell_{\rm mfp}}$$

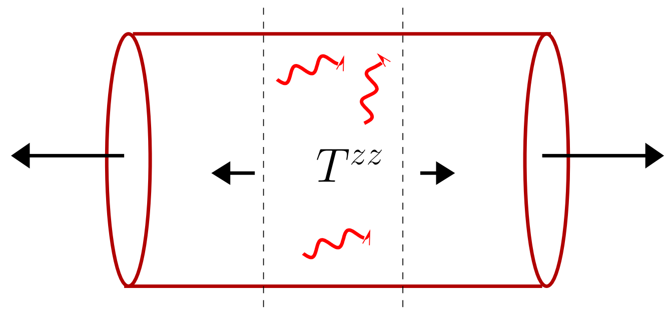
$$\frac{1}{\tau} \ll \frac{1}{\tau}\frac{1}{\sqrt{\epsilon}} \ll \frac{1}{\tau}\frac{1}{\epsilon}$$

These inequalities are the same and hold whenever hydro applies $\ell_{\rm mfp} \sim \eta/(e+p)$

Want to develop a set of hydro-kinetic equations for $k \sim k_*$

using the scale separation $\epsilon \ll \sqrt{\epsilon} \ll 1$

Estimate of longitudinal pressure from non-equilibrium modes



$$\Delta T^{zz} \sim \underbrace{\tfrac{1}{2}T}_{\text{energy / mode}} \times \underbrace{k_*^3}_{\text{number of non-eq modes}}$$

- Using $e + p = sT$ and $k_* = 1/\sqrt{\gamma_\eta \tau}$ we estimate

$$\frac{\Delta T^{zz}}{e+p} \sim \frac{1}{s} \frac{1}{(\gamma_\eta \tau)^{3/2}}$$

- The full result will be:

$$\frac{\langle T^{zz} \rangle}{e+p} = \Big[\underbrace{\frac{p}{e+p}}_{\sim 1} - \underbrace{\frac{4}{3}\frac{\gamma_\eta}{\tau}}_{\text{1st order}} + \underbrace{\frac{1.08318}{s\,(4\pi\gamma_\eta\tau)^{3/2}}}_{\text{3/2 order}} + \underbrace{\frac{(\lambda_1 - \eta\tau_\pi)}{e+p}\frac{8}{9\tau^2}}_{\text{2nd order}} \Big]$$

The correction is suppressed by $s =$ the number of degrees of freedom

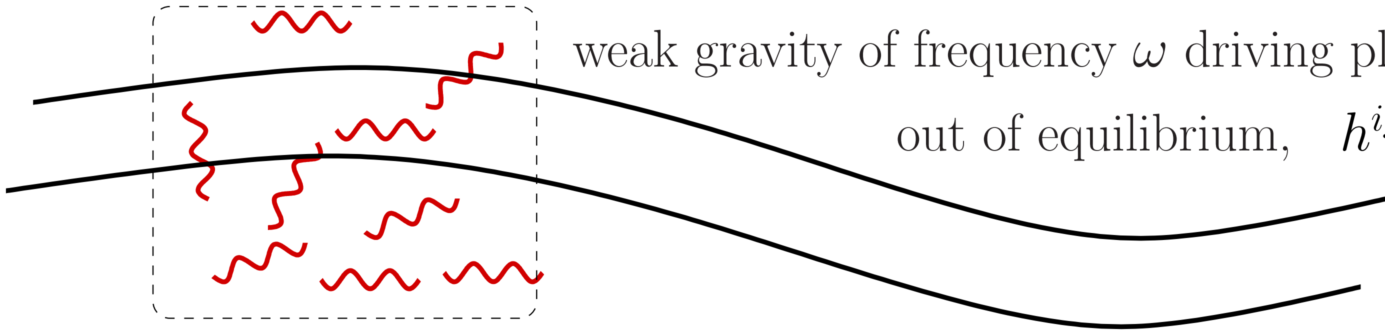
Outline

1. Hydrodynamic Linear Response – develop hydro-kinetics

2. Bjorken Expansion

Equilibrium Linear Response

Kovtun, Yaffe; Kovtun, Moore, Romatschke



weak gravity of frequency ω driving plasma out of equilibrium, $h^{ij}(\omega)$

Hydro prediction

$$\langle T^{ij}\rangle = ph^{ij} - \eta\big(\overbrace{\nabla^i u^j}^{\text{cov-deriv}} + \nabla^j u^i - \frac{2}{3}\nabla\cdot u\big) + \text{2nd order}$$

So

$$\langle T^{xy}(\omega)\rangle = \Big[p - \overbrace{i\omega\eta}^{\text{1st order}} + \overbrace{(\eta\tau_\pi - \tfrac{1}{2}\kappa)\omega^2}^{\text{2nd order}}\Big] h^{xy}(\omega)$$

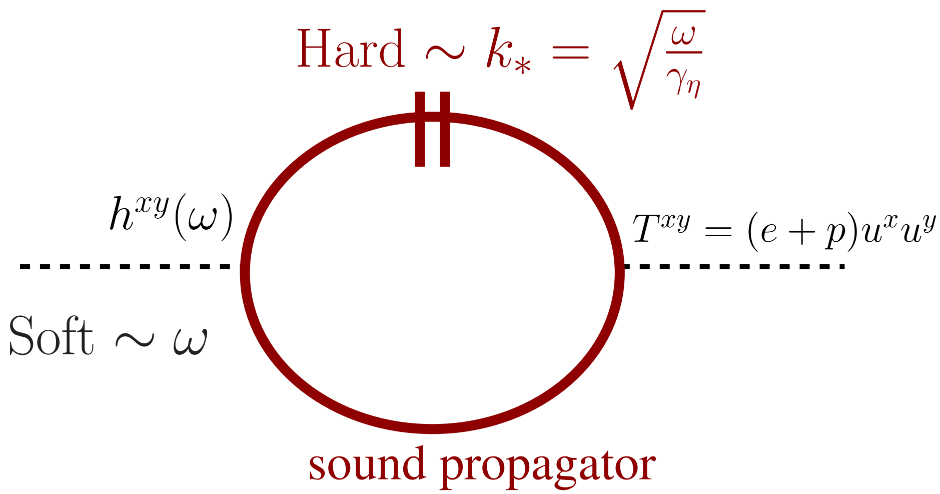
Thermal flucts. are not included, and are driven slightly out of equilibrium for $k \sim k_*$

$$\gamma_\eta k_*^2 \sim \omega \qquad \text{and they are hard} \qquad \omega \ll \; k_* \sim \sqrt{\frac{\omega}{\gamma_\eta}} \;\ll \frac{1}{\ell_{\rm mfp}}$$

Include hard thermal fluctuations with $k_* \sim \sqrt{\omega/\gamma_\eta}$ as loops

Hard Hydro Thermal Loops (HHTLs)

Kovtun, Yaffe; Kovtun, Moore, Romatschke



Evaluate the "Hard Hydro Thermal Loop"

$$\langle T^{xy}(\omega)\rangle = \Big[p - \underbrace{i\omega\eta}_{\text{1st order}} + \underbrace{\frac{(7+(3/2)^{3/2})}{240\pi} T \left(\frac{\omega}{\gamma_\eta}\right)^{3/2}}_{\text{3/2 order}} + \mathcal{O}(\omega^2)\Big] h^{xy}(\omega)$$

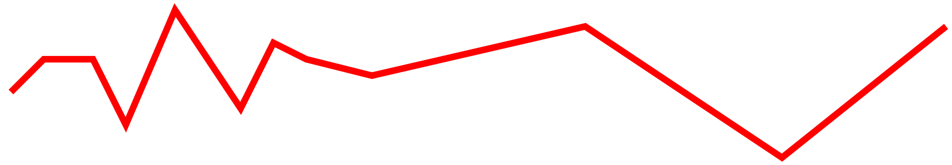
The correction is of order

$$\Delta T^{xy} \sim \tfrac{1}{2} T\, k_*^3\, h^{xy}$$

We will derive HHTLs from hydro-kinetic theory!

Developing hydro-kinetics – Brownian motion

Random Walk



$$\frac{dp}{dt} = \underbrace{-\eta p}_{\text{drag}} + \underbrace{\xi}_{\text{noise}} \qquad \langle \xi(t)\xi(t') \rangle = 2TM\eta\, \delta(t-t')$$

1. Then we want to calculate

$$N(t) = \langle p^2(t) \rangle$$

2. Integrate the equation for short times

$$p(t+\Delta t) = -\eta\, p(t)\Delta t + \int_t^{t+\Delta t} \xi(t')dt'$$

3. Compute $\langle p(t+\Delta t)\, p(t+\Delta t) \rangle$ and find an equation

$$\frac{\Delta N}{\Delta t} = -2\eta\left[N - \underbrace{TM}_{\text{equilibrium}}\right]$$

Developing hydro-kinetics – linearized hydro in a uniform system

1. Evolve fields of linearized hydro with bare parameters $p_0(\Lambda)$, $\eta_0(\Lambda)$, $s_0(\Lambda)$ etc

$$\phi_a(\boldsymbol{k}) \equiv \Big(e(\boldsymbol{k}), g^x(\boldsymbol{k}), g^y(\boldsymbol{k}), g^z(\boldsymbol{k})\Big)$$

2. Then the equations are schematically exactly the same

$$\frac{d\phi_a(\boldsymbol{k})}{dt} = \underbrace{\mathcal{L}_{ab}(\boldsymbol{k})}_{\text{ideal} \sim c_s k} \phi_b(\boldsymbol{k}) + \underbrace{D_{ab}\phi_b}_{\text{visc} \sim -\eta_0 k^2} + \xi_a \qquad \langle \xi_a \xi_b \rangle = 2T\mathcal{D}_{ab}(\boldsymbol{k})\delta_{tt'}$$

3. Break up the equations into eigen modes of $\mathcal{L}_{ab}$, and analyze exactly same way:

$$\underbrace{\text{right moving sound}}_{\lambda_+ = +ic_s k} \qquad \underbrace{\text{left moving sound}}_{\lambda_- = -ic_s k} \qquad \underbrace{\text{two diffusion modes}}_{\lambda_T = 0}$$

So for k in the z direction, work with the following linear combos (eigenvects)

$$\phi_A \equiv \Big[\underbrace{c_s e(\boldsymbol{k}) \pm g^z(\boldsymbol{k})}_{\phi_+ \text{ and } \phi_-} \;,\; \underbrace{g^x(\boldsymbol{k})}_{\equiv \phi_{T_1}} \;,\; \underbrace{g^y(\boldsymbol{k})}_{\equiv \phi_{T_2}} \Big]$$

Sound modes in the eigen basis:

$$\frac{d\phi_+}{dt} = \underbrace{-ic_s k\,\phi_+}_{\text{rapid phase}} - \underbrace{\tfrac{2}{3}\gamma_\eta k^2\,\phi_+}_{\text{drag}} + \underbrace{\xi_L}_{\text{noise}}$$

Diffusion modes in the eigen basis:

$$\frac{d\phi_{T_1}}{dt} = \underbrace{-\gamma_\eta k^2\,\phi_{T_1}}_{\text{drag}} + \underbrace{\xi_T}_{\text{noise}}$$

The kinetic equations in flat space

1. Want to compute how the density of sound modes (squared amplitude) evolve:

$$N_{++}(\boldsymbol{k},t) = \left\langle \phi^*_+(\boldsymbol{k},t)\phi_+(\boldsymbol{k},t)\right\rangle \qquad N_{T_1T_1} = \left\langle \phi^*_{T_1}(\boldsymbol{k},t)\phi_{T_1}(\boldsymbol{k},t)\right\rangle$$

2. Thus following the Brownian example:

$$\frac{dN_{++}}{dt} = -\tfrac{4}{3}\gamma_\eta k^2 \left[N_{++} - N^{\rm eq}_{++}\right]$$

$$\frac{dN_{T_1T_1}}{dt} = -2\gamma_\eta k^2 \left[N_{T_1T_1} - N^{\rm eq}_{T_1T_1}\right]$$

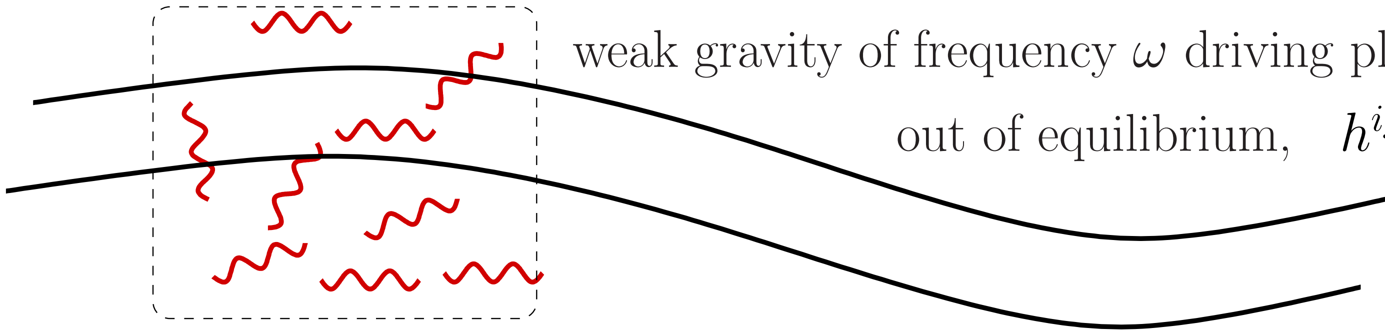
   and similar equations for N_{--} and $N_{T_2T_2}$. Here

$$N^{\rm eq}_{T_1T_1} \equiv (e+p)T \qquad \text{and} \quad N^{\rm eq}_{++} \equiv (e+p)T$$

3. Neglect off diagonal components of density matrix in eigen-basis

Now we will do the same for a perturbed and expanding system

Case 1: Kinetic equations for perturbed system (HHTLs)



Hydro equations become $\phi_a \equiv \Big(e(\boldsymbol{k}), g^x(\boldsymbol{k}), g^y(\boldsymbol{k}), g^z(\boldsymbol{k})\Big)$

$$\frac{d\phi_a(\boldsymbol{k})}{dt} = \underbrace{\mathcal{L}_{ab}(\boldsymbol{k})\phi_b(\boldsymbol{k})}_{\text{ideal}} + \underbrace{D_{ab}\phi_b}_{\text{visc}} + \underbrace{\xi_a}_{\text{noise}} + \underbrace{\mathcal{P}_{ab}\phi_b}_{\text{perturbation}}$$

with

$$\mathcal{P}_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\partial_t h_{ij} \end{pmatrix}, \qquad h_{ij}(t) = \text{metric perturbation}$$

Case 1: Kinetic equations for perturbed system (HHTLs)

1. Turn on a weak gravitational perturbations, $h_{ij} = h(t)\,\mathrm{diag}(1,1,-2)$

$$\partial_t N_{++}(k) = -\underbrace{\tfrac{4}{3}\gamma_\eta k^2 \left[N_{++} - N^{\rm eq}_{++}\right]}_{\text{damping}} + \underbrace{\partial_t h\,(\sin^2\theta_k - 2\cos^2\theta_k)}_{\text{perturbation}\quad h_{ij}\hat{k}^i\hat{k}^j} N_{++}$$

2. Solve the equations to first order in the gravitational, $h(t) = he^{-i\omega t}$

$$\delta N_{++} = \frac{i\omega h\,(\sin^2\theta_k - 2\cos^2\theta_k)}{-i\omega + \frac{4}{3}\gamma_\eta K^2} \qquad \Longleftarrow \qquad \text{solution}$$

3. Calculate the stress tensor

$$\delta T^{ij} = (e+p)\left\langle v^i v^j \right\rangle = \int \frac{d^3K}{(2\pi)^3}\frac{\left\langle g^i(\boldsymbol{k}) g^j(-\boldsymbol{k})\right\rangle}{e+p}$$

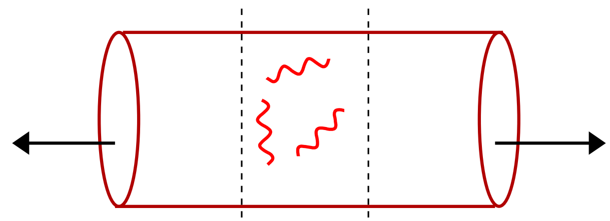
4. Find an HTL like expression

$$\langle \delta T^{xx} + \delta T^{yy} - 2\delta T^{zz}\rangle \supset h\int \frac{d^3K}{(2\pi)^3}\,\delta N_{++}\,\underbrace{(\sin^2\theta - 2\cos^2\theta)}_{\hat{k}^x\hat{k}^x + \hat{k}^y\hat{k}^y - 2\hat{k}^z\hat{k}^z}$$

Precisely reproduces Yaffe-Kovtun hard hydro loop calculation

$$\langle T^{xy}(\omega)\rangle = \Big[p - \underbrace{i\omega\eta}_{\text{1st order}} + \underbrace{\frac{(7+(3/2)^{3/2})}{240\pi}T\left(\frac{\omega}{\gamma_\eta}\right)^{3/2}}_{\text{3/2 order}} + \mathcal{O}(\omega^2)\Big]h^{xy}(\omega)$$

Case 2: Kinetic equations for a Bjorken expansion – Hard Hydro Expanding Loops (HHELs)



- The hydrodynamic field fields $\phi_a = (c_s e, g^x, g^y, \tau g^\eta)$ are:

$$\phi_a(\tau, \boldsymbol{k}_\perp, \kappa) = \int d^2\boldsymbol{x} \int d\eta\, e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp + i\kappa\,\eta}\, \phi_a(\tau, \boldsymbol{x}_\perp, \eta)$$

- The equations take the form:

$$\frac{d}{d\tau}\phi_a(\boldsymbol{k}_\perp, \kappa) = \underbrace{\mathcal{L}_{ab}}_{\text{ideal}}\,\phi_b + \underbrace{D_{ab}\phi_b}_{\text{viscous}} + \underbrace{\mathcal{P}_{ab}\phi_b}_{\text{perturb}} + \underbrace{\xi_a}_{\text{noise}}$$

The previous analysis goes through with a no complications, $\lambda = \pm i c_s k, 0$

$$\mathcal{P}_{ab} = \frac{1}{\tau}\begin{pmatrix} 1 + c_s^2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 2 \end{pmatrix}$$

The kinetic equations and approach to equilibrium:

- The kinetic equations and approach to equilibrium

$$\frac{\partial}{\partial \tau} N_{++} = -\frac{1}{\tau}\underbrace{\left[2 + c_{s0}^2 + \frac{\kappa^2/\tau^2}{k_\perp^2 + \kappa^2/\tau^2}\right]}_{\text{perturbation } = 2\mathcal{P}_{++}} N_{++} - \underbrace{\frac{\frac{4}{3}\eta_0}{s_0 T_0}\left(k_\perp^2 + \frac{\kappa^2}{\tau^2}\right)\left[N_{++} - \frac{s_0 T_0^2}{2c_{s0}^2 \tau}\right]}_{\text{damping to equilibrium}},$$

$$\frac{\partial}{\partial \tau} N_{T_2T_2} = -\frac{2}{\tau}\underbrace{\left[1 + \frac{k_\perp^2}{k_\perp^2 + \kappa^2/\tau^2}\right]}_{\text{perturbation } = 2\mathcal{P}_{T_2T_2}} N_{T_2T_2} - \underbrace{\frac{2\eta_0}{s_0 T_0}\left(k_\perp^2 + \frac{\kappa^2}{\tau^2}\right)\left[N_{T_2T_2} - \frac{s_0 T_0^2}{\tau}\right]}_{\text{damping to equilibrium}}.$$

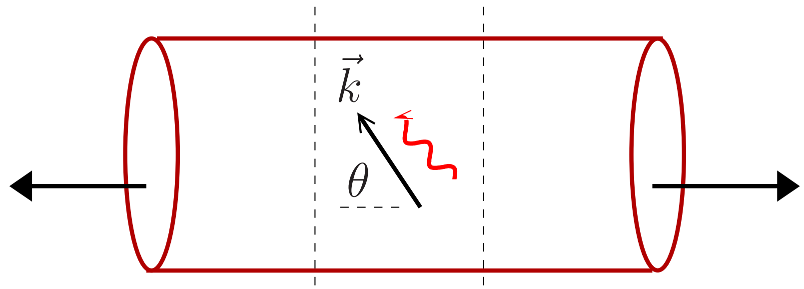
and similar equations for the other modes

- For large k, we solve, and the modes approximately equilibrate:

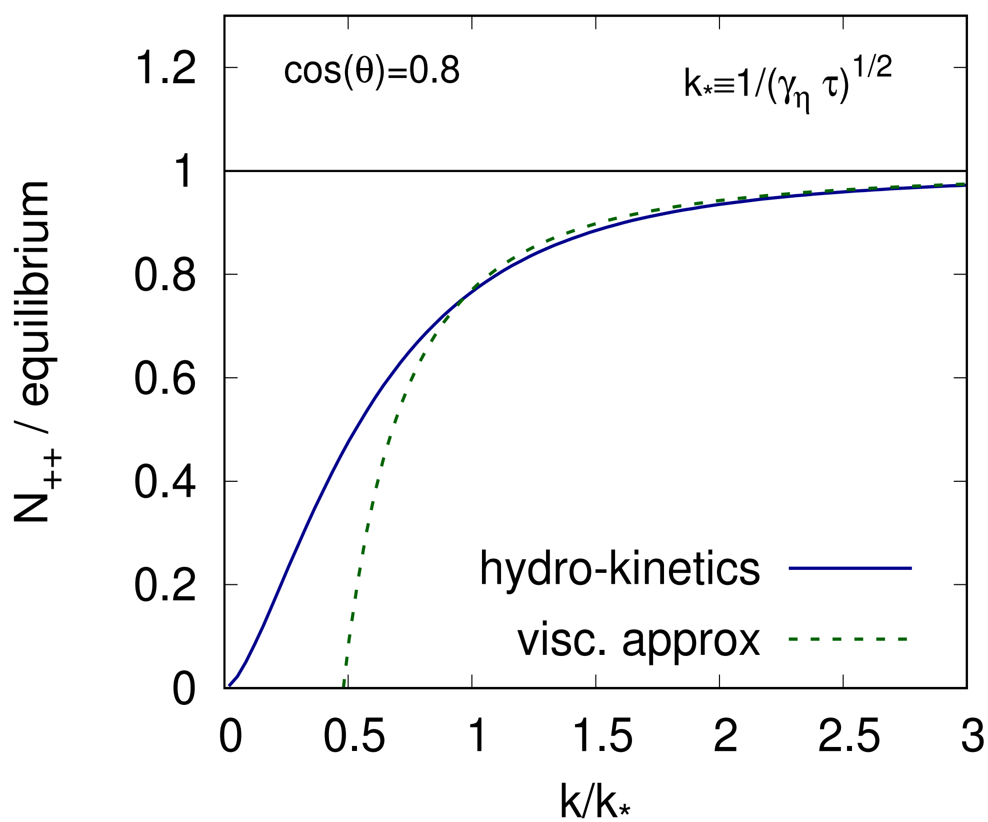
$$N_{++} \simeq \frac{s_0 T_0^2}{2c_{s0}^2\tau}\left[\underbrace{1}_{\text{equilibrium}} + \underbrace{\frac{s_0 T_0}{\frac{4}{3}\eta_0(k_\perp^2 + \kappa^2/\tau^2)}\left(c_{s0}^2 - \frac{\kappa^2/\tau^2}{k_\perp^2 + \kappa^2/\tau^2}\right)}_{\text{first viscous correction analogous to } \delta f} + \ldots\right]$$

Now we solved these kinetic equations numerically

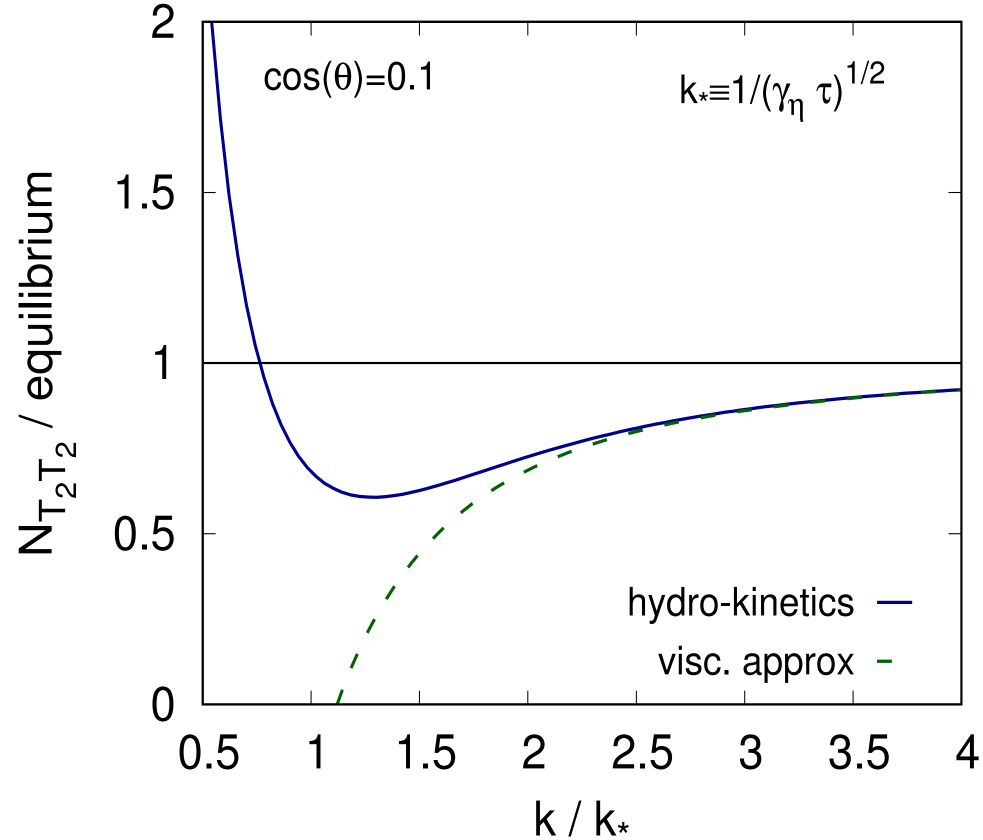
The non-equilibrium steady state at late times:



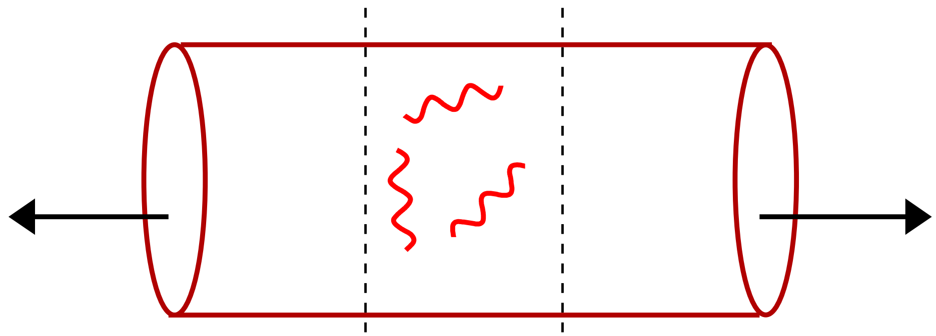
Sound Modes



Transverse modes



The evolution of the background



$$\frac{dT^{\tau\tau}}{d\tau} = -\frac{T^{\tau\tau} + T^{zz}}{\tau}$$

where

$$T^{zz}_{\rm hydro} = \underbrace{p_0(\Lambda)}_{\text{Ideal}} - \underbrace{\frac{\frac{4}{3}\eta_0(\Lambda)}{\tau}}_{\text{first order}} + \underbrace{(\lambda_1 - \eta\tau_\pi)\frac{8}{9\tau^2}}_{\text{second order}} + \ldots$$

In addition the fluctuations give another contribution:

$$T^{zz}_{\rm flucts} = (e+p)\left\langle v^z v^z \right\rangle$$

Evaluating the fluctuation contribution:

$$\frac{T^{zz}_{\rm flucts}}{e+p} = \langle v^z v^z \rangle$$

$$= \int \frac{d^2k_\perp d\kappa}{(2\pi)^3} \frac{1}{(e_o+p_o)^2} \left[N_{++}\cos^2\theta + N_{T_2T_2}\sin^2\theta\right]$$

$$= \underbrace{\frac{T_o\Lambda^3}{6\pi^2}}_{\text{from equilib}} - \underbrace{\left(\frac{17\Lambda}{120\pi^2}\frac{s_oT_o^2}{\eta_o(\Lambda)}\right)\frac{\frac{4}{3}}{\tau}}_{\text{from first viscous correction}} + \text{finite}$$

Thus the full stress is then:

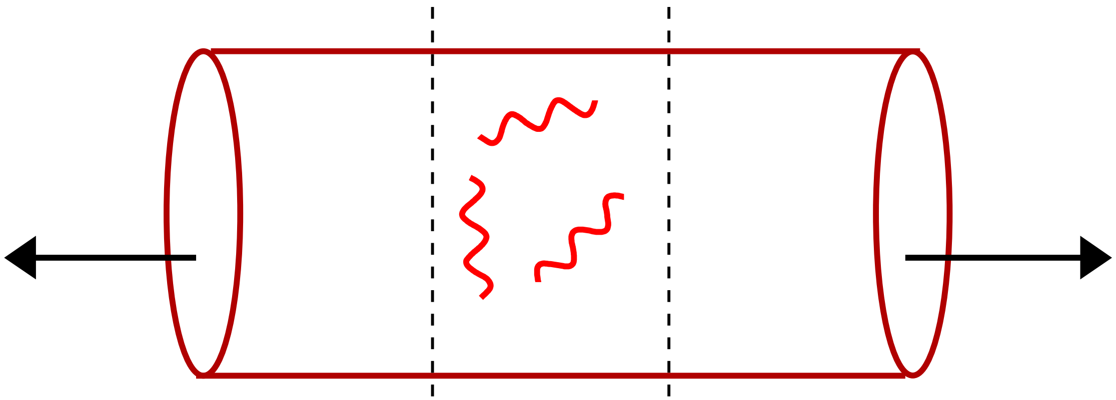
compare Kovtun, Moore, Romatschke

$$T^{zz} = T^{zz}_{\rm hydro} + T^{zz}_{\rm flucts}$$

$$= \underbrace{\left[p_0(\Lambda) + \frac{T_o\Lambda^3}{6\pi^2}\right]}_{p_{\rm phys}} - \frac{4}{3\tau}\underbrace{\left[\eta_0(\Lambda) + \left(\frac{17\Lambda}{120\pi^2}\frac{s_oT_o^2}{\eta_o(\Lambda)}\right)\right]}_{\eta_{\rm phys}} + \text{finite}$$

where the physical quantities, $p_{\rm phys}$ and $\eta_{\rm phys}$, are independent of Λ

Final result for a Bjorken expansion:



$$\frac{dT^{\tau\tau}}{d\tau} = -\frac{T^{\tau\tau} + T^{zz}}{\tau}$$

$$\frac{\langle T^{zz} \rangle}{e+p} = \Big[\underbrace{\frac{p}{e+p}}_{\sim 1} - \underbrace{\frac{4}{3}\frac{\gamma_\eta}{\tau}}_{\text{1st order}} + \underbrace{\frac{1.08318}{s\,(4\pi\gamma_\eta\tau)^{3/2}}}_{\text{3/2 order}} + \underbrace{\frac{(\lambda_1 - \eta\tau_\pi)}{e+p}\frac{8}{9\tau^2}}_{\text{2nd order}} \Big]$$

From which much can be wrought or wrung . . .

Numerical results:

Take representative numbers

$$\frac{(\lambda_1 - \eta\tau\pi)}{e+p} \simeq -0.8\left(\frac{\eta}{e+p}\right)^2 \qquad \frac{T^3}{s} \simeq \frac{1}{13.5}$$

For $\eta/s = 1/4\pi$ find:

$$\frac{T^{zz}}{e+p} = \frac{1}{4}\left[1. - \underbrace{0.092}_{\text{first}}\left(\frac{4.5}{\tau T}\right) + \underbrace{0.034}_{\text{3/2 order} \simeq 30\%}\left(\frac{4.5}{\tau T}\right)^{3/2} - \underbrace{0.0009}_{\text{second}}\left(\frac{4.5}{\tau T}\right)^2\right]$$

while for $\eta/s = 2/4\pi$ we have:

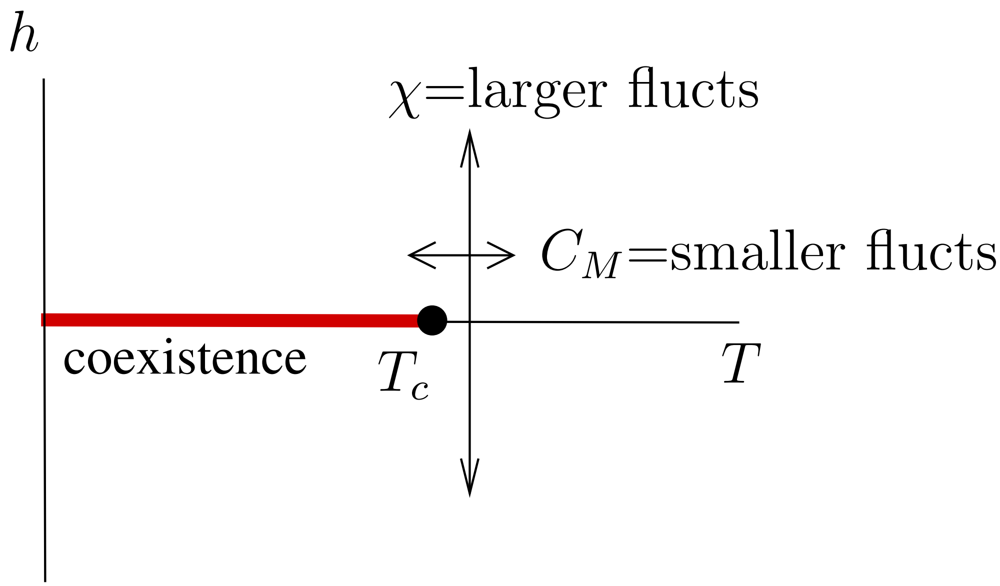
$$\frac{T^{zz}}{e+p} = \frac{1}{4}\left[1. - \underbrace{0.185}_{\text{first}}\left(\frac{4.5}{\tau T}\right) + \underbrace{0.013}_{\text{3/2 order} \simeq 10\%}\left(\frac{4.5}{\tau T}\right)^{3/2} - \underbrace{0.0034}_{\text{second}}\left(\frac{4.5}{\tau T}\right)^2\right]$$

Fluctuation contribution is a correction to first order hydro

but larger than second order in practice!

Hydrodynamic fluctuations and the critical point

Ising Model Fluctuations:



- Thermodynamic variables and their equilibrium fluctuations

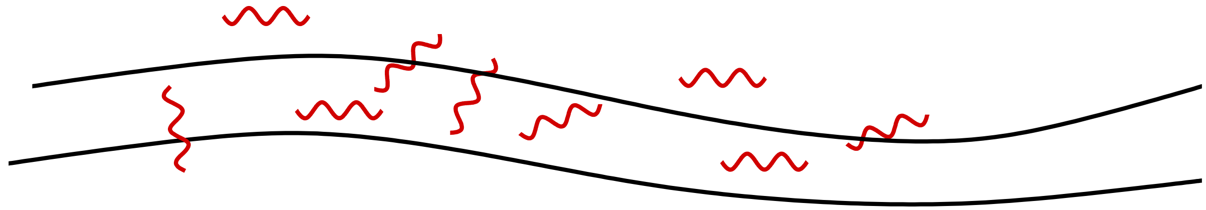
$$x^A \equiv \underbrace{(\mathcal{M}, \delta e_{\rm is})}_{\text{magnetization and energy density}} \qquad \mathcal{X}^{AB}_{\rm is} = \underbrace{\langle \delta x^A \delta x^B \rangle}_{\text{fluctuations}}$$

- Largest and smallest fluctuations, $\det \mathcal{X}_{\rm is} = \chi C_M$

$$\chi \equiv \mathcal{X}^{11}_{\rm is} = \frac{\partial \overline{\mathcal{M}}}{\partial h} = \text{largest fluctuations, } \delta T_{\rm is}^{-\gamma}$$

$$C_M \equiv \mathcal{X}^{22}_{\rm is} - \frac{(\mathcal{X}^{12}_{\rm is})^2}{\mathcal{X}^{11}_{\rm is}} = \text{smallest fluctuations, } \delta T_{\rm is}^{-\alpha}$$

QCD hydrodynamic fluctuations:



1. Thermodynamic variables and their conjugates

$$x^a = \underbrace{(e(\boldsymbol{k}), n(\boldsymbol{k})),\ g^i(\boldsymbol{k})}_{\text{energy, density, momentum}} \qquad \delta X_a(\boldsymbol{k}) = -\frac{\partial S}{\partial x^a} = \underbrace{(-\beta, \hat{\mu}),\ \beta u^i}_{\text{conjugates}}$$

2. We will study

$$\mathcal{X}^{ab}(k,t) = \left\langle x^a(k) x^b(-k) \right\rangle \Big|_{\text{equilibrium}}$$

3. Also study pressure fluctuations:

$$\delta p = p^a \delta X_a \qquad (p^e, p^n) = (T(e+p),\ Tn)$$

which determine the speed of sound

$$\left\langle (\delta p)^2 \right\rangle = T(e+p)c_s^2 = \underbrace{p^a \mathcal{X}^{-1}_{ab} p^b}_{\text{"nice little formula"}}$$

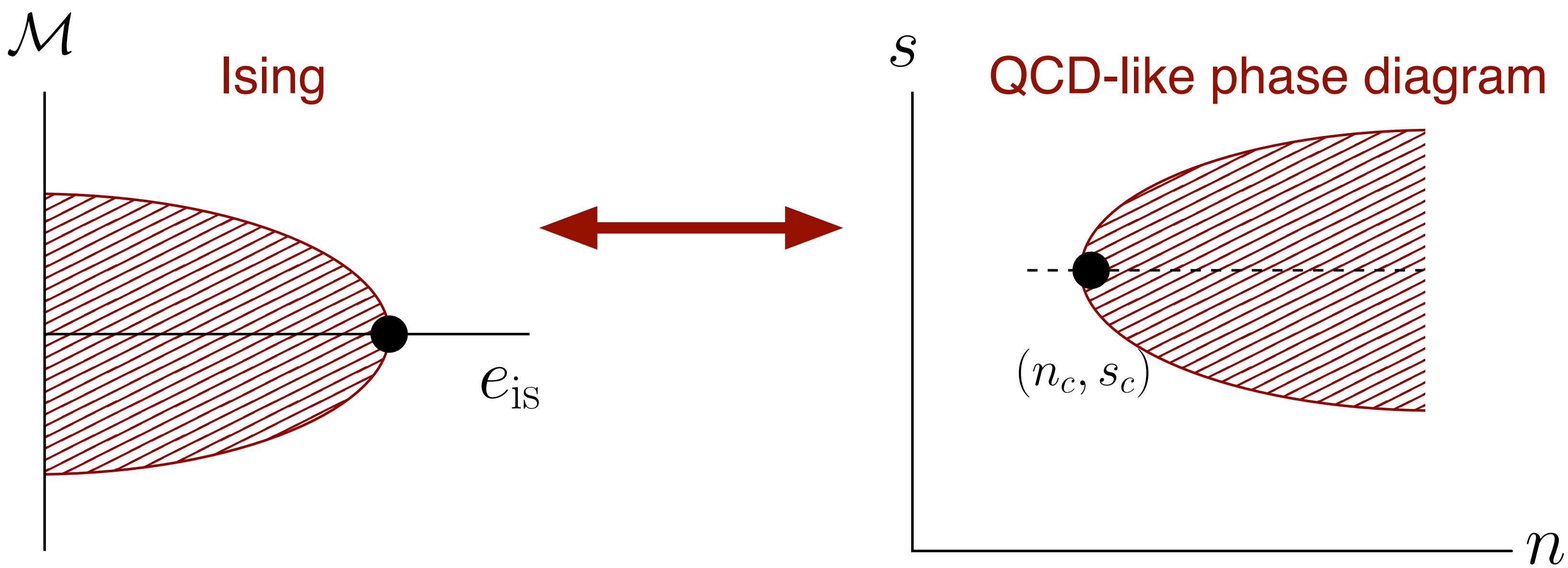
From QCD to Ising and back

Onuki phase transition dynamics

Assume a linear relation between reduced parameters, e.g. $\left(\frac{\delta T_{\rm is}}{T_{\rm isc}}, h\right) \Leftrightarrow \left(\frac{\delta\mu}{\mu_c}, \frac{\delta T}{T_c}\right)$

$$\underbrace{\delta x^A}_{\text{Ising fields}} = M^A{}_b \underbrace{\delta x^b}_{\text{QCD fields}}$$

Thermodynamic conjugates obey the inverse linear map, $X_{\rm is} = M^{-1} X_{\rm QCD}$



We will take the simplest mapping:

$$\underbrace{\delta\mathcal{M}}_{\text{magnetization}} = M^h_e \underbrace{\frac{\delta s}{s_c}}_{\text{entropy}} \quad \text{and} \quad \underbrace{\delta e_{\rm is}}_{\text{ising-edense}} = M^\tau_n \underbrace{\frac{\delta n}{n_c}}_{\text{b-density}}$$

Hydrodynamic Fluctuations and Dynamics

Linearized equations of motion for e, n, g

$$\frac{dx^a(\boldsymbol{k})}{dt} = \underbrace{\mathcal{L}^{ab}(\boldsymbol{k})X_b(\boldsymbol{k})}_{\text{ideal}} + \underbrace{\Lambda^{ab}X_b}_{\text{viscosity+conductivity}} + \underbrace{\xi_a}_{\text{noise}}$$

1. Two sound modes with eigenvalues $\pm c_s k$
2. One diffusive (zero) mode for the *entropy per baryon* fluctuations

$$\delta\sigma \equiv \delta e - \frac{e+p}{n}\,\delta n = Tn\,\delta\left(\frac{s}{n}\right)$$

   which satisfies $\langle \delta p \delta\sigma \rangle = 0$.

3. Fluctuations of σ obey a relaxation type equation, $N^{\sigma\sigma} = \langle \delta\sigma(\boldsymbol{k},t)\delta\sigma(-\boldsymbol{k},t)\rangle$

$$\frac{dN^{\sigma\sigma}}{dt} = -\underbrace{\frac{2T(e+p)\lambda k^2}{\mathcal{X}^{\sigma\sigma}}}_{\text{relaxation controlled by } \lambda \,\equiv \text{conductivity}} \left[N^{\sigma\sigma} - \mathcal{X}^{\sigma\sigma}\right] ,$$

   where $\mathcal{X}^{\sigma\sigma} = c_s^2 \det \mathcal{X}^{ab}$ is the static susceptibility for $\delta\sigma$.

Mapping entropy/baryon fluctuations onto the ising

(Onuki - Phase transition dynamics)

1. The speed of sound approaches zero like the smallest ising susceptibility

$$T(e+p)c_s^2 = p^a \mathcal{X}^{-1}_{ab} p^b$$

$$\text{sound} = p^A \mathcal{X}^{-1}_{AB} p^B \simeq \underbrace{\left(\frac{dp}{d\tau}\right)^2 \frac{1}{C_M}}_{\text{the smallest susceptibility}}$$

2. The susceptibility matrix also transforms $\det \mathcal{X} = (\det M)^2 \det \mathcal{X}_{\rm is} \propto \chi C_M$

3. The fluctuation in σ diverge as the largest susceptibility

$$\mathcal{X}^{\sigma\sigma} = c_s^2 \det \mathcal{X} \propto \underbrace{\chi}_{\text{largest ising susceptibility}}$$

The fluctuations in the entropy per baryon diverge maximally like χ

(independently of how the mapping to the ising variables is done!)

Summary of equation for fluctuations in the specific entropy $\sigma \equiv n\delta(s/n)$

$$\frac{d\bar{N}^{\sigma\sigma}(k,t)}{dt} = -\frac{2\lambda_{\rm eff} k^2}{\chi(k)}\left[\bar{N}^{\sigma\sigma} - \chi(k)\right]$$

1. Definitions:

$$\bar{N}^{\sigma\sigma} = \underbrace{N^{\sigma\sigma}}_{\text{flucts of } \sigma} \quad \underbrace{(M_e^h)^2}_{\text{mapping params}}$$

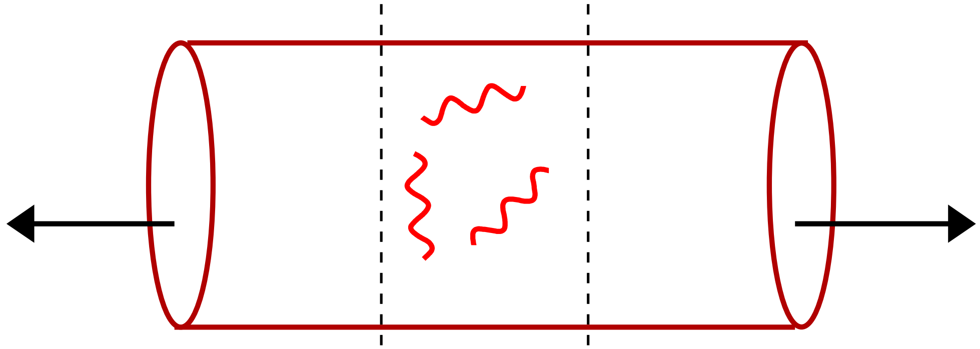
$$\lambda_{\rm eff} = \underbrace{\lambda}_{\text{conductivity}} \left(\frac{e+p}{nT}\right)^2 \underbrace{(M_e^h)^2}_{\text{mapping params}}$$

2. Model susceptibility near critical point as a function of k with correlation length ξ

$$\bar{N}^{\sigma\sigma}(\boldsymbol{k},t)\Big|_{\rm equil} = \chi(k) = \underbrace{\frac{\chi_o(\xi/\ell_o)^{2-\eta}}{1+(k\xi)^{2-\eta}}}_{\text{susceptibility } \chi(\boldsymbol{k})}$$

We will solve this equation to monitor the equilibration of various wavenumbers

Transiting the critical point



1. Pass right through the critical point at late time $\tau = \tau_Q$, define $t \equiv \tau - \tau_Q$:

$$\partial_\tau \overline{n} = -\frac{n_c}{\tau_Q} \qquad \Longrightarrow \qquad \frac{\delta \overline{n}}{n_c} = -\frac{t}{\tau_Q}$$

$$\partial_\tau \overline{s} = -\frac{s_c}{\tau_Q} \qquad \qquad \frac{\delta \overline{s}}{s_c} = -\frac{t}{\tau_Q} + \underbrace{\Delta}_{\text{set to zero}}$$

Set $\Delta = 0$ to go directly through the critical point.

2. The (ising) reduced $T_{\rm is}$ and correlation length behaves $a\nu \equiv \nu/(1-\alpha) \simeq 0.71$

$$\delta T_{\rm is} \propto \left(\frac{|t|}{\tau_Q}\right)^{1-\alpha} \qquad \text{and} \qquad \xi = \ell_o \left(\frac{\tau_Q}{|t|}\right)^{a\nu}$$

Dynamical critical exponents

Son and Stephanov most helpful

1. The fluctuations of $\delta\sigma \equiv n\delta(s/n)$ satisfy:

$$\partial_t \bar{N}^{\sigma\sigma} = -\frac{2\lambda_{\rm eff} k^2}{\chi(k)} \left[\bar{N}^{\sigma\sigma} - \chi(k)\right]$$

$$= -\frac{2\lambda_{\rm eff}}{\chi_o \ell_o^2 (\xi/\ell_o)^{4-\eta}} (k\xi)^2 (1 + (k\xi)^{2-\eta}) \left[\bar{N}^{\sigma\sigma} - \chi(k)\right]$$

2. Then the equilibration time for $k\xi \sim 1$:

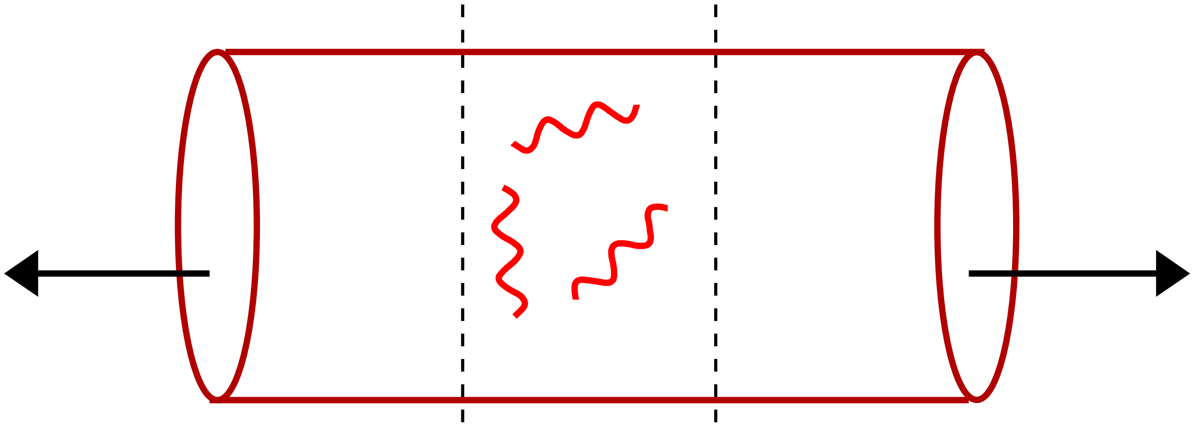
$$\underbrace{\tau_{\rm eq}(\xi) \equiv \tau_o \left(\frac{\xi}{\ell_o}\right)^z}_{\text{equilibration time}} \quad \text{with} \quad \underbrace{z \equiv 4 - \eta}_{\text{dynamic critical exponent}} \quad \text{and} \quad \underbrace{\tau_o \equiv \frac{\chi_o \ell_o^2}{\lambda_{\rm eff}}}_{\text{micro relax-time}}$$

The equation to be solved is :

$$\partial_t \bar{N}^{\sigma\sigma}(\boldsymbol{k}, t) = -\frac{2(k\xi)^2 (1 + (k\xi)^{2-\eta})}{\tau_{\rm eq}(\xi)} \left[\bar{N}^{\sigma\sigma}(\boldsymbol{k}, t) - \chi(\boldsymbol{k}, t)\right]$$

Kibble-Zurek Scaling of Equation

Kible,Zurek; Berdnikov, Rajagopal; Mukherjee, Venugopalan, Yin



$$\xi(t) = \ell_o \left(\frac{\tau_Q}{|t|}\right)^{a\nu}$$

$$\tau_{\rm eq}(\xi) = \tau_o \left(\frac{\xi(t)}{\ell_o}\right)^{z}$$

1. There is a timescale, $t = t_{\rm kz}$, where the relaxation rate can't keep up with $\xi(t)$

$$\underbrace{\frac{1}{\tau_{\rm eq}(\xi(t_{\rm kz}))}}_{\text{relaxation rate}} = \underbrace{\frac{\partial_t \xi(t_{\rm kz})}{\xi(t_{\rm kz})} = \frac{a\nu}{t_{\rm kz}}}_{\text{rate-of change of } \xi(t)}$$

2. Find a Kibble-Zurek time scale, $t_{\rm kz}$, and length, $\ell_{\rm kz} = \xi(t_{\rm kz})$

$$t_{\rm kz} \equiv \tau_o \left(\frac{\tau_Q}{\tau_o}\right)^{a\nu z/(a\nu z+1)} \simeq \tau_o \left(\frac{\tau_Q}{\tau_o}\right)^{0.74} \gg \tau_o$$

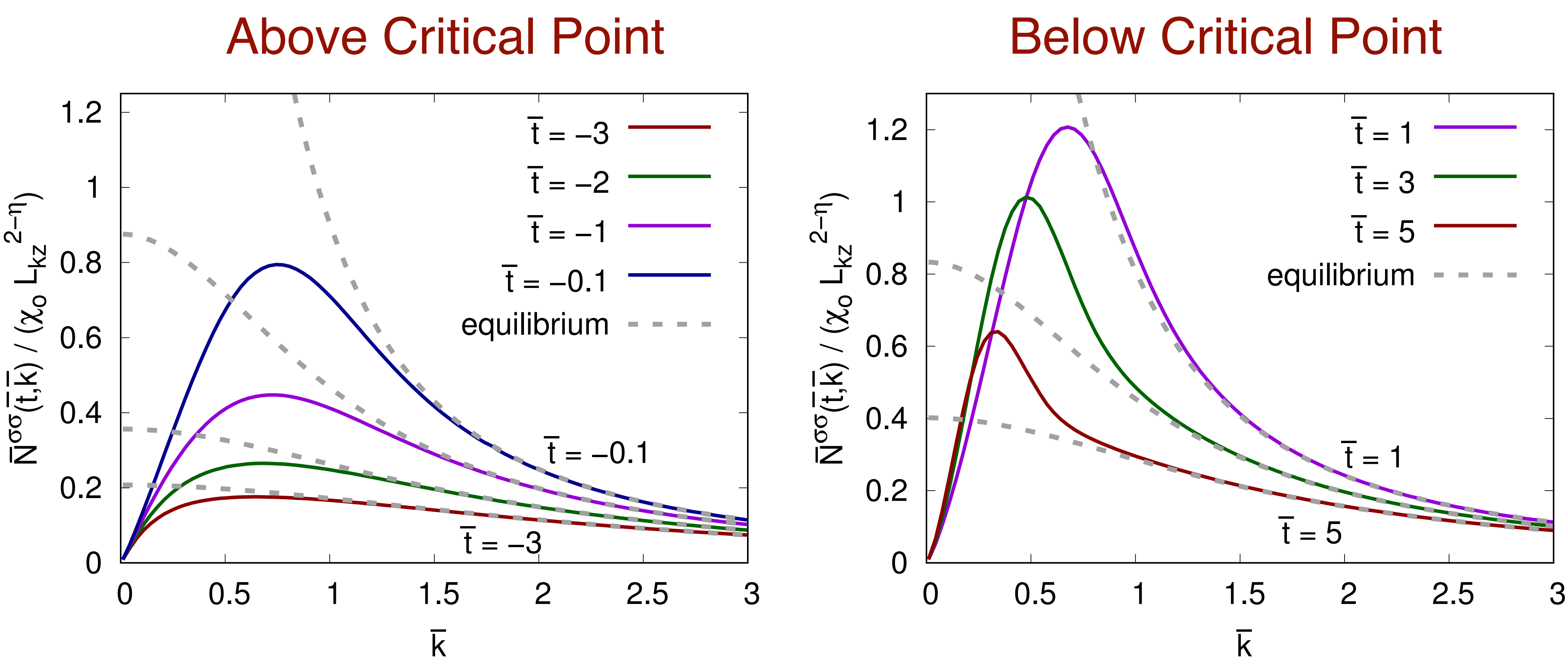
$$\ell_{\rm kz} = \ell_o \left(\frac{\tau_Q}{\tau_o}\right)^{a\nu/(a\nu z+1)} \simeq \ell_o \left(\frac{\tau_Q}{\tau_o}\right)^{0.19} \gg \ell_o$$

Kibble-Zurek rescaled equation:

1. Measure all lengths, wavenumbers, and times in terms of $\ell_{\rm kz}$ and $t_{\rm kz}$

$$\bar{t} = \frac{t}{t_{\rm kz}} \quad \text{and} \quad \bar{k} = k\ell_{\rm kz} \quad \text{and} \quad \bar{\xi} = \frac{\xi}{\ell_{\rm kz}}$$

2. Also rescale the correlator, $\bar{N}^{\sigma\sigma} \to \bar{N}^{\sigma\sigma}/\chi_o \ell_{\rm kz}^{2-\eta}$, motivated by equilibrium:



Summary of Scales

1. The small parameter is the ratio of microscopic length to system size:

$$\epsilon = \frac{\tau_o}{\tau_Q} = \frac{\text{micro scale}}{\text{macro scale}} \simeq \frac{1}{7}$$

2. Hierarchy of scales:

$$\underbrace{k_{\rm hydro}}_{\sim v_2} \ll \underbrace{k_*}_{\text{hyd-kinetics}} \ll \underbrace{k_{\rm kz}}_{\text{longest critical fluct}} \ll \underbrace{\frac{1}{\ell_o}}_{\text{microlength}}$$

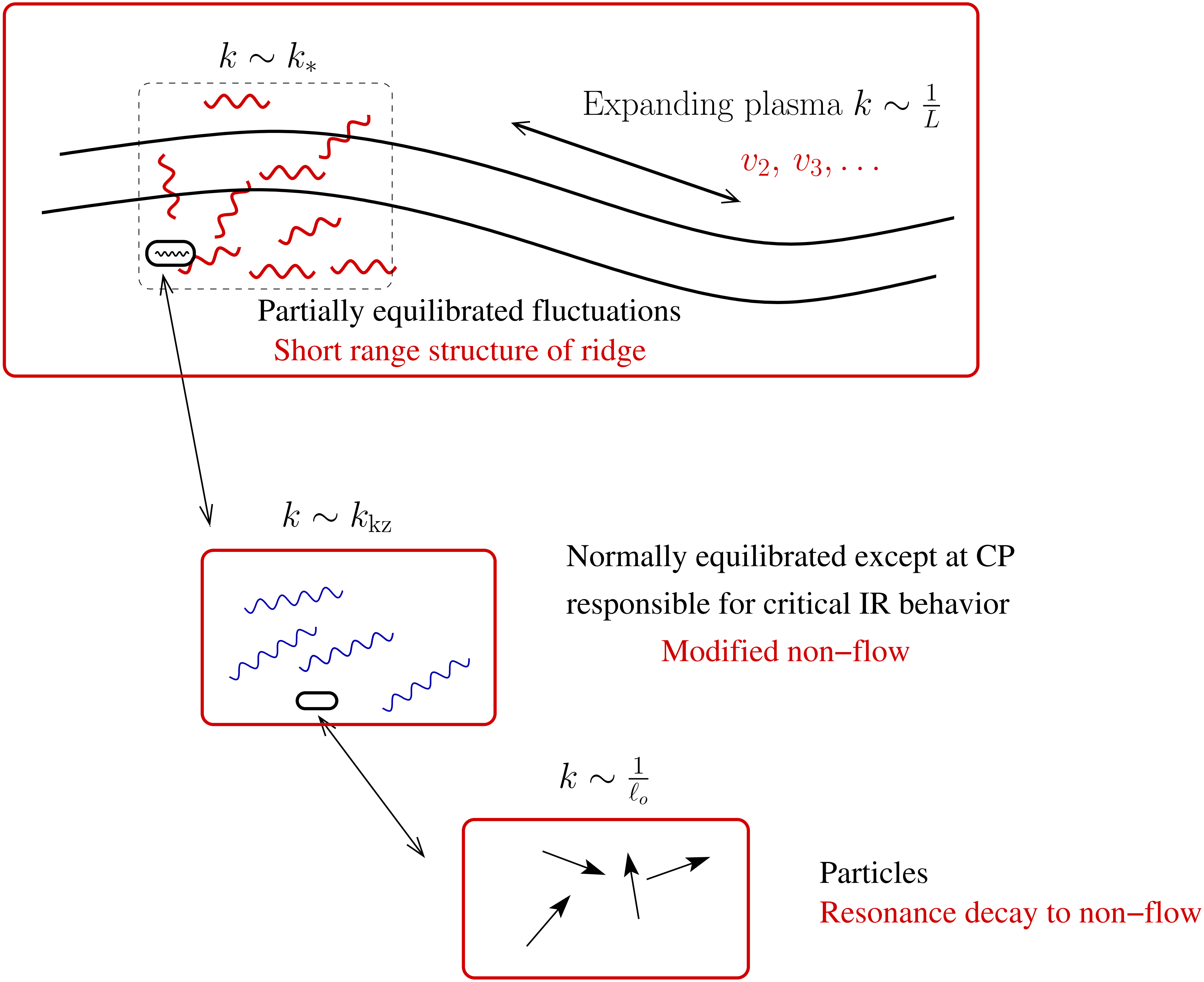
which are of relative order

$$\epsilon \ll \sqrt{\epsilon} \ll \epsilon^{0.18} \ll 1 \qquad \text{or} \qquad 0.14 \ll 0.38 \ll 0.70 \ll 1$$

3. The duration of the KZ regime is short compared to τ_Q (parametrically only)

$$\tau_o \ll t_{\rm kz} \ll \tau_Q \qquad \text{or} \qquad \epsilon \ll \underbrace{\epsilon^{0.26}}_{\sim 0.6} \ll 1$$

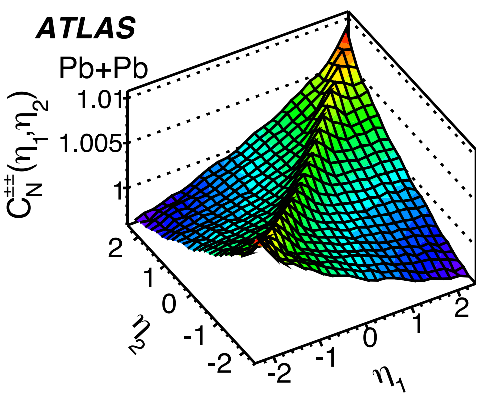
May not have a clear separation of scales in practice


$k \sim k_*$
Expanding plasma $k \sim \frac{1}{L}$
$v_2,\ v_3, \ldots$
Partially equilibrated fluctuations
Short range structure of ridge
$k \sim k_{\rm kz}$
Normally equilibrated except at CP
responsible for critical IR behavior
Modified non–flow
$k \sim \frac{1}{\ell_o}$
Particles
Resonance decay to non–flow

Real correlation functions at high energies

$$C(\eta_1, \eta_2) = \frac{\left\langle \frac{dN}{d\eta_1} \frac{dN}{d\eta_2} \right\rangle}{\left\langle \frac{dN}{d\eta_1} \right\rangle \left\langle \frac{dN}{d\eta_2} \right\rangle}$$

Correlation function = Short Range = "Non-flow" + Long range rapidity flucts



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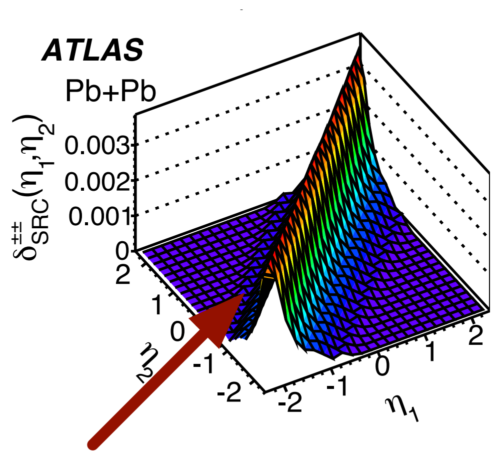

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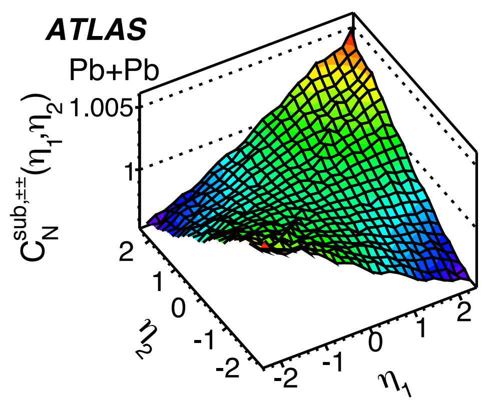
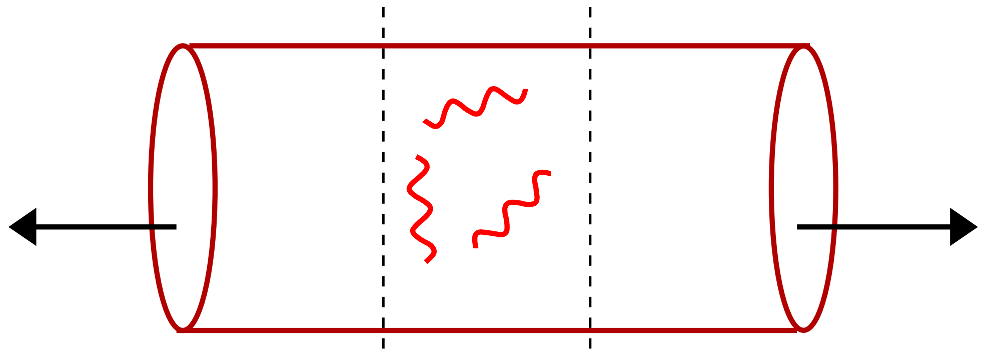


Find the CP in here
at lower energy

Transiting close to the critical point



1. Pass close to the critical point at late time $\tau = \tau_Q$, define $t \equiv \tau - \tau_Q$:

$$\partial_\tau \overline{n} = -\frac{n_c}{\tau_Q} \qquad \Longrightarrow \qquad \frac{\delta \overline{n}}{n_c} = -\frac{t}{\tau_Q}$$

$$\partial_\tau \overline{s} = -\frac{s_c}{\tau_Q} \qquad \qquad \frac{\delta \overline{s}}{s_c} = -\frac{t}{\tau_Q} + \underbrace{\Delta}_{\text{small}}$$

2. The "detuning" Δ acts like a magnetic field regulating critical dynamics

$$\Delta = \underbrace{\frac{n_c}{s_c}\delta\left(\overline{s}/\overline{n}\right)}_{\text{a small detuning}}$$

The detuning limits the rate of change of critical fluctuations

Time-scale for the maximal equilibrium fluctuations:

see Berdnikov,Rajagopal hep-ph/9912274

1. The correlation length is a function of the scaling variable, $\xi = \bar{h}^{-\nu/\beta\delta} f(z)$

$$\underbrace{z}_{\text{scaling-var}} = \underbrace{\bar{\tau}}_{\text{reduced } T_{\text{is}}} \times \underbrace{\bar{h}^{-1/\beta\delta}}_{(\text{reduced field})^{-1/\beta\delta}}$$

2. The correlation length is maximal for $z \sim 1$. With

$$\frac{\delta n}{n_c} \sim -\frac{t_{\text{cross}}}{\tau_Q} \quad \text{and} \quad \frac{\delta s}{s_c} \sim \Delta - \frac{t_{\text{cross}}}{\tau_Q}$$

we find the timescale for the maximal correlation length

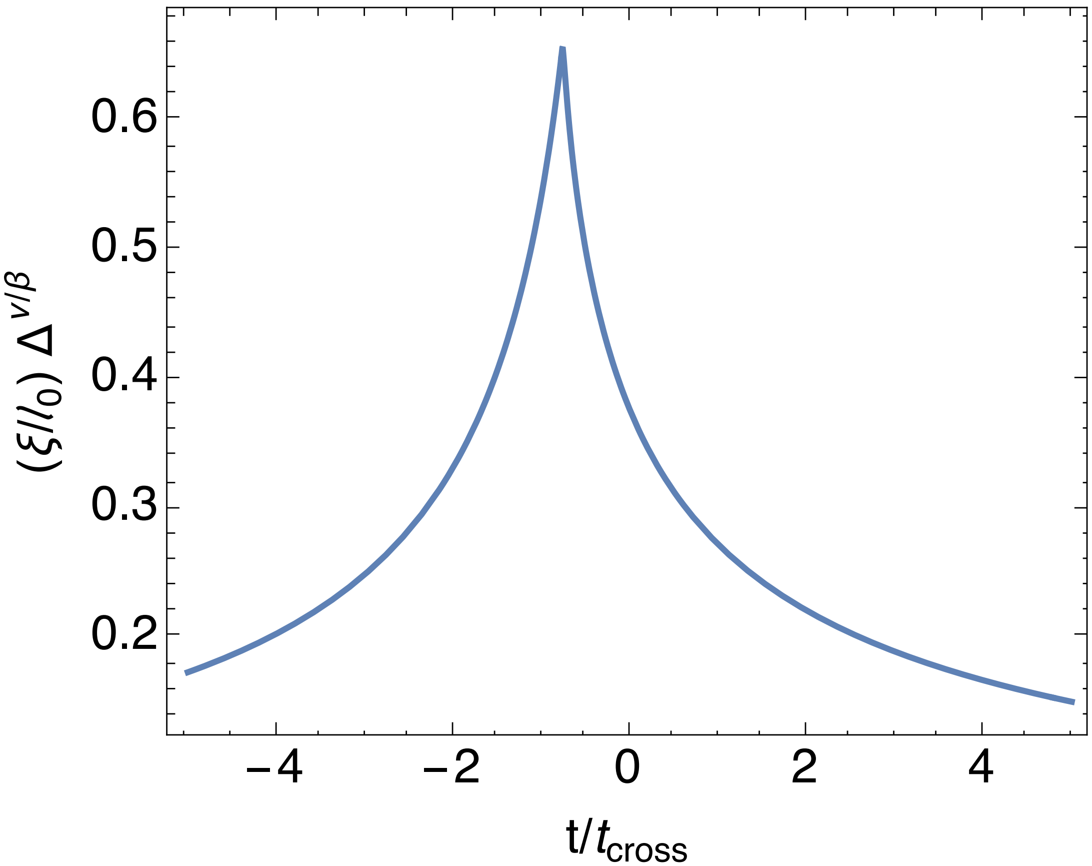
$$t_{\text{cross}} \sim \underbrace{\tau_Q}_{\text{only timescale}} \times \underbrace{\Delta^{(1-\alpha)/\beta}}_{\text{only dimensionless number}}$$

For $t \sim t_{\text{cross}}$ the correlation length is regulated by the detuning Δ

The correlation length:

numerical data Engels,Fromme,Seniuch, cond-mat/0209492

$$t_{\rm cross} \propto \tau_Q \Delta^{(1-\alpha)/\beta}$$



If the system is sufficiently detuned (i.e. $t_{\rm cross} \gg t_{kz}$) we remain in equilibrium

Comparing the Kibble-Zurek and crossing time-scales

1. We will remain in equilibrium for

$$t_{\rm cross} \gg t_{\rm kz}$$

2. Find that

$$\Delta \gg \left(\frac{\tau_o}{\tau_Q}\right)^{\beta/(\nu z+1-\alpha)}$$

or

$$\Delta \gg \underbrace{\left(\frac{\tau_o}{\tau_Q}\right)^{0.096}}_{\text{A very small power}}$$

The Kibble-Zurek mechanism is probably the dominant regulator of critical dynamics since the power 0.096 is small.

Summary

1. For wavenumbers of order

$$k \sim k_* \equiv \sqrt{\frac{e+p}{\eta\tau}}$$

   the system transitions to equilibrium away from the critical point

2. Worked out an alternate description of hydro with noise:

   - Hydro + hydro-kinetics

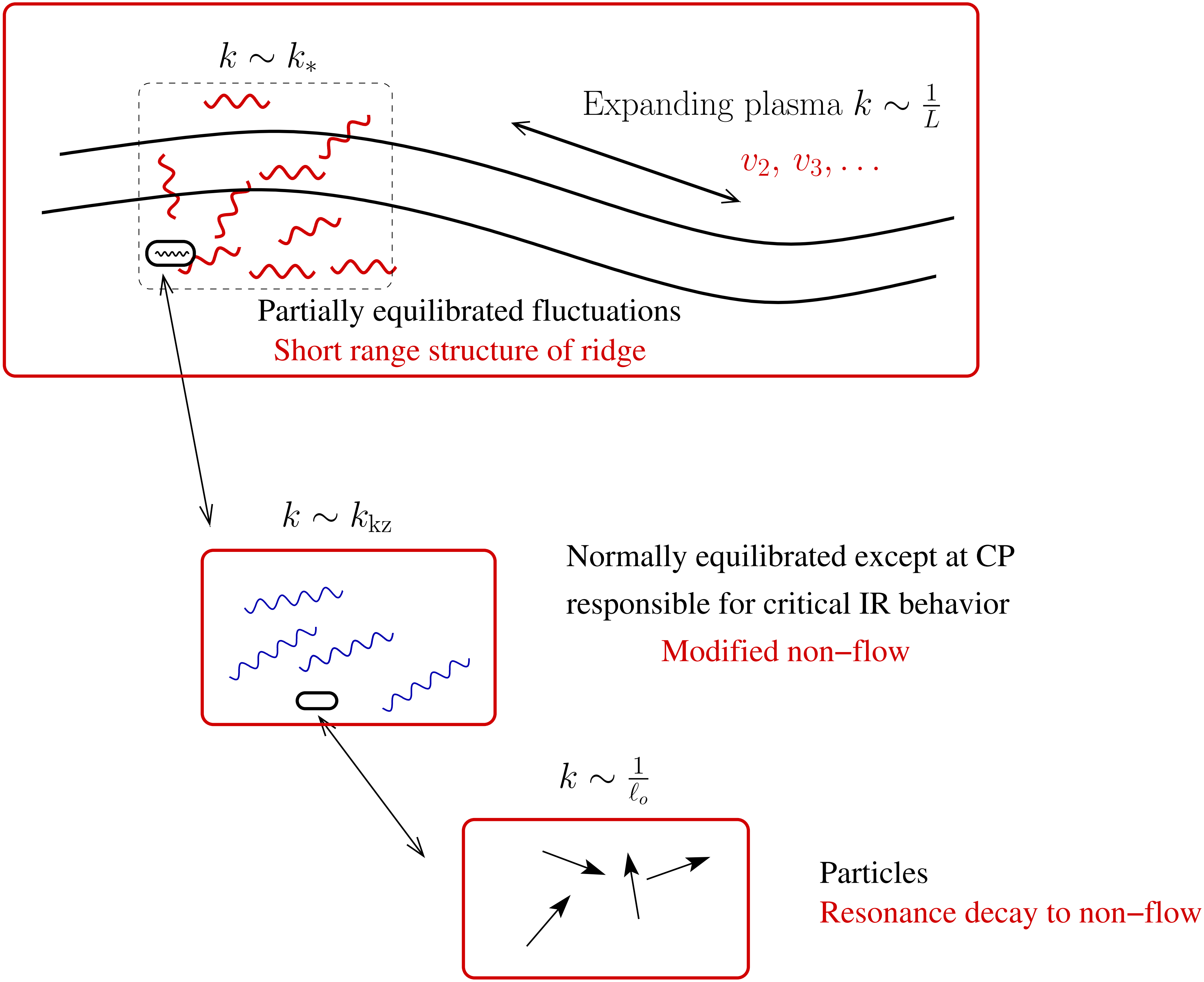
$$\partial_\mu (T^{\mu\nu}_{\rm hydro} + T^{\mu\nu}_{\rm flucts}) = 0$$
$$\partial_\tau N_{\rm flucts}(\boldsymbol{k},\tau) = \ldots$$

   This should be generalized to a general flows.

3. Fluctuating hydro is much more important than second order hydro in practice!

4. Clarified where critical fluctuations are relevant

$$\underbrace{k_{\rm hydro}}_{\sim v_2} \ll \underbrace{k_*}_{\text{hyd-kinetics}} \ll \underbrace{k_{\rm kz}}_{\text{longest critical fluct}} \ll \underbrace{\frac{1}{\ell_o}}_{\text{microlength}}$$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